

Chapter 1. Combinatorial Theory

1.1: So You Think You Can Count?

(From “Probability & Statistics with Applications to Computing” by Alex Tsun)

Before we jump into probability, we must first learn a little bit of combinatorics, or more informally, counting. You might wonder how this is relevant to probability, and we’ll see how very soon. You might also think that counting is for kindergarteners, but it is actually a lot harder than you think!

To motivate us, let’s consider how easy or difficult it is for a robber to randomly guess your PIN code. Every debit card has a PIN code that their owners use to withdraw cash from ATMs or to complete transactions. How secure are these PINs, and how safe can we feel?

1.1.1 Sum Rule

First though, we will count baby outfits. Let’s say that a baby outfit consists of *either* a top *or* a bottom (but not both), and we have 3 tops and 4 bottoms. How many baby outfits are possible? We simply add $3 + 4 = 7$, and we have found the answer using the sum rule.

Theorem 1.1.1: Sum Rule

If an experiment can either end up being one of N outcomes, or one of M outcomes (where there is no overlap), then the number of possible outcomes of the experiment is:

$$N + M$$

More formally, if A and B are sets with no overlap ($A \cap B = \emptyset$), then $|A \cup B| = |A| + |B|$.

Example(s)

Suppose you must take a natural science class this year to graduate at any of the three UW campuses: Seattle, Bothell, and Tacoma. Seattle offers 4 different courses, Bothell offers 7, and Tacoma only 2. How many choices of class do you have in total?

Solution By the sum rule, it is simply $4 + 7 + 2 = 13$ different courses (since there is no overlap)! □

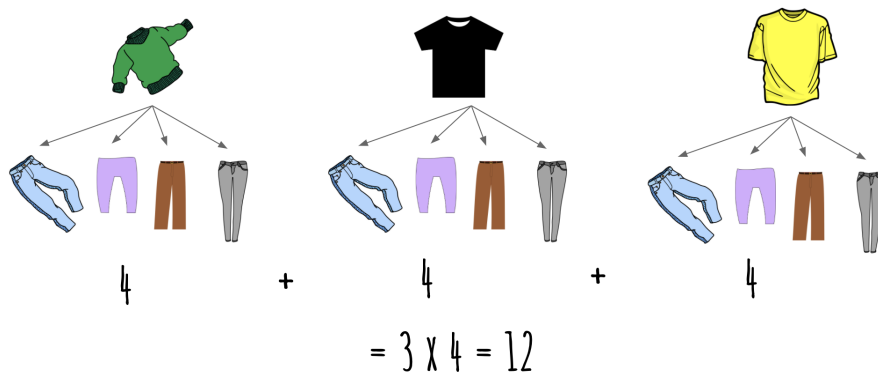
We’ll see some examples of the Sum Rule combined with the Product Rule (next), so that they can be a bit more complex!

1.1.2 Product Rule

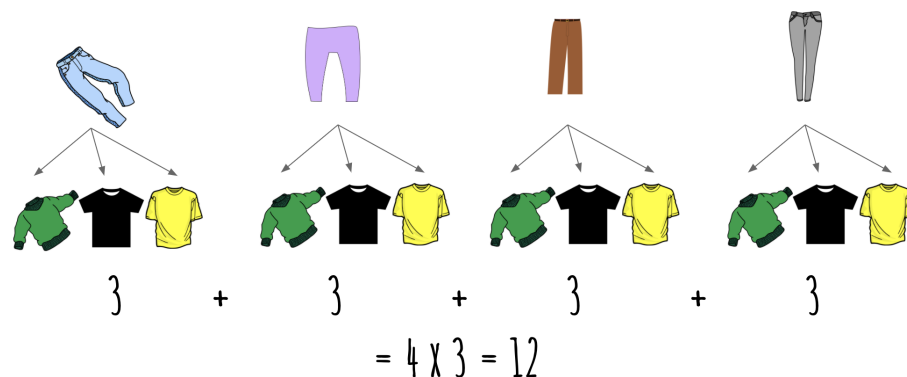
Now we will count real outfits. Let’s say that a real outfit consists of *both* a top *and* a bottom, and again, we still have 3 tops and 4 bottoms. then how many outfits are possible?

Well, we can consider this from first picking out a top. Once we have our top, we have 4 choices for our bottom. This means we have 4 choices of bottom for each top, which we have 3 of. So, we have a total of

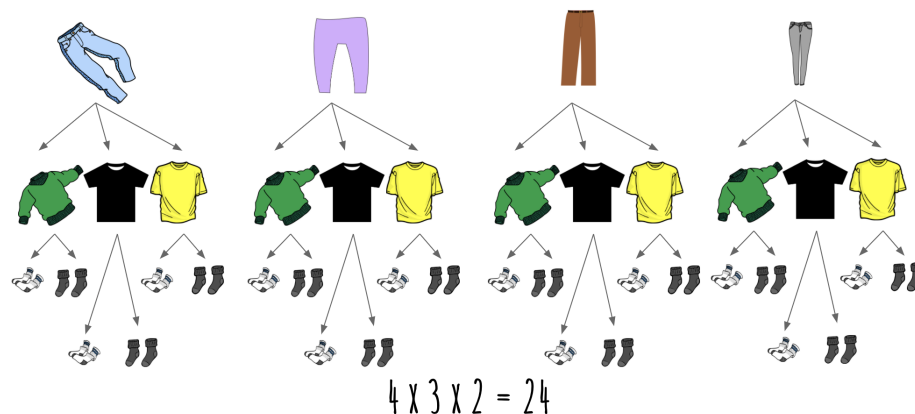
$4 + 4 + 4 = 3 \cdot 4 = 12$ outfit choices.



We could also do this in reverse and first pick out a bottom. Once we have our bottom, we have 3 choices for our top. This means we have 3 choices of top for each bottom, which we have 4 of. So, we still have a total of $3 + 3 + 3 + 3 = 4 \cdot 3 = 12$ outfit choices. (This makes sense - the number of outfits should be the same no matter how I count!)



What if we also wanted to add socks to the outfit, and we had 2 different pairs of socks? Then, for each of the 12 choices outlined above, we now have 2 choices of sock. This brings us to a total of 24 possible outfits.



This could be calculated more directly rather than drawing out each of these unique outfits, by multiplying our choices: $3 \text{ tops} \cdot 4 \text{ bottoms} \cdot 2 \text{ socks} = 24 \text{ outfits}$.

Theorem 1.1.2: Product Rule

If an experiment has N_1 outcomes for the first stage, N_2 outcomes for the second stage, $\dots$, and N_m outcomes for the m^{th} stage, then the total number of outcomes of the experiment is $N_1 \cdot N_2 \cdots N_m$.

More formally, if A and B are sets, then $|A \times B| = |A| \cdot |B|$ where $A \times B = \{(a, b) : a \in A, b \in B\}$ is the Cartesian product of sets A and B .

If this still sounds “simple” to you or you just want to practice, see the examples below! There are some pretty interesting scenarios we can count, and they are more difficult than you might expect.

Example(s)

1. How many outcomes are possible when flipping a coin n times? For example, when $n = 2$ there are four possibilities: HH, HT, TH, TT.
2. How many subsets of the set $[n] = \{1, 2, \dots, n\}$ are there?

Solution

1. The answer is 2^n : for the first flip, there are two choices: H or T. Same for the second flip, the third, and so on. Multiply 2 together n times to get 2^n .
2. This may be hard to think about at first. But think of the subset $\{2, 4, 5\}$ of the set $\{1, 2, 3, 4, 5, 6, 7\}$ as follows: for each number in the set, either it is in the subset or not. So there are two choices for the first element (in or out), and for each of them. This gives 2^n as well!

□

Example(s)

Flamingos Fanny and Freddy have three offspring: Happy, Glee, and Joy. These five flamingos are to be distributed to seven different zoos so that no zoo gets both a parent and a child :(. It is not required that every zoo gets a flamingo. In how many different ways can this be done?

Solution There are two disjoint (mutually exclusive) cases we can consider that cover every possibility. We can use the sum rule to add them up since they don’t overlap!

1. **Case 1: The parents end up in the same zoo.** There are 7 choices of zoo they could end up at. Then, the three offspring can go to any of the 6 other zoos, for a total of $7 \cdot 6 \cdot 6 \cdot 6 = 7 \cdot 6^3$ possibilities (by the product rule).
2. **Case 2: The parents end up in different zoos.** There are 7 choices for Fanny and 6 for Freddy. Then, the three offspring can go to any of the 5 other zoos, for a total of $7 \cdot 6 \cdot 5^3$ possibilities.

The result, by the sum rule, is $7 \cdot 6^3 + 7 \cdot 6 \cdot 5^3$. (Note: This may not be the only way to solve this problem. Often, counting problems have two or more approaches, and it is instructive to try different methods to get the same answer. If they differ, at least one of them is wrong, so try to find out which one and why!) □

1.1.3 Permutations

Back to the example of the debit card. There are 10 possible digits for each of the 4 digits of a PIN. So how many possible 4-digit PINs are there? This can be solved as $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10,000$. So, there is a one in ten thousand chance that a robber can guess your pin code (randomly).

Let's consider a stronger case where you must use each digit exactly once, so the PIN is exactly 10 digits long. How many such PINs exist?

Well, we have 10 choices for the first digit, 9 choices for the second digit, and so forth, until we only have 2 choices for the ninth digit, and 1 choice for the tenth digit. This means there are 362,880 possible PINs in this scenario as follows:

$$10 \cdot 9 \cdot \dots \cdot 2 \cdot 1 = \prod_{i=1}^{10} i = 362,880$$

This formula/pattern seems like it would appear often! Wouldn't it be great if there were a shorthand for this?

Definition 1.1.1: Permutation

The number of orderings of N *distinct* objects, is called a **permutation**, and mathematically defined as:

$$N! = N \cdot (N - 1) \cdot (N - 2) \cdot \dots \cdot 3 \cdot 2 \cdot 1 = \prod_{j=1}^N j$$

$N!$ is read as " N factorial". It is important to note that $0! = 1$ since there is one way to arrange 0 objects.

Example(s)

A standard 52-card deck consists of one of each combination of: 13 different ranks (Ace, 2, 3,...,10, Jack, Queen, King) and 4 different suits (clubs, diamonds, hearts, spades), since $13 \cdot 4 = 52$. In how many ways a 52-card deck be dealt to thirteen players, four to each, so that every player has one card of each suit?

Solution This is a great example where we can try two equivalent approaches. Each person usually has different preferences, and sometimes one way is significantly easier to understand than another. Read them both, understand why they both make sense and are equal, and figure out which approach is more intuitive for you!

Let's assign each player one at a time. The first player has 13 choices for the club, 13 for the heart, 13 for the diamond, and 13 for the spade, for a total of 13^4 ways. The second player has 12^4 choices (since there are only 12 of each suit remaining). And so on, so the answer is $13^4 \cdot 12^4 \cdot 11^4 \cdot \dots \cdot 2^4 \cdot 1^4$.

Alternatively, we can assign each suit one at a time. For the clubs suit, there are $13!$ ways to distribute them to the 13 different players. Then, the diamonds suit can be assigned in $13!$ ways as well, and same for the other two suits. By the product rule, the total number of ways is $(13!)^4$. Check that this different order of

assigning cards gave the same answer as earlier! (Expand the factorials.) $\square$

Example(s)

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

Solution We first choose the ordering of the families, of which there are $n!$. Then, in the first family, we have $m!$ ways to arrange them. The second family also has $m!$ ways to be arranged. And so on. By the product rule, the number of orderings is $n! \cdot (m!)^n$. $\square$

1.1.4 Complementary Counting

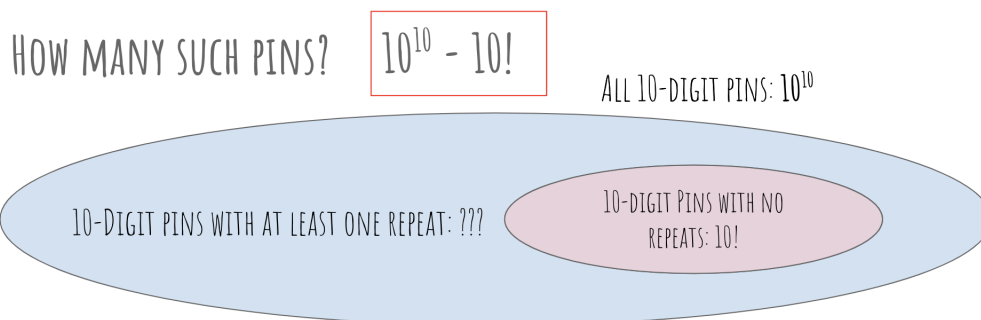
Now, let's consider an even trickier PIN requirement. Suppose we are still making a 10-digit PIN, but now at least one digit has to be repeated at least once. How many such PINs exist?

Some examples of this PIN would be 1111111111, 01234556788, or 9876598765, but the list goes on!

Let's try our "normal" approach. If we try this, we'll end up getting stuck. Consider placing the first digit - we have 10 choices. How many choices do we have for the second digit? Is this a repeated digit or not? We can try to find a product of counts of choices for each digit in different scenarios but this can become complicated as we move around which digits are repeated...

Another approach might be to count how many PINs **don't** satisfy this property, and subtract it from the total number of PINs. This strategy is called complementary counting, as we are counting the size of the complement of the set of interest. The number of possible 10-digit PINs, with no stipulations, is 10^{10} (from the product rule, multiplying 10 choices with itself for each of 10 positions). Then, we found above that the 10-digit PINs with no repeats has $10!$ possibilities (each digit used exactly once). Well, consider that the 10-digit PINs with at least one repeat will be all other possibilities (they could have one, two, or more repeats but certainly won't have none). This means that we can count this by taking the difference of all the possible 10-digit PINs and those with no repeats. That is:

$$10^{10} - 10!$$



Definition 1.1.2: Complementary Counting

Let $\mathcal{U}$ be a (finite) universal set, and S a subset of interest. Let $S^C = \mathcal{U} \setminus S$ denote the set difference (complement of S). Then,

$$|S| = |\mathcal{U}| - |S^C|$$

Informally, to find the number of ways to do something, we could count the number of ways to NOT to do that thing, and subtract it from the total. That is, the complement of the subset of interest is also of interest! This technique is called **complementary counting**.

Think about how this is just the Sum Rule rephrased, using the diagram above!

1.1.5 Exercises

1. Suppose we have 6 people who want to line up in a row for a picture, but two of them, A and B, refuse to sit next to each other. How many ways can they sit in a row?

Solution: There are two equivalent approaches. The first approach is to solve it directly. However, depending on where A sits, B has a different number of options (whether A sits at the end or the middle). So we have two disjoint (non-overlapping) cases:

- (a) Case 1: A sits at one of the two end seats. Then, A has 2 choices for where to sit, and B has 4. (See this diagram where A sits at the right end: $- - - - A$.) Then, there are $4!$ ways for the remaining people to sit, for a total of $2 \cdot 4 \cdot 4!$ ways.
- (b) Case 2: A sits in one of the middle 4 seats. Then, A has 4 choices of seat, but B only has three choices for where to sit. (See this diagram where A sits in a middle seat: $- A - - -$.) Again, there are $4!$ ways to seat the rest, for a total of $4 \cdot 3 \cdot 4!$ ways.

Hence our total by the sum rule is $2 \cdot 4 \cdot 4! + 4 \cdot 3 \cdot 4! = 480$.

The alternative approach is complementary counting. We can count the total orderings, of which there are $6!$, and subtract the cases where A and B *do* sit next to each other. There's a trick we can do to guarantee this: let's treat A and B as a *single entity*. Then, along with the remaining 4 people, there are only 5 entities. We order the entities in $5!$ ways, but also multiply by $2!$ since we could have the seating AB or BA. Hence, the number of ways they do sit together is $2 \cdot 5! = 240$, and the ways they do not sit together is $6! - 240 = 720 - 240 = 480$.

Decide which approach you liked better - oftentimes, one method will be easier than another!

2. You love playing the 5 racket sports: tennis, badminton, ping pong, squash, and racquetball. You plan a week of sports at a time, from Sunday to Saturday. Each day you want to play *one* of these sports with a friend, but to avoid getting bored, you don't ever play the same sport two days in a row. If your mom is visiting town and wants to play tennis with you on Wednesday, how many possible "sports schedules" for the week can you create?

Solution: If you try to start from Sunday (which is a very natural thing to do since it is the first day), you will run into some trouble. You could have 5 choices for Sunday, and 4 for the Monday (since you can't play the same sport as Sunday). But Tuesday is interesting because you can't choose tennis because of Wednesday, and you don't know what Monday's choice was...

We should try a different approach. Why don't we start by assigning Wednesday to tennis first (1 way) and work outwards. Then, let's plan Tuesday (4 ways), then Monday (4 ways), and Sunday (4 ways). Then, similarly plan the rest of the week Thurs-Sat. The total number of ways is just 4^6 then because you have 4 choices for each of the other 6 days!

The goal of this problem is to show you that you don't always have to start left to right or right to left - as long as it works!

3. Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Solution: This is hard to tackle directly. A lot of these problems require some interesting modeling, which you'll get used to through practice!

There are two disjoint (non-overlapping) cases for your friend and you, so we can use the sum rule.

- (a) **Case 1:** You are next to your friend. Then, there are 7 sets of positions you and your friend can occupy (positions 1/2, 2/3, ..., 7/8), and for each set of positions, there are $2!$ ways to arrange you and your friend. So there are $7 \cdot 2!$ ways to pick positions for you and your friend.
- (b) **Case 2:** There is exactly 1 person between you and your friend. Then, there are 6 sets of positions you and your friend can occupy (positions 1/3, 2/4, ..., 6/8), and for each set of positions, there are again $2!$ ways to arrange you and your friend. So there are $6 \cdot 2!$ ways to pick positions for you and your friend.

Note that in both cases, there are then $6!$ ways to arrange the remaining people, so we multiply both cases by $6!$ by the product rule. This gives $(7 \cdot 2! + 6 \cdot 2!) \cdot 6!$ ways in total.