



**CS109: Probability for
Computer Scientists**

Why are we learning counting?

- The setup of probability involves a `set` of possible things that can happen
- We are learning to figure out the sizes of `sets` given some verbal description of them (examples from Monday)
 - sum rule of counting
 - product rule of counting



Why are we learning counting?

- The setup of probability involves a `set` of possible things that can happen
- We are learning to figure out the sizes of `sets` given some verbal description of them (examples from Monday)
 - sum rule of counting
 - product rule of counting
- We can use the sizes of sets & subsets to come up with a formal notion of probability
 - More on this Friday!
- Counting the sizes of sets is called combinatorics!





Permutations I

Sort n distinct objects



Piech + Cain, CS109, Stanford University



Sort n distinct objects



Ayesha



Tim



Irina



Joey



Waddie



Sort n distinct objects



Ayesha

1st



Tim

2nd



Irina

3rd



Joey

4th



Waddie

5th



Sort n distinct objects



Ayesha

1st



Tim

2nd



Irina

3rd



Joey

4th



Waddie

5th

Steps:

1. Choose 1st can 5 options
2. Choose 2nd can 4 options
- ...
5. Choose 5th can 1 option

$$\begin{aligned} \text{Total} &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$



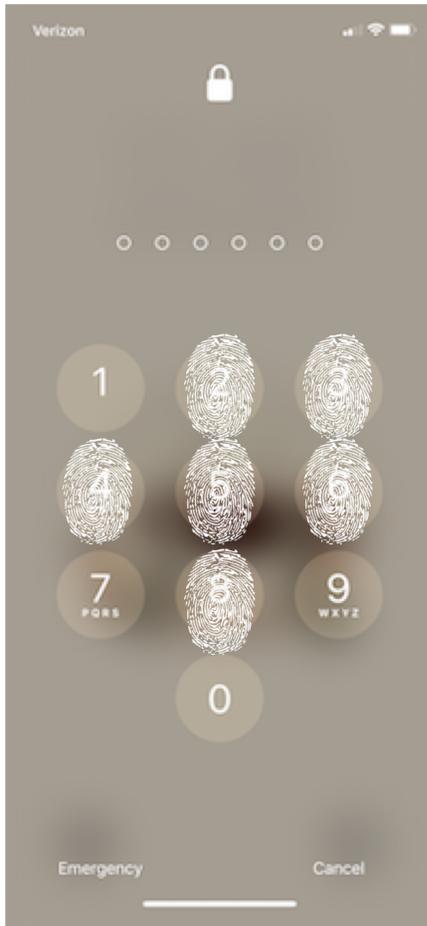
Permutations

- A **permutation** is an ordered arrangement of objects.
- The number of unique orderings (**permutations**) of n distinct objects is

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.$$



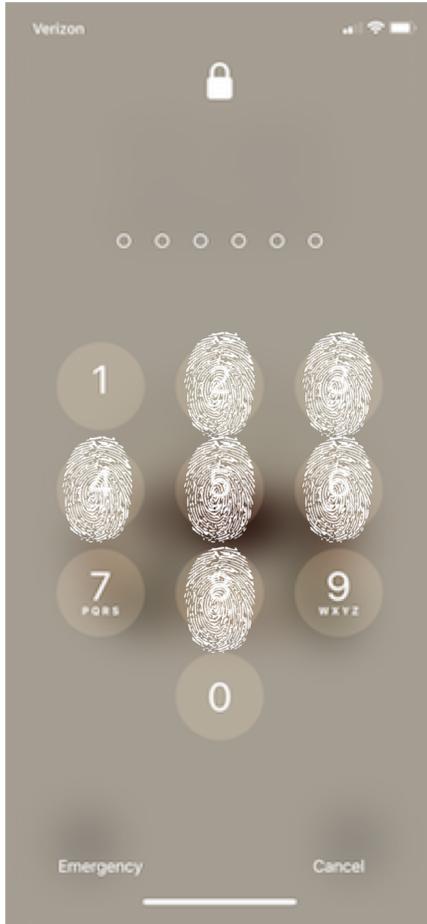
Unique 6-digit passcodes with **six** smudges



How many unique passcodes are possible if a phone password uses **six distinct** numbers out of **ten** available digits?



Unique 6-digit passcodes with **six** smudges



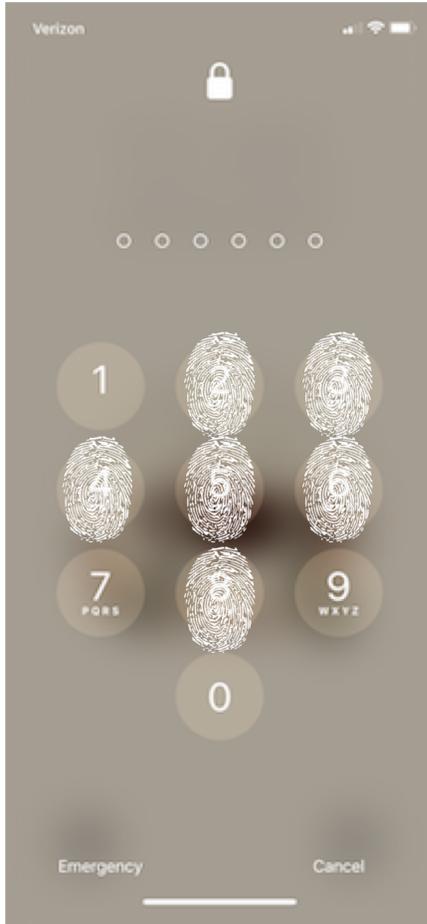
How many unique passcodes are possible if a phone password uses **six distinct** numbers out of **ten** available digits?

$$\text{Total} = 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

$$\frac{10!}{(10-6)!} = \frac{10!}{4!} = 151200 \text{ passcodes}$$



Unique 6-digit passcodes with **six** smudges



How many unique passcodes are possible if a phone password uses **six distinct** numbers out of **ten** available digits?

$$\text{Total} = 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

$$\frac{10!}{(10-6)!} = \frac{10!}{4!} = 151200 \text{ passcodes}$$

This technique is called “n permute k”. In this question, we have “10 permute 6”





Permutations II

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)



Sort n distinct objects



Ayesha



Tim



Irina



Joey



Waddie

of permutations = $n!$

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
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Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

$n!$



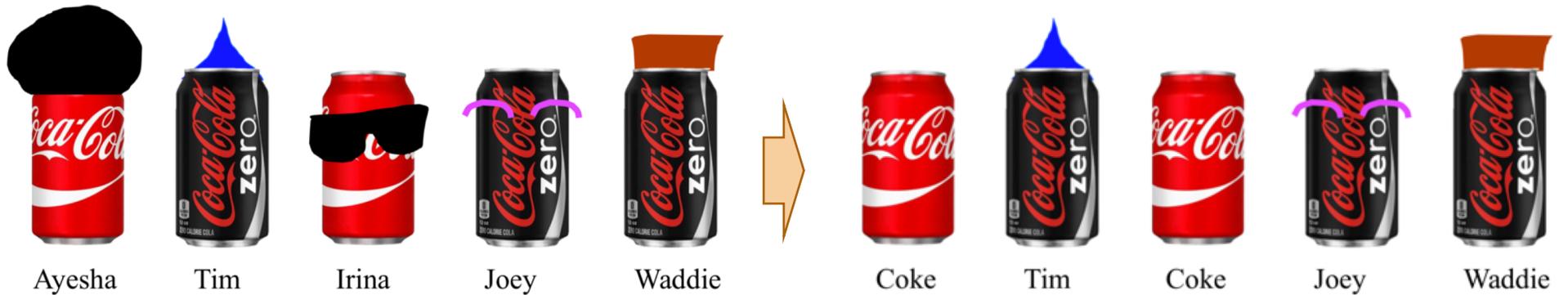
Sort semi-distinct objects

Order n
distinct objects

$n!$

All distinct

Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\begin{array}{l} \text{permutations} \\ \text{of distinct objects} \end{array} = \begin{array}{l} \text{permutations} \\ \text{considering some} \\ \text{objects are indistinct} \end{array} \times \begin{array}{l} \text{Permutations} \\ \text{of just the} \\ \text{indistinct objects} \end{array}$$

Sort semi-distinct objects

How do we find **the number of permutations considering some objects are indistinct?**

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or **indistinct**), and
 n_2 are the same, and

...

n_r are the same,

The number of unique orderings (**permutations**) is

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many permutations?



Coke



Coke0



Coke



Coke0



Coke0

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
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Distinct
(distinguishable)

Some
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$$n!$$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Strings

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many letter orderings are possible for the following strings?

1. CHRISDADDY

2. MISSISSIPPI



Strings

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many letter orderings are possible for the following strings?

1. CHRISDADDY

$$\frac{10!}{1!1!1!1!1!3!1!1!} = \frac{10!}{3!} = 604,800$$

2. MISSISSIPPI

$$\frac{11!}{1!4!4!2!} = \frac{11!}{4!4!2!} = 34,650$$

Unique 6-digit passcodes with **six** smudges

Order n semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

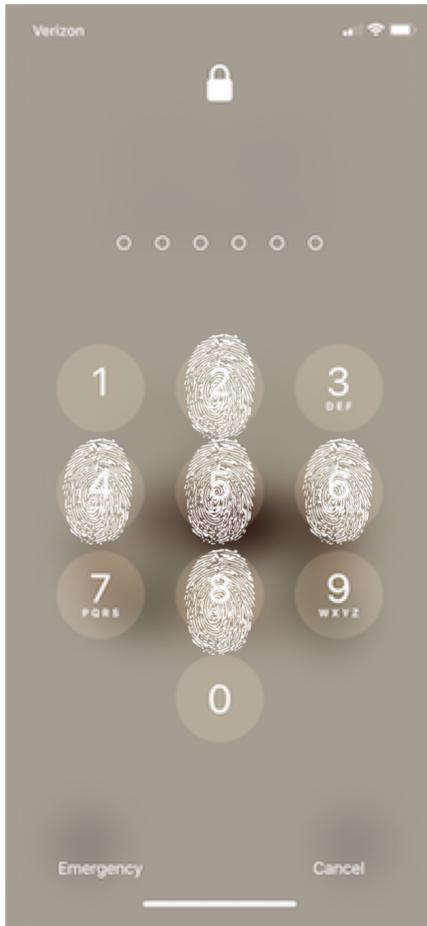


How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

Total = $6!$
= 720 passcodes

Unique 6-digit passcodes with **five** smudges

Order n semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of **five** distinct numbers?

Steps:

1. Choose digit to repeat 5 outcomes
2. Create passcode (sort 6 digits: 4 distinct, 2 indistinct)

$$\begin{aligned} \text{Total} &= 5 \times \frac{6!}{2!} \\ &= 1,800 \text{ passcodes} \end{aligned}$$

Combinations I

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

Distinct



$$n!$$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

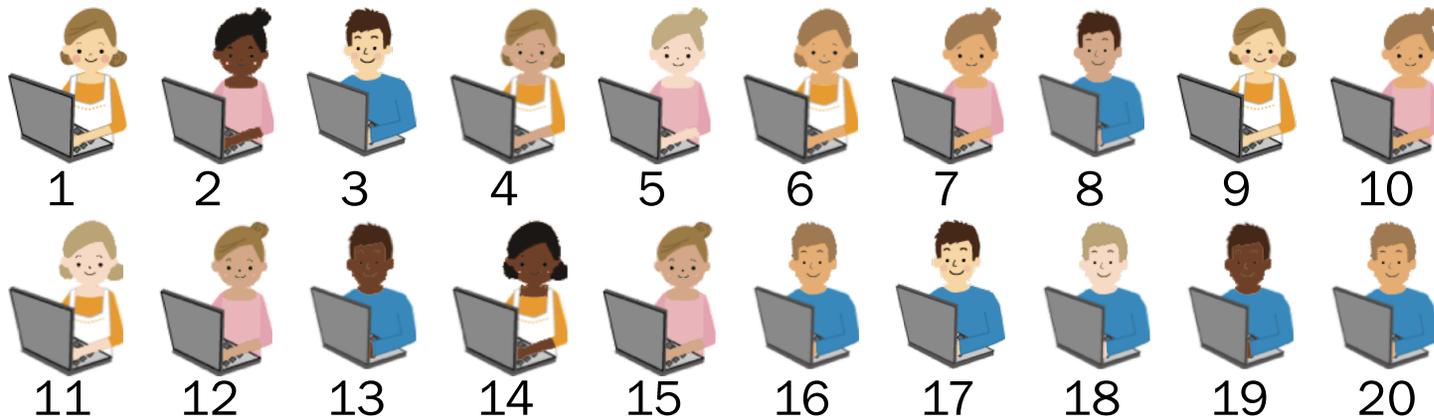


Consider the following generative process...

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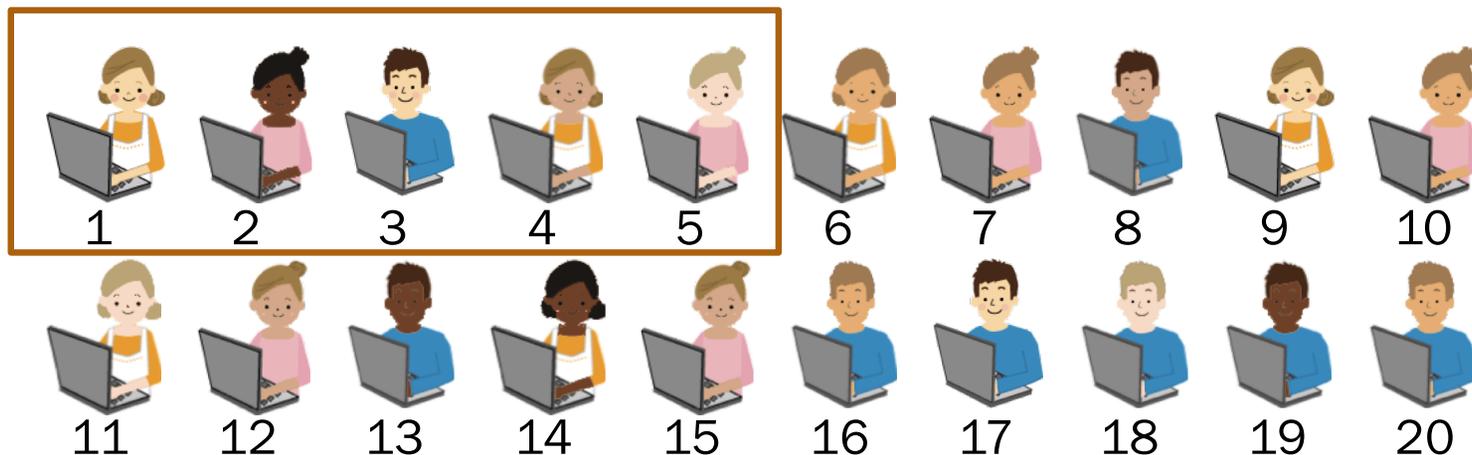
1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?



1. n people
get in line

$n!$ ways

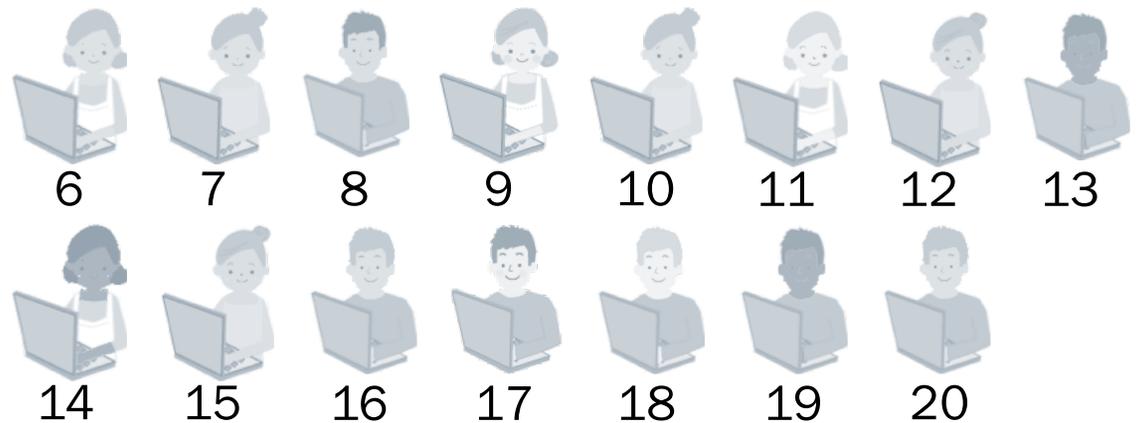
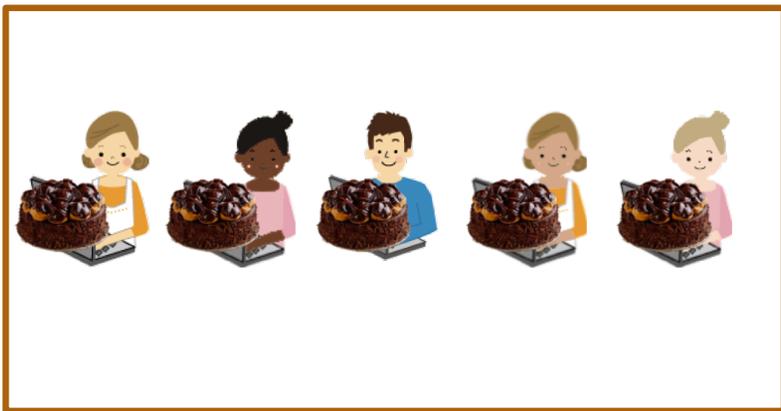
2. Put first k
in cake room

1 way

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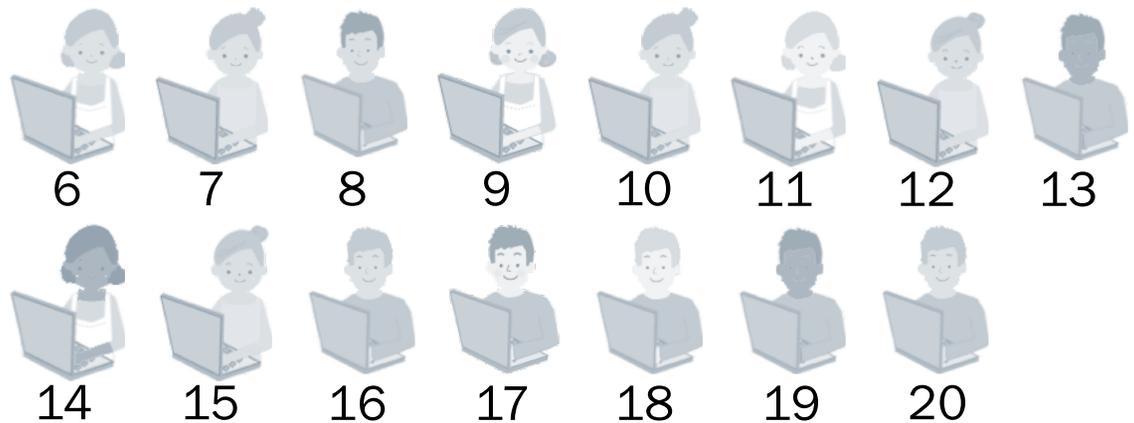
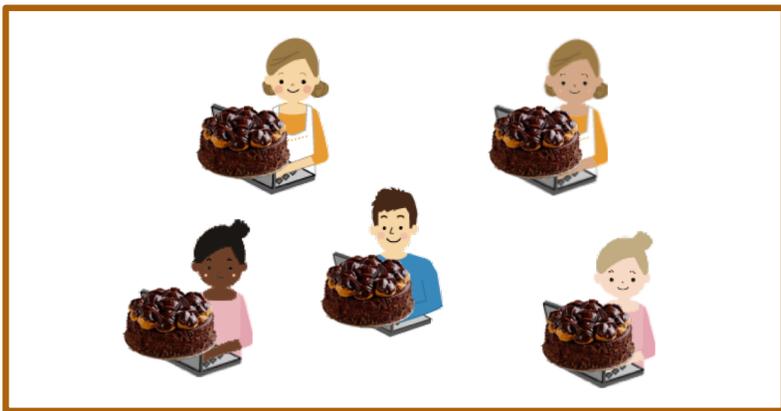
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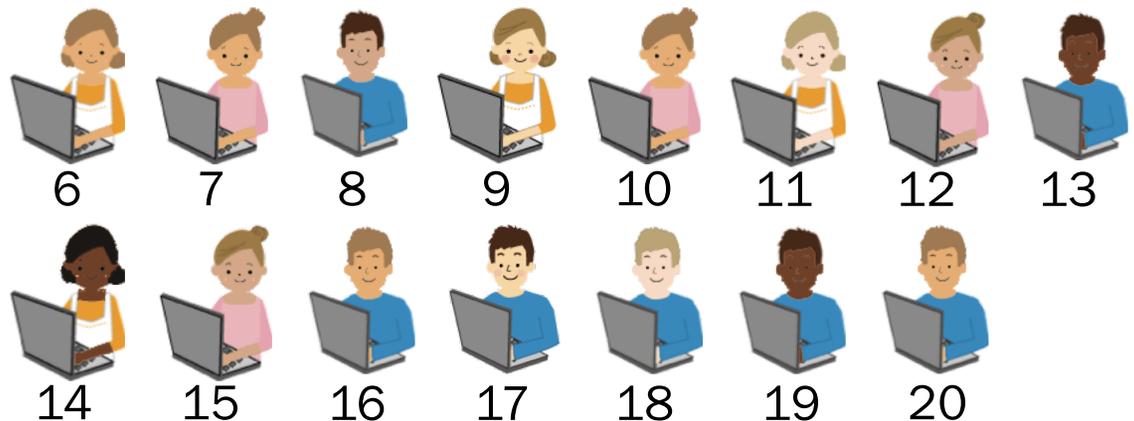
3. Allow cake group to mingle

$k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

2. Put first k in cake room

1 way

3. Allow cake group to mingle

$k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.

How many ways can we **choose** $k = 5$ people to get cake?



1. n people
get in line

$n!$ ways

2. Put first k
in cake room

1 way

3. Allow cake
group to mingle

$k!$ different
permutations lead to
the same mingle

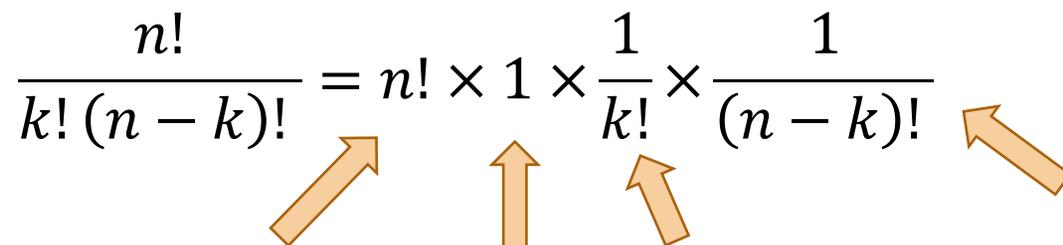
4. Allow non-cake
group to mingle

$(n - k)!$ different
permutations lead to the
same mingle

Combinations

A **combination** is an unordered selection of k objects from a set of n **distinct** objects.

The number of ways of making this selection is

$$\frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$


1. Order n distinct objects

2. Take first k as chosen

3. Overcounted: any ordering of chosen group is same choice

4. Overcounted: any ordering of unchosen group is same choice

Combinations

A **combination** is an unordered selection of k objects from a set of n **distinct** objects.

The number of ways of making this selection is

$$\frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k} \text{ Binomial coefficient}$$

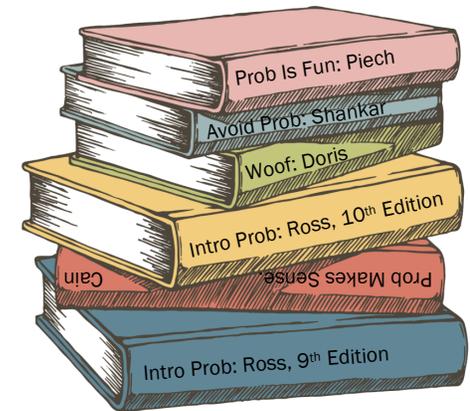
Fun Fact: $\binom{n}{k} = \binom{n}{n-k}$ This is read “n choose k” or “n choose n-k”

Probability textbooks

Choose k of
 n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways}$$



Probability textbooks

Choose k of
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1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

D. $\binom{6}{3} - \binom{4}{1} = 16$

B. $\frac{6!}{3!3!2!} = 10$

E. Both C and D

C. $\binom{4}{2} + \binom{4}{2} + \binom{4}{3} = 16$

F. Something else



Probability textbooks

Choose k of
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1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!3!} = 20$ ways
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Strategy 1: Sum Rule

Probability textbooks

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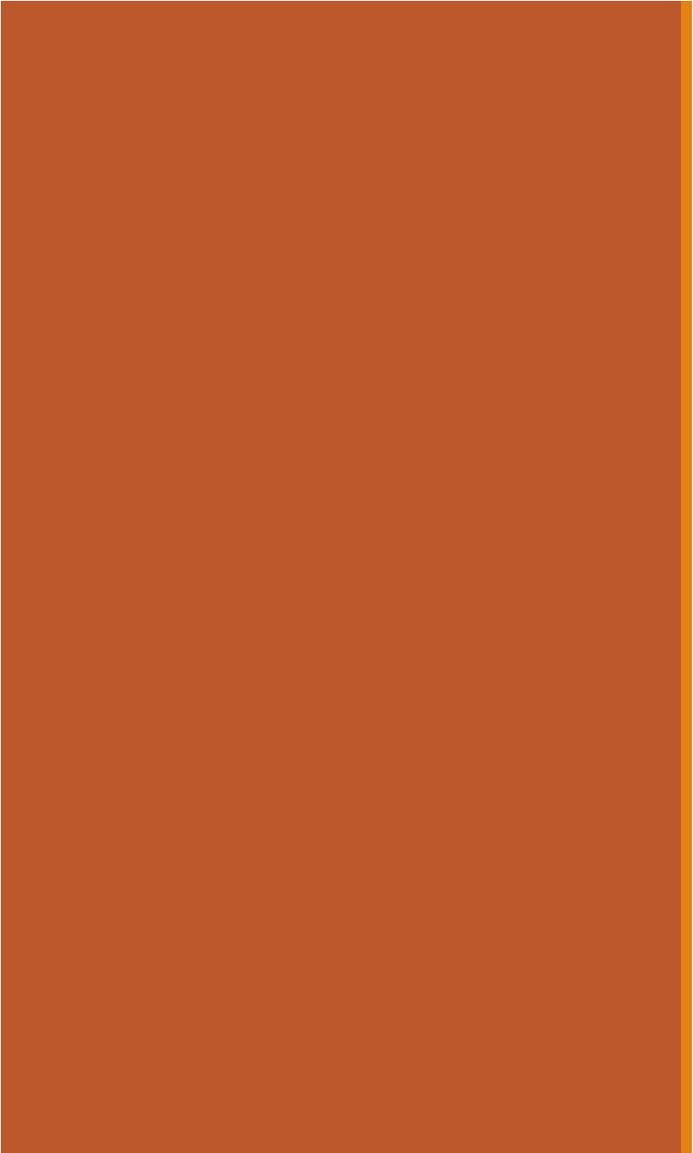
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Strategy 2: “Forbidden method” (unofficial name)

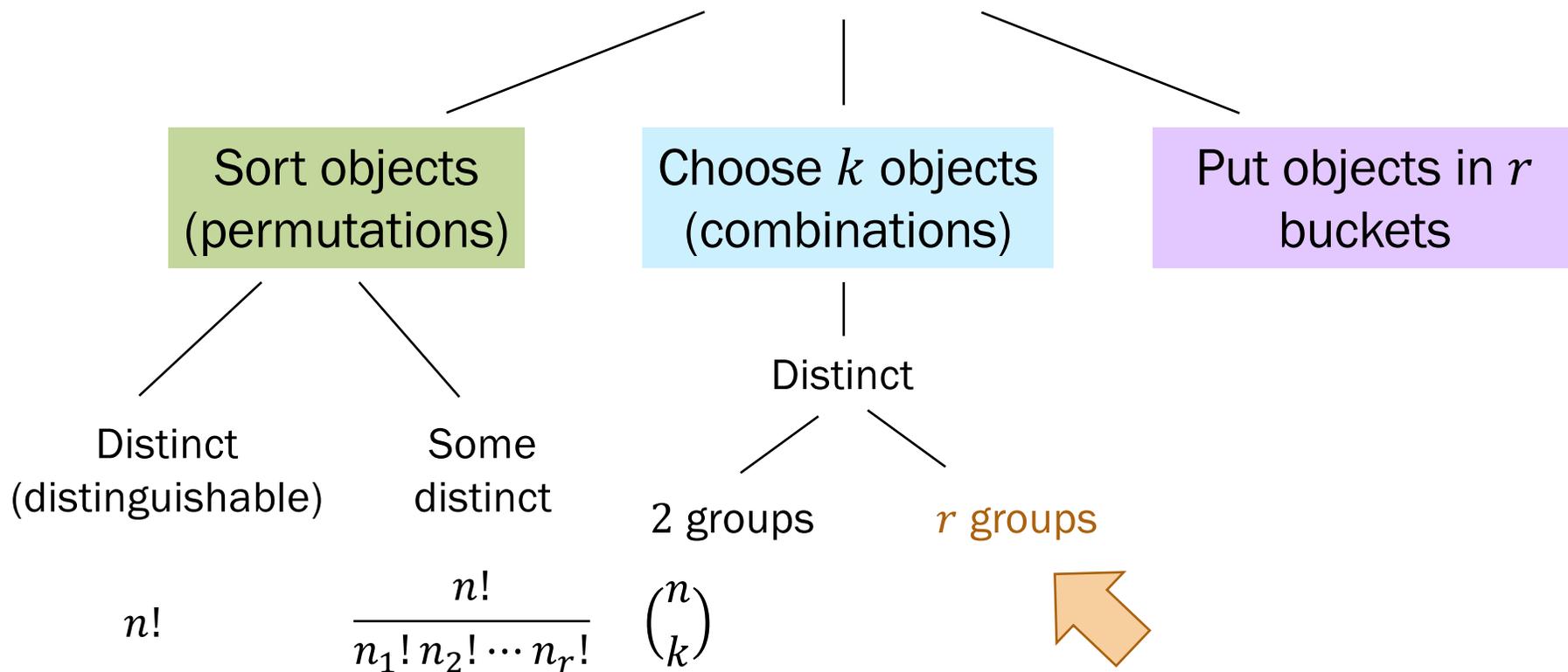
Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.



Combinations II

Summary of Combinatorics

Counting tasks on n objects



General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i , and

$\sum_{i=1}^r n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1! n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \dots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects
into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

13 different computers are to be allocated to
3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

- A. $\binom{13}{6,4,3} = 60,060$
- B. $\binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060$
- C. $6 \cdot 1001 \cdot 10 = 60,060$
- D. A and B
- E. All of the above



Datacenters

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A. $\binom{13}{6,4,3} = 60,060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datacenters

Choose k of n distinct objects into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

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Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Datacenters

Choose k of n distinct objects into r groups of size n_1, \dots, n_r $\binom{n}{n_1, n_2, \dots, n_r}$

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Group 1 (datacenter A): $n_1 = 6$

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B. $\binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

Multinomial vs Binomial

The binomial is a special case of the multinomial

$$\frac{7!}{2!5!} = \binom{7}{2} = \binom{7}{2} \binom{5}{5} = \binom{7}{2,5} = \binom{7}{5} \binom{2}{2} = \binom{7}{5} = \frac{7!}{5!2!}$$

In practice, we rarely write multinomial notation $\binom{7}{2,5}$ for two groups

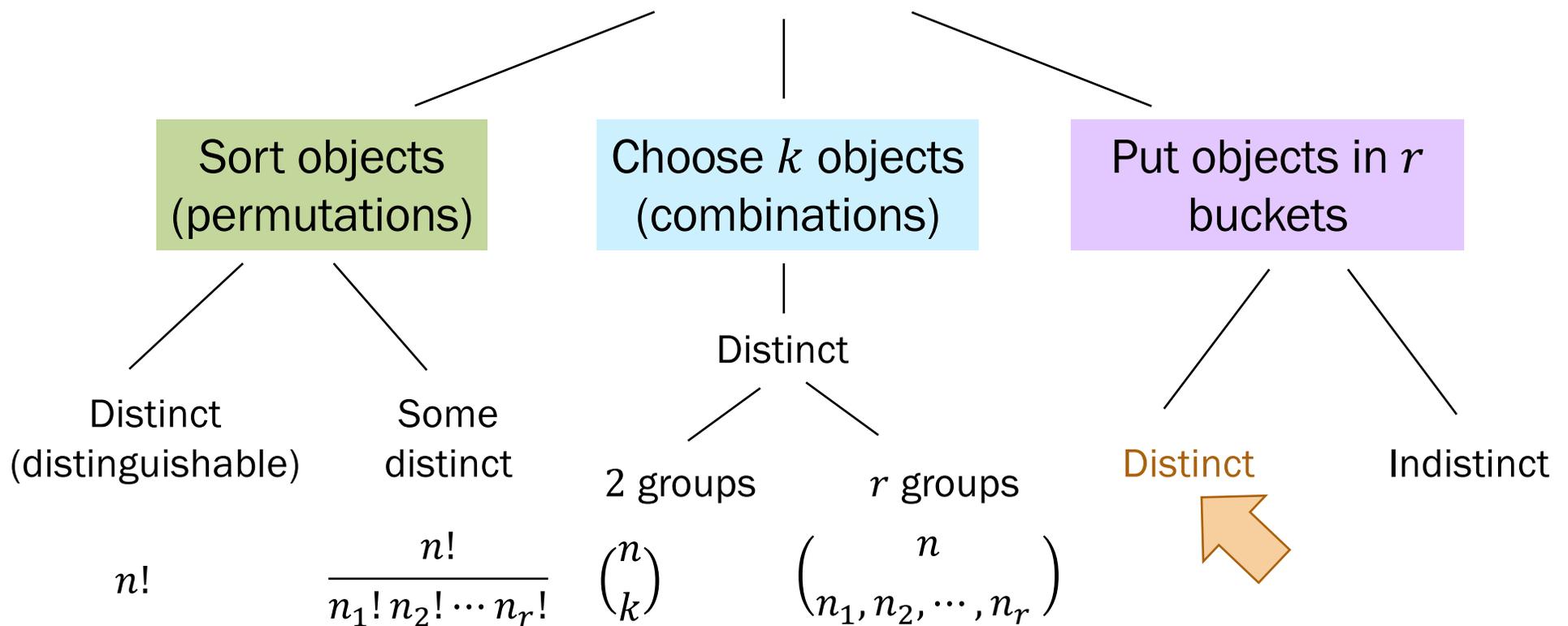
We will tend to use either $\binom{7}{2}$ or $\binom{7}{5}$ when there are two groups.



Buckets and The Divider Method

Summary of Combinatorics

Counting tasks on n objects

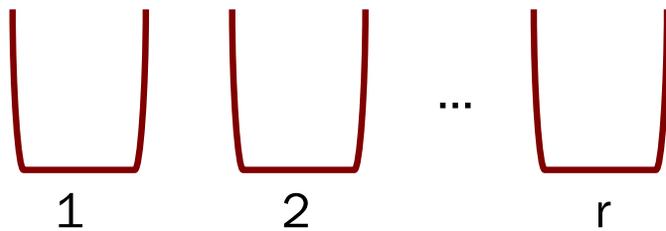
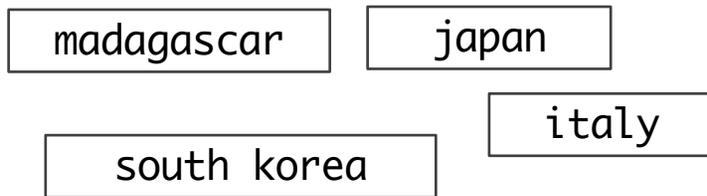


Balls and urns Hash tables and **distinct** strings

How many ways are there to hash n **distinct** strings to r buckets?

Steps:

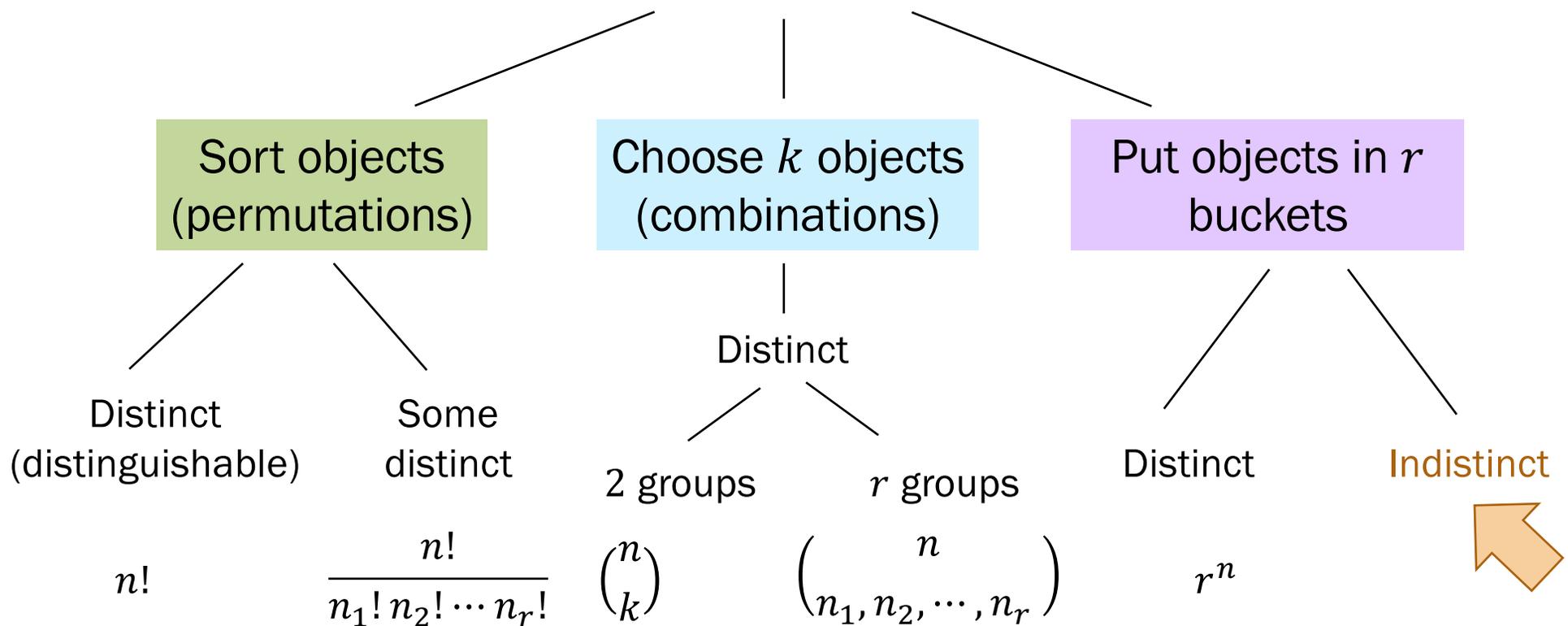
1. Bucket 1st string
2. Bucket 2nd string
- ...
- n . Bucket n^{th} string



r^n outcomes

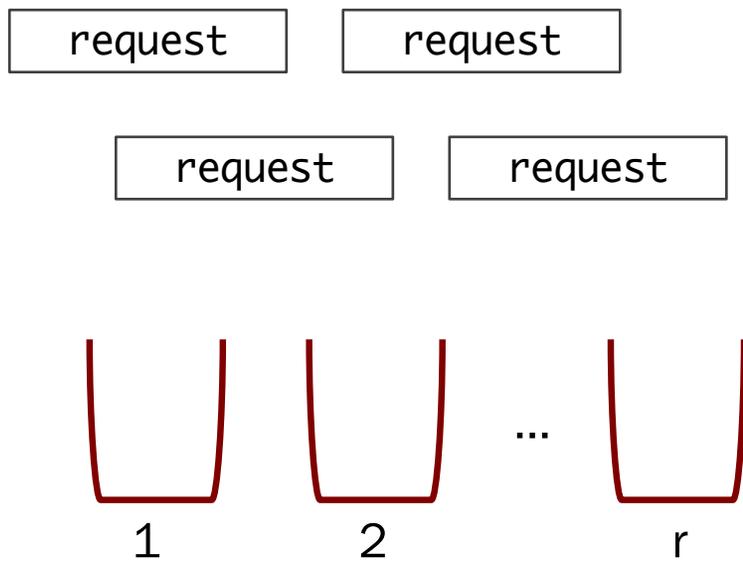
Summary of Combinatorics

Counting tasks on n objects



Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** web requests to r servers?



Goal

Server 1 has x_1 requests,

Server 2 has x_2 requests,

...

Server r has x_r requests

$$\text{constraint: } \sum_{i=1}^r x_i = n$$

Simple example: $n = 3$ requests and $r = 2$ servers

Bicycle helmet sales

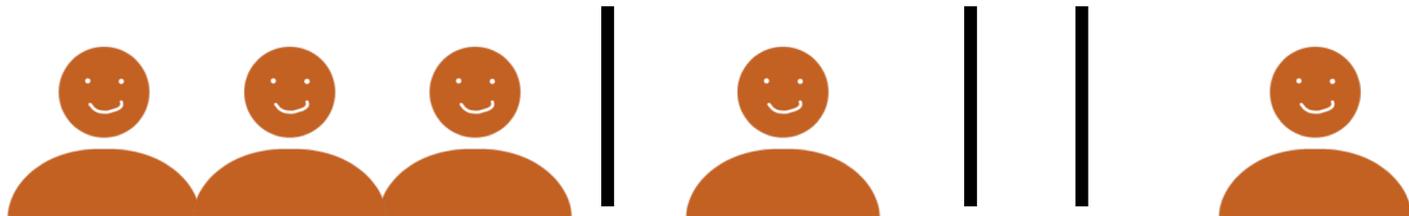
How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?



Bicycle helmet sales

1 possible assignment outcome:

Goal Order n indistinct objects and $r - 1$ indistinct dividers.



Consider the following generative process...

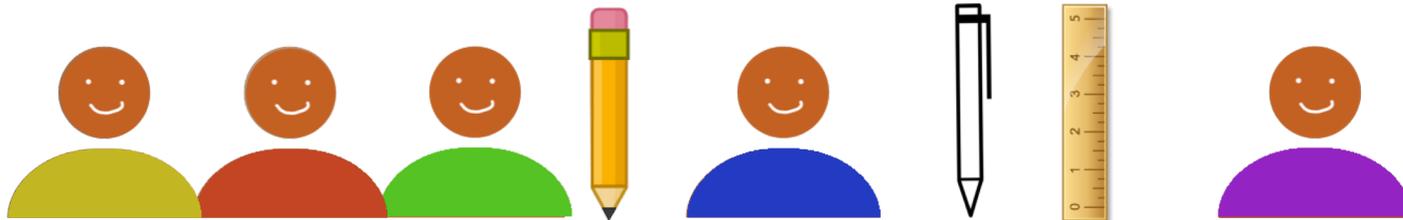


The divider method: A generative proof

How many ways can we **assign** $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n **indistinct** objects and $r - 1$ **indistinct** dividers.

0. Make objects and dividers distinct

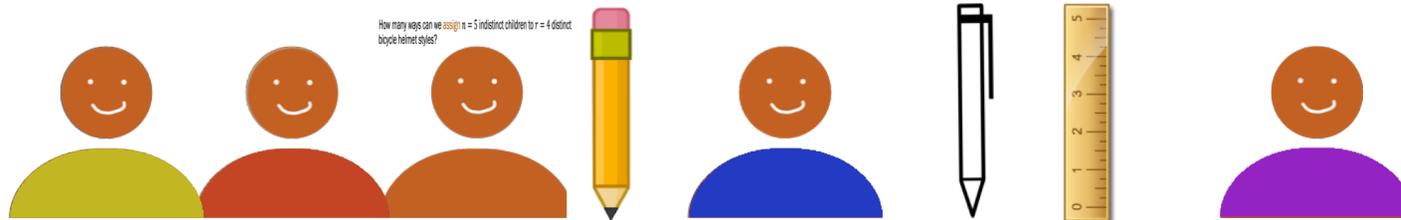


The divider method: A generative proof

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1. Order n distinct objects and $r - 1$ distinct dividers

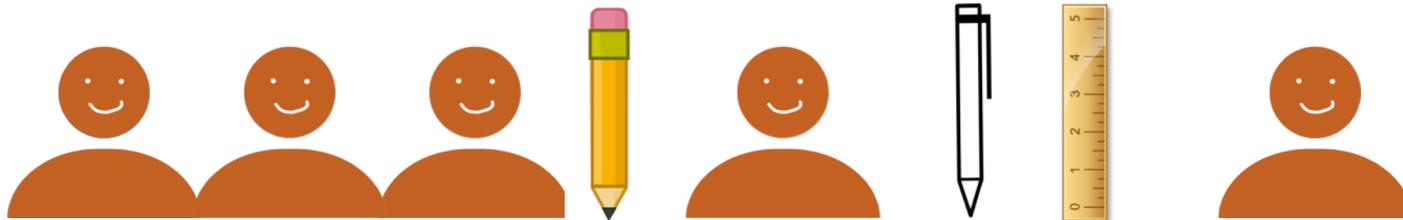
$$(n + r - 1)!$$

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

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1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

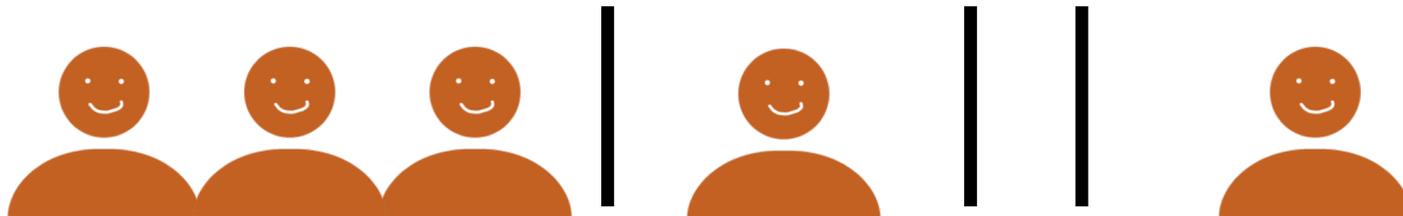
$$\frac{1}{n!}$$

The divider method: A generative proof

How many ways can we **assign** $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n **indistinct** objects and $r - 1$ **indistinct** dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r - 1)!}$$

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that n are indistinct objects, and $r - 1$ are indistinct dividers:

$$\begin{aligned} \text{Total} &= (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!} \\ &= \binom{n + r - 1}{r - 1} \text{ outcomes} \end{aligned}$$

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i , x_i is an integer such that $0 \leq x_i \leq n$?

Positive integer equations can be solved with the divider method.

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?



Venture capitalists. #1

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

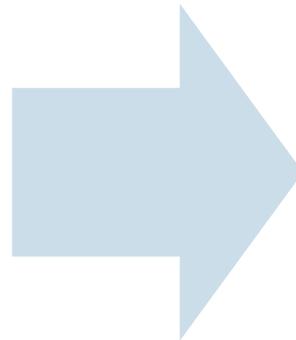
1. How many ways can you fully allocate your \$10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i : amount invested in company i

$$x_i \geq 0$$



Solve

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

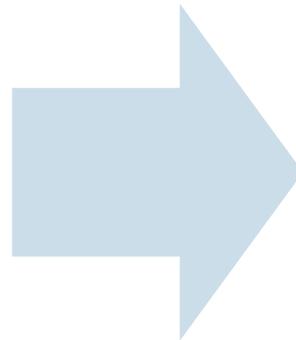
1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i : amount invested in company i

$$x_1 \geq 3, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$



Solve

Venture capitalists. #3

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in \$1 million increments).

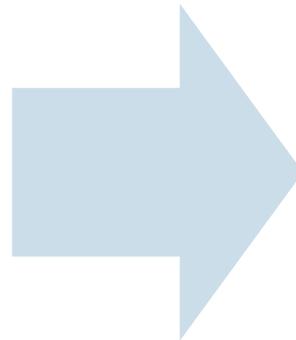
1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

Set up

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i : amount invested in company i

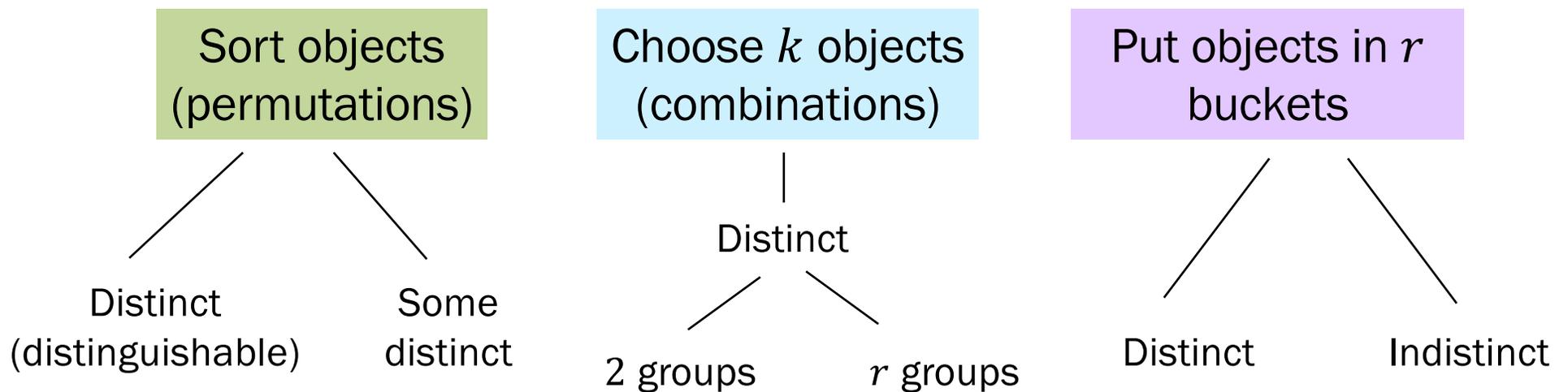
$$x_i \geq 0$$



Solve

Summary of Combinatorics

Counting tasks on n objects



- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases

Summary of Combinatorics

