Conditional Probability and Bayes

Announcements

- Pset #1 is due on Friday
- Office hours, wahoo!
- First sections this week!
- Section assignments will be sent out tonight.
- Auditing? Cool, but just lectures and online resources.
- Don’t forget concept checks

Review, Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Identity 3: $P(E^c) = 1 - P(E)$

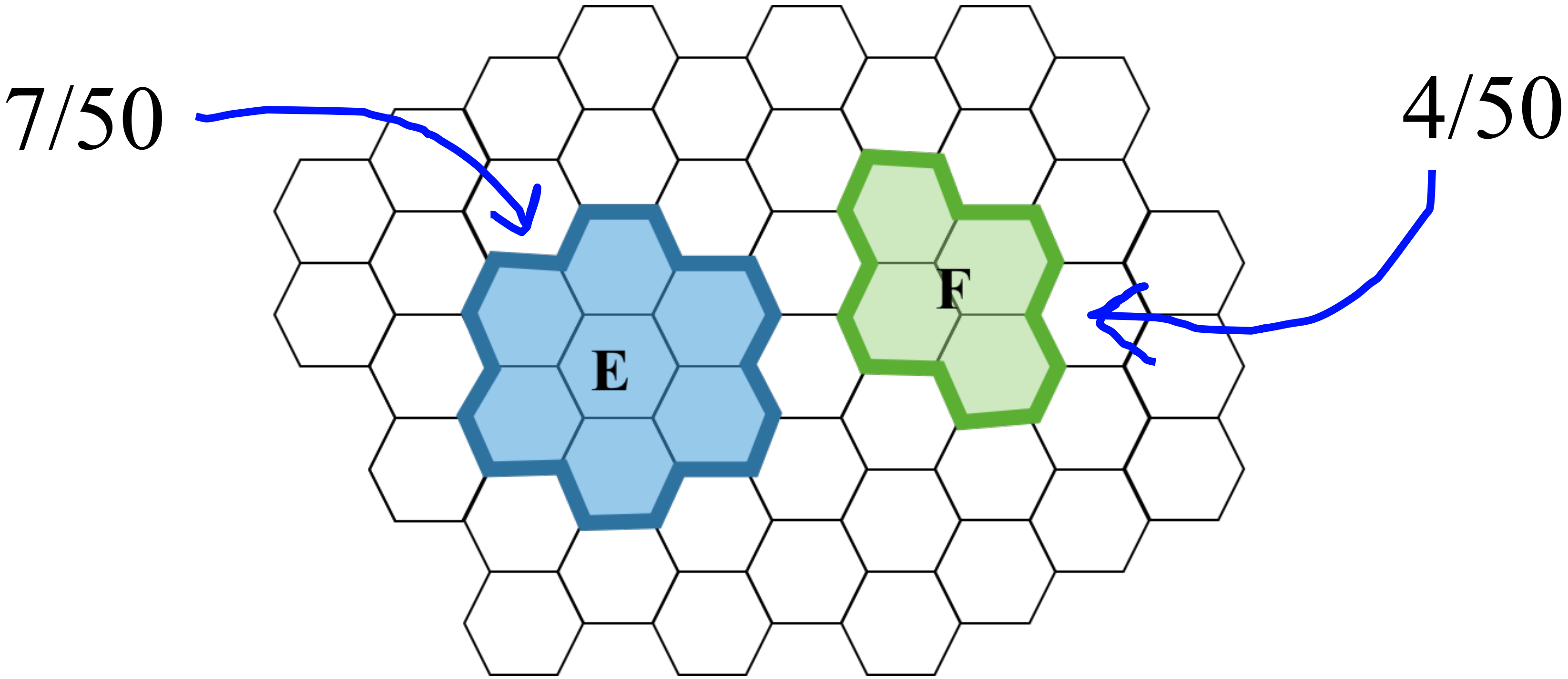
Review, Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \le P(E) \le 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$

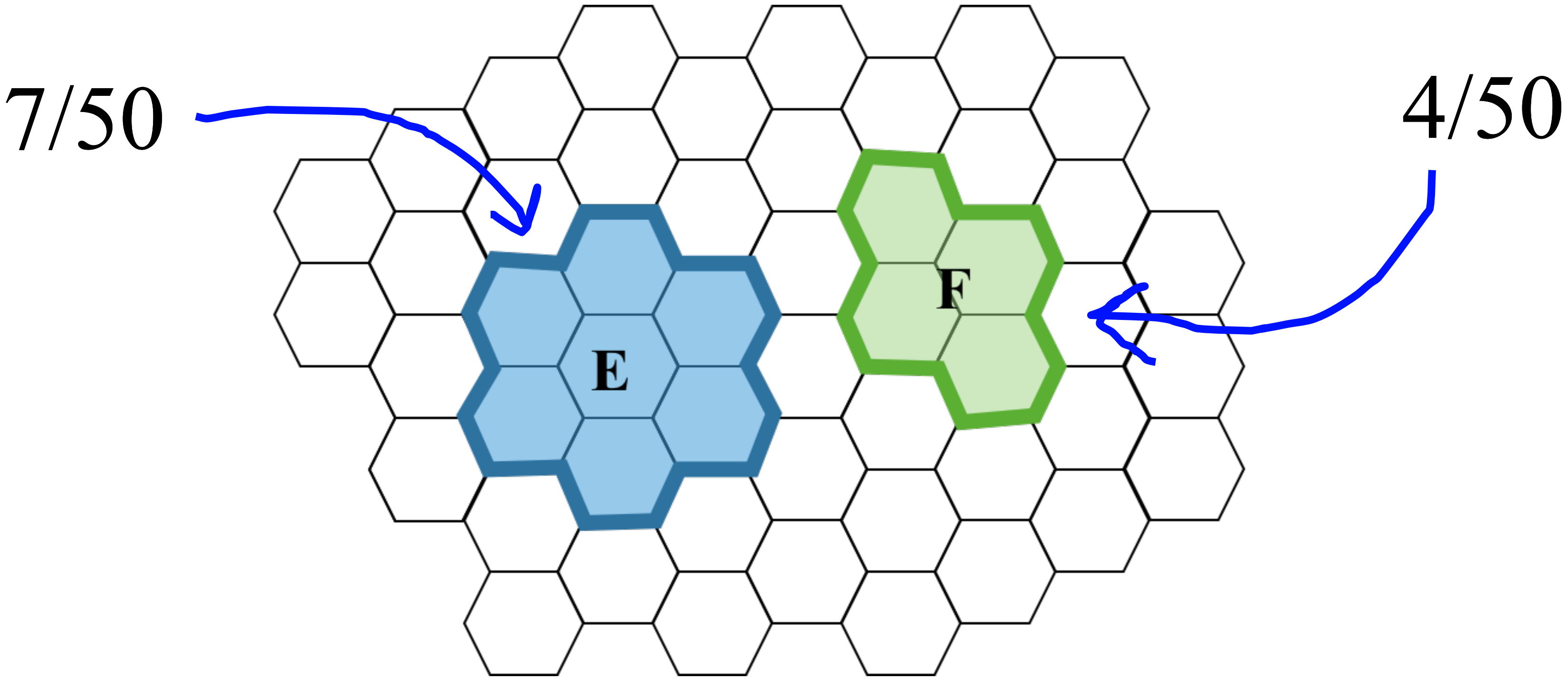
Review, Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

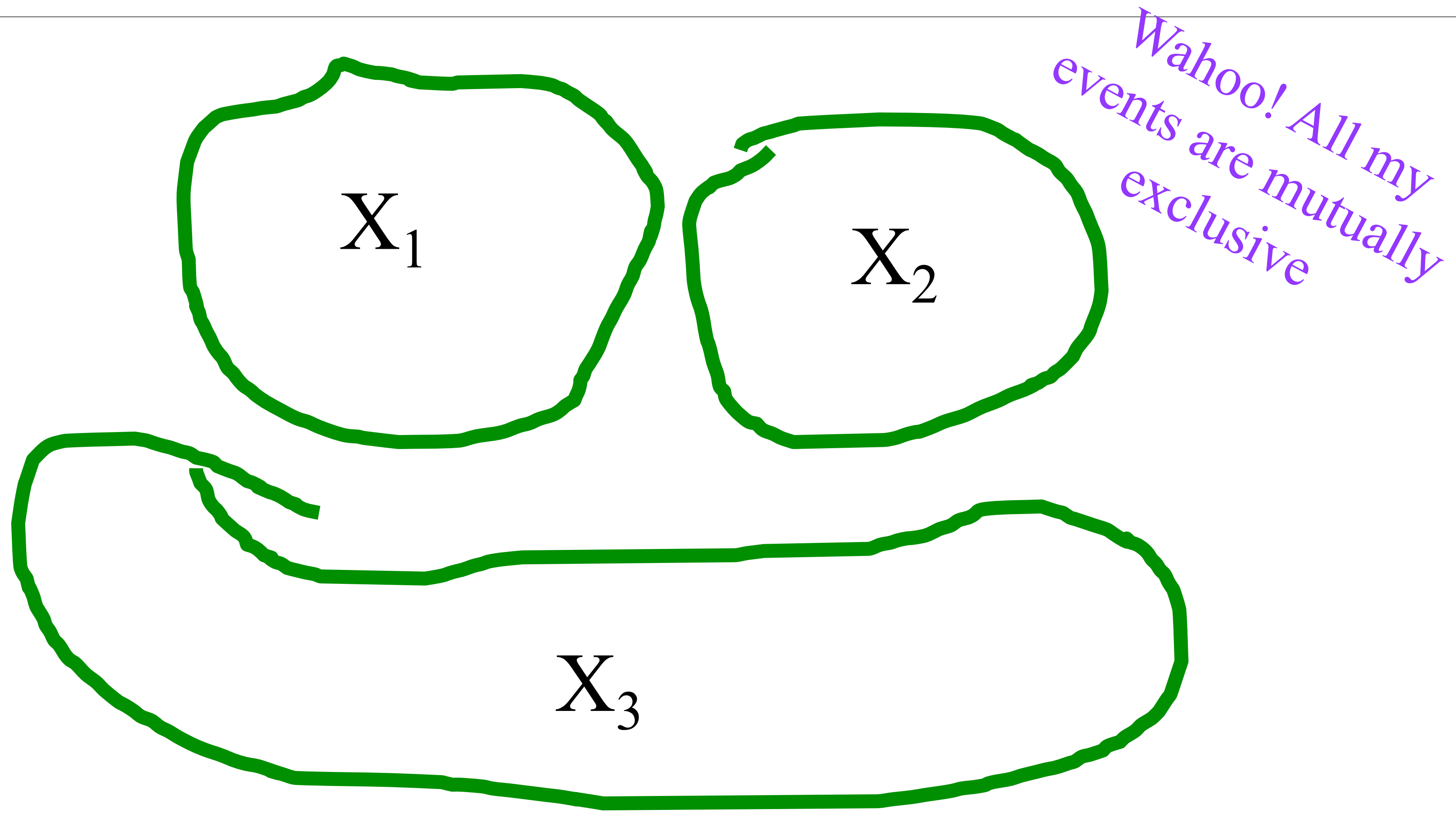
Review, Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

Review, Mutually Exclusive Events



$$P(X_1 \cup X_2 \cup \cdots \cup X_n) = \sum_{i=1}^{n} P(X_i)$$

Review, Mutually Exclusive Events

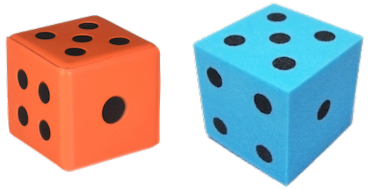
If events are *mutually exclusive* probability of OR is easy!

Conditional Probability

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided fair dice. What is P(sum = 7)?

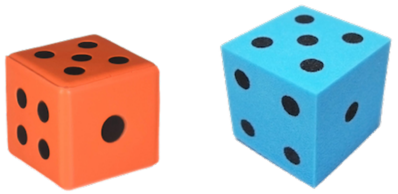


S = {(1,1) (1,2) (1,3) (1,4) (1,5) **(1,6)**
(2,1) (2,2) (2,3) (2,4) **(2,5)** (2,6)
(3,1) (3,2) (3,3) **(3,4)** (3,5) (3,6)
(4,1) (4,2) **(4,3)** (4,4) (4,5) (4,6)
(5,1) **(5,2)** (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

$E =$ *In blue*

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.
You want them to sum to 4.



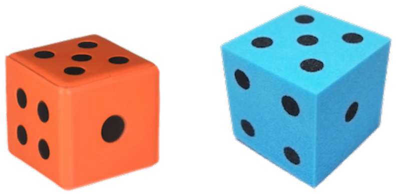
What is the best outcome for $P(D_1)$?

Your Choices:

- A. 1 and 3 tie for best
- B. 1, 2 and 3 tie for best
- C. 2 is the best
- D. Other/none/more than one

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = 2$.

What is $P(E$, given F already observed$)$?

$S = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)\}$

$E = \{(2,2)\}$

$P(E) = 1/6$

Conditional Probability

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Written as: $P(E|F)$

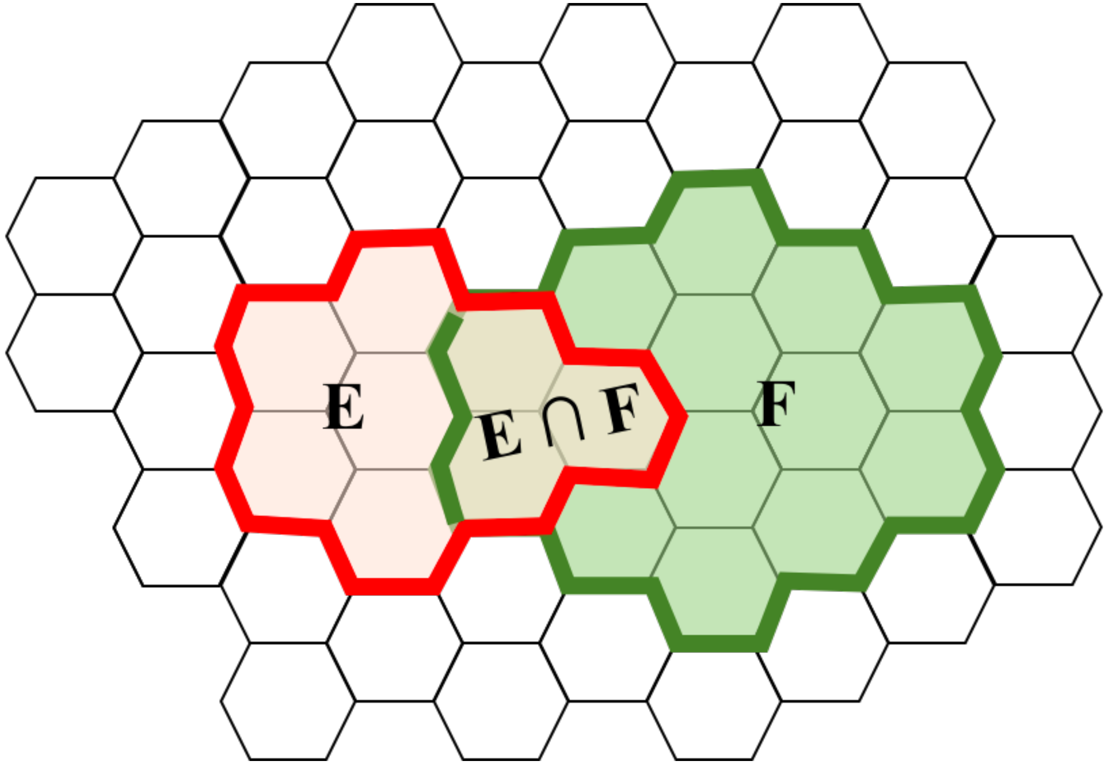
Means: "$P(E,$ given F already observed)"

Sample space → all possible outcomes consistent with F (i.e. $S \cap F$)

Event → all outcomes in E consistent with F (i.e. $E \cap F$)

Conditional Probability, visual intuition

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$

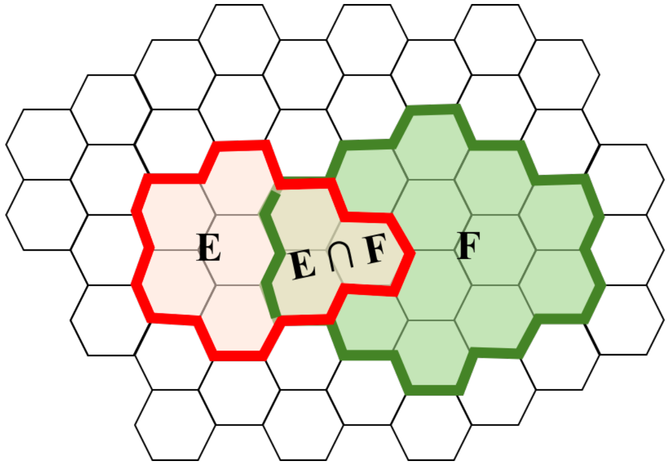
Conditional Probability, equally likely outcomes

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Shorthand notation for set intersection (aka set "and")

With equally likely outcomes:

$$\Pr(E|F) = \frac{\#\text{ of outcomes in } E \text{ consistent with } F}{\#\text{ of outcomes in } S \text{ consistent with } F} = \frac{|EF|}{|SF|} = \frac{|EF|}{|F|}$$



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$

Conditional probability in general

These properties hold even when outcomes are not equally likely.

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

What if P(F) = 0?

- P(E | F) undefined
- *Congratulations! Observed impossible*

Generalized Chain Rule

$$\Pr(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots E_n)$$

$$= \Pr(E_1) \cdot \Pr(E_1|E_2) \cdot \Pr(E_3|E_1, E_2) \cdots \Pr(E_n|E_1, E_2 \ldots E_{n_1})$$

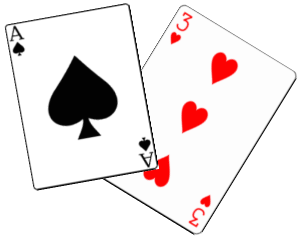
Typo, should say E2 | E1

Typo, should say n - 1

This class going forward

<u>Last week</u>

Equally likely events

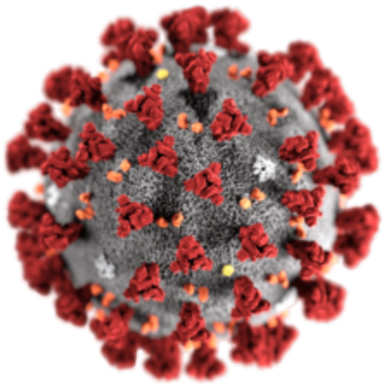


$P(E \cap F)$ $P(E \cup F)$

(counting, combinatorics)

<u>Today and for most of this course</u>

Not equally likely events



NETFLIX

and Learn

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

What is the probability that a user will watch Life is Beautiful?

$$P(E)$$

S = {Watch, Not Watch}

E = {Watch}

P(E) = ½ ?

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$$P(E)$$

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$$P(E)$$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\text{\#people who watched movie}}{\text{\#people on Netflix}}$$

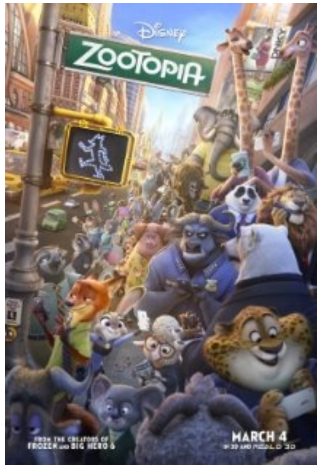
$P(E)$ = 10,234,231 / 50,923,123 = 0.20

Netflix and Learn

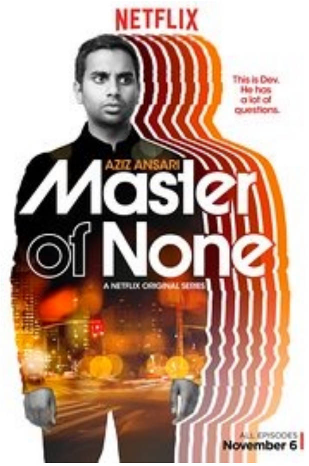
$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.

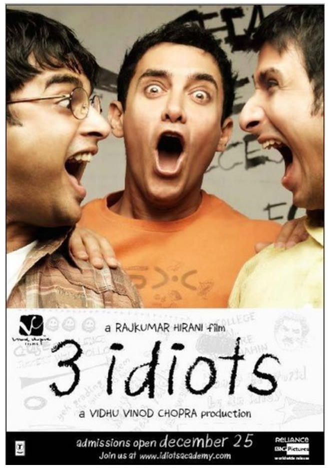
$P(E) = 0.19$



$P(E) = 0.32$



$P(E) = 0.20$



$P(E) = 0.09$

$P(E) = 0.20$

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

Let F = a user watches CODA.

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F)$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{\text{\# people who have watched both}}{\text{\# people on Netflix}}}{\frac{\text{\# people who have watched Amelie}}{\text{\# people on Netflix}}}$$

$$= \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}}$$

$$\approx 0.42$$

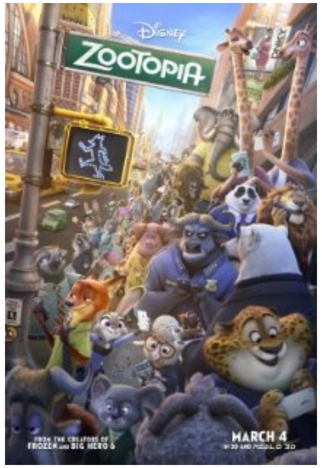
Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches CODA (2021).

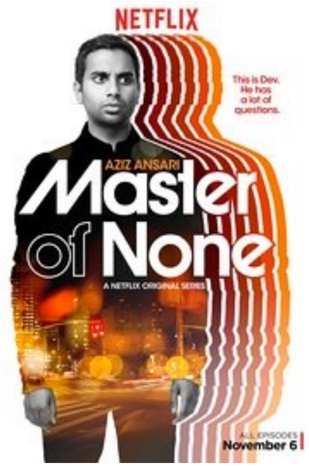
$P(E) = 0.19$

$P(E|F) = 0.14$



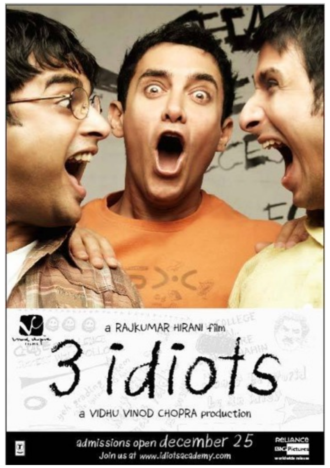
$P(E) = 0.32$

$P(E|F) = 0.35$



$P(E) = 0.20$

$P(E|F) = 0.20$



$P(E) = 0.09$

$P(E|F) = 0.72$

$P(E) = 0.20$

$P(E|F) = 0.42$

Machine Learning

Machine Learning is:
Probability + Data + Computers

Law of Total Probability

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule (Product rule)

Definition of conditional probability

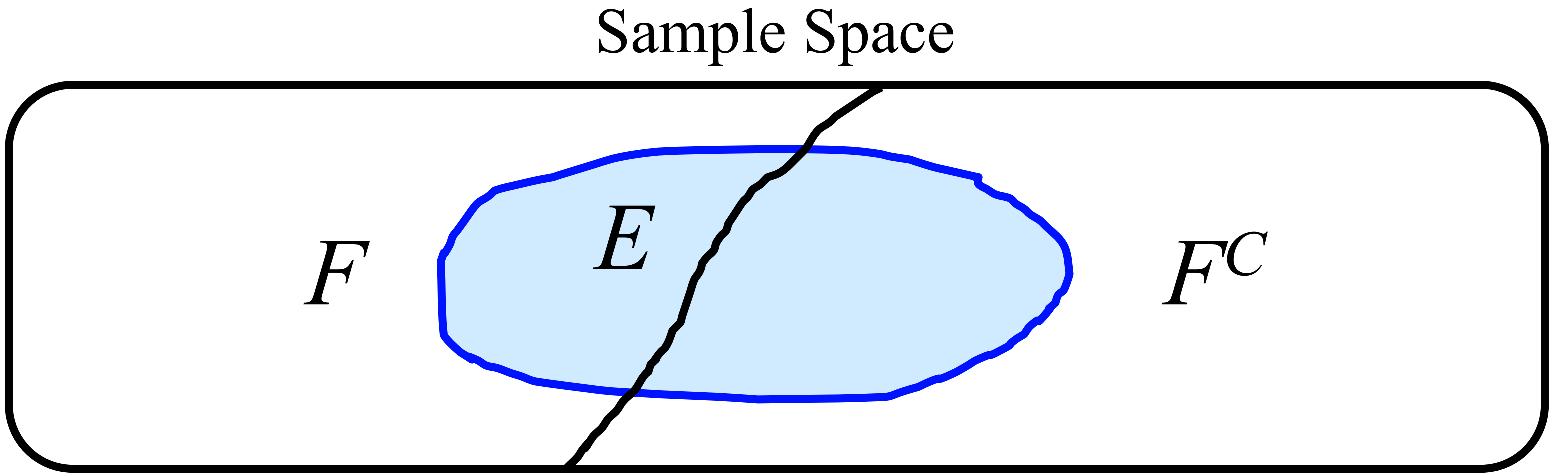
$$P(E|F)$$

Law of Total Probability

$$P(E)$$

Law of Total Probability

Say E and F are events in S



$$E = EF \cup EF^c$$

$$P(E) = P(EF) + P(EF^c)?$$

Law of Total Probability

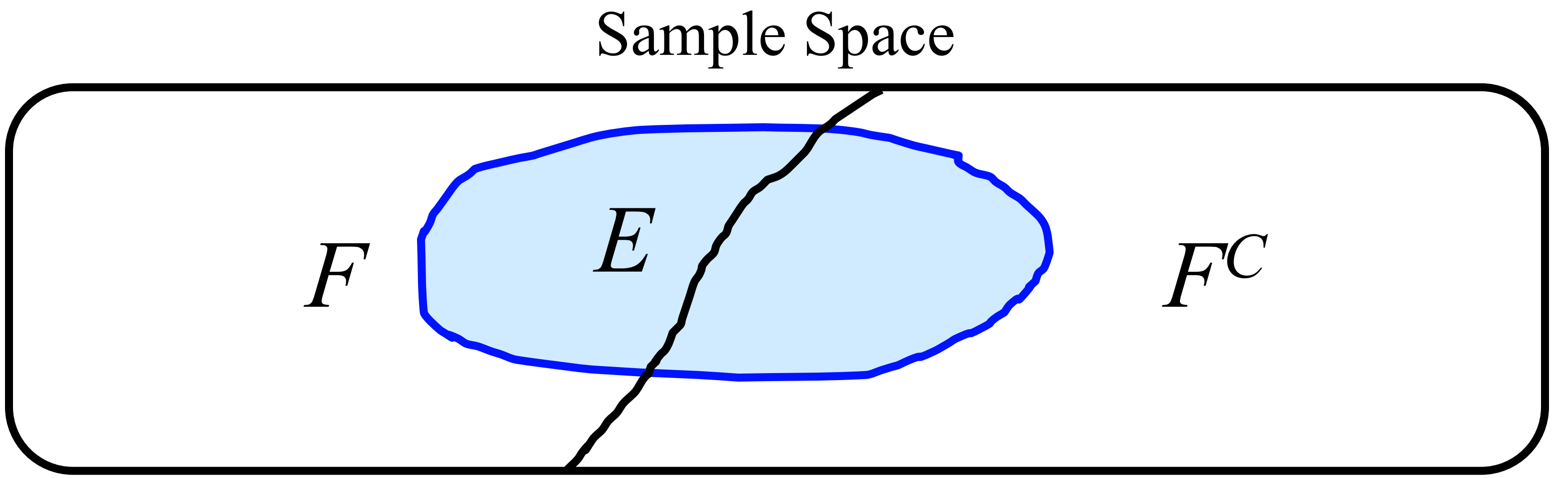
Thm Let F be an event where $P(F) > 0$. For any event E,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

Proof

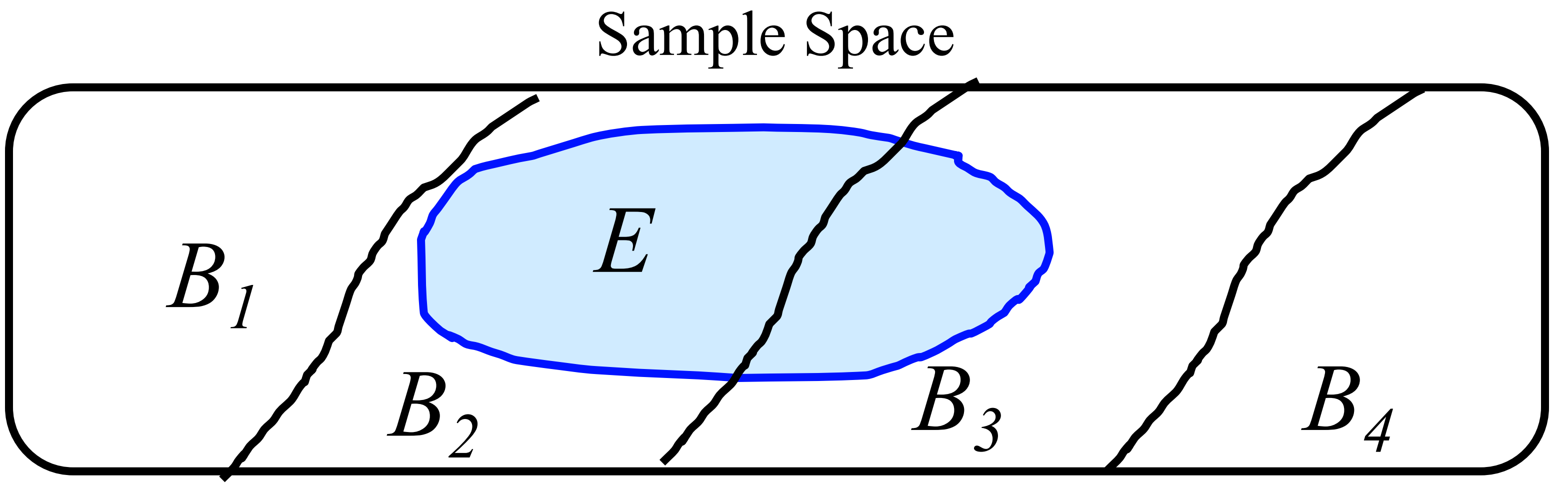
1. $E = (EF)$ or (EF^C) — Since F and F^C are disjoint
2. $P(E) = P(EF) + P(EF^C)$ — Probability of **or** for disjoint
3. $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ — Chain rule (product rule)

Law of Total Probability



$$P(E) = P(EF) + P(EF^C)$$
$$= P(E|F)P(F) + P(E|F^C)P(F^C)$$

Law of Total Probability



Thm

For mutually exclusive events $B_1, B_2, \ldots, B_n$
s.t. $B_1 \cup B_2 \cup \cdots \cup B_n = S$,

$$P(E) = \sum_i P(B_i \cap E)$$

$$= \sum_i P(E|B_i)P(B_i)$$

Evolution of Bacteria

$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ Law of Total Probability

You have bacteria in your gut which is causing a disease.
10% have a mutation which makes them resistant to anti-biotics
You take half a course of anti-biotics...

Probability a bacteria survives given it has the mutation: 20%
Probability a bacteria survives given it doesn't have the mutation: 1%
What is the probability that a randomly chosen bacteria survives?

Let E be the event that our bacterium survives. Let M be the event that a bacteria has the mutation. By the By Law of Total Probability (LOTP):

$$\begin{aligned}
\Pr(E) &= \Pr(E \text{ and } M) + \Pr(E \text{ and } M^C) && \text{LOTP} \\
&= \Pr(E|M)\Pr(M) + \Pr(E|M^C)\Pr(M^C) && \text{Chain Rule} \\
&= 0.20 \cdot 0.10 + 0.01 \cdot 0.90 && \text{Substituting} \\
&= 0.029
\end{aligned}$$

Real question. What is the probability that a surviving bacteria has the mutation?

$$Pr(\text{Mutation} \mid \text{Survives})$$

$$Pr(M \mid E)$$

Real Question: Pr(*M* | *E*)?

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&= 0.029
\end{aligned}
$$

Real Question: Pr(*M* | *E*)?

You have bacteria in your gut which is causing a disease.
10% have a mutation which makes them resistant to anti-biotics
You take half a course of anti-biotics…

$\Pr(E \mid M) = 0.20$
$\Pr(E \mid M^C) = 0.01$
What is the probability that a randomly chosen bacteria survives?

Let E be the event that our bacterium survives. Let M be the event that a bacteria has the mutation. By the By Law of Total Probability (LOTP):

$$\begin{aligned}
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\end{aligned}$$

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule (Product rule) ↑

↓ Definition of conditional probability

$$P(E|F)$$

Law of Total Probability ↙

$$P(E)$$

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule (Product rule) ↑ ↓ Definition of conditional probability

$$P(E|F)$$

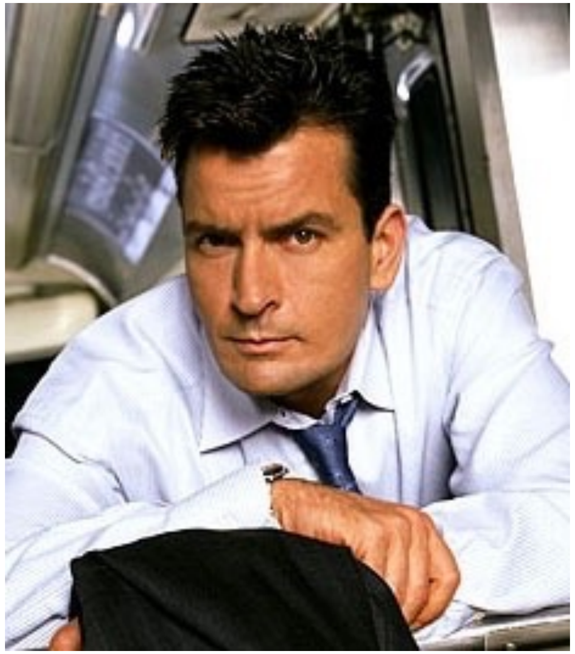
Law of Total Probability ↙ ↘ Bayes' Theorem

$$P(E) \qquad P(F|E)$$

Bayes' Theorem

Thomas Bayes

Rev. Thomas Bayes (~1701-1761):
British mathematician and Presbyterian minister



He looked remarkably similar to Charlie Sheen

(but that's not important right now)

Thomas Bayes

$P(F \mid E)$

I want to calculate
P(State of the world F | Observation E)
It seems so tricky!…

The other way around is easy
P(Observation E | State of the world F)
What options to I have, chief?

$P(E \mid F)$

Thomas Bayes

Want $P(F \mid E)$. Know $P(E \mid F)$

$$P(F|E) = \frac{P(EF)}{P(E)} \quad \text{Def. of Conditional Prob.}$$

A little while later…

$$= \frac{P(E|F)P(F)}{P(E)} \quad \text{Chain Rule}$$

Bayes' Theorem

$P(E|F)$ ⇒ $P(F|E)$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Proof

2 steps! See board

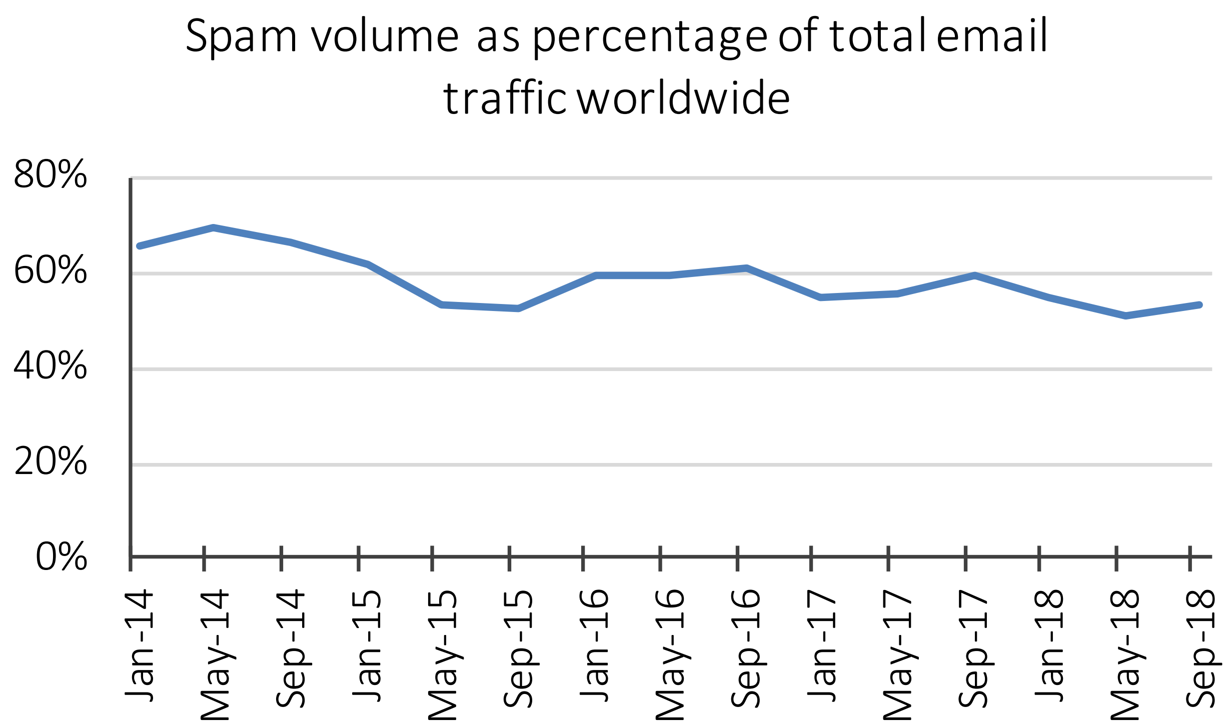
Expanded form:

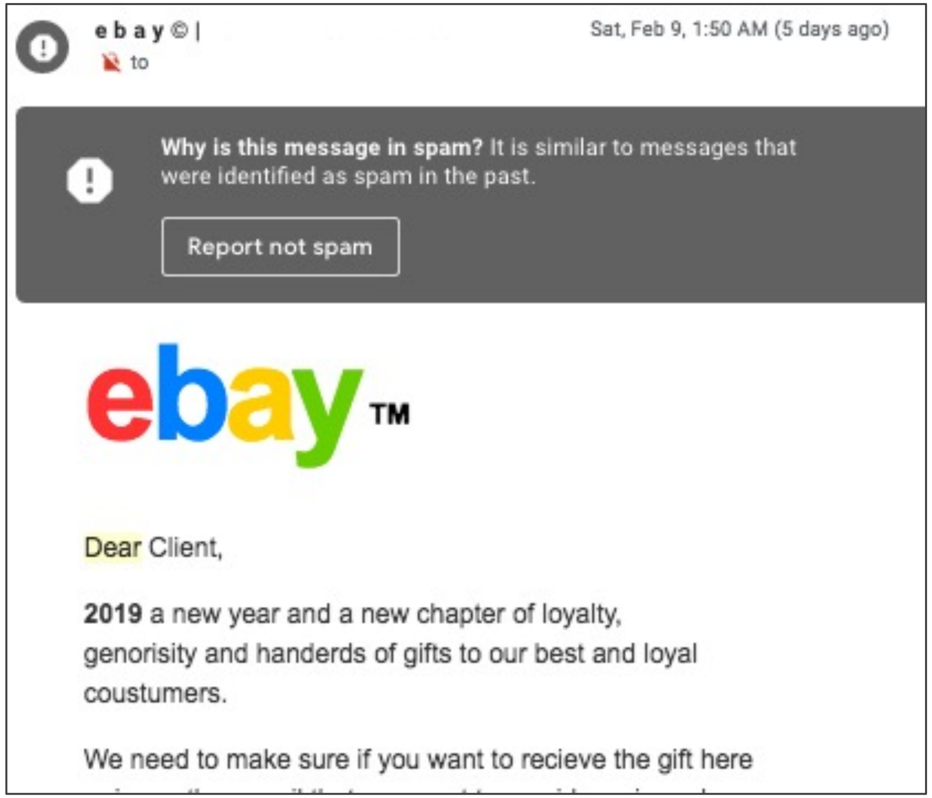
$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

Proof

1 more step! See board

Detecting spam email





We can easily calculate how many spam emails contain "Dear":

$$P(E|F) = P\left(\text{“Dear”} \,\middle|\, \begin{matrix}\text{Spam}\\ \text{email}\end{matrix}\right)$$

But what is the probability that an email containing "Dear" is spam?

$$P(F|E) = P\left(\begin{matrix}\text{Spam}\\ \text{email}\end{matrix} \,\middle|\, \text{“Dear”}\right)$$

(silent drumroll)

Detecting spam email

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- 60% of all email in 2016 is spam.
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear" in it.

What is the probability that the email is spam?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Let: E: "Dear", F: spam

Want: P(spam | "Dear")

$= P(F|E)$

Bayes' Theorem terminology

- 60% of all email in 2016 is spam. $P(F)$
- 20% of spam has the word "Dear" $P(E|F)$
- 1% of non-spam (aka ham) has the word "Dear" $P(E|F^C)$

You get an email with the word "Dear" in it.

What is the probability that the email is spam? Want: $P(F|E)$

likelihood prior

posterior

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

normalization constant

SARS Virus Testing

A test is 98% effective at detecting SARS

- However, test has a "false positive" rate of 1%
- 0.5% of US population has SARS
- Let E = you test positive for SARS with this test
- Let F = you actually have SARS
- What is P(F | E)?

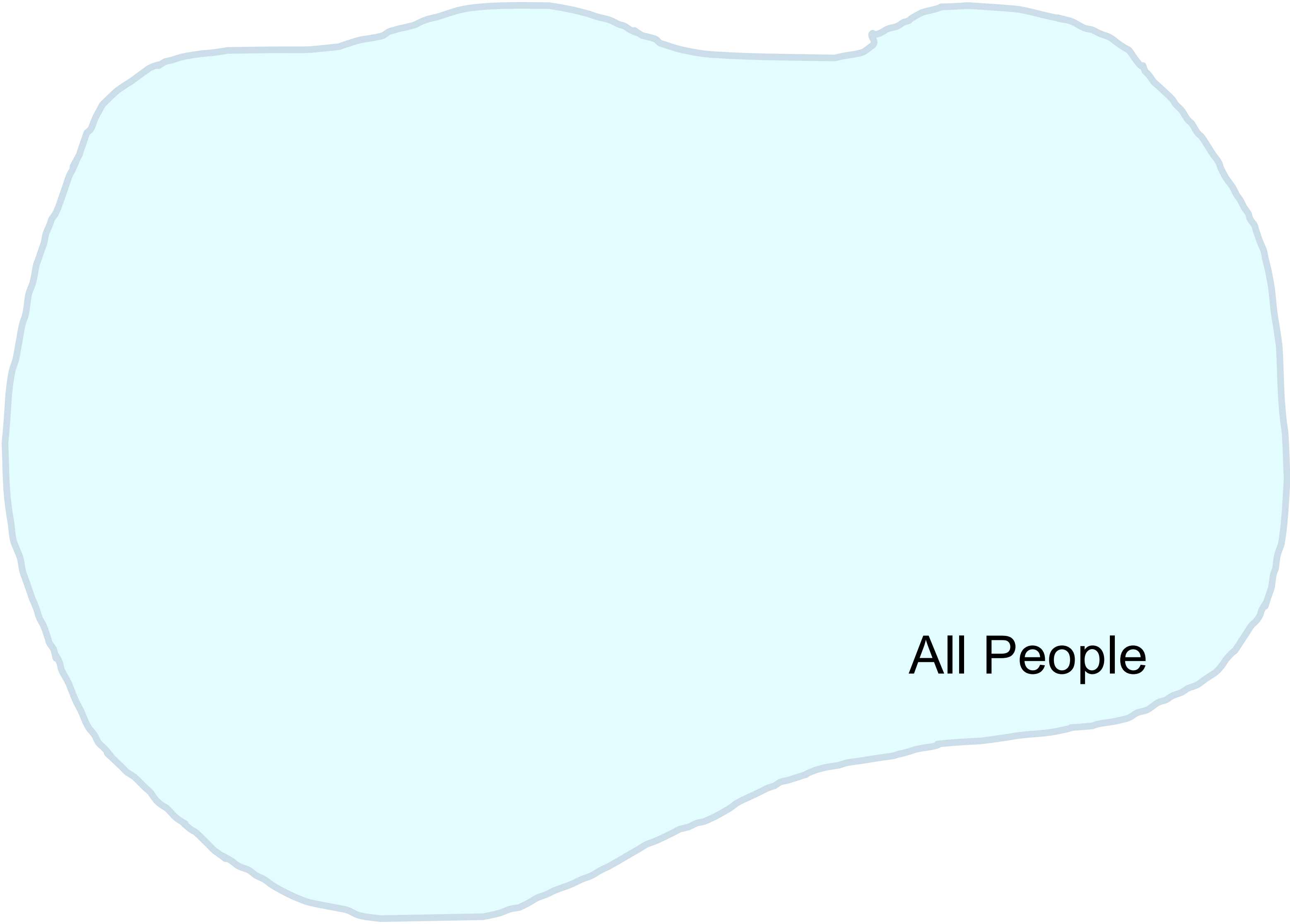
Solution:

$$P(F \mid E) = \frac{P(E \mid F)\, P(F)}{P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c)}$$

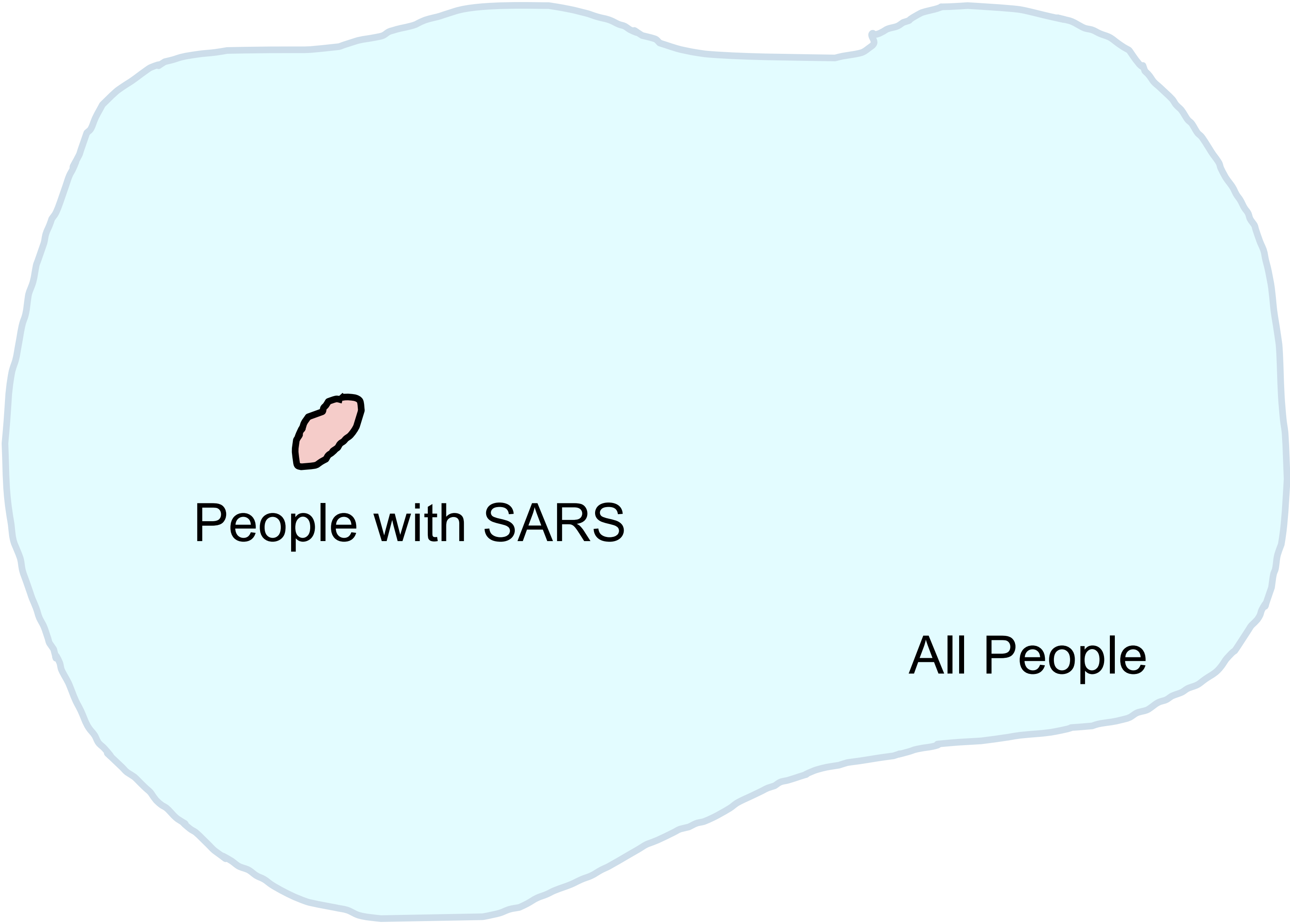
$$P(F \mid E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \approx 0.330$$

Intuition Time

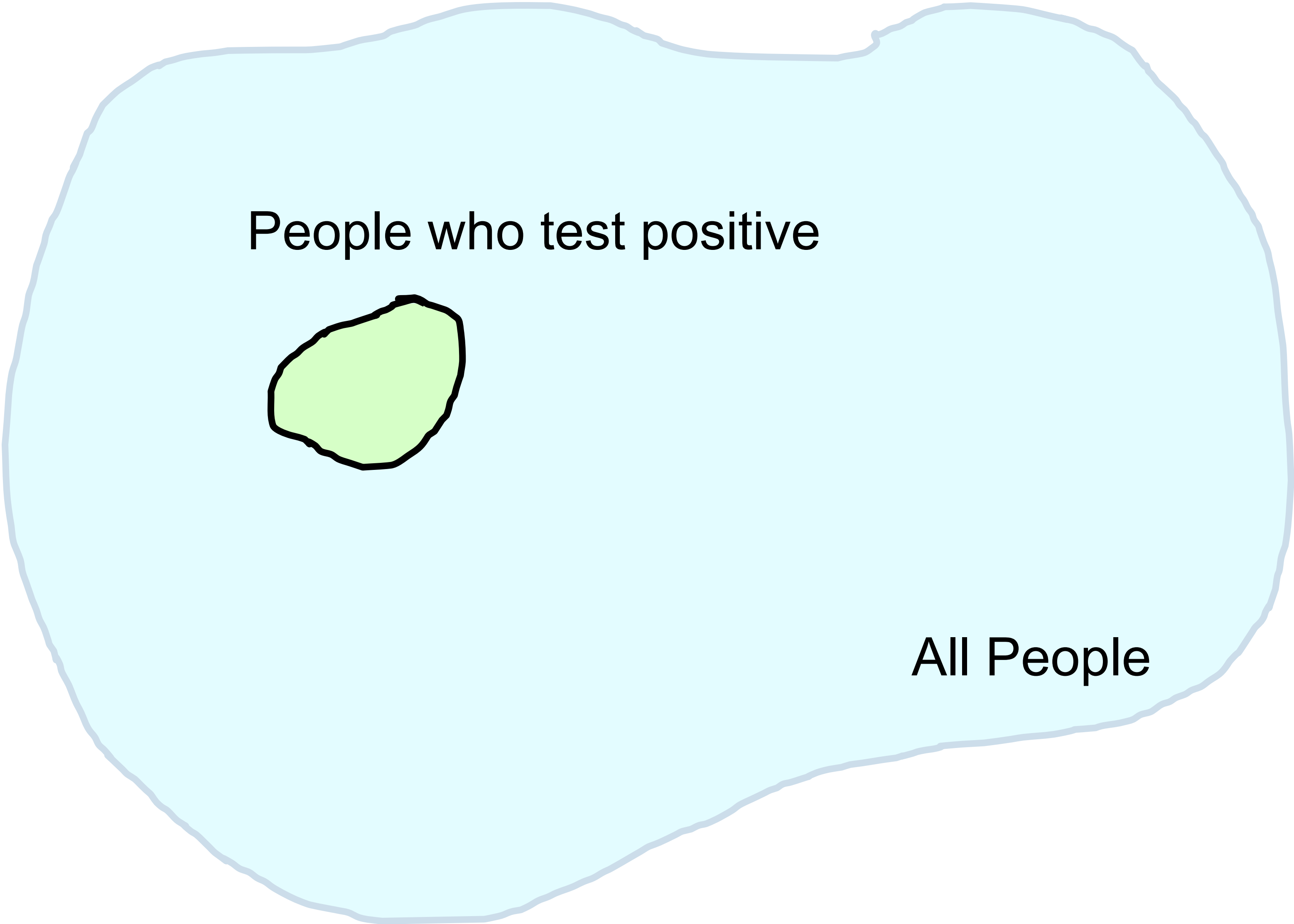
Bayes Thorem Intuition



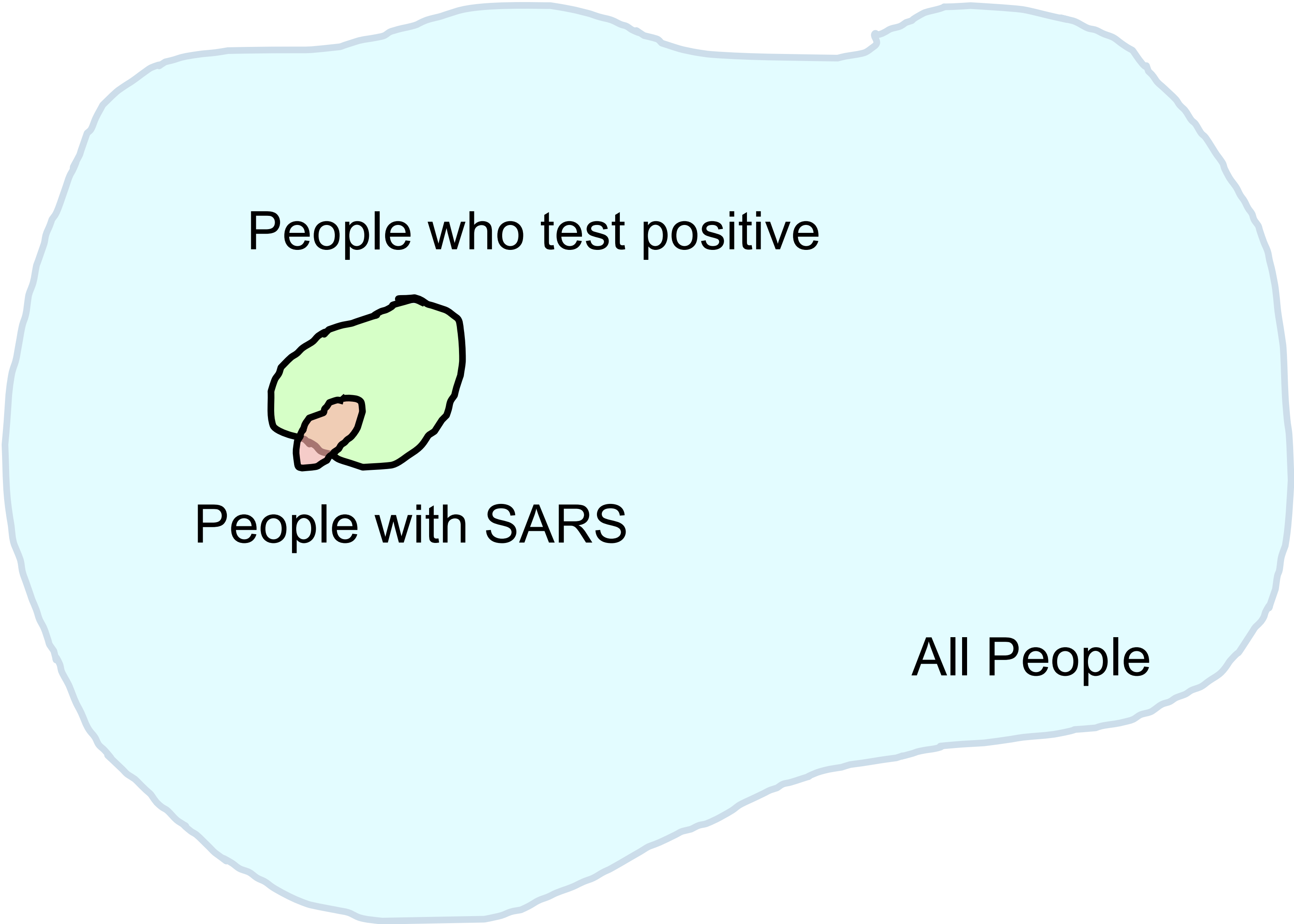
Bayes Thorem Intuition



Bayes Thorem Intuition

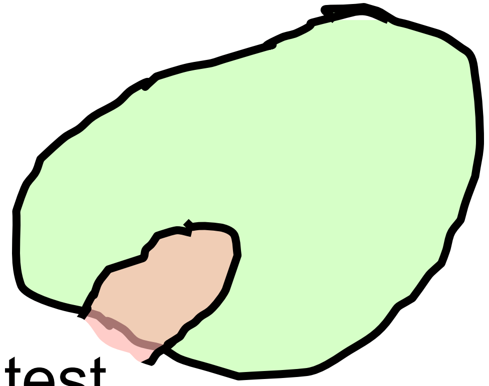


Bayes Thorem Intuition



Bayes Thorem Intuition

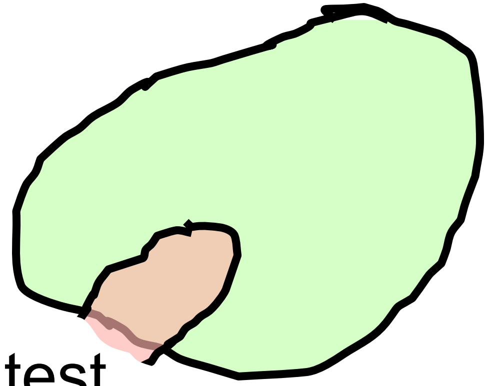
Conditioning on a positive result changes the sample space to this:



≈ 0.330

Bayes Thorem Intuition

Conditioning on a positive result changes the sample space to this:



People who test positive

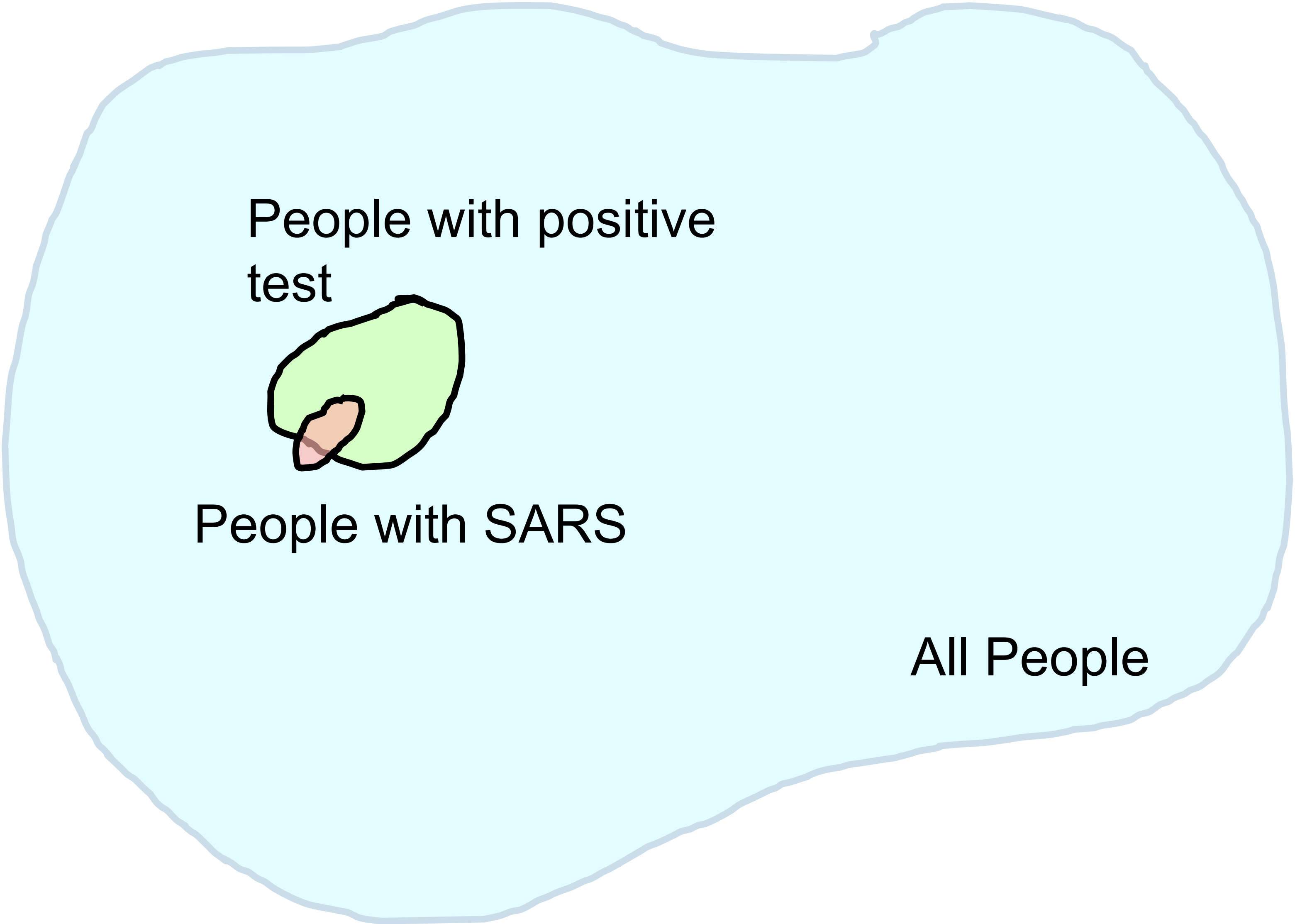
$$P(F)P(E|F) + P(F^c)P(E|F^c)$$

People who test positive and have SARS

$$P(F)P(E|F)$$

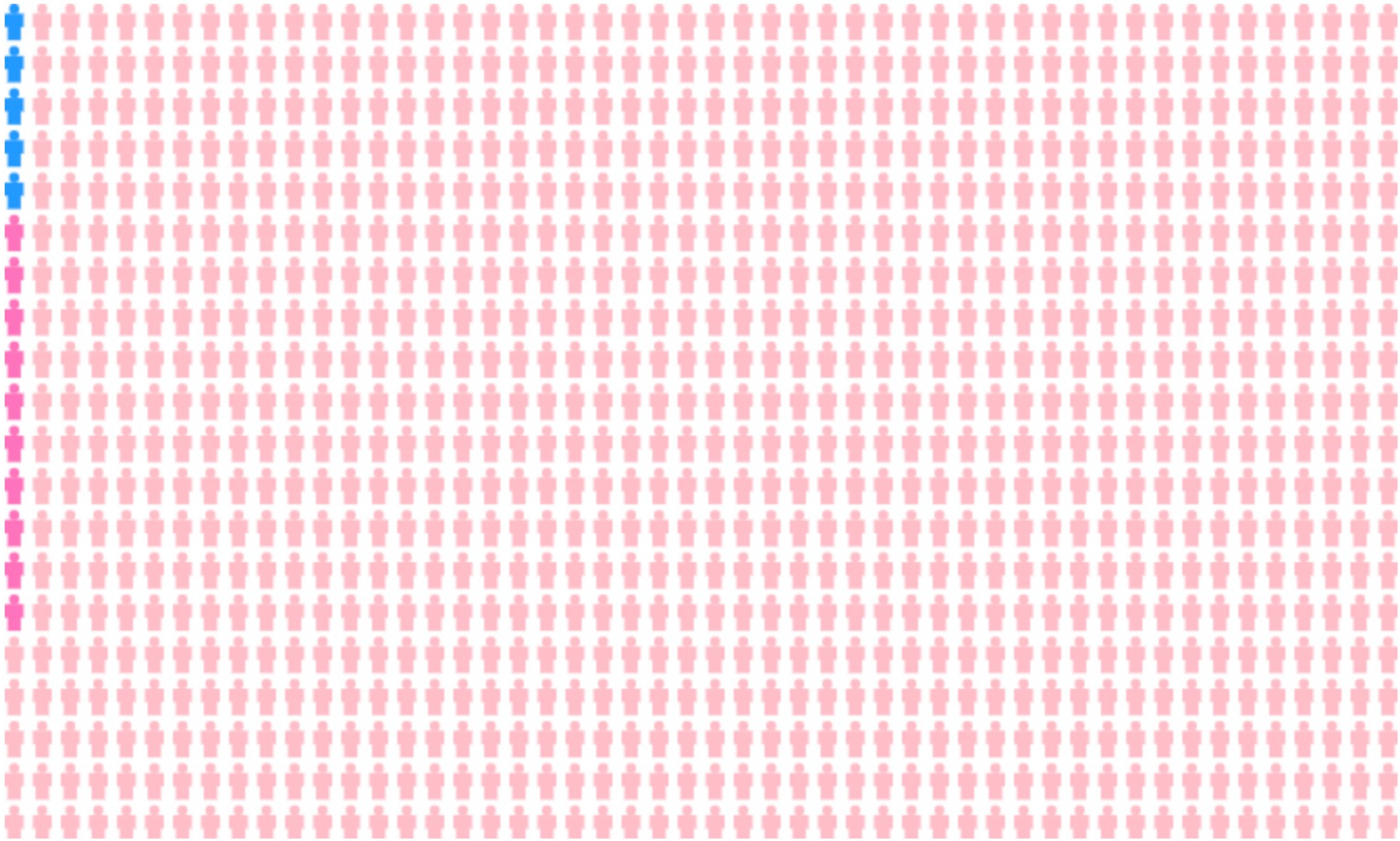
≈ 0.330

Bayes Thorem Intuition



Bayes Thorem Intuition

Say we have 1000 people:



5 have SARS and test positive, 985 **do not** have SARS and test negative.

10 **do not** have SARS and test positive.

≈ 0.333

Bayes Thorem Intuition

Conditioned on just those that test positive:

Notice that all the people with SARS are here, but the group is still mainly folks without SARS

5 have SARS and test positive, 985 **do not** have SARS and test negative.
10 **do not** have SARS and test positive.

≈ 0.333

Why it is still good to get tested

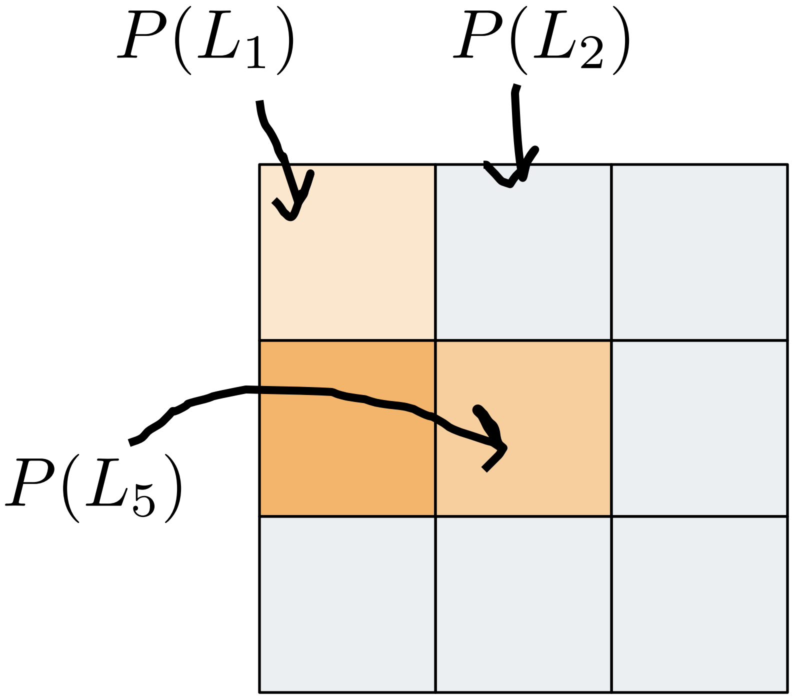
	SARS +	SARS –
Test +	$0.98 = P(E \mid F)$	$0.01 = P(E \mid F^c)$
Test –	$0.02 = P(E^c \mid F)$	$0.99 = P(E^c \mid F^c)$

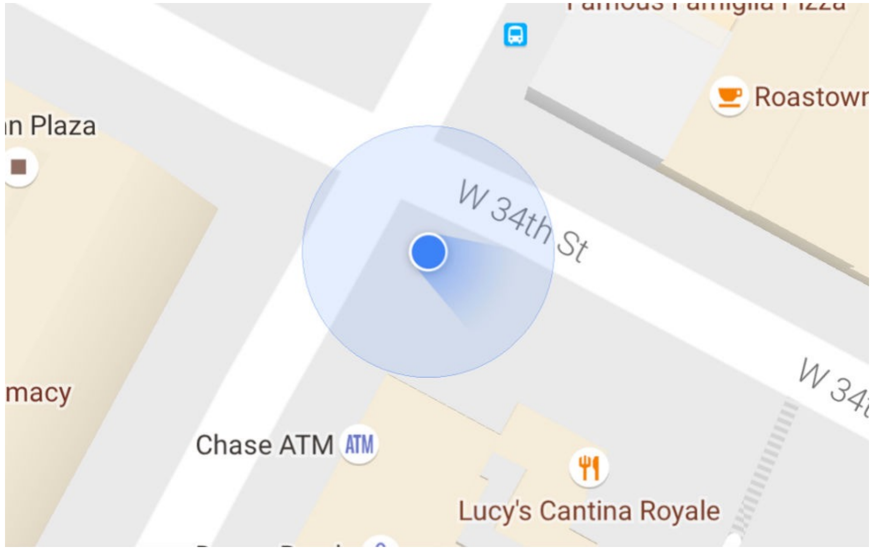
- Let E^c = you test <u>negative</u> for SARS with this test
- Let F = you actually have SARS
- What is $P(F \mid E^c)$?

$$P(F \mid E^c) = \frac{P(E^c \mid F)\, P(F)}{P(E^c \mid F)\, P(F) + P(E^c \mid F^c)\, P(F^c)}$$

$$P(F \mid E^c) = \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \approx 0.0001$$

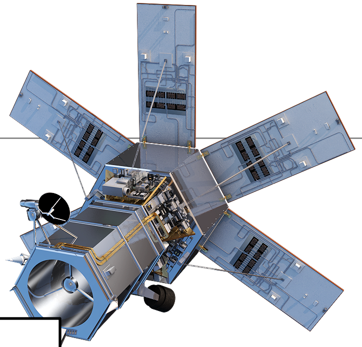
Bayes' Theorem and Location



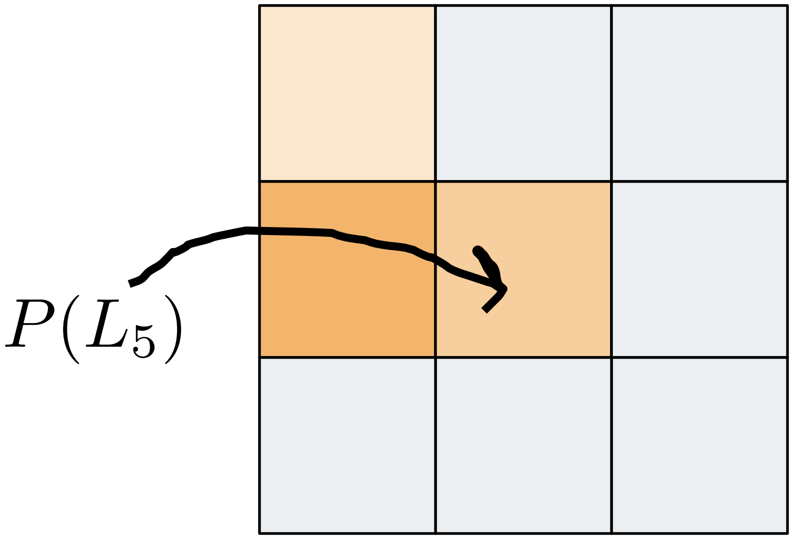


Before Observation

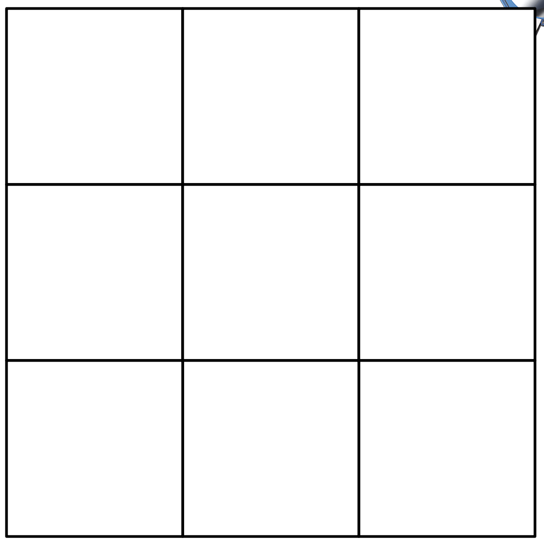
Bayes' Theorem and Location



Know: $P(O|L_i)$

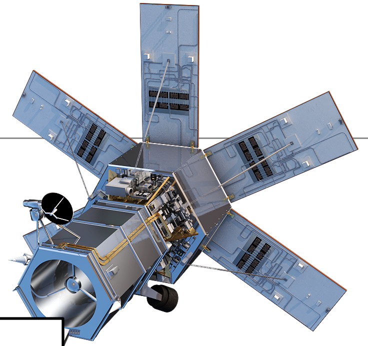


Before Observation

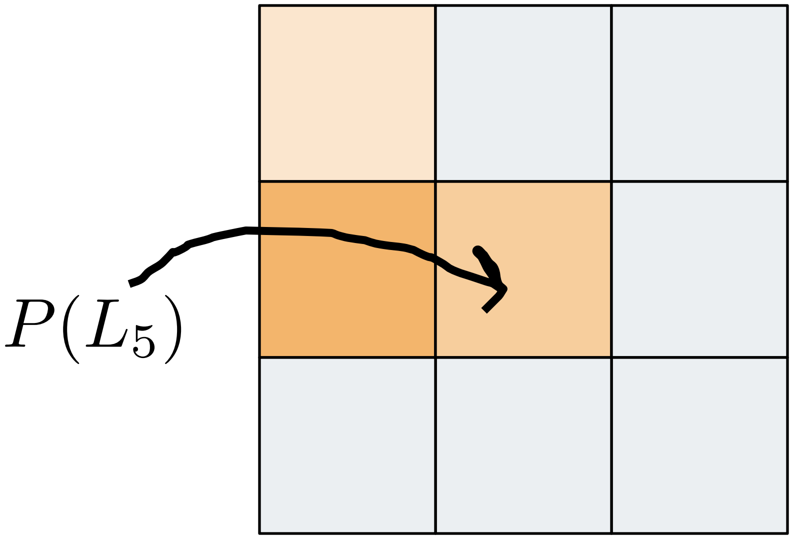


After Observation

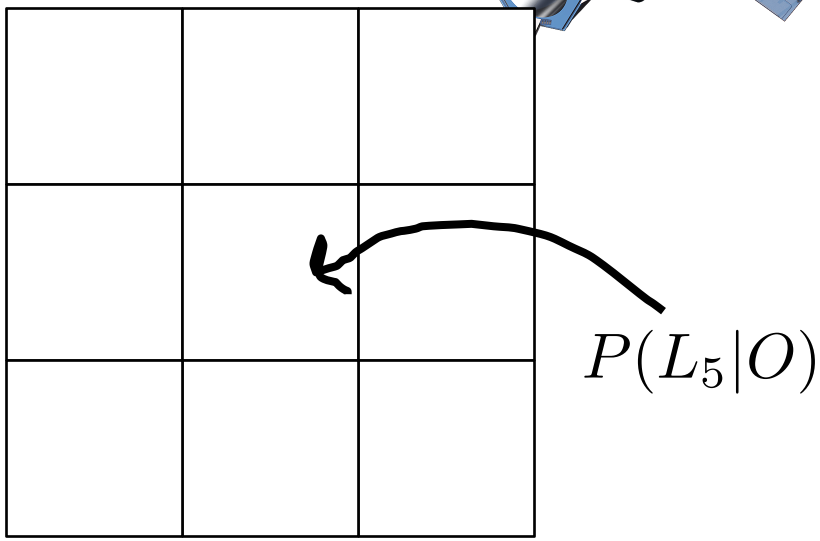
Bayes' Theorem and Location



Know: $P(O|L_i)$



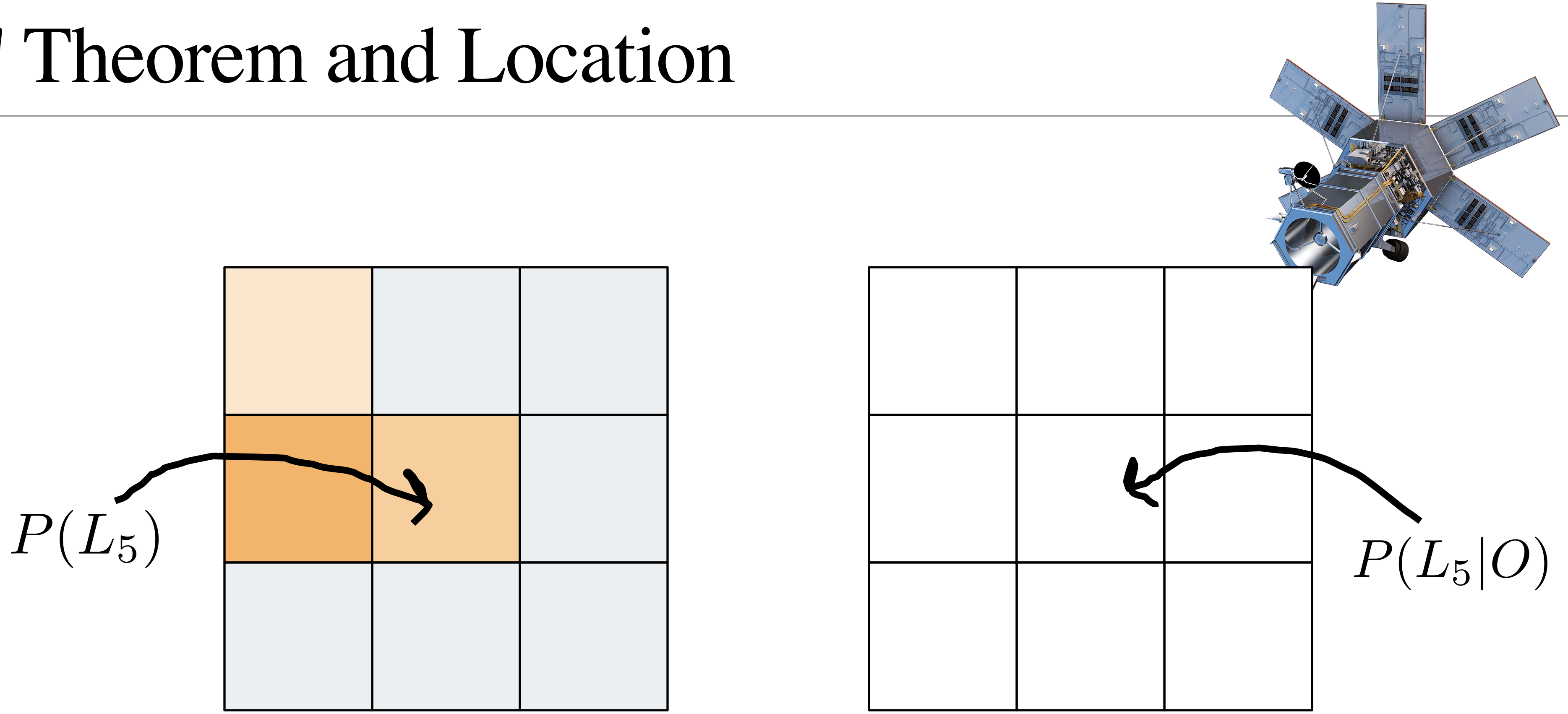
Before Observation



After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{P(O)}$$

Bayes' Theorem and Location

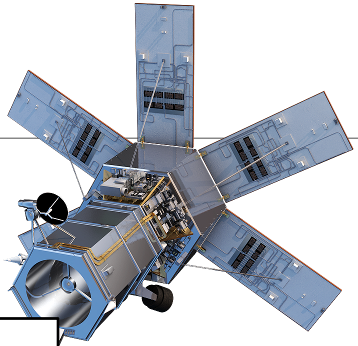


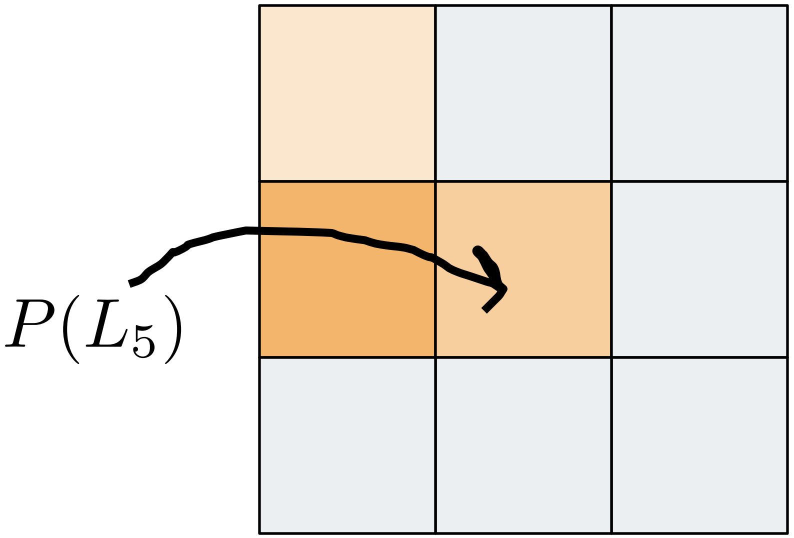
Before Observation

After Observation

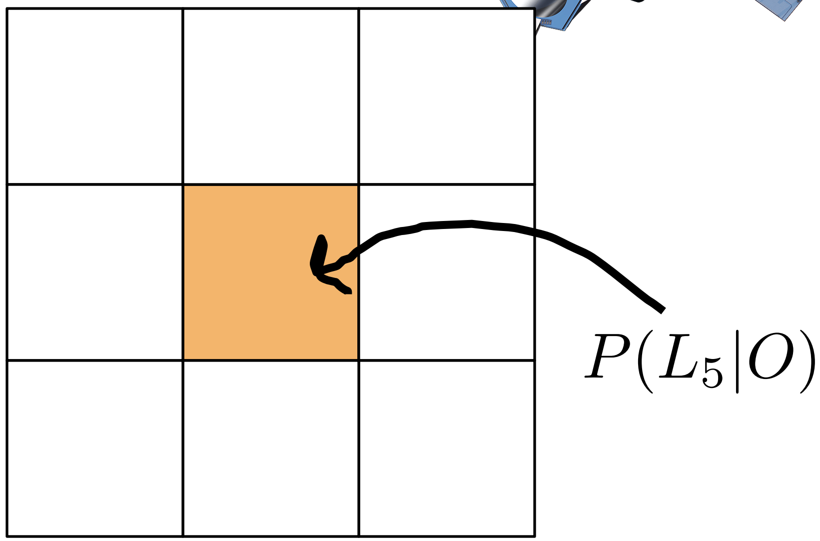
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Bayes' Theorem and Location





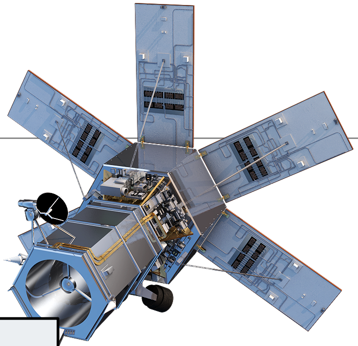
Before Observation

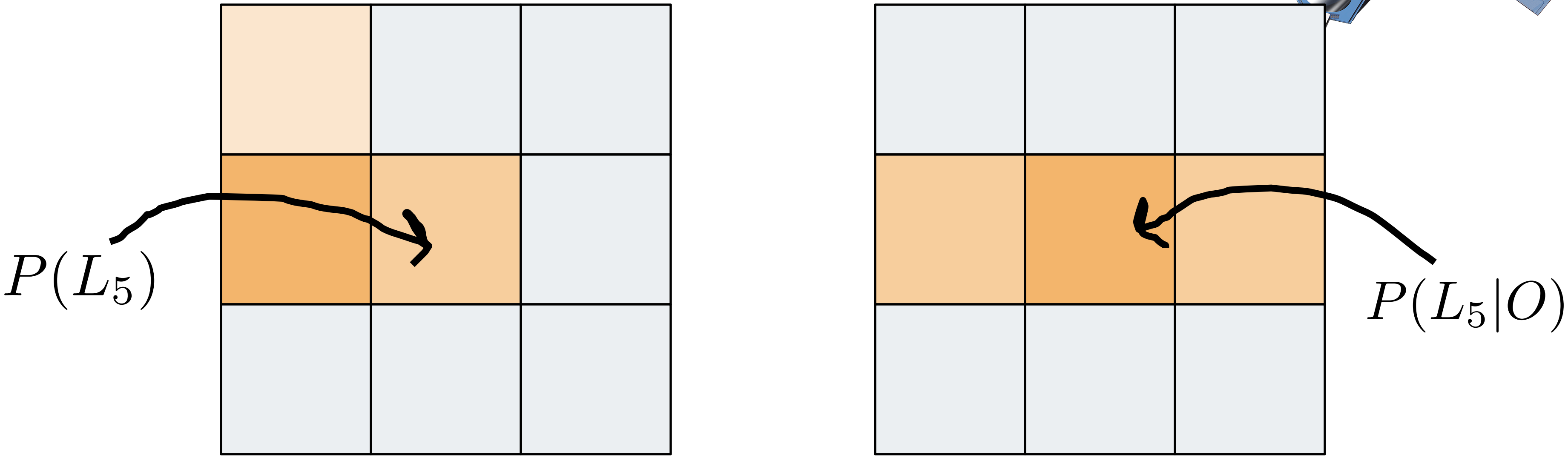


After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Bayes' Theorem and Location



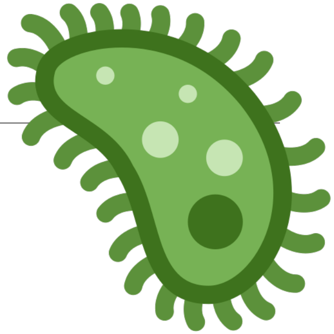


Before Observation

After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Smell Test for Coronavirus?



What is the probability of having corona virus given that you **can** still smell? Leave your answer in terms of the probability of covid.

The probability of **Anosmia given covid** is 0.7,

The probability of **Anosmia before covid** was 0.01 (assumed to be the probability of Anosmia given no Covid).

Let A be the event of Anosmia,

Let S be the event that you can smell, aka A^C

C be the event of Covid

Monty Hall Problem

Monty Hall Problem

and Wayne Brady

Monty Hall Problem aka Let's Make a Deal

Behind one door is a prize (equally likely to be any door).

Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Should we switch?

Doors A,B,C

Note: If we don't switch, P(win) = 1/3 (random)

We are comparing P(win) and P(win|switch).

If we switch

Without loss of generality, say we pick A (out of Doors A,B,C).

1/3 1/3 1/3

A = prize

- Host opens B or C
- We switch
- We always lose

P(win | A prize, picked A, switched) = 0

B = prize

- Host must open C
- We switch to B
- We always win

P(win | B prize, picked A, switched) = 1

C = prize

- Host must open B
- We switch to C
- We always win

P(win | C prize, picked A, switched) = 1

P(win | picked A, switched) = 1/3 * 0 + 1/3 * 1 + 1/3 * 1 = 2/3

You should switch.

Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

$$\begin{cases} \frac{1}{1000} = P(\text{envelope is prize}) \\ \frac{999}{1000} = P(\text{other 999 envelopes have prize}) \end{cases}$$

2. I open 998 of remaining 999 (showing they are empty).

$$\frac{999}{1000} = P(\text{998 empty envelopes had prize}) + P(\text{last other envelope has prize})$$

$$= P(\text{last other envelope has prize})$$

3. Should you switch?

No: P(win without switching) = $\frac{1}{\text{original \# envelopes}}$

Yes: P(win with new knowledge) = $\frac{\text{original \# envelopes - 1}}{\text{original \# envelopes}}$

Marilyn Vos Savant