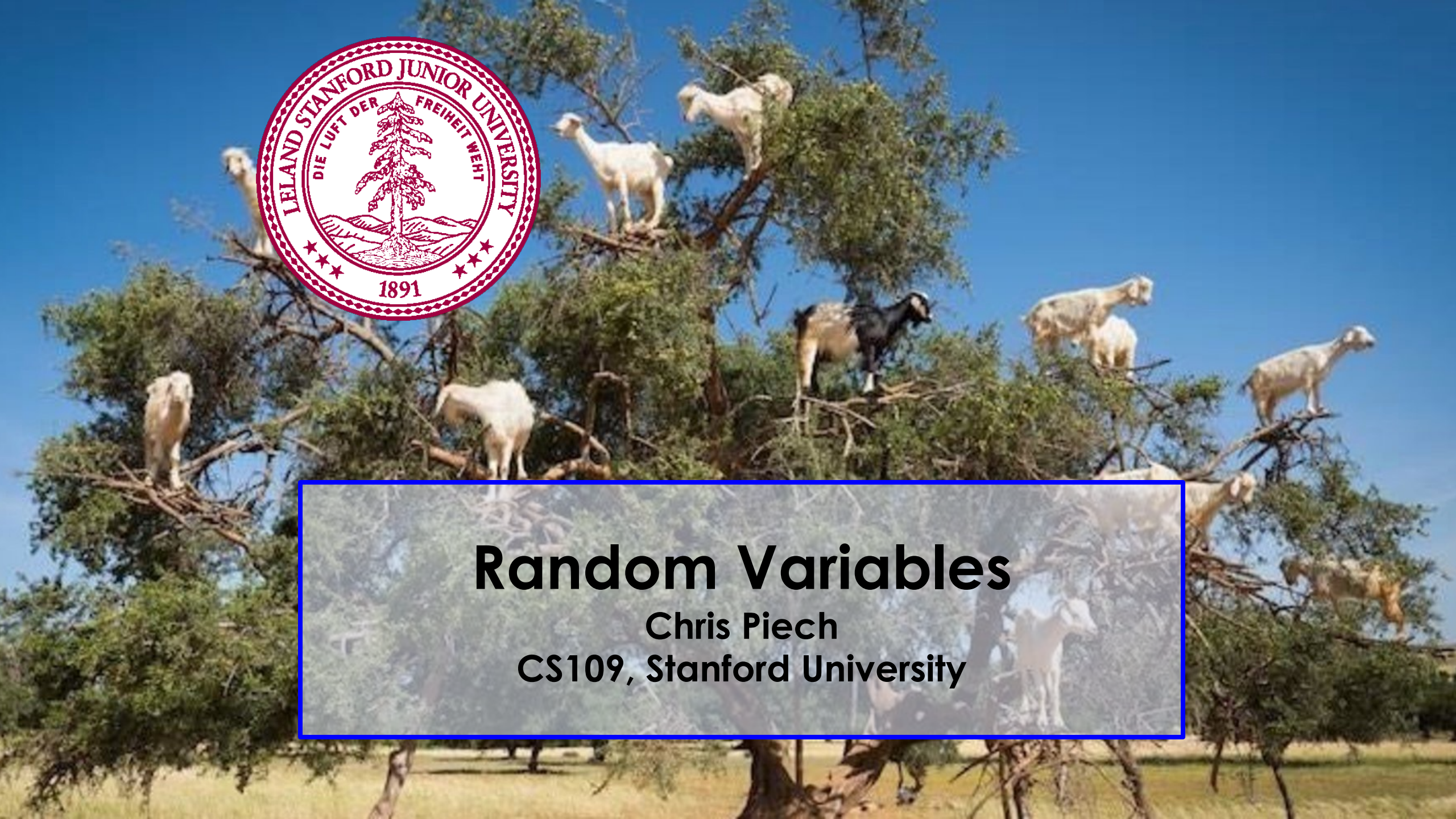
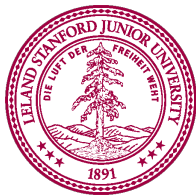


Random Variables

Chris Piech
CS109, Stanford University

Announcements

- Pset #1 in today
- Pset #2 out today
- Concept Checks (and recordings)
- Happy Friday!

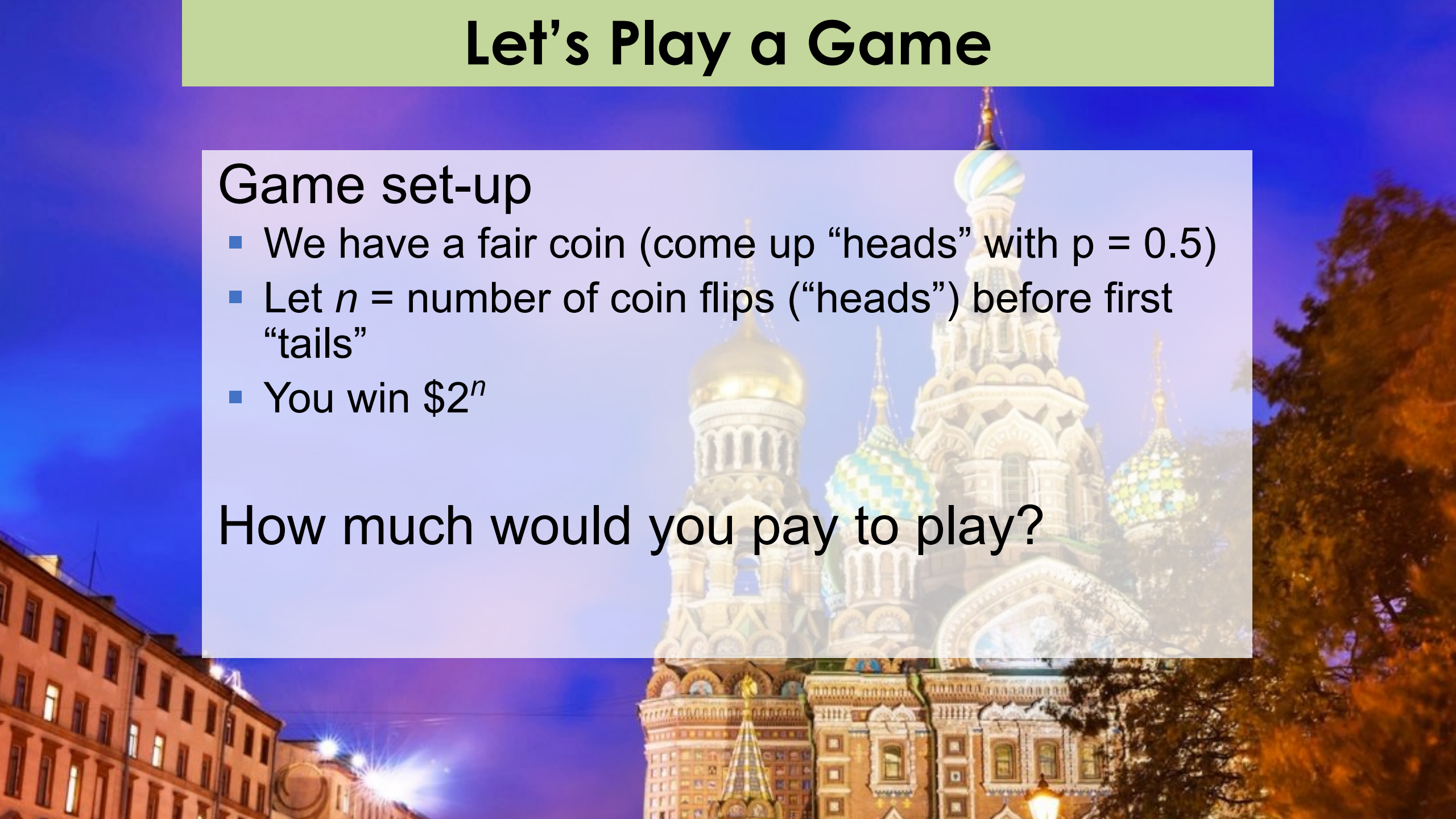


Story: Ultimate Probability

https://www.maikaisogawa.com/ultimate-frisbee-probability/

https://www.youtube.com/watch?v=H2IfTwGisOg

Let's Play a Game



Game set-up

- We have a fair coin (come up "heads" with p = 0.5)
- Let *n* = number of coin flips ("heads") before first "tails"
- You win $\$2^n$

How much would you pay to play?

Review!

1. Review: Axiom Probability Tools

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Identity 3: $P(E^c) = 1 - P(E)$

2. Review: Conditional Probability Tools

$$P(E \text{ and } F)$$

Chain rule
(Product rule)

Definition of
conditional probability

$$P(E|F)$$

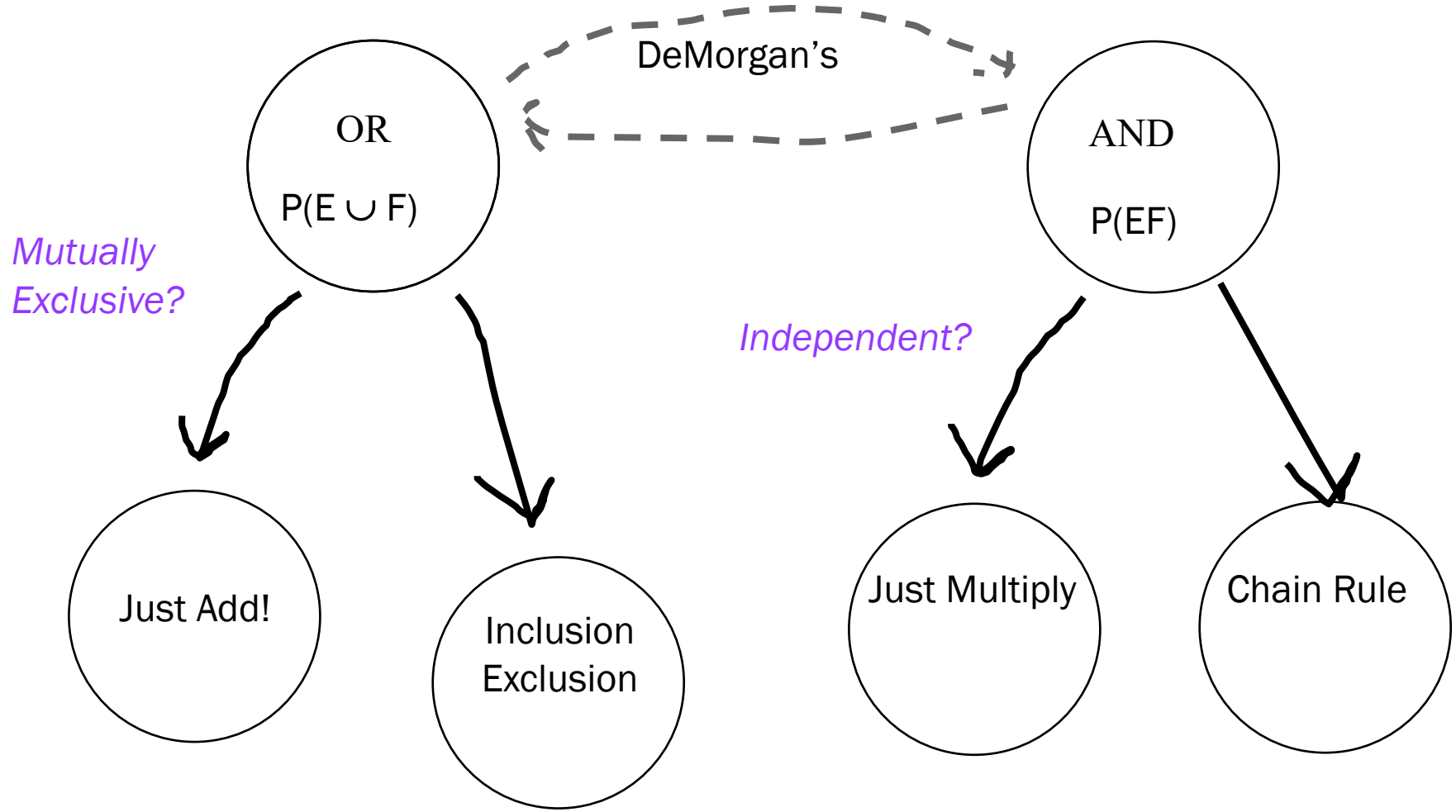
Law of Total
Probability

Bayes'
Theorem

$$P(E)$$

$$P(F|E)$$

3. Review: Probability of Or and And



Skill: Art form of Defining Events

m strings are hashed (not uniformly) into a hash table with n buckets. Each string hash is an **independent trial** with probability p_i of getting hashed into bucket i

What is the probability that **at least 1** of buckets 1 to k has ≥ 1 string hashed into it?

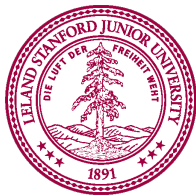
Define: F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$

Practice: Lets do the Frisbee Problem.

You flip **two frisbees**. For each frisbee, the probability that it lands "**heads**" is **0.6**. The two frisbees are considered "**even**" if both frisbees are heads **or** both frisbees are tails.

What is the probability that the frisbees are even?



Early Pedagogical Pause!

Advanced Concept:
Conditional Independence

Conditional Paradigm: In the conditional paradigm, the ***formulas*** of probability are preserved.

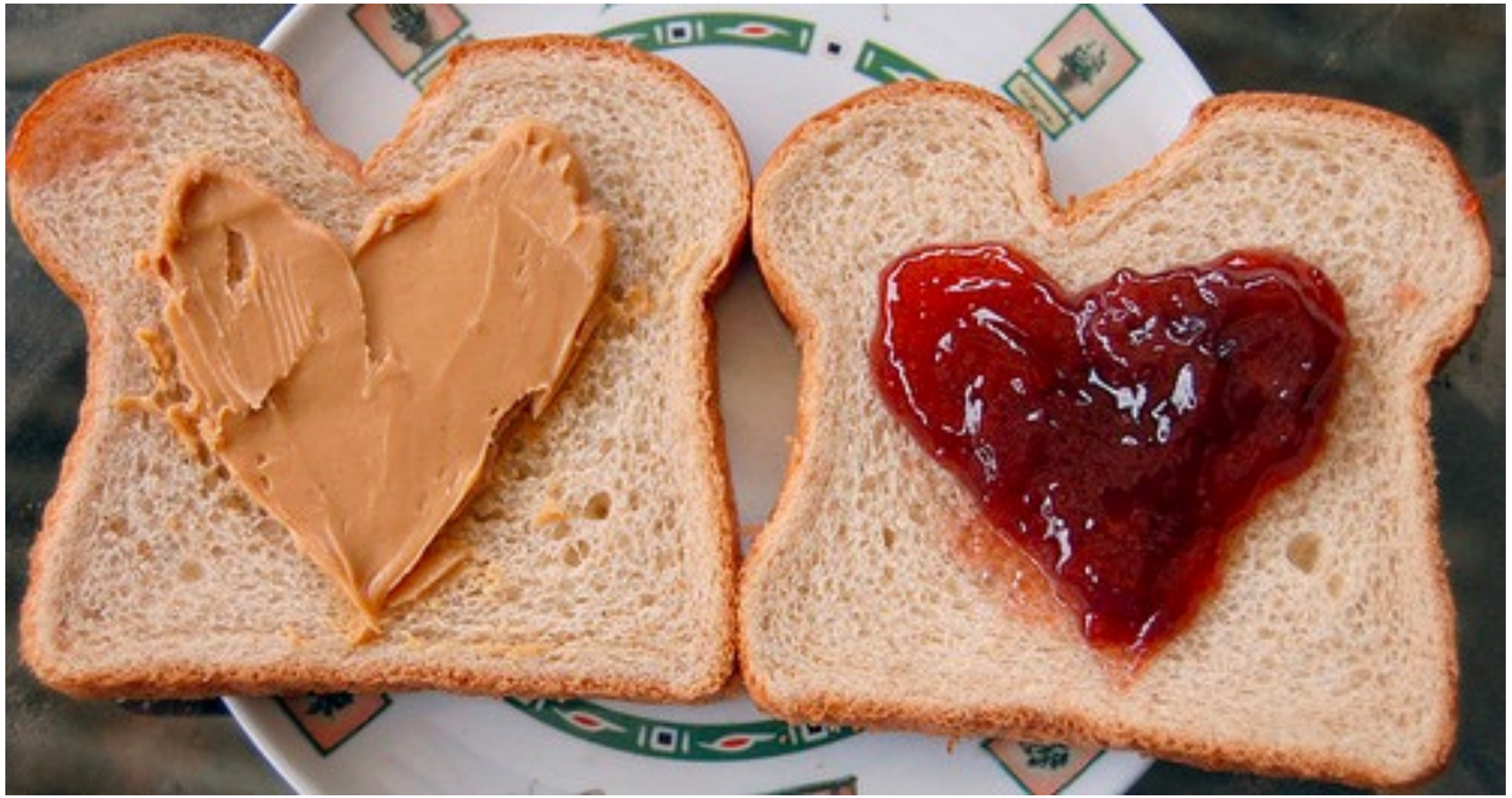
BAE's Theorem

$$P(A \mid B\ E) = \frac{P(B \mid A\ E)\ P(A \mid E)}{P(B \mid E)}$$

Two Great Tastes

Conditional Probability

Independence

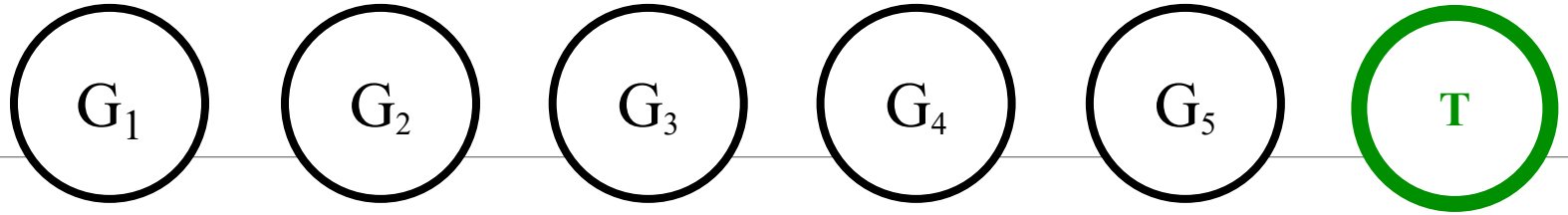


Conditional Paradigm: In the conditional paradigm, the ***formulas*** of probability are preserved.

Independence relationships can change with conditioning.

If E and F are independent, that does not mean they will still be independent given another event G.

G_1
G_2
G_3
G_4
G_5
T



dna.txt — dna

dna.txt

```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
True,False,False,True,False,False
False,True,False,True,True,False
True,False,False,True,False,False
True,False,True,True,False,False
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False,False,True,True,False,False
True,True,False,False,True,True
True,False,True,True,False,False
True,True,True,True,True,True
True,False,True,False,False,True
False,True,False,True,True,True
False,False,True,False,False,False
False,False,False,True,True,False
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True,True,False,True,True,True
False,True,False,True,True,False
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False,False,True,True,False,True
False,False,False,True,False,False
False,True,True,False,False,True
False,True,False,False,True,True
False,False,False,False,False,True
False,True,False,True,True,False
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True,True,False,True,True,True
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True,True,False,True,True,True
False,False,False,True,False,False
```
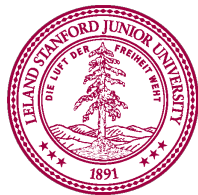
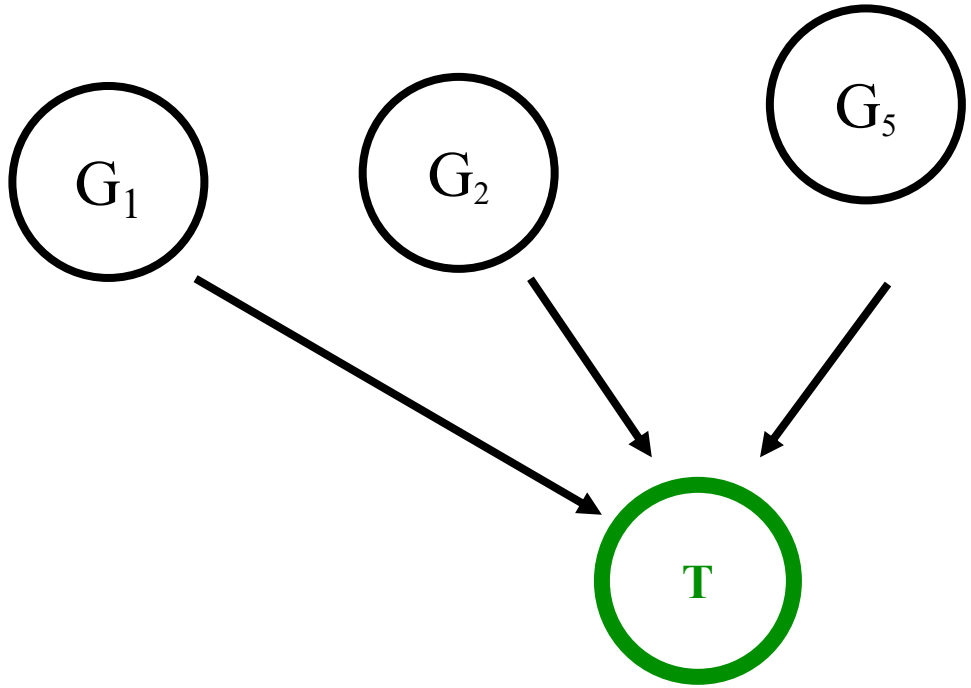

6 observations per sample

100,000 samples

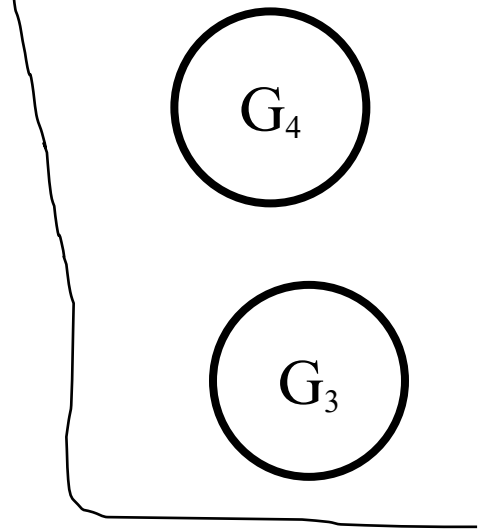


Independence Rules out Causality Structures

These genes do impact T (are not independent)



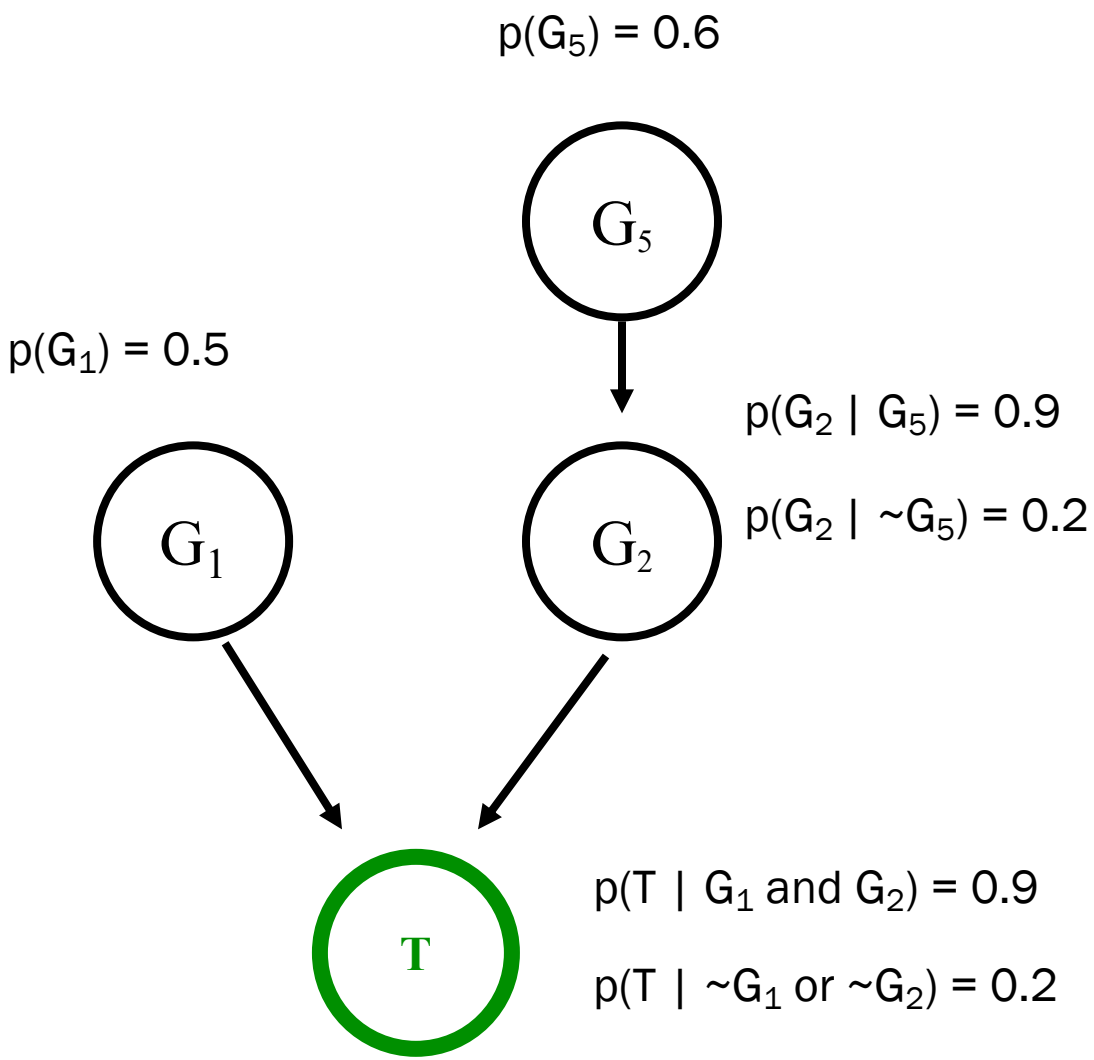
These genes don't impact T (are independent)



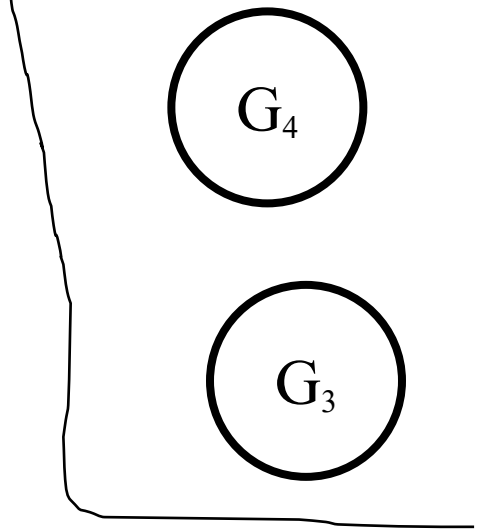
Teaser for Future Classes: Conditional Independence

```
Piech-2:dna piech$ python findStructure.py
size data =  100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234

...

p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

Only Causal Structure That Fits



These genes don't impact T (are independent)



Advanced: Conditional Independence

Two events E and F are called **conditionally independent given G**, if

$$P(EF|G) = P(E|G)P(F|G)$$

Or, equivalently if:

$$P(E|FG) = P(E|G)$$

NETFLIX

And Learn

Netflix Learning

What is the probability that a user will watch Life is Beautiful?

$P(E)$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

$P(E) = 10{,}234{,}231 / 50{,}923{,}123 = 0.20$

Netflix Learning

What is the probability that a user will watch Life is Beautiful, given they watched Coda?

$P(E|F)$

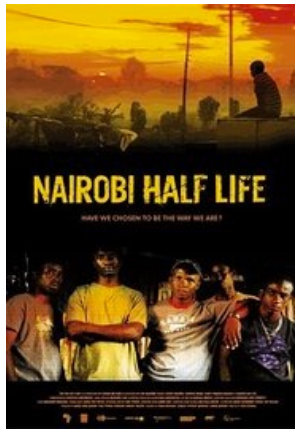
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\#\text{people who watched both}}{\#\text{people who watched } F}$$

$$P(E|F) = 0.42$$

Conditioned on liking a set of movies?

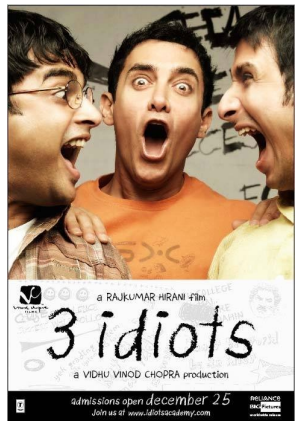
Netflix Learning

Each event corresponds to liking a particular movie



E_1

E_2



E_3

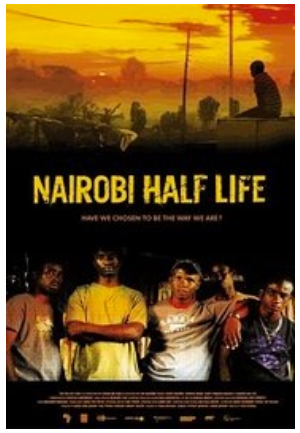
E_4

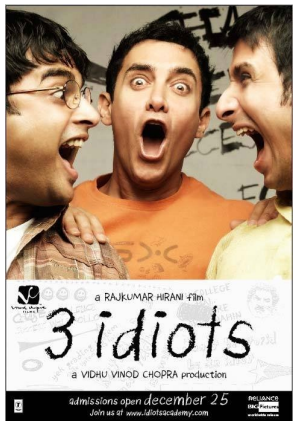
$$P(E_4|E_1, E_2, E_3)?$$

Is E_4 independent of E_1,E_2,E_3?

Netflix Learning

Is E_4 independent of E_1,E_2,E_3?



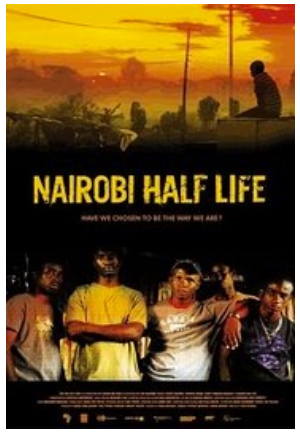


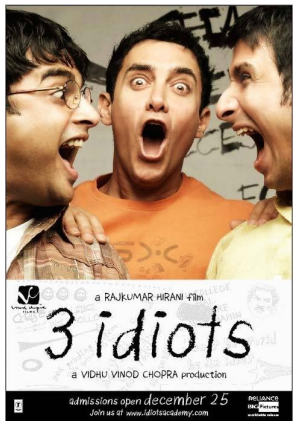
E_1 E_2 E_3 E_4

$$P(E_4|E_1, E_2, E_3) \stackrel{?}{=} P(E_4)$$

Netflix Learning

Is E_4 independent of E_1,E_2,E_3?





E_1 E_2 E_3 E_4

$$P(E_4|E_1, E_2, E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)}$$

Netflix Learning

What is the probability that a user watched four particular movies?

- There are 13,000 titles on Netflix
- The user watches 30 random titles.
- E = movies watched include the given four.

Solution:

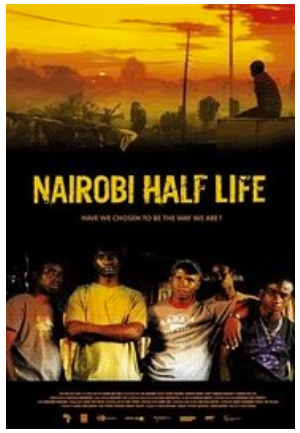
Watch those four

Choose 24 movies not in the set

$$P(E) = \frac{\binom{4}{4}\binom{12996}{24}}{\binom{13000}{30}} = 10^{-11}$$

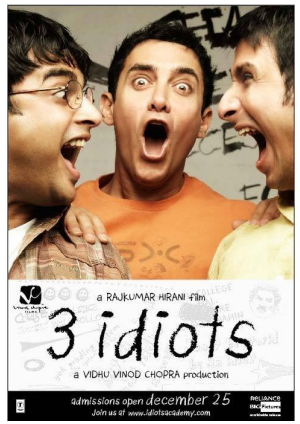
Choose 30 movies from netflix

Netflix Learning



E_1

E_2



E_3

E_4

Netflix Learning: Advanced, Conditional Independence

K_1

Like foreign emotional comedies

E_1 E_2 E_3 E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix Learning: Advanced, Conditional Independence

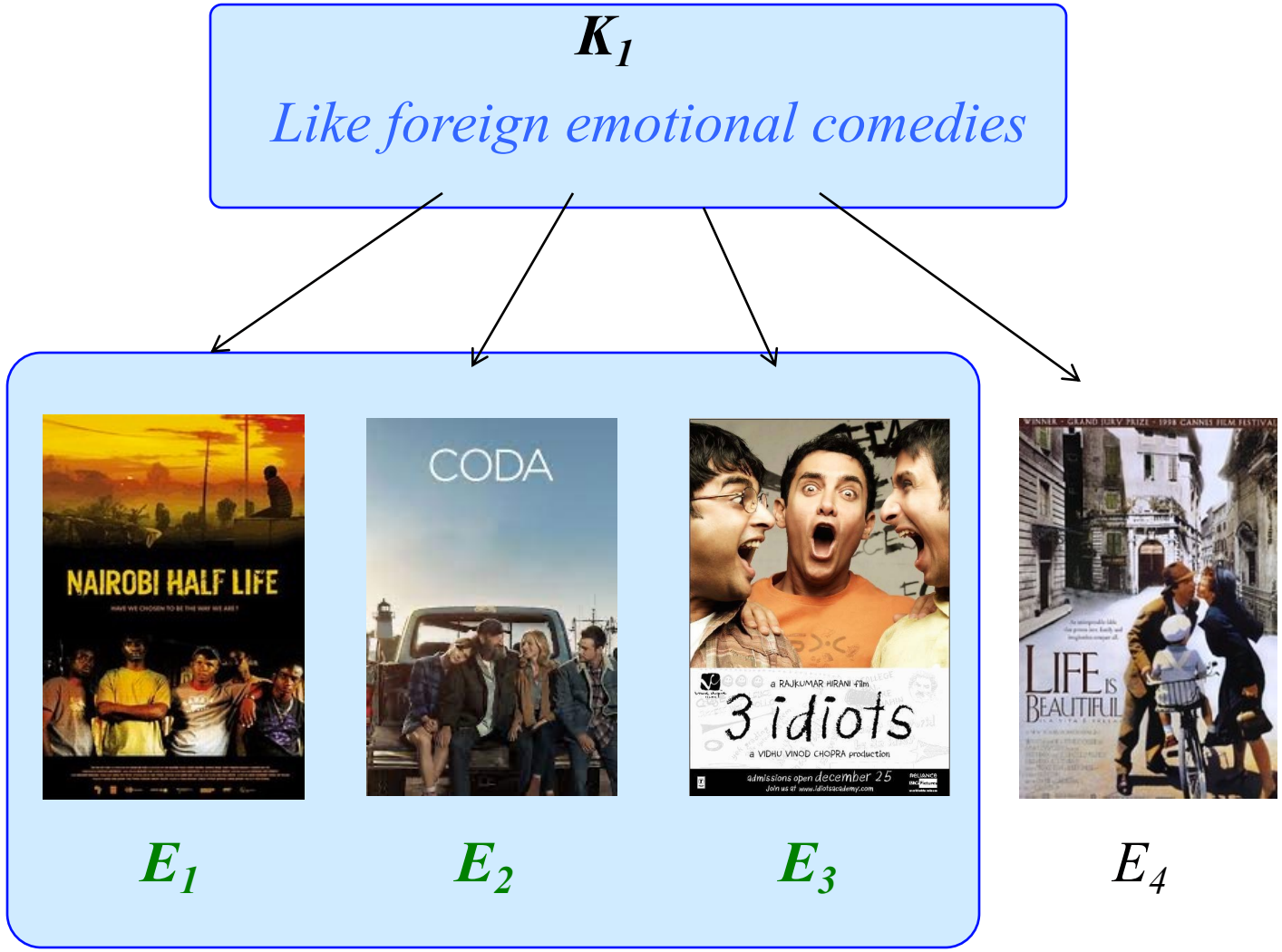
K_1

Like foreign emotional comedies

E_1 E_2 E_3 E_4

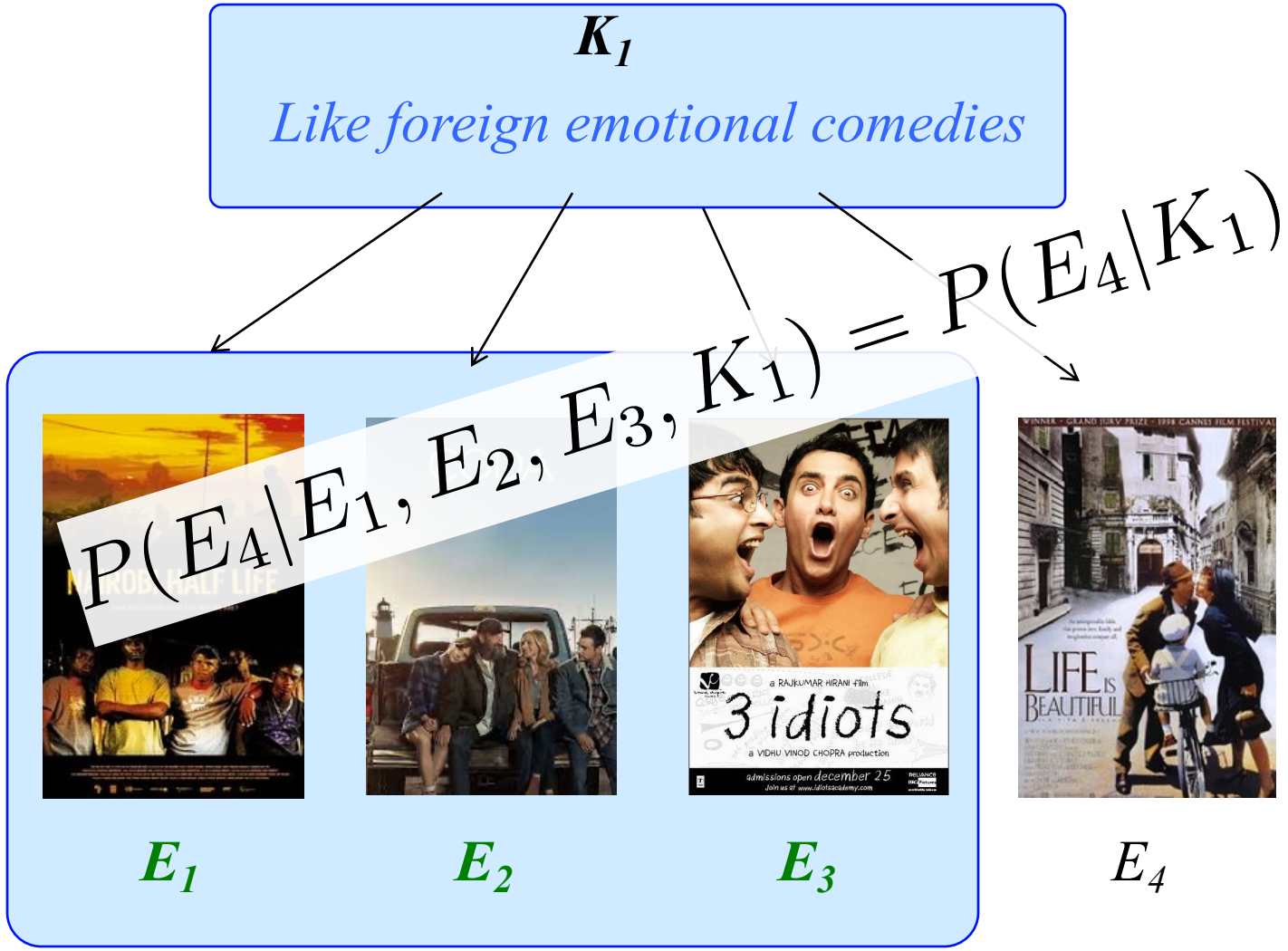
Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix Learning: Advanced, Conditional Independence



Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix Learning: Advanced, Conditional Independence



Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Conditional independence is a practical, real world way of decomposing hard probability questions.

Conditional Independence

If E and F are dependent,

that does not mean E and F will be dependent when another event is observed.

Conditional Dependence

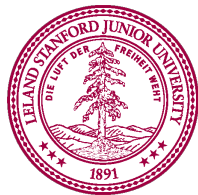
If E and F are independent,

that does not mean E and F will be independent when another event is observed.

Big Deal!

"Exploiting ***conditional independence*** to generate fast probabilistic computations is one of the main contributions CS has made to probability theory"

-Judea Pearl wins 2011 Turing Award, *"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"*



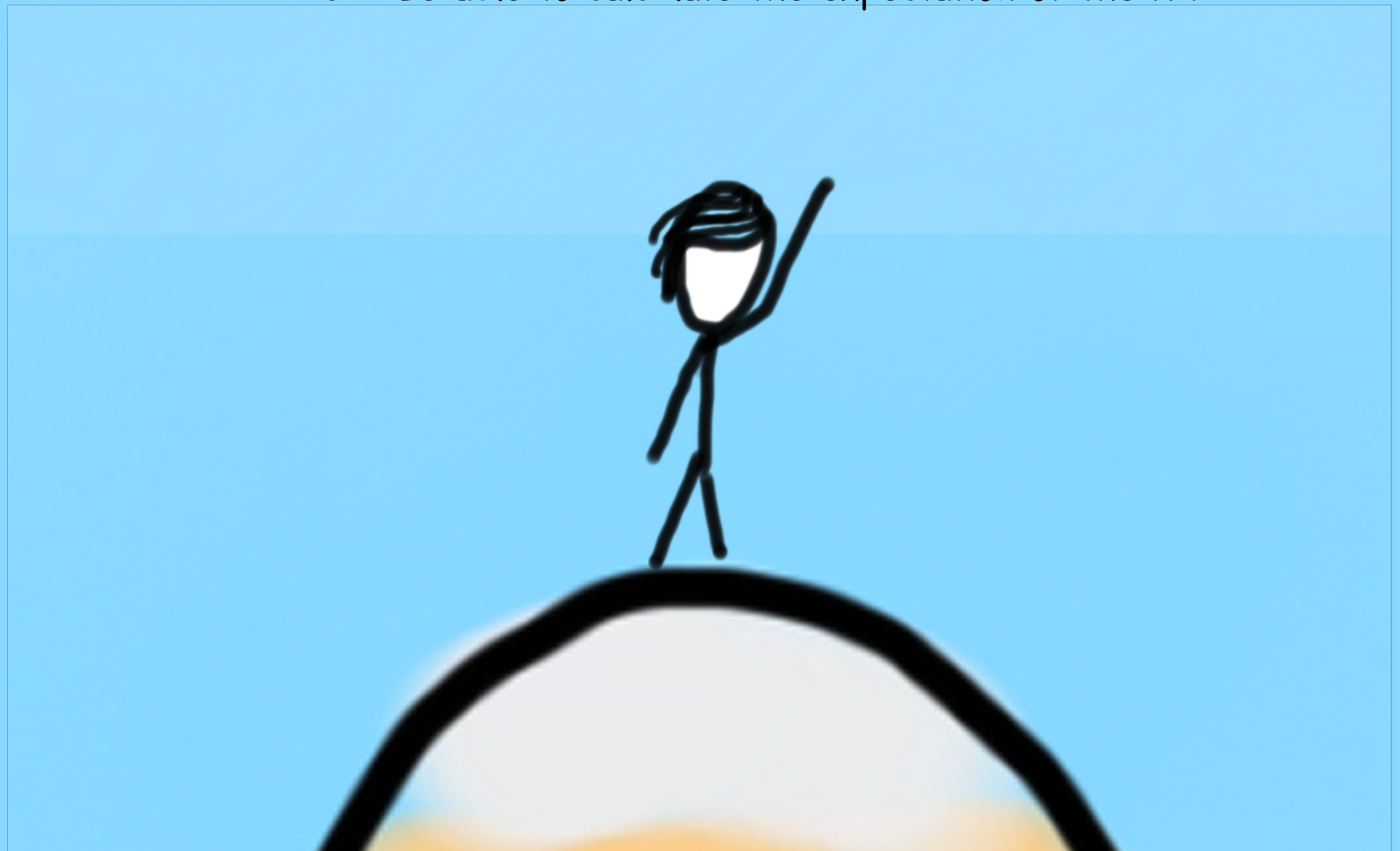
Ready for the next (cs109) episode

Another Pause!

Random Variables

Learning Goals

1. Be able to define a random variable (R.V.)
2. Be able to use and produce a PMF of a R.V.
3. Be able to calculate the expectation of the R.V.



Remember Learning to Code

name

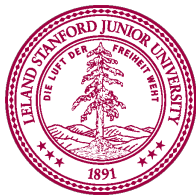
value

type

```
int a = 5;
double b = 4.2;
bit c = 1;
choice d = medium;
```

$$z \in \{\text{high}, \text{medium}, \text{low}\}$$

Random variables are like programming variables, with uncertainty



Pirates of the Variables

```
int a = 5;
```

A is the number of pirate ships in our *future* armada.

$$A \in \{1, 2, \ldots, 10\}$$

```
double b = 4.2;
```

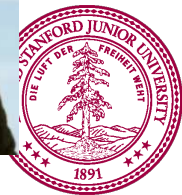
B is the amount of money we get *after* we defeat Blackbeard.

$$B \in \mathbb{R}^+$$

```
bit c = 1;
```

C is 1 *if* we successfully raid Isla de Muerta. 0 otherwise.

$$C \in \{0, 1\}$$



Random Variable

A **Random Variable** is a variable will have a value.
But there is uncertainty as to what value.

Example:

- 3 fair coins are flipped.
- Y = number of “heads” on 3 coins
- **Y is a random variable**
- P(Y = 0) = 1/8 (T, T, T)
- P(Y = 1) = 3/8 (H, T, T), (T, H, T), (T, T, H)
- P(Y = 2) = 3/8 (H, H, T), (H, T, H), (T, H, H)
- P(Y = 3) = 1/8 (H, H, H)
- P(Y ≥ 4) = 0

It is confusing that both random variables and events use the same notation

Random variables and events are two *different* things

We can define an event to be a particular assignment to a random variables

Example of Random Variables

Consider 5 coin flips, each which independently come up heads with probability *p*

- Recall:

$$P(2 \text{ heads}) = \binom{5}{2} p^2 (1-p)^3$$

$$P(3 \text{ heads}) = \binom{5}{3} p^3 (1-p)^2$$

- Y = number of “heads” on 5 flips

$$Y \in \{1, 2, \ldots, 5\}$$

$$P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$$

* Pro tip: Not really about coins. Many real world binary events work like this.

Properties of Random Variables

Probability Mass Function:

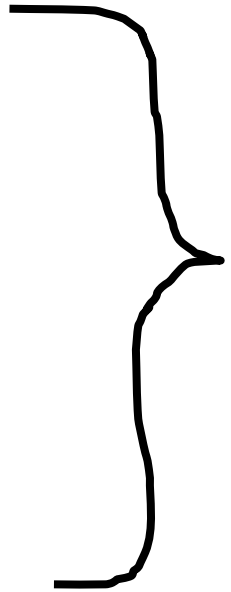
$$P(X = a)$$

Expectation:

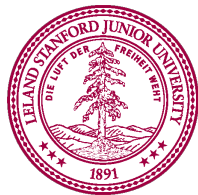
$$E[X]$$

Variance:

$$\mathrm{Var}(X)$$



Learning goals for today



1. Probability Mass Function

The relationship between values a random variable can take on, and the corresponding probability, is a ***function***!

Let Y be a random variable

$$Y$$

For example Y is the number of heads in 5 coin flips

$$Y = 2$$

It is an event when
Y takes on a value

For example Y is the number of heads in 5 coin flips

If this is a number

$$P(Y = 2)$$

Then this is a number
(between 0 and 1)

For example Y is the number of heads in 5 coin flips

If this is a variable

$$P(Y = k)$$

Then this is a function

For example Y is the number of heads in 5 coin flips

Random Variables -> Functions

$$P(Y = k)$$

$$k = 5 \qquad 0.03125$$

For example Y is the number of heads in 5 coin flips

Random Variables -> Functions

$$P(Y = k)$$

```
def event_probability(k):
    # probability mass function of Y in python
    ways = choose(N, k);
    prob_heads = Math.pow(P, k)
    prob_tails = Math.pow(P, N-k)
    return ways * a * b


N = 5    # number of coin flips
P = 0.5  # probability of heads
```

For example Y is the number of heads in 5 coin flips

If a random variable is discrete we call this function the **Probability Mass *Function***

Probability Mass Function (PMF)

Let X be a random variable that represents the result of a **single dice roll**. X can take on the values $\{1, 2, 3, 4, 5, 6\}$

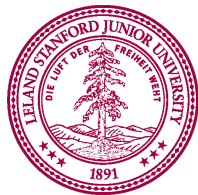
$$P(X = x)$$

$$p(x)$$

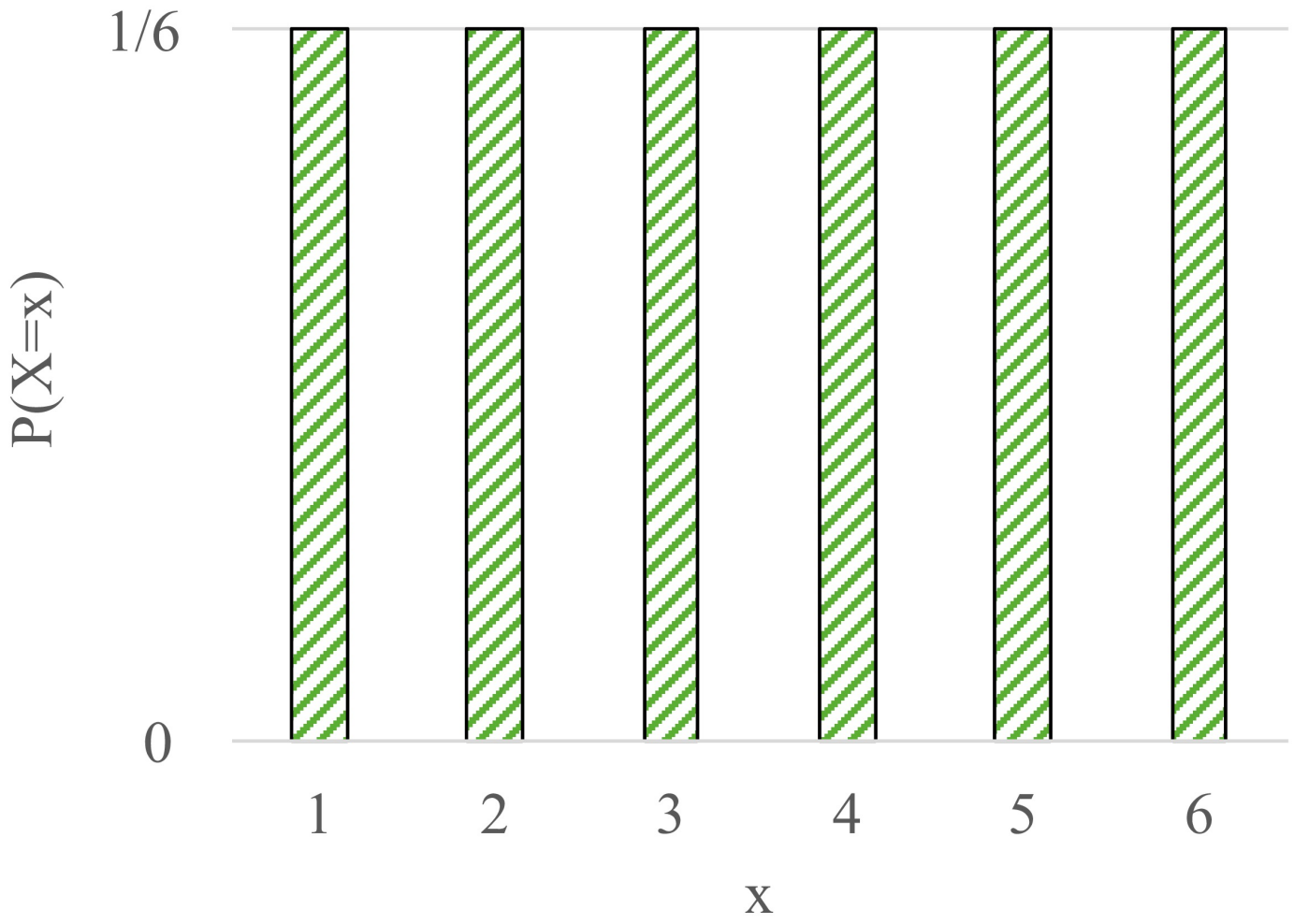
This is shorthand notation for the PMF

$$p_X(x)$$

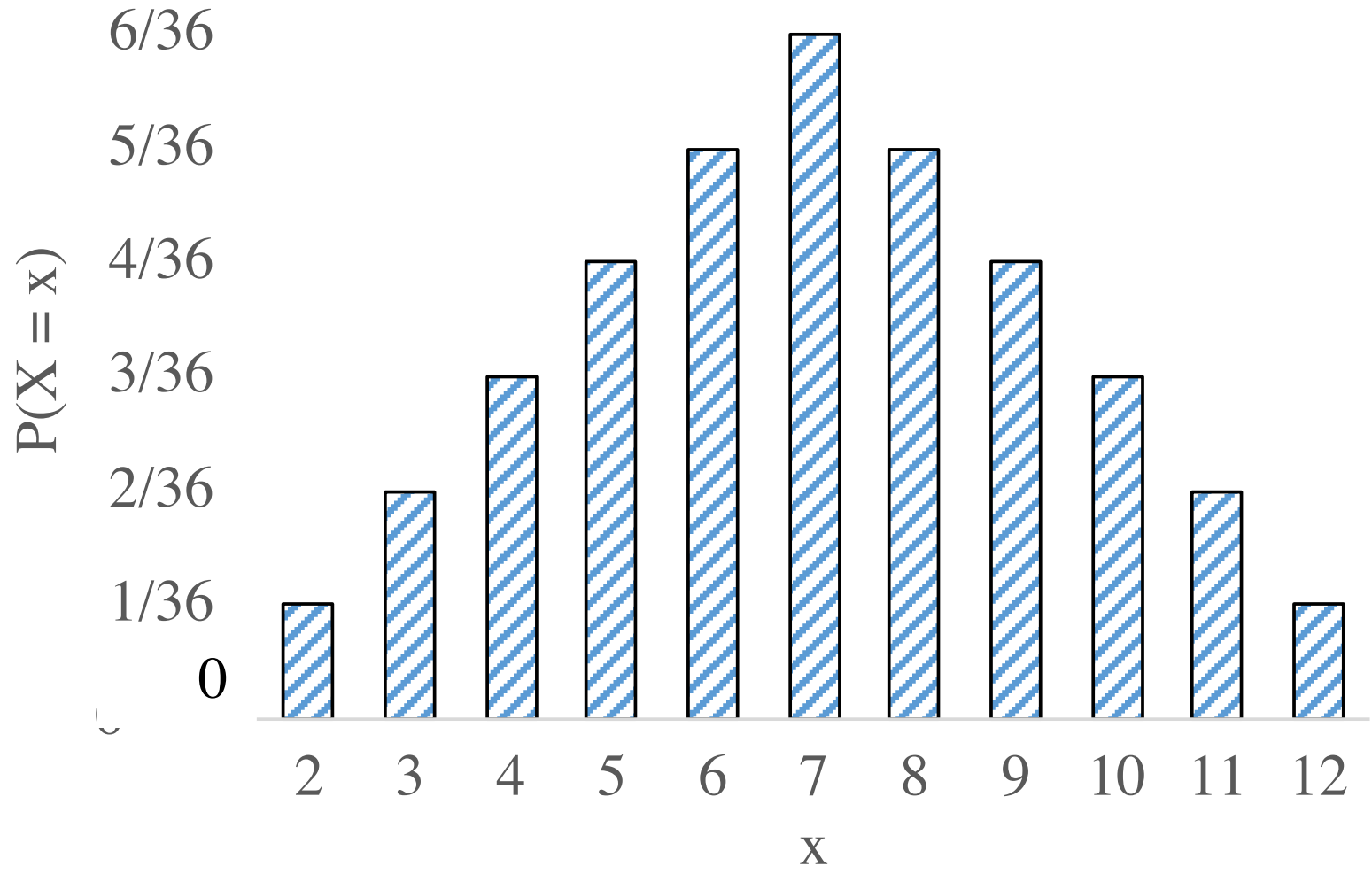
This is also shorthand notation for the PMF



PMF for *X the outcome of a die roll*



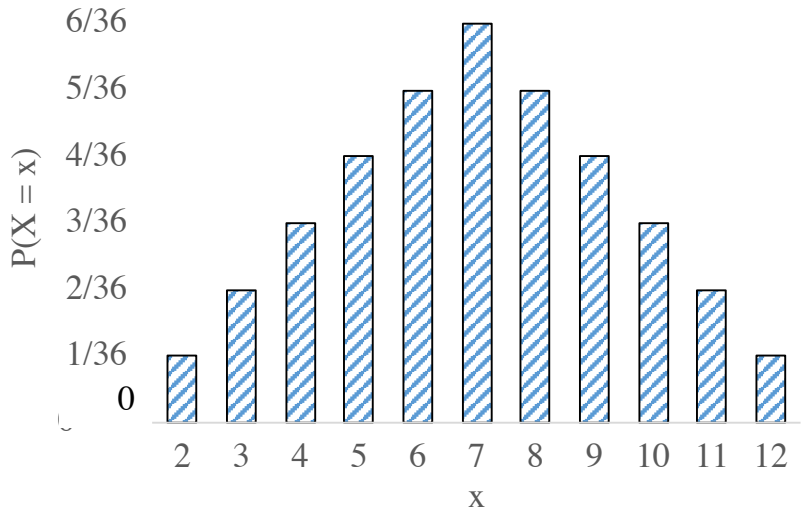
PMF for X *the sum of two dice rolls*



PMF as an equation

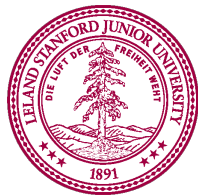
$$p(X = x) = \begin{cases} \frac{x-1}{36} & \text{if } x \in \mathbb{Z} \text{ , } 1 \leq x \leq 6 \\ \frac{13-x}{36} & \text{if } x \in \mathbb{Z} \text{ , } 7 \leq x \leq 12 \\ 0 & \text{else} \end{cases}$$

Again, this is the probability for the sum of two dice



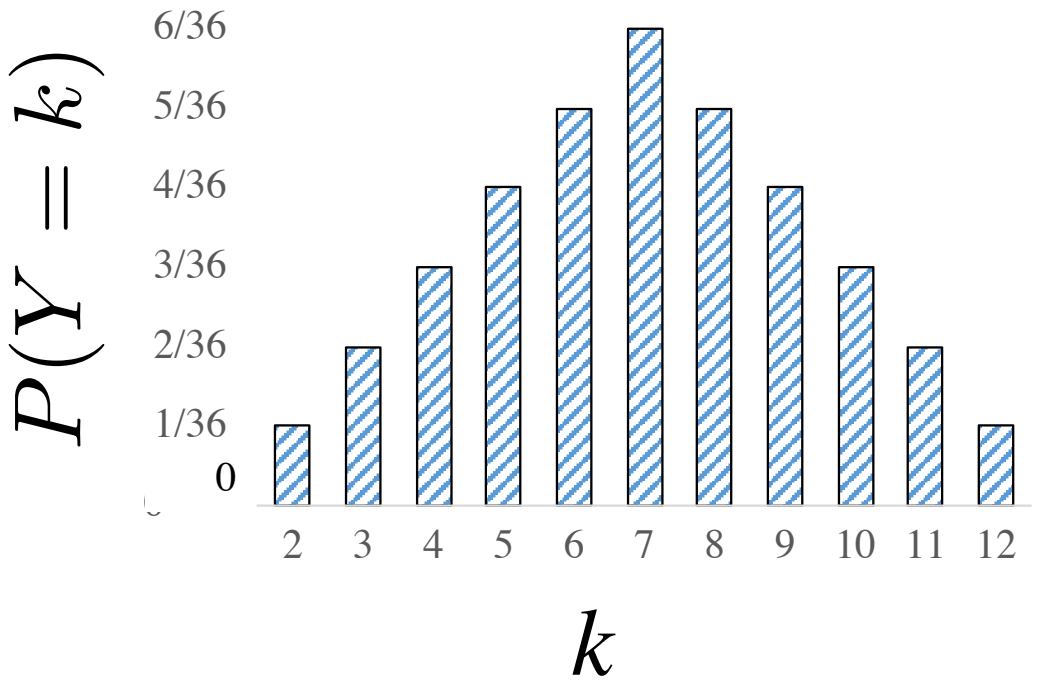
This is Fine Check

$$\sum_{\text{all } k} P(Y = k) \stackrel{?}{=}$$



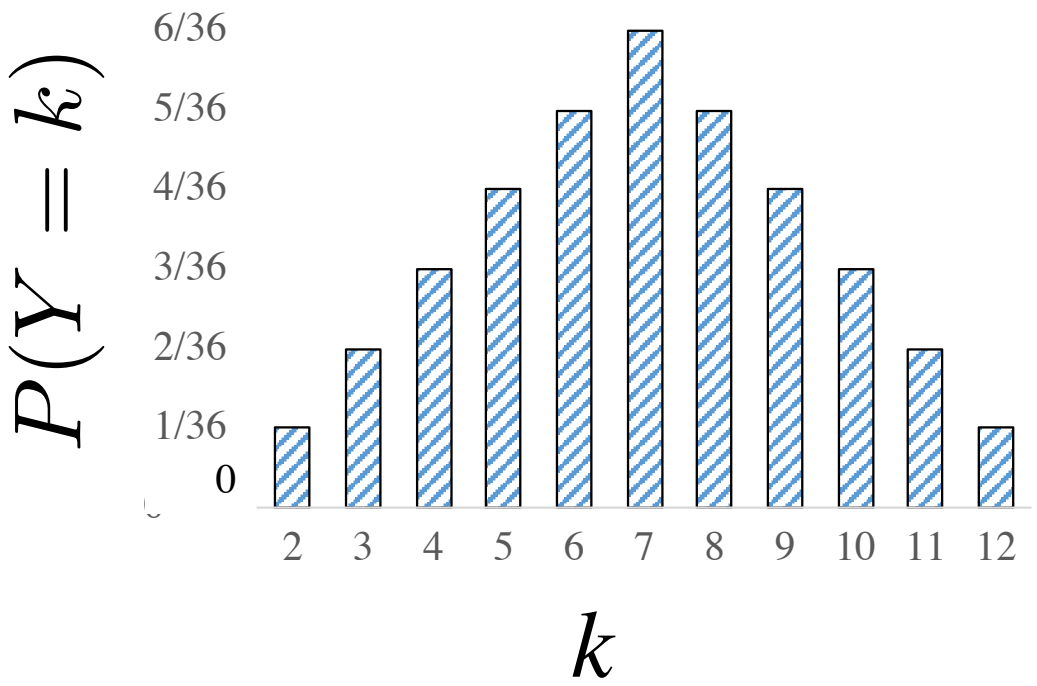
This is Fine Check

$$\sum_{\text{all } k} P(Y = k) \stackrel{?}{=}$$



This is Fine Check

$$\sum_k P(Y = k) = 1$$



2. Expectation

Properties of Random Variables

Probability Mass Function:

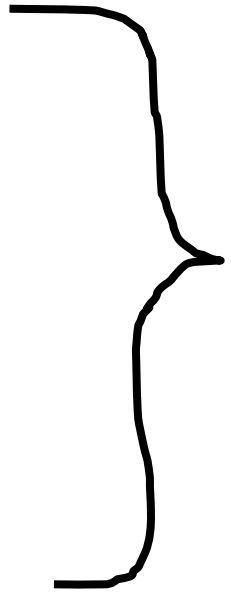
$$P(X = a)$$

Expectation:

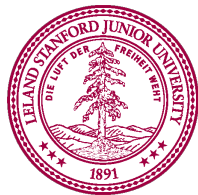
$$E[X]$$

Variance:

$$\mathrm{Var}(X)$$



Learning goals for today



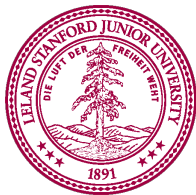
Expected Value

The Expected Values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} x \cdot p(x)$$

Note: sum over all values of x that have $p(x) > 0$.

Expected value also called: ***Mean***, *Expectation*, ***Weighted Average***, ***Center of Mass***, *1st Moment*



Expected Value

Roll a 6-Sided Die. X is outcome of roll

- p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6

E[X] = $1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{7}{2}$

Y is random variable

- P(Y = 1) = 1/3, P(Y = 2) = 1/6, P(Y = 3) = 1/2

E[Y] = 1 (1/3) + 2 (1/6) + 3 (1/2) = 13/6

Lying With Statistics

"There are three kinds of lies:
lies, damned lies, and statistics"

– *Mark Twain*

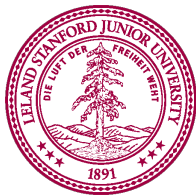
School has 3 classes with 5, 10 and 150 students

Randomly choose a class with equal probability

X = size of chosen class

What is E[X]?

- E[X] = 5 (1/3) + 10 (1/3) + 150 (1/3)
  = 165/3 = 55



Lying With Statistics Part 2

"There are three kinds of lies:
lies, damned lies, and statistics"

– *Mark Twain*

School has 3 classes with 5, 10 and 150 students

Randomly choose a student with equal probability

Y = size of class that student is in

What is E[Y]?

- E[Y] = 5 (5/165) + 10 (10/165) + 150 (150/165)
  = 22635/165 ≈ 137

Note: E[Y] is students' perception of class size

- But E[X] is what is usually reported by schools!

Properties of Expectation (more on this later)

Linearity:

$$E[aX + b] = aE[X] + b$$

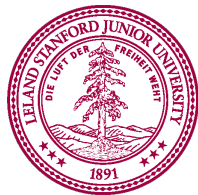
- Consider X = 6-sided die roll, Y = 2X – 1.
- E[X] = 3.5 E[Y] = 6

Expectation of a sum is the sum of expectations

$$E[X + Y] = E[X] + E[Y]$$

Unconscious statistician:

$$E[g(x)] = \sum_x g(x)p(x)$$



Properties of Random Variables

Probability Mass Function:

$$P(X = a)$$

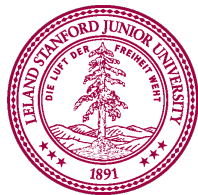
Expectation:

$$E[X]$$

Variance:

$$\mathrm{Var}(X)$$

Learning goals for today



Wonderful

St Petersburg

Game set-up

- We have a fair coin (come up “heads” with p = 0.5)
- Let *n* = number of coin flips (“heads”) before first “tails”
- You win $\$2^n$

How much would you pay to play?

St Petersburg

Game set-up

- We have a fair coin (come up “heads” with p = 0.5)
- Let *n* = number of coin flips (“heads”) before first “tails”
- You win $\$2^n$

How much would you pay to play?

Solution

- Let X = your winnings
- E[X] = $\left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \ldots = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i$

$$= \sum_{i=0}^{\infty} \frac{1}{2} = \infty$$

- I’ll let you play for $1 thousand... but just once! Takers?

St Petersburg + Reality

What if Chris has only $65,536?

- Same game
- If you win over $65,536 I leave the country.

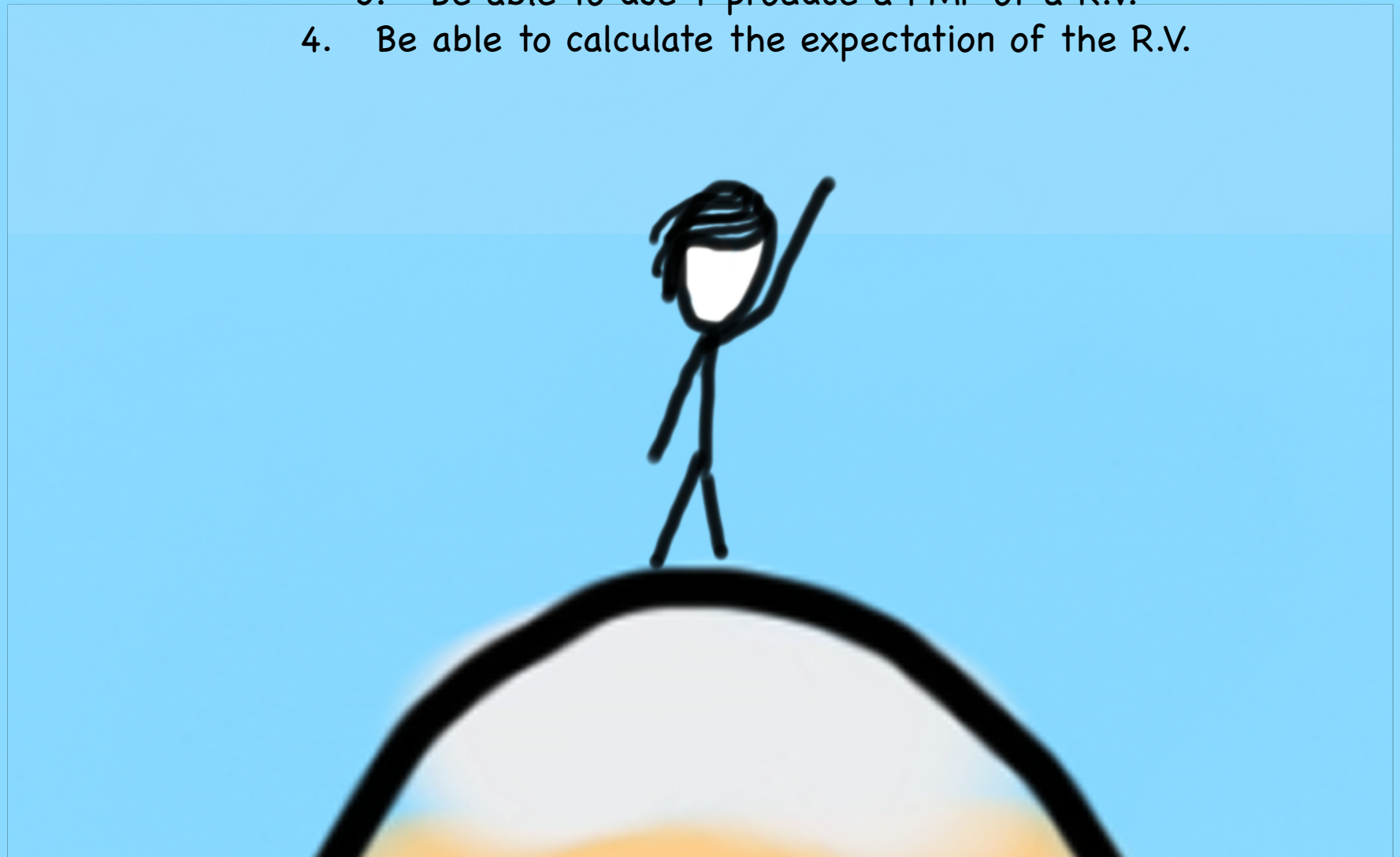
Solution

- Let X = your winnings
- E[X]

$$= \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \dots$$

$$= \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i+1} 2^i \text{ s.t. } k = \log_2(65,536)$$

$$= \sum_{i=0}^{16} \frac{1}{2} = 8.5$$

Learning Goals

1. Know what is meant by Conditional Independence
2. Be able to define a random variable (R.V.)
3. Be able to use + produce a PMF of a R.V.
4. Be able to calculate the expectation of the R.V.