Continuous Variables

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Announcements

- Happy Friday
- Lecture Videos?
- Concept checks: correct / incorrect?
- Coding: from scratch?

ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE

Learning Goals

1. Comfort using new discrete random variables
2. Integrate a density function (PDF) to get a probability
3. Use a cumulative function (CDF) to get a probability

Goal: Be Able to Use a New Random Variable

Don’t have to derive all of the following distributions. We want you to get a sense of how random variables work.

Here are a few more Random Variables

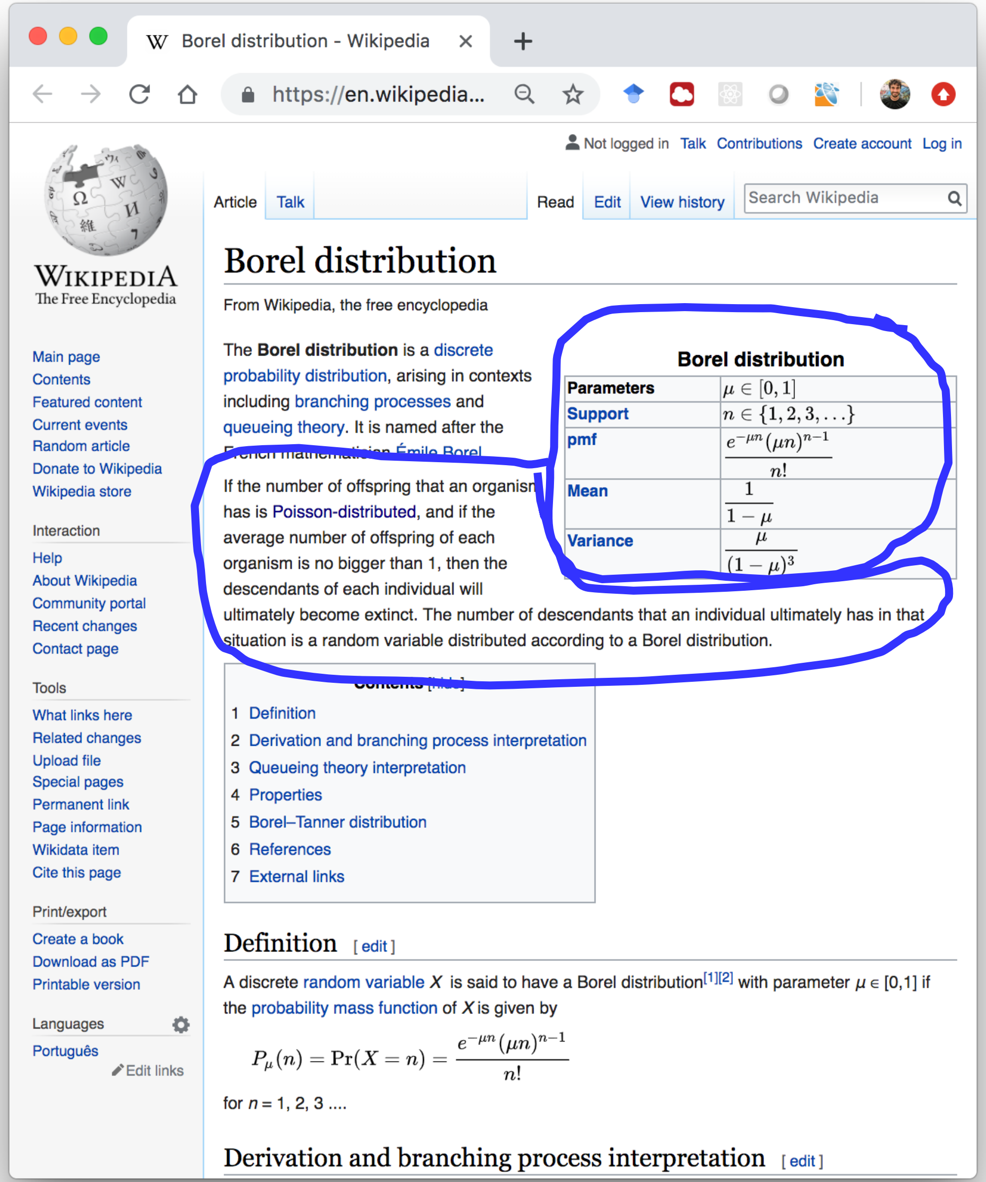
	number of successes	time to get successes	
One trial	$X \sim \mathrm{Ber}(p)$	$X \sim \mathrm{Geo}(p)$	One success
	↑ $n = 1$	↑ $r = 1$	
Several trials	$X \sim \mathrm{Bin}(n, p)$	$X \sim \mathrm{NegBin}(r, p)$	Several successes
Interval of time	$X \sim \mathrm{Poi}(\lambda)$	$X \sim \mathrm{Exp}(\lambda)$	One success

Goal: Be Able to Use a New Random Variable

You are learning about servers…

You read about the MD1 queue…

You find a paper that says the server "busy period" is distributed as a Borel with parameter $\mu = 0.2$ …



Geometric Random Variable

X is **Geometric** Random Variable: $X \sim \text{Geo}(p)$

- X is number of independent trials until first success
- p is probability of success on each trial
- X takes on values 1, 2, 3, …, with probability:

$$P(X = n) = (1 - p)^{n-1} p$$

$$E[X] = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

Negative Binomial Random Variable

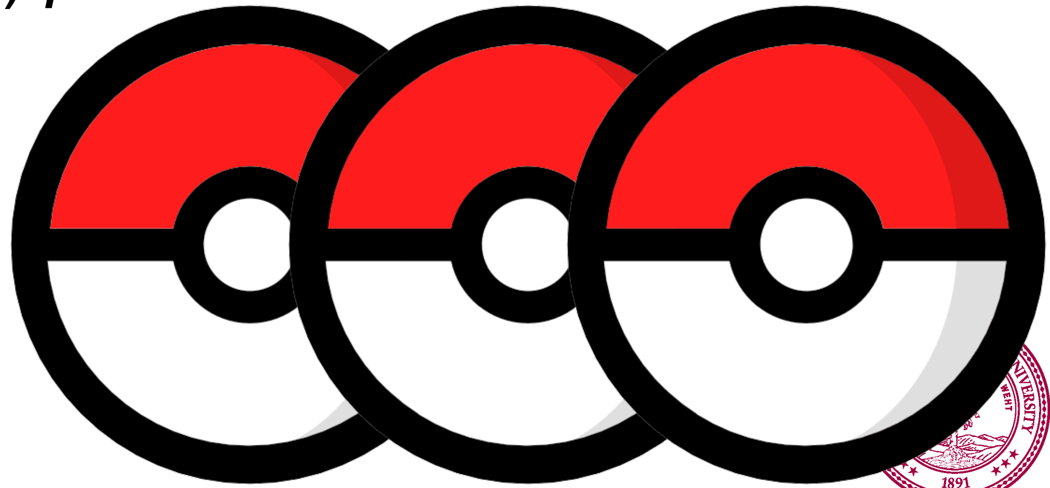
X is **Negative Binomial** RV: X ~ NegBin(r, p)

- X is number of independent trials until r successes
- p is probability of success on each trial
- X takes on values r, r + 1, r + 2…, with probability:

$$P(X=n)=\binom{n-1}{r-1}p^r(1-p)^{n-r},\ \text{where } n=r,r+1,\ldots$$

- E[X] = r/p Var(X) = $r(1-p)/p^2$

Note: Geo(p) ~ NegBin(1, p)



Discrete Distributions

Bernoulli:
- indicator of coin flip X ~ Ber(*p*)

Binomial:
- # successes in *n* coin flips X ~ Bin(*n*, *p*)

Poisson:
- # successes in *n* coin flips X ~ Poi(λ)

Geometric:
- # coin flips until success X ~ Geo(*p*)

Negative Binomial:
- # trials until *r* successes X ~ NegBin(*r*, *p*)

Bitcoin Mining

SHA-256 Hash(Data *Fixed* Salt *Choice*)

↓

Number that looks like random bits

↓

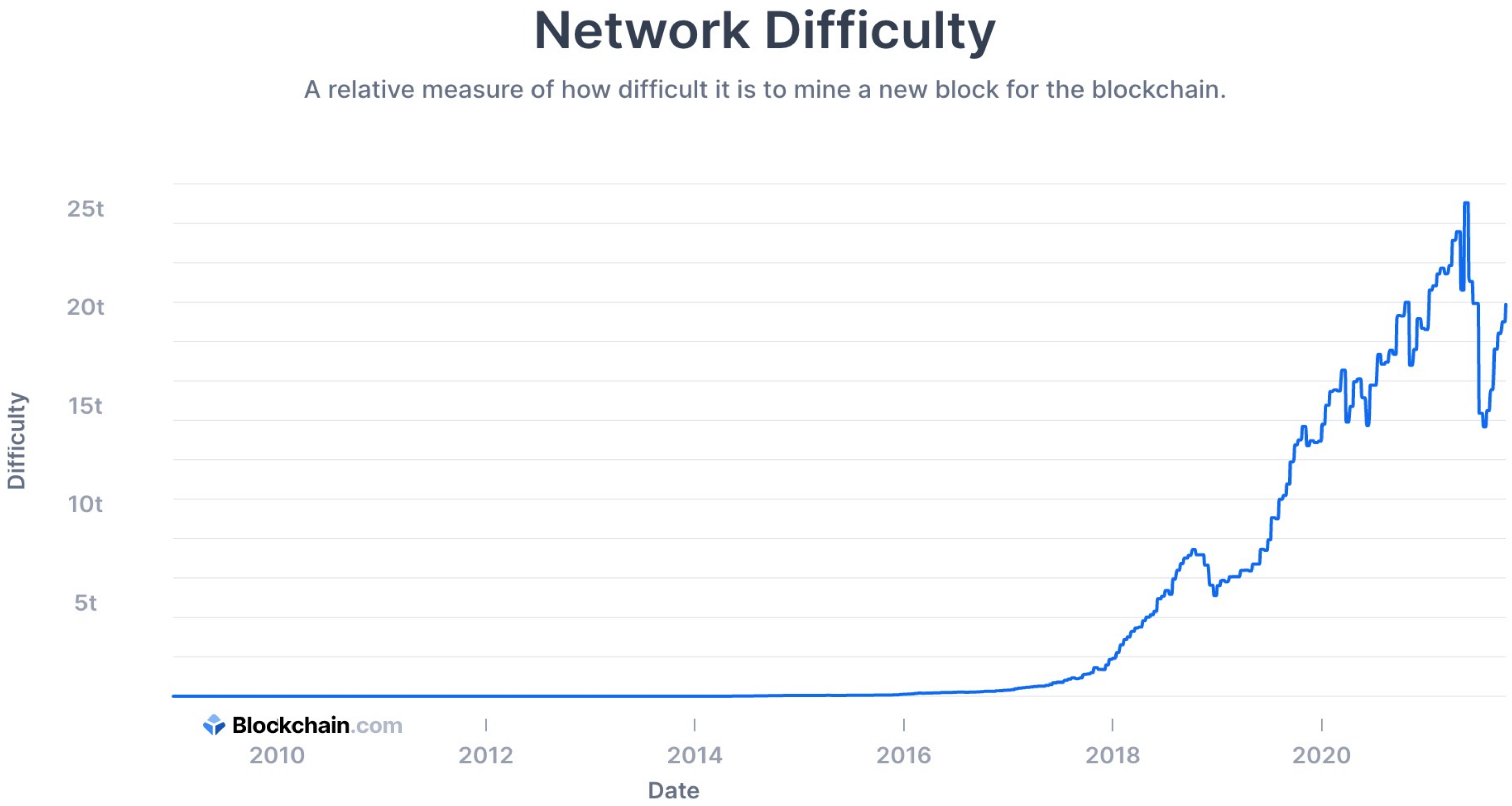
You "mine a bitcoin" if, for given data *D*, you find a salt number *N* such that Hash(*D*, *N*) produces a string that starts with *g* zeroes.

You "mine a bitcoin" if, for given data D, you find a number N such that Hash(D, N) produces a string that starts with g zeroes.

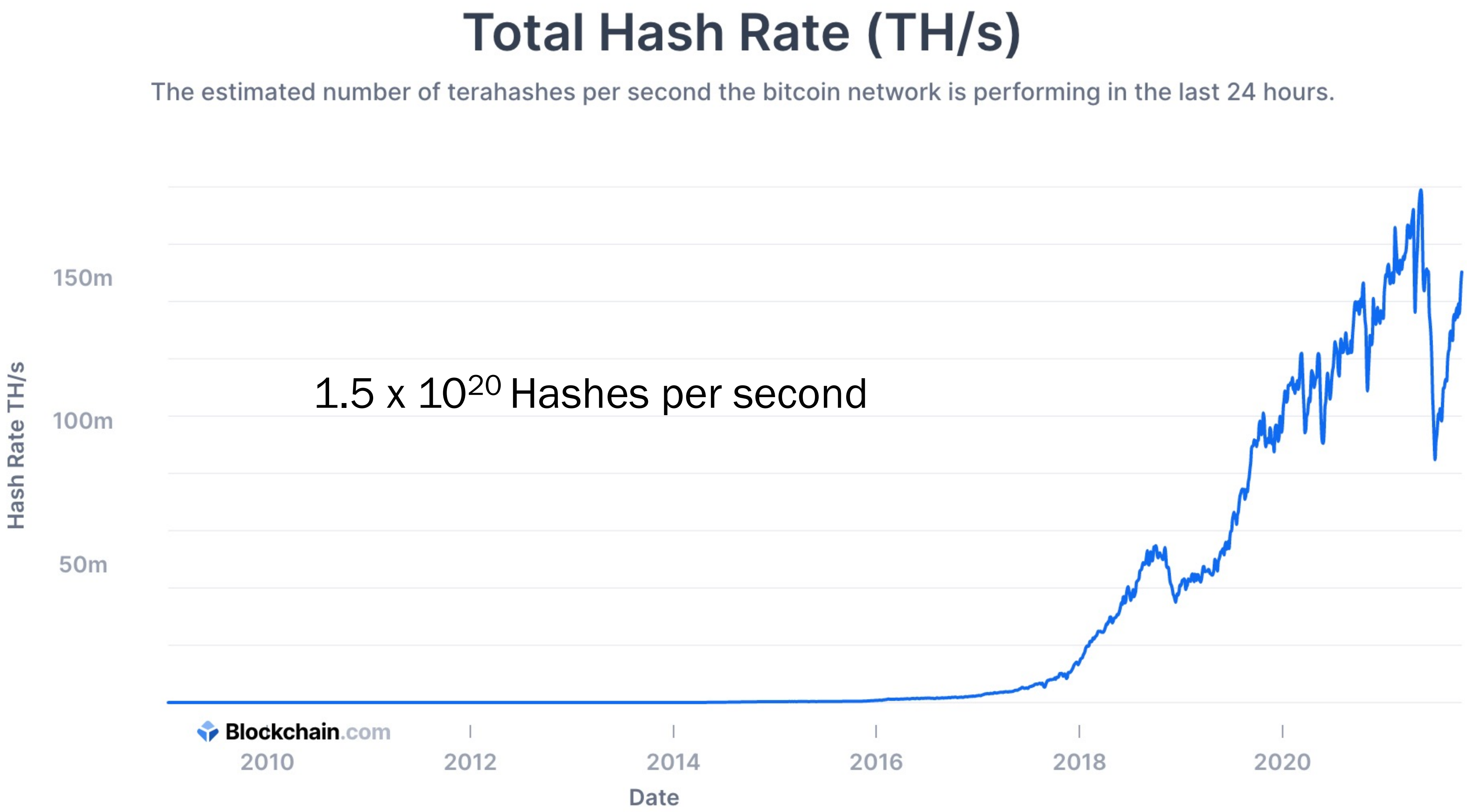
(a) What is the probability that the first number you try will produce a bit string which starts with g zeroes (in other words you mine a bitcoin)?

(b) How many different numbers do you expect to have to try before you mine five bitcoins?

Bitcoin Mining Real Life



Bitcoin Mining Real Life



1.5×10^{20} Hashes per second

Dating at Stanford

Each person you date has a 0.2 probability of being someone you spend your life with. What is the average number of people one will date? What is the standard deviation?

Equity in the Courts

Berghuis v. Smith

If a group is underrepresented in a jury pool, how do you tell?

- Article by Erin Miller –January 22, 2010
- Thanks to (former CS109er) Josh Falk for this article

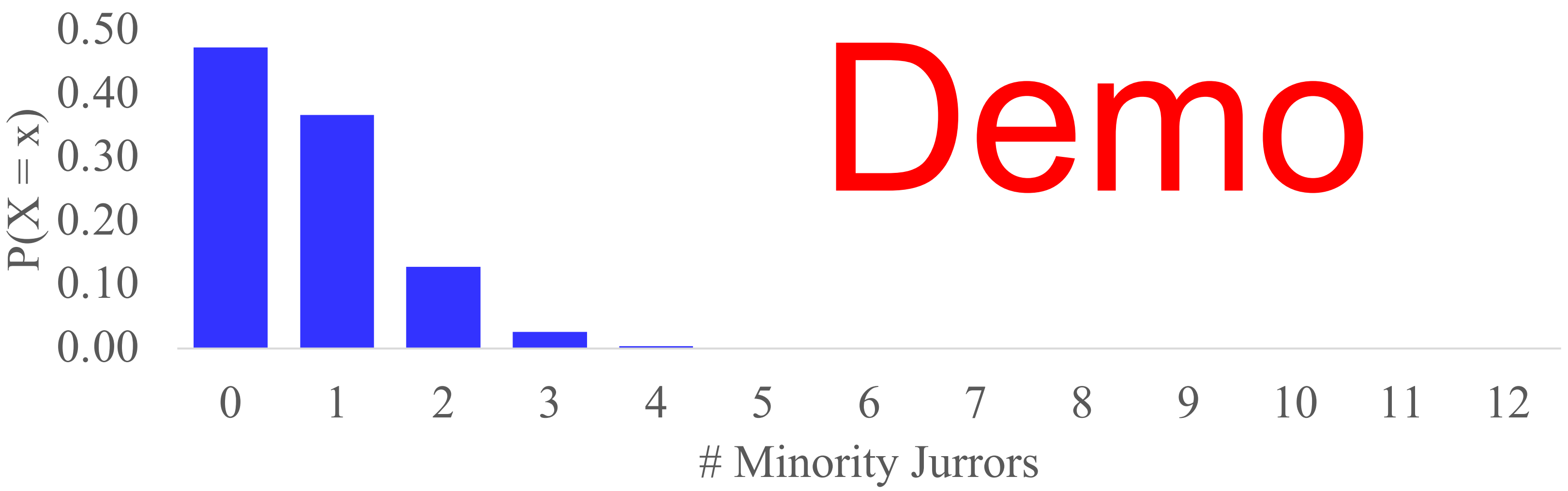
Justice Breyer [Stanford Alum] opened the questioning by invoking the binomial theorem. He hypothesized a scenario involving **"an urn with a thousand balls, and sixty are blue, and nine hundred forty are purple, and then you select them at random… twelve at a time."** According to Justice Breyer and the binomial theorem, if the purple balls were under represented jurors then **"you would expect… something like a third to a half of juries would have at least one minority person"** on them.

Justin Breyer Meets CS109

Approximation using Binomial distribution

- Assume P(blue ball) constant for every draw = 60/1000
- X = # blue balls drawn. $X \sim Bin(12, 60/1000 = 0.06)$
- $P(X \geq 1) = 1 - P(X = 0) \approx 1 - 0.4759 = 0.5240$

In Breyer's description, should actually expect just <u>over half</u> of juries to have at least one non-white person on them

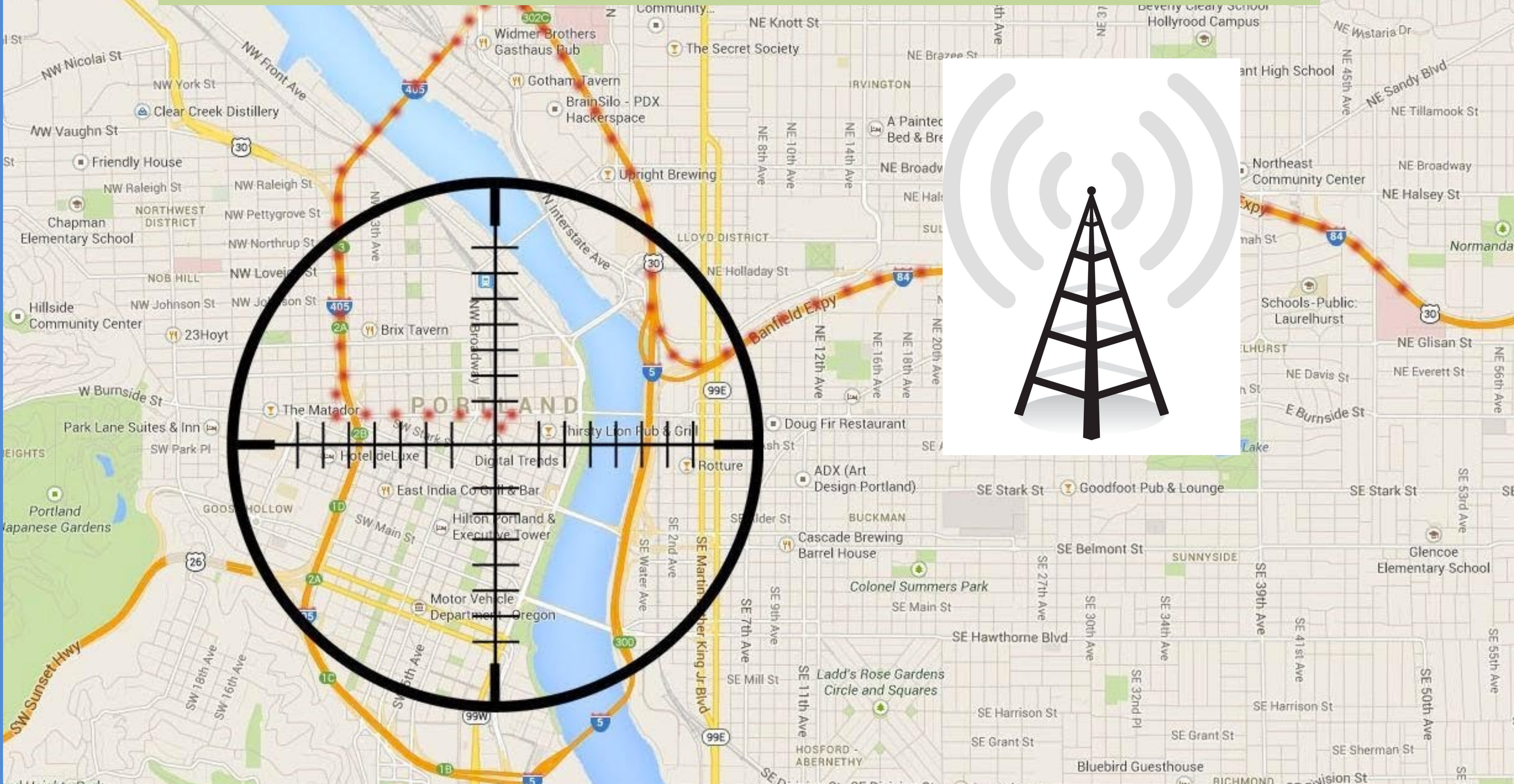


Demo

Pedagogic Pause

Big hole in our knowledge

Not all values are discrete



```
random()?
```

Riding the Marguerite

Riding the Marguerite

You are running to the bus stop. You don't know exactly when the bus arrives. You have a distribution of probabilities.

You show up at 2:20pm.

What is P(wait < 5 minutes)?

What is the probability that the bus arrives at:
2:17pm and 12.1233391102389234 seconds?

Riding the Marguerite

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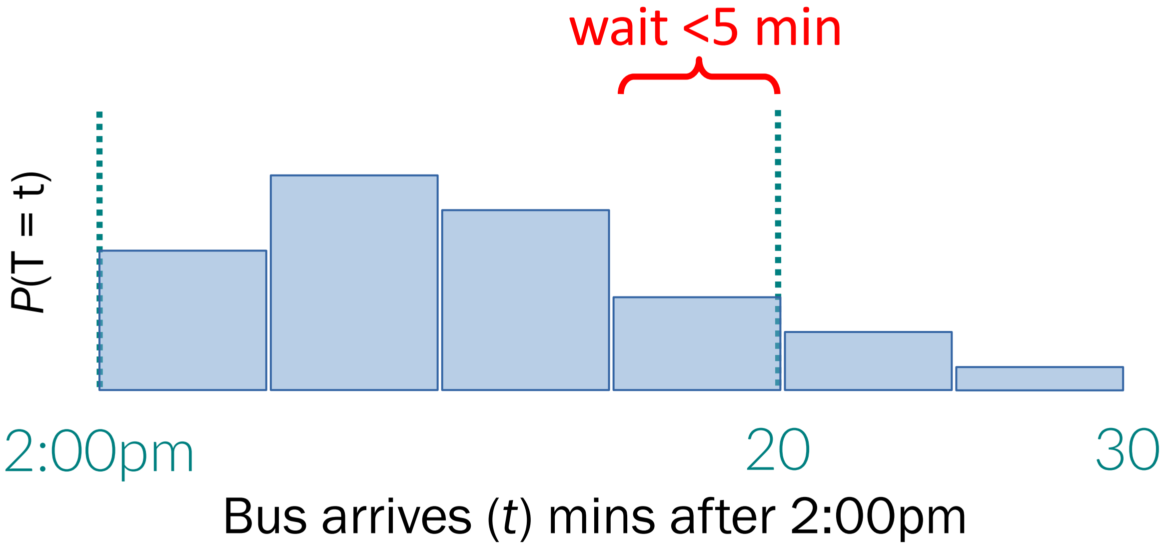

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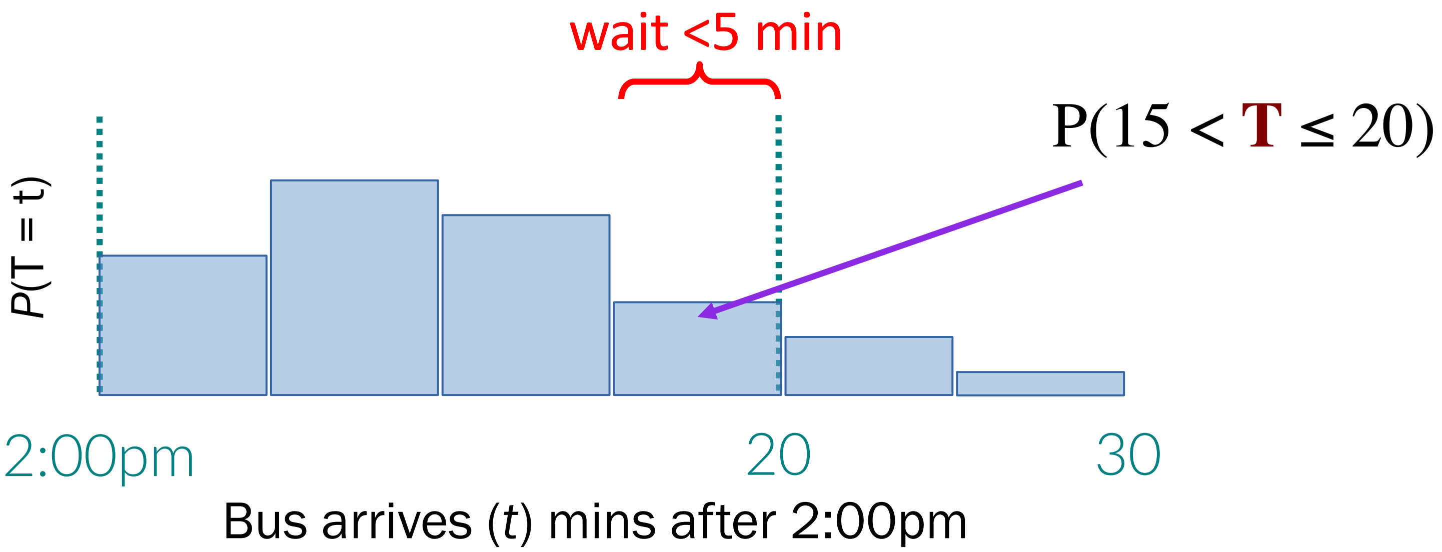

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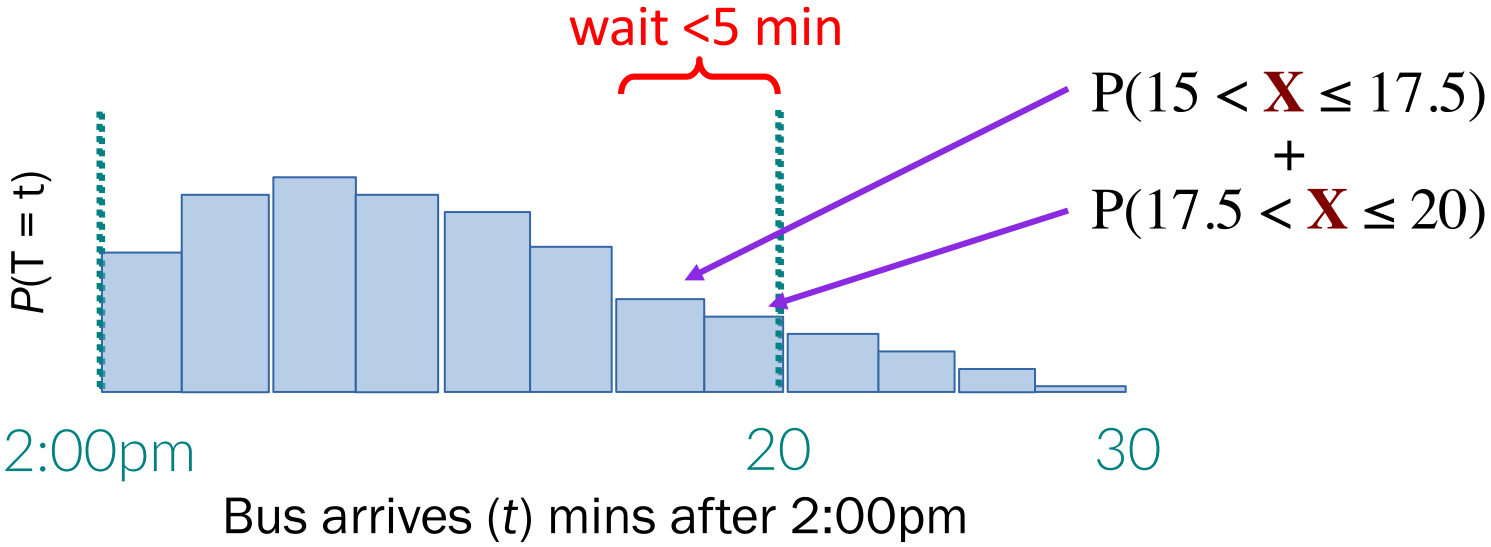

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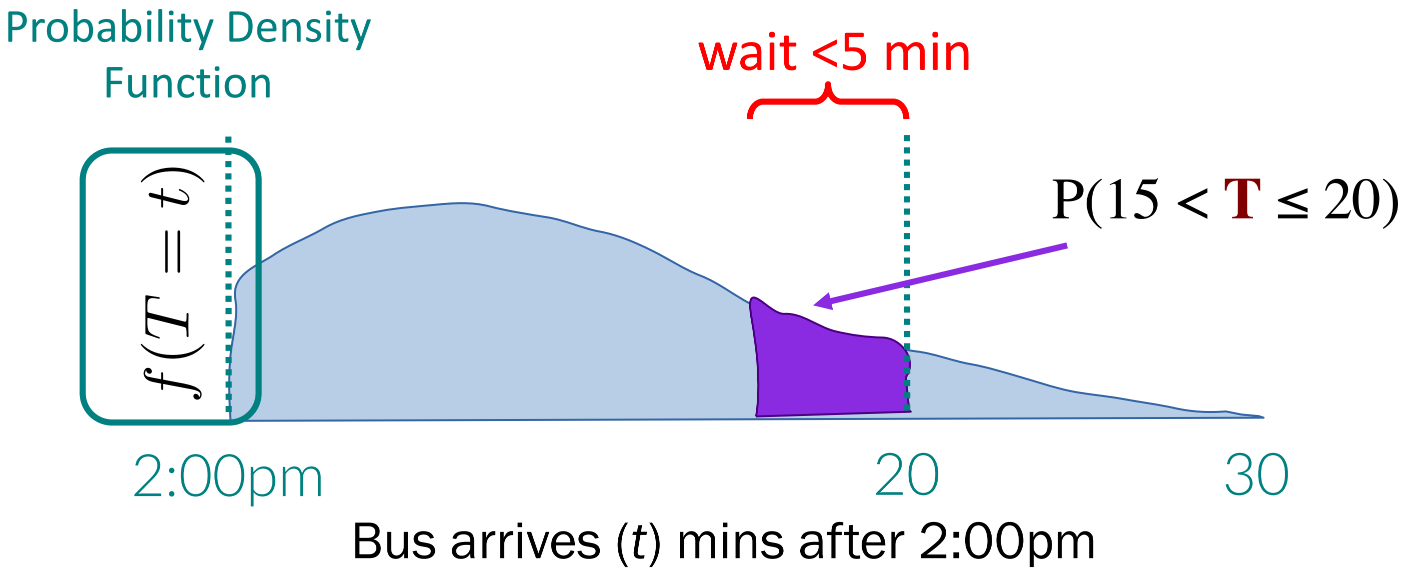

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Probability Density Function

The **probability density function** (PDF) of a continuous random variable represents the relative likelihood of various values.

Units of probability *divided by units of X*.
Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} f(X = x)\,dx$$

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Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} \boxed{f_X(x)}\, dx$$

This is another way to write the PDF

Probability Density Function

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Integrate it to get probabilities!

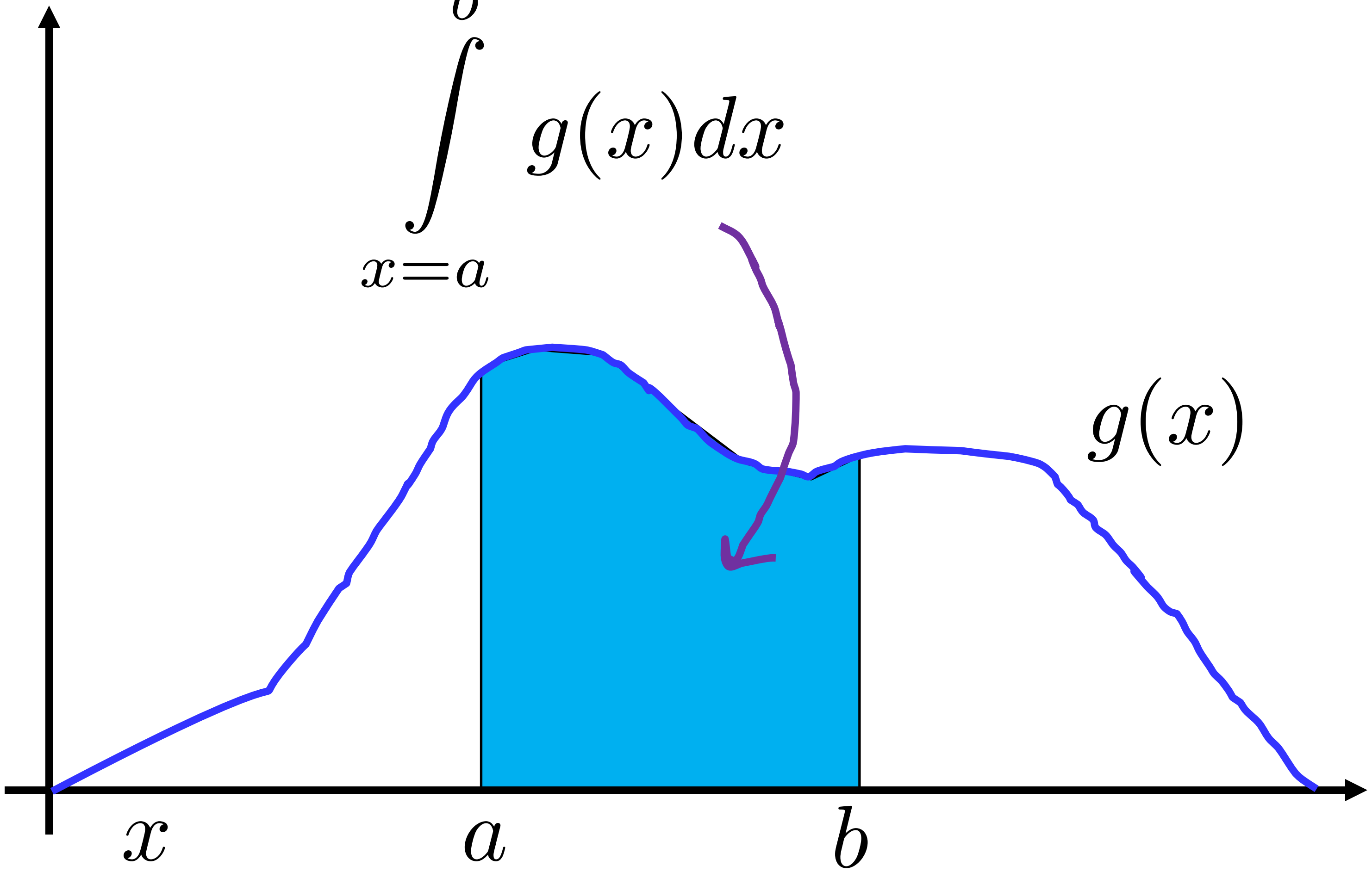
$$P(a < X < b) = \int_{x=a}^{b} f(X = x)dx$$

Integrals!

*loving, not scary

Integrals

$$\int_{x=a}^{b} g(x)\,dx$$



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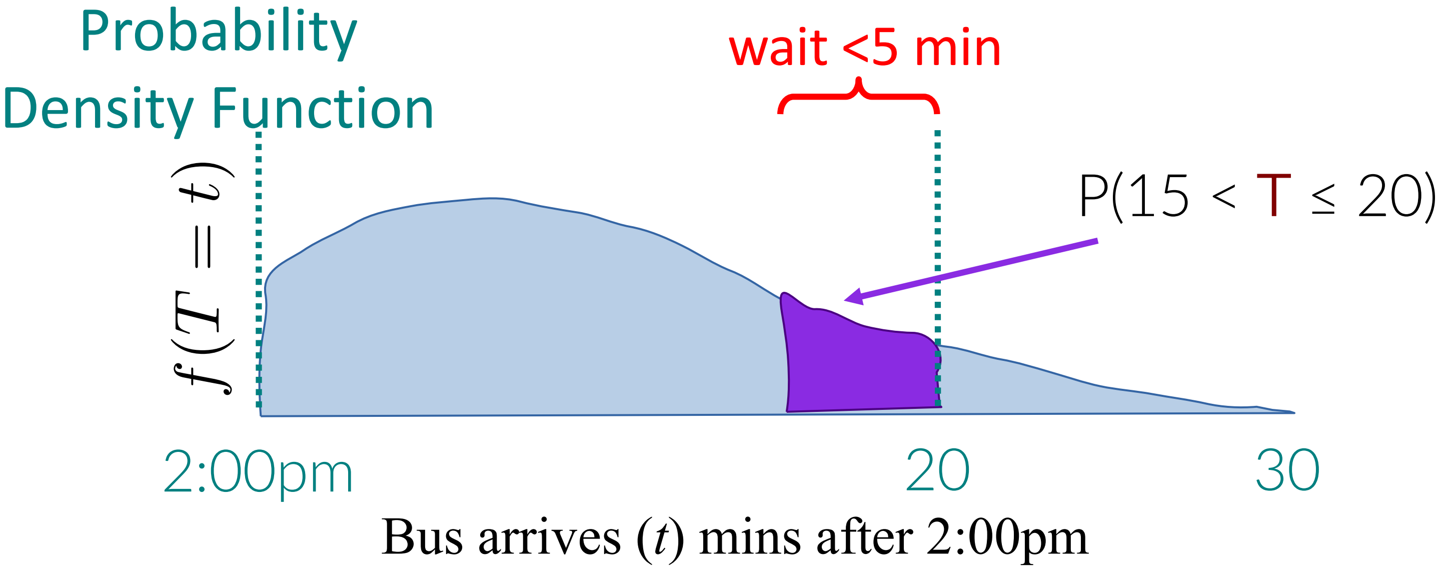

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Properties of PDFs

The integral of a PDF gives a probability. Thus:

$$0 \leq \int_{x=a}^{b} f(X = x)\, dx \leq 1$$

$$\int_{x=-\infty}^{\infty} f(X = x)\, dx = 1$$

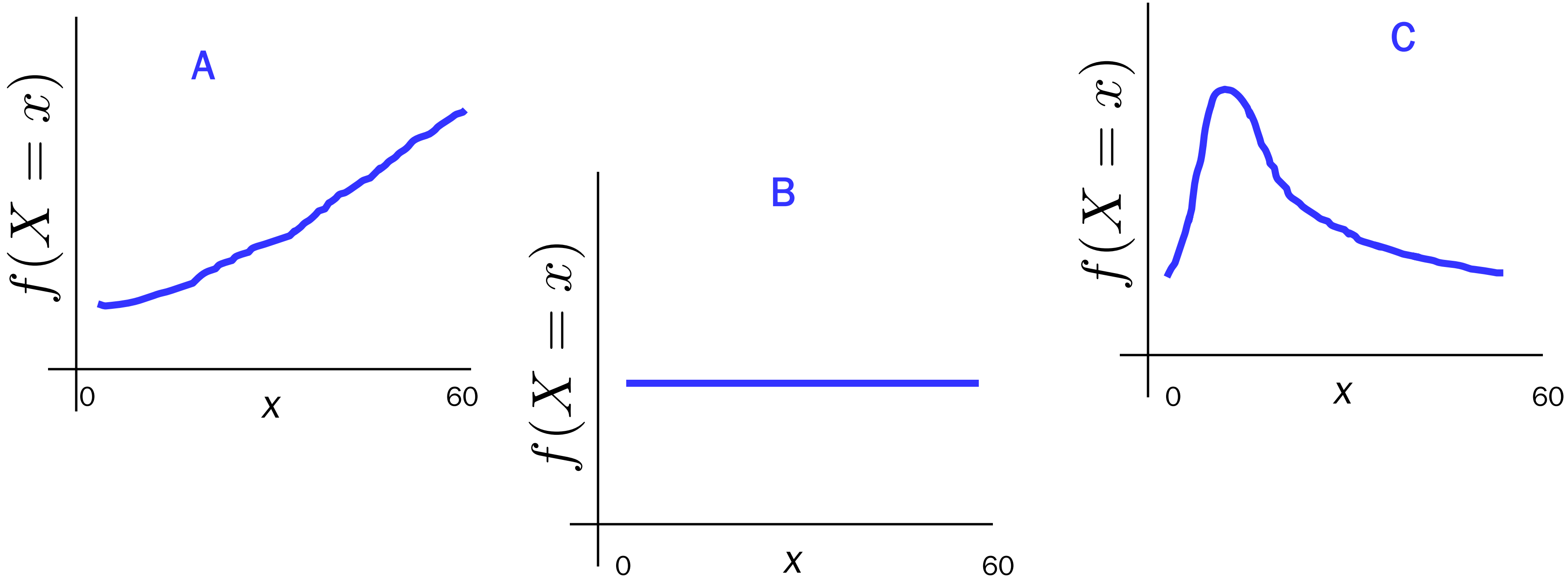
What do you get if you integrate over a probability *density* function?

A probability!

Probability Density Function

Probability density functions articulate ***relative*** belief.

Let X be a random variable which is the # of minutes after 2pm that the bus arrives at the stop:

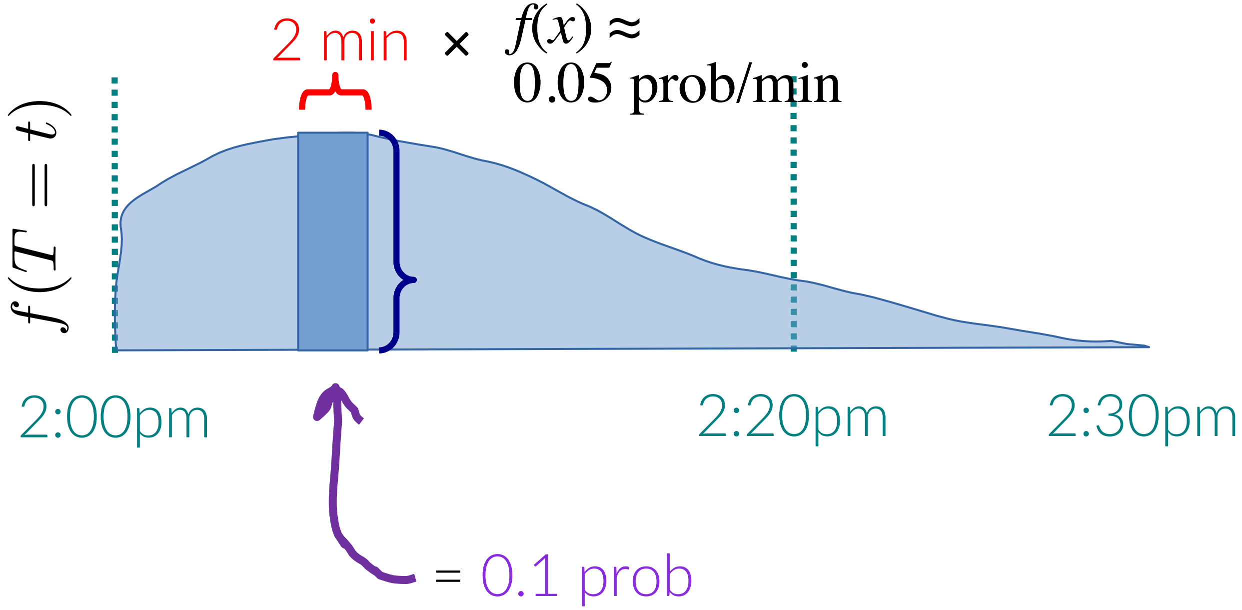


Which of these represent that you think the arrival is more likely to be close to 3:00pm

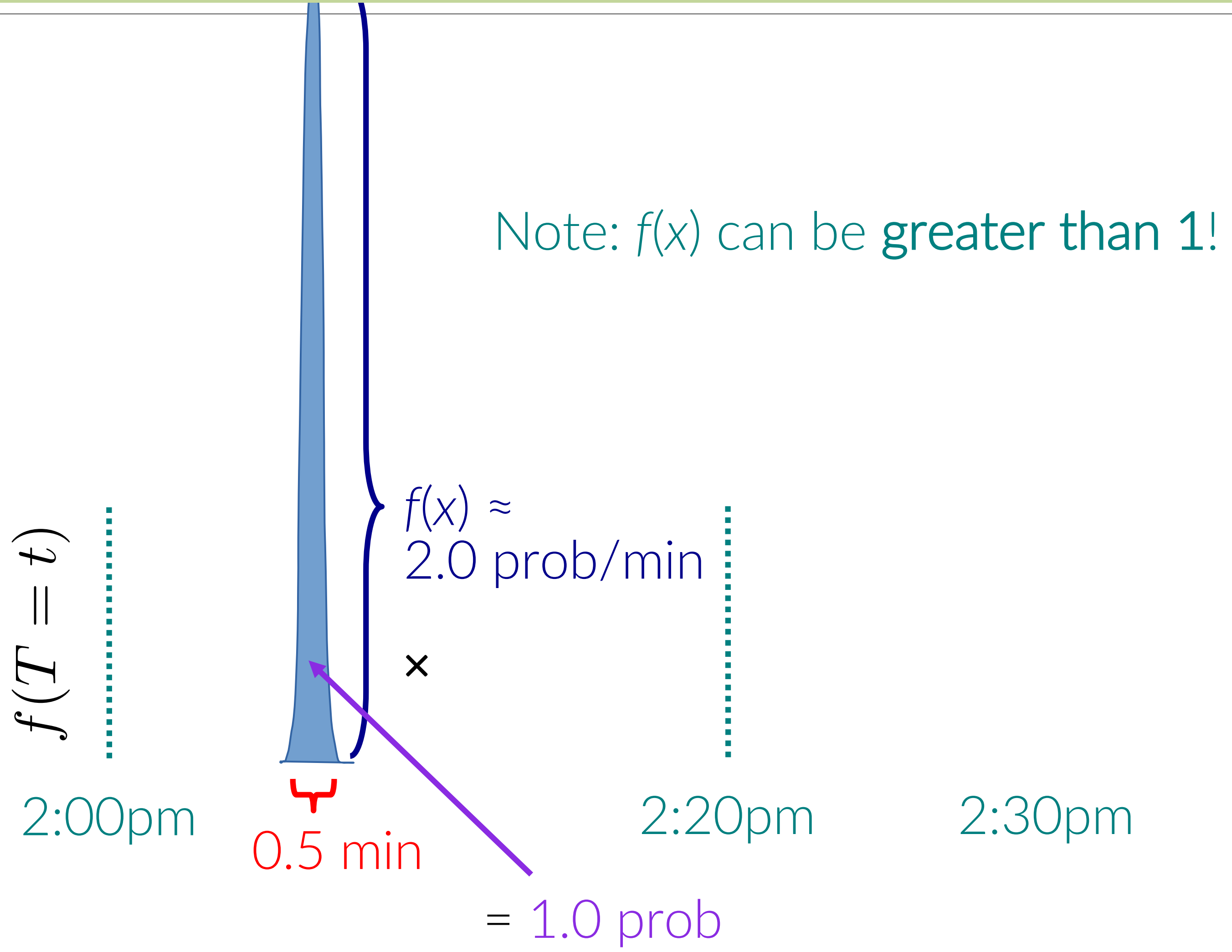
The ratio of probability densities is meaningful

$f(X = x)$ is *Not* a Probability

Rather, it has “units” of:
probability divided by units of *X*.



$f(X = x)$ is *Not* a Probability



$f(X = x)$ vs $P(X = x)$

"The probability that a discrete random variable X takes on the value little *x*."

$$P(X = x)$$

Aka the PMF

"The *derivative* of the probability that a continuous random variable X takes on the value little *x*."

$$f(X = x)$$

Aka the PDF

They are both measures of how likely X is to take on the value x.
Sometimes called the distribution function. Sometimes called the likelihood functions.

Simple Example

Consider a random 5000x5000 matrix, where each element in the matrix is Uniform(0,1). What is the probability that a selected eigenvalue (λ) of the matrix is greater than 0?*

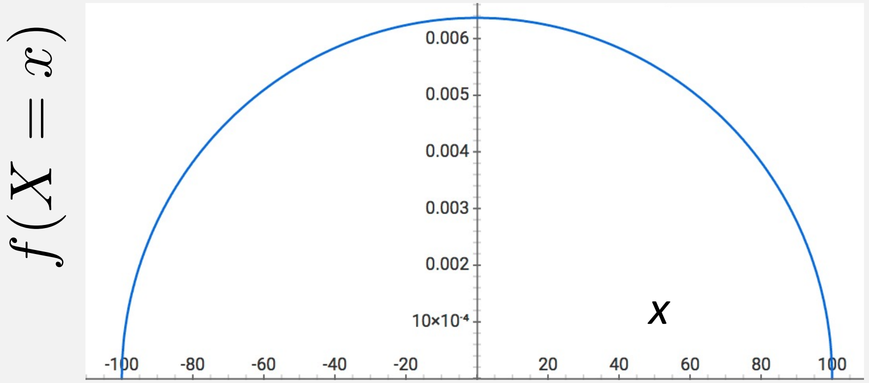
* With help from Wigner, Chris is going to rephrase this problem

Simple Example from Quantum Physics

Let X be a continuous random variable[1]

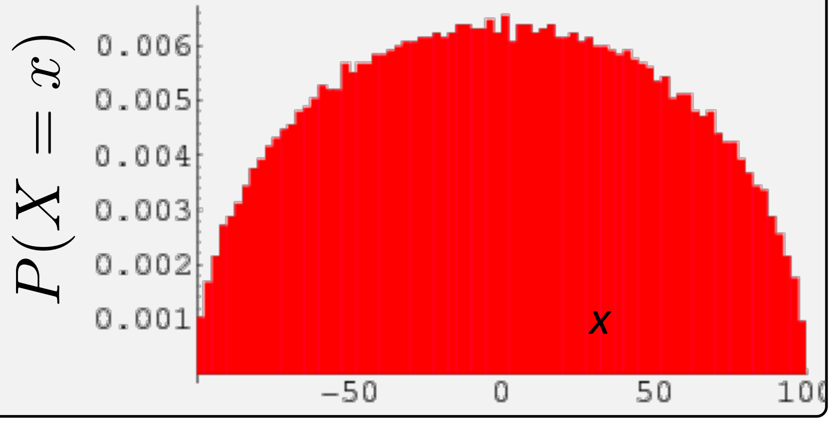
Theory

$$f(X = x) = \frac{1}{15708}\sqrt{100^2 - x^2}$$



Practice

From simulations



$$P(X > 0) = ?$$

[1] X represents the eigenvalue of a 5000x5000 matrix of uniform random variables

https://en.wikipedia.org/wiki/Wigner_semicircle_distribution

Simple Example from Quantum Physics

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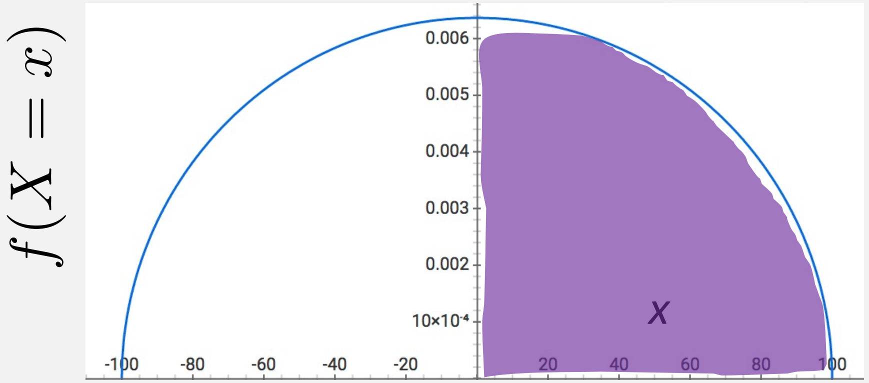

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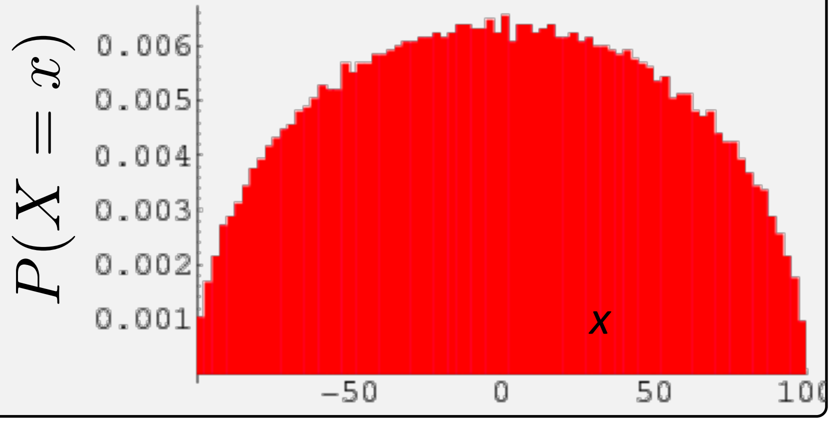


Approach #1: Integrate over the PDF

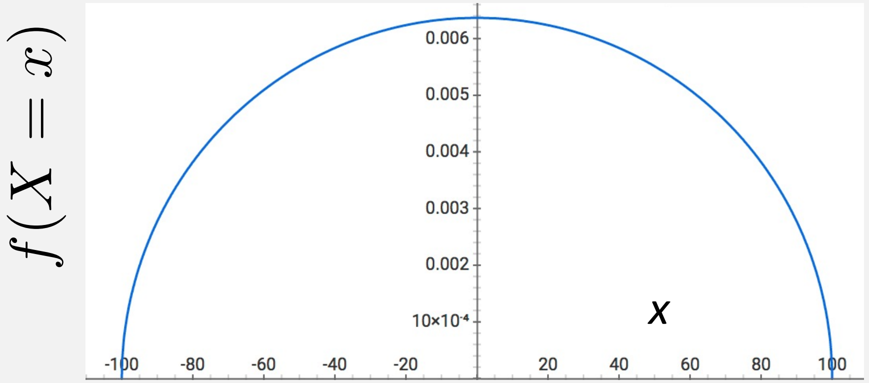
$$P(X > 0) = \int_{0}^{100} f(X = x)\, dx$$

Simple Example from Quantum Physics

Let X be a continuous random variable[1]

Theory

$$f(X = x) = \frac{1}{15708}\sqrt{100^2 - x^2}$$



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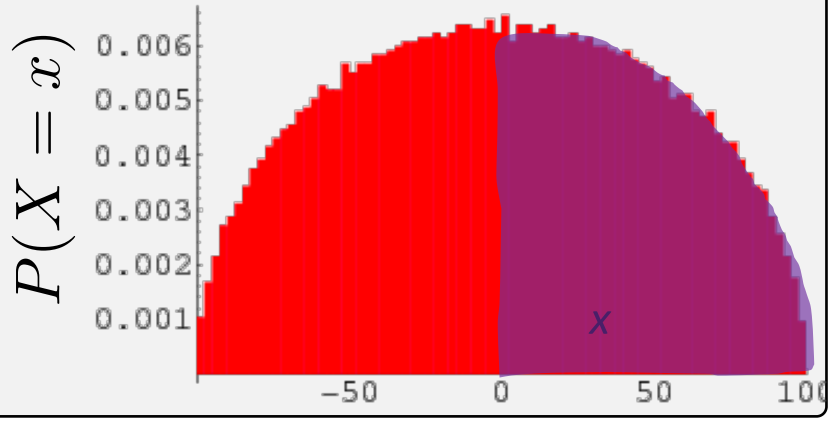


Approach #2: Discrete Approximation

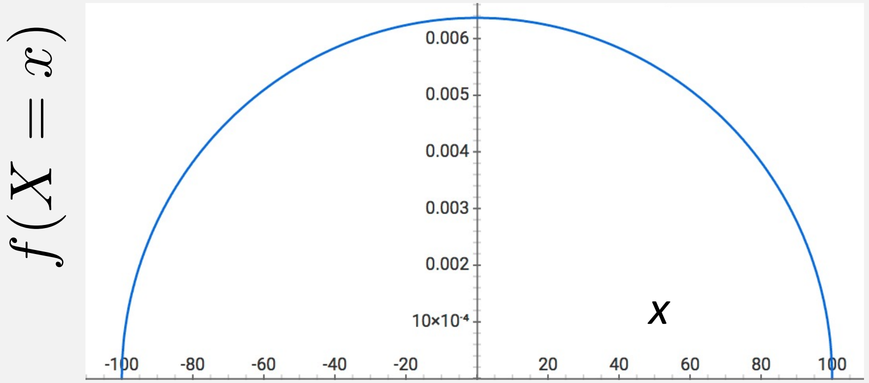
$$P(X > 0) \approx \sum_{i=0}^{100} P(X = i)$$

Simple Example from Quantum Physics

Let X be a continuous random variable[1]

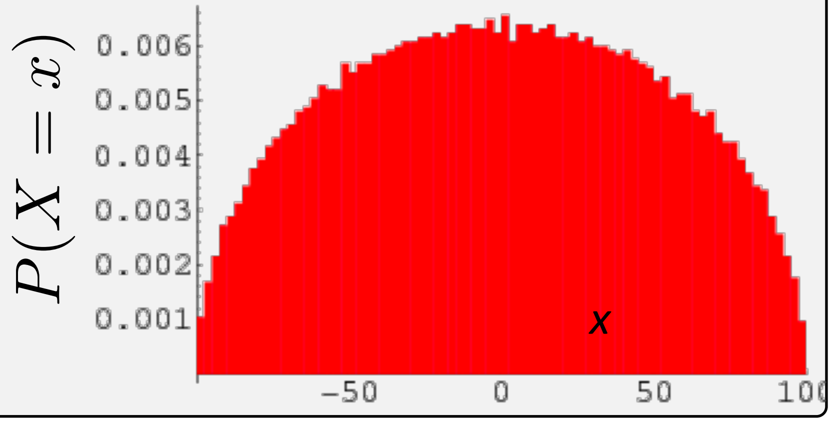
Theory

$$f(X = x) = \frac{1}{15708}\sqrt{100^2 - x^2}$$



Practice

From simulations



Approach #3: Know Semi-Circles

$$P(X > 0) = \frac{1}{2}$$

What do you get if you integrate over a probability *density* function?

A probability!

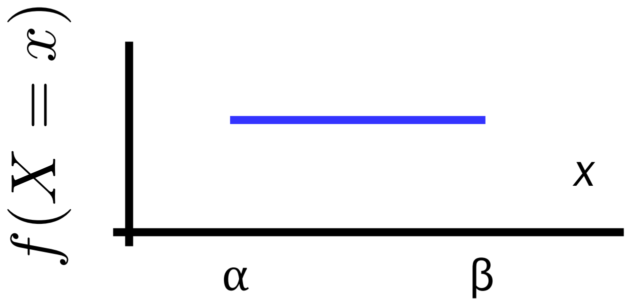
Uniform Random Variable

A **uniform** random variable is **equally likely** to be any value in an interval.

$$X \sim \text{Uni}(\alpha, \beta)$$

Probability Density

$$f(X = x) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$



Properties

$$E[X] = \frac{\beta - \alpha}{2}$$

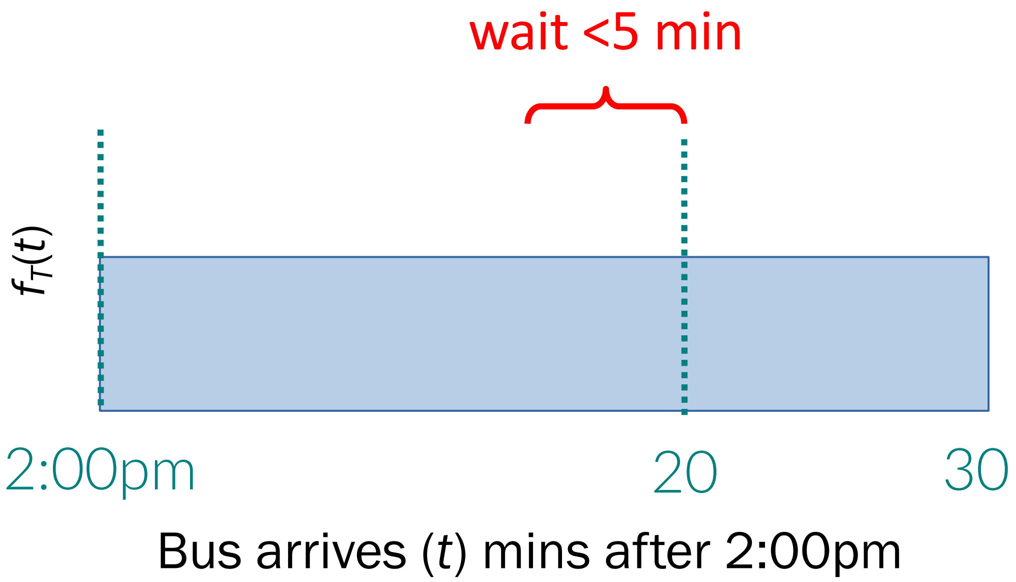
$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

Uniform Bus

You are running to the bus stop. You don't know exactly when the bus arrives. **You believe all times between 2 and 2:30 are equally likely.**

You show up at 2:15pm. What is P(wait < 5 minutes)?

$$T \sim \text{Uni}(\alpha = 0, \beta = 30)$$



$$P(\text{Wait} < 5) = \int_{15}^{20} \frac{1}{\beta - \alpha}\, dx$$

$$= \frac{x}{\beta - \alpha}\Bigg|_{15}^{20}$$

$$= \frac{x}{30 - 0}\Bigg|_{15}^{20} = \frac{5}{30}$$

Expectation and Variance

For <u>discrete</u> RV *X*:

$$E[X] = \sum_x x \cdot p(X = x)$$

$$E[g(X)] = \sum_x g(x) \cdot p(X = x)$$

$$E[X^n] = \sum_x x^n \cdot p(X = x)$$

For <u>continuous</u> RV *X*:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)$$

$$E[X^n] = \int_{-\infty}^{\infty} x^n \cdot f_X(x)$$

For <u>both</u> discrete and continuous RVs:

$$E[aX + b] = aE[X] + b$$

$$\text{Var}(X) = E[(x - \mu)^2] = E[X^2] - (E[X]^2)$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Expectation of Uniform $X \sim \text{Uni}(\alpha, \beta)$

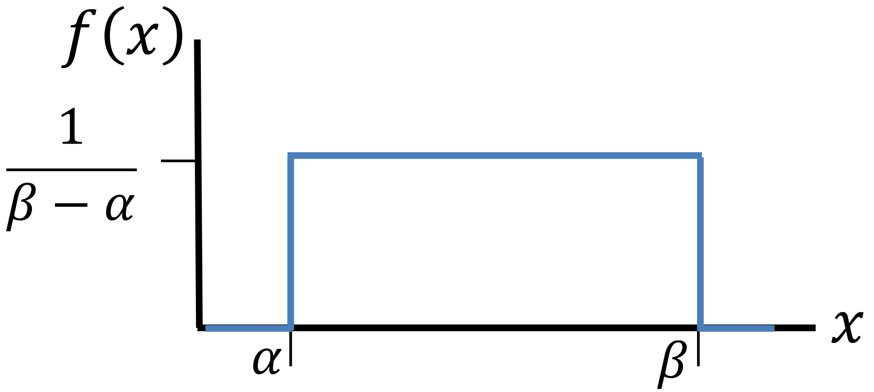
$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\ dx$$

$$= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha}\ dx$$

$$= \frac{1}{\beta - \alpha}\left[\frac{1}{2}x^2\right]_{\alpha}^{\beta}$$

$$= \frac{1}{\beta - \alpha}\left[\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right]$$

$$= \frac{1}{2}\frac{1}{\beta - \alpha}(\beta + \alpha)(\beta - \alpha)$$



just average the start and end!

$$= \frac{1}{2}(\alpha + \beta)$$

Exponential Random Variable

Consider an experiment that lasts a duration of time until success occurs.

def An **Exponential** random variable X is the amount of time until success.

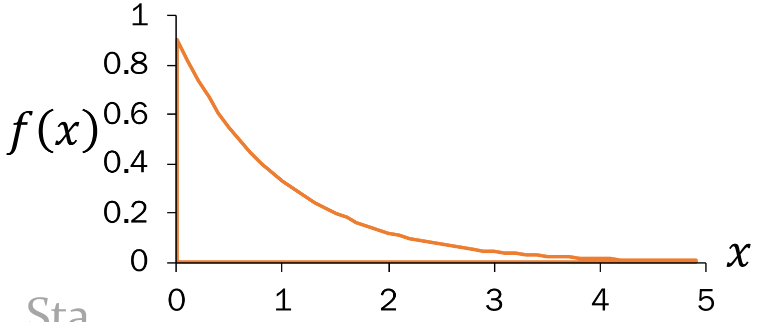
$X \sim \text{Exp}(\lambda)$

Support: $[0, \infty)$

PDF	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
Expectation	$E[X] = \frac{1}{\lambda}$
Variance	$\text{Var}(X) = \frac{1}{\lambda^2}$

Examples:

- Time until next earthquake
- Time for request to reach web server
- Time until end of cell phone contract



1906 Earthquake
Magnitude 7.8

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How Many Earthquakes

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the probability of **zero major earthquakes magnitude next year?**

X = Number of major earthquakes next year

$$X \sim \text{Poi}(\lambda = 0.002)$$

$$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = \frac{0.002^0 e^{-0.002}}{0!} \approx 0.998$$

*In California, according to the USGS, 2015 ty

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the probability of **a major earthquake in the next 30 years?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$f_Y(y) = \lambda e^{-\lambda y}$$
$$= 0.002^{-0.002y}$$

$$P(Y < 30) = \int_0^{30} 0.002e^{-0.002y} \; dy$$

*In California, according to the USGS, 2015

Integral Review

$$\int e^{cx} \, dx = \frac{1}{c} e^{cx}$$

How Long Until the Next Earthquake

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Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002) \qquad f_Y(y) = \lambda e^{-\lambda y}$$

$$= 0.002^{-0.002y}$$

$$P(Y < 30) = \int_0^{30} 0.002 e^{-0.002y} \, dy$$

$$= 0.002 \Big[-500 e^{-0.002y} \Big]_0^{30}$$

$$= \frac{500}{500}(-e^{-0.06} + e^0) \qquad \approx 0.06$$

*In California, according to the USGS, 2015

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the **expected number of years until the next earthquake?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$E[Y] = \frac{1}{\lambda} = \frac{1}{0.002} = 500$$

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the **standard deviation of years until the next earthquake?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$\text{Var}(Y) = \frac{1}{\lambda^2} = \frac{1}{0.002^2} = 250,000 \text{ years}^2$$

$$\text{Std}(Y) = \sqrt{\text{Var}(X)} = 500 \text{ years}$$

*In California, according to the USGS, 2015

Is there a way to avoid integrals?

Cumulative Density Function

A cumulative density function (CDF) is a "closed form" equation for the probability that a random variable is less than a given value

$$F(x) = P(X < x)$$

If you learn how to use a cumulative density function, you can avoid integrals!

$$F_X(x)$$

This is also shorthand notation for the PMF

Cumulative Density Function

$$F(x) = P(X < x)$$

$$x = 2$$

$$0.03125$$

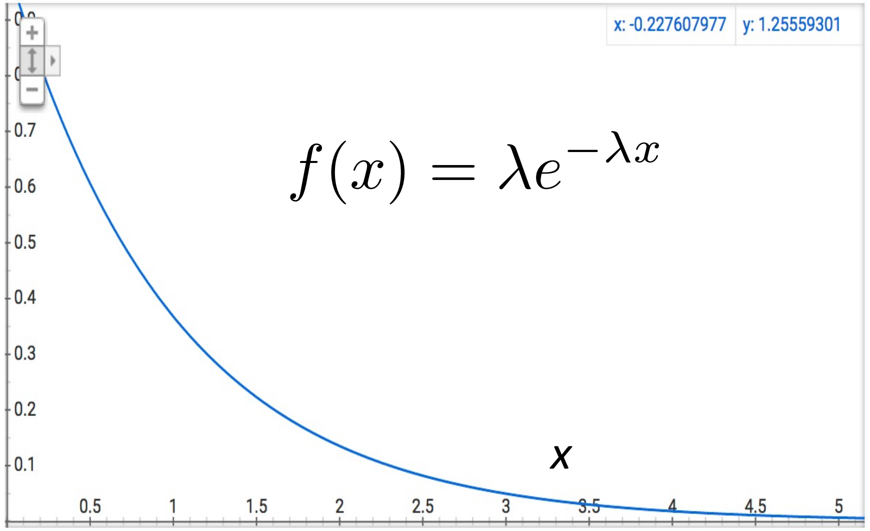
CDF of an Exponential

$$F_X(x) = 1 - e^{-\lambda x}$$

$$\begin{aligned}
P(X < x) &= \int_{y=-\infty}^{x} f(y)\, dy \\
&= \int_{y=0}^{x} \lambda e^{-\lambda y}\, dy \\
&= \frac{\lambda}{\lambda}\Big[-e^{-\lambda y}\Big]_0^x \\
&= [-e^{-\lambda x}] - [-e^{\lambda 0}] \\
&= 1 - e^{-\lambda x}
\end{aligned}$$

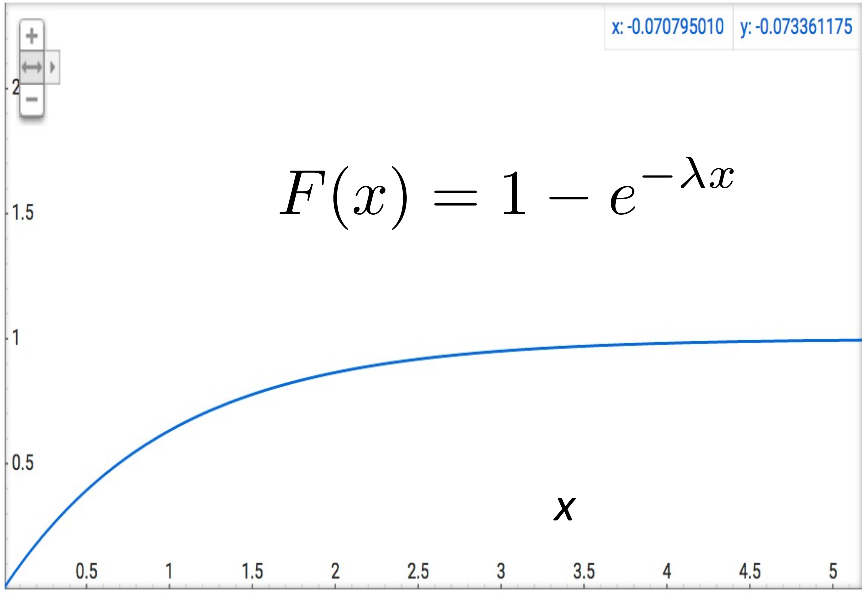
Using CDF Example. X is Exp(λ = 1)

Probability density function



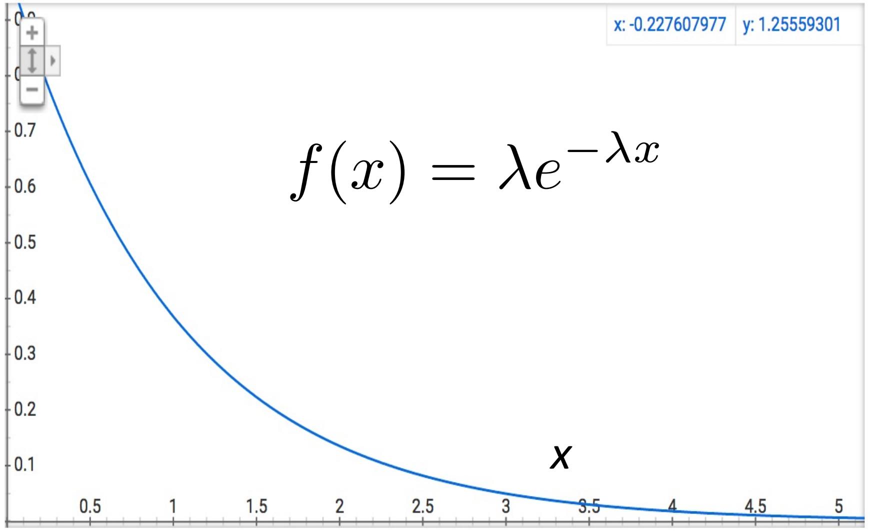
Cumulative density function

$$F_X(x) = P(X < x)$$
$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability density function

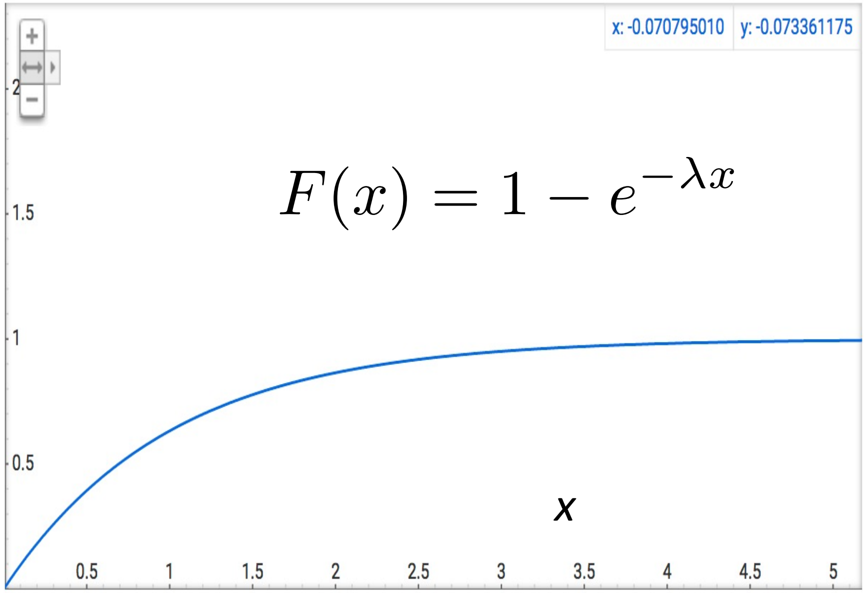


P(X < 2)

Cumulative density function

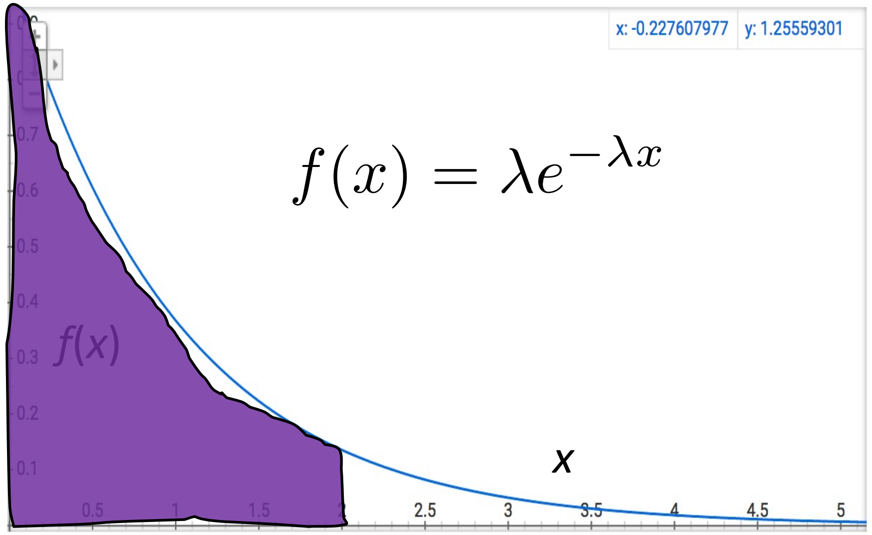
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability
density
function



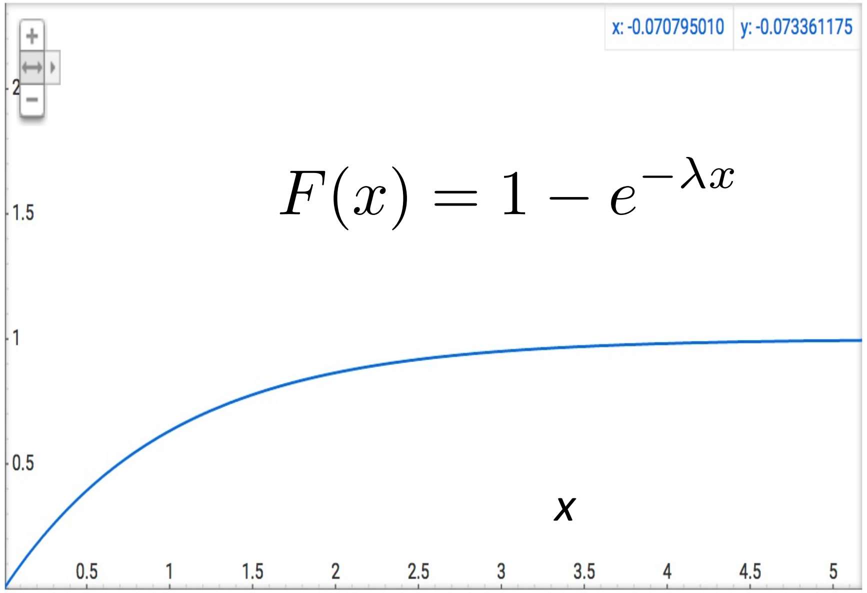
P(X < 2)

$$= \int_{x=-\infty}^{2} f(x)\, dx$$

Cumulative
density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

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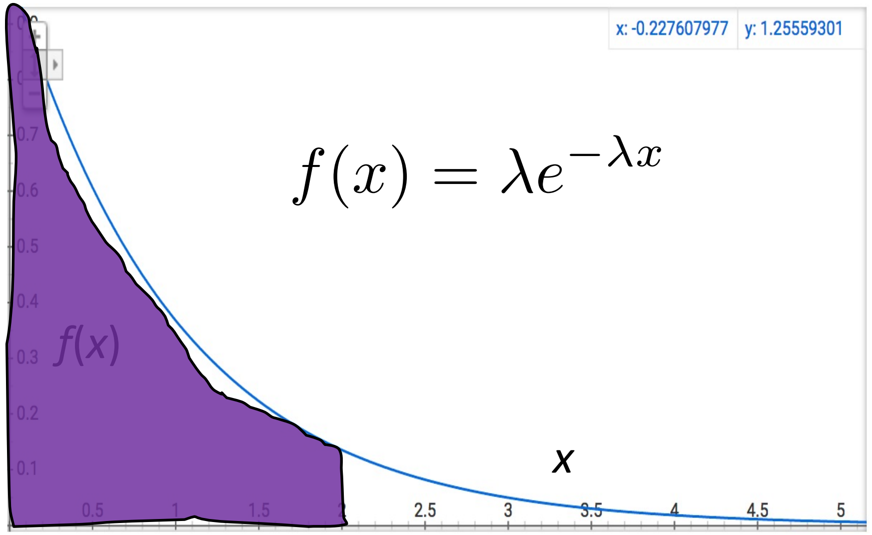


P(X < 2)

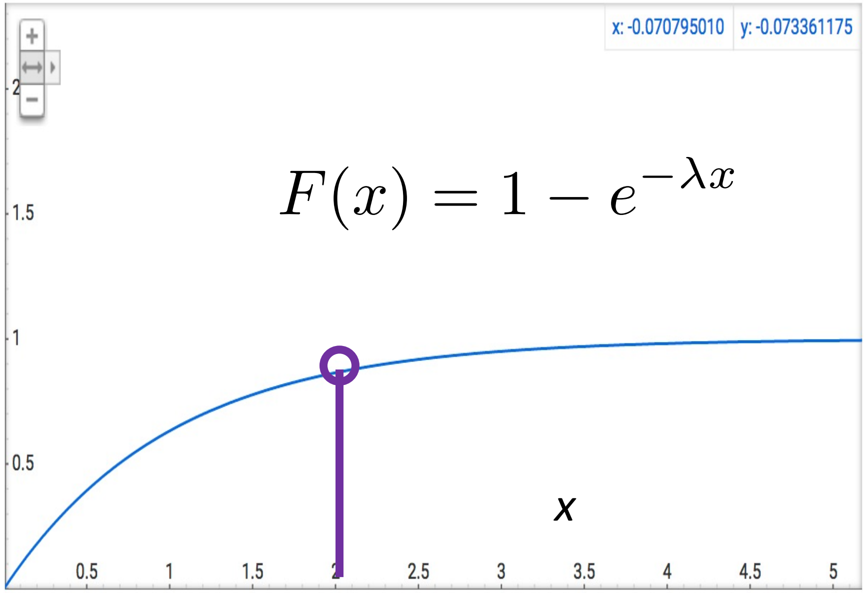
$$= \int_{x=-\infty}^{2} f(x)\, dx$$

or

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



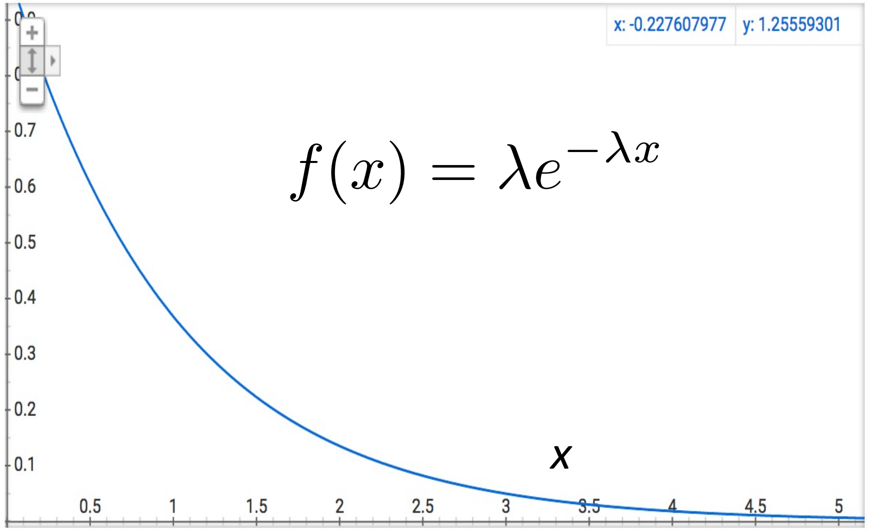
$$= F(2)$$

$$= 1 - e^{-2}$$

$$\approx 0.84$$

Using CDF Example. X is Exp($\lambda = 1$)

Probability density function



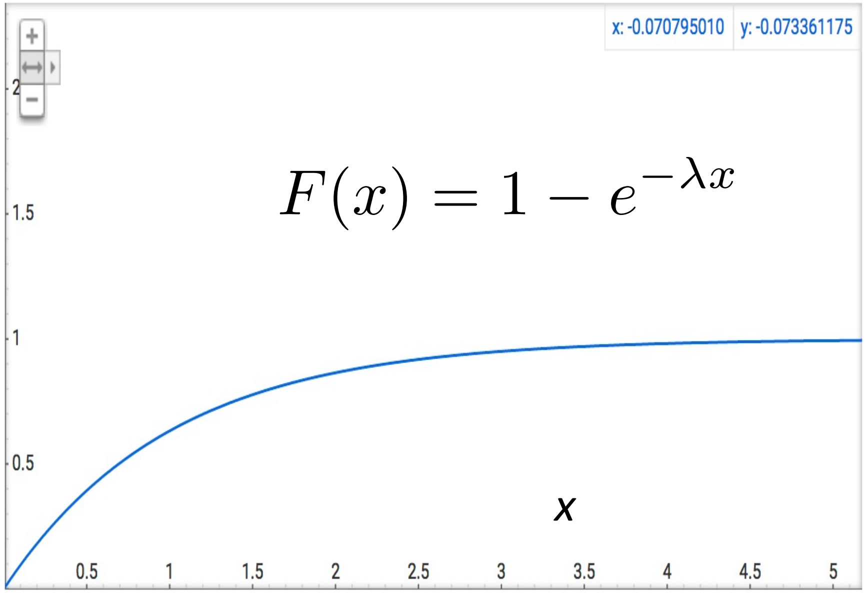
P(X > 1)

____

Cumulative density function

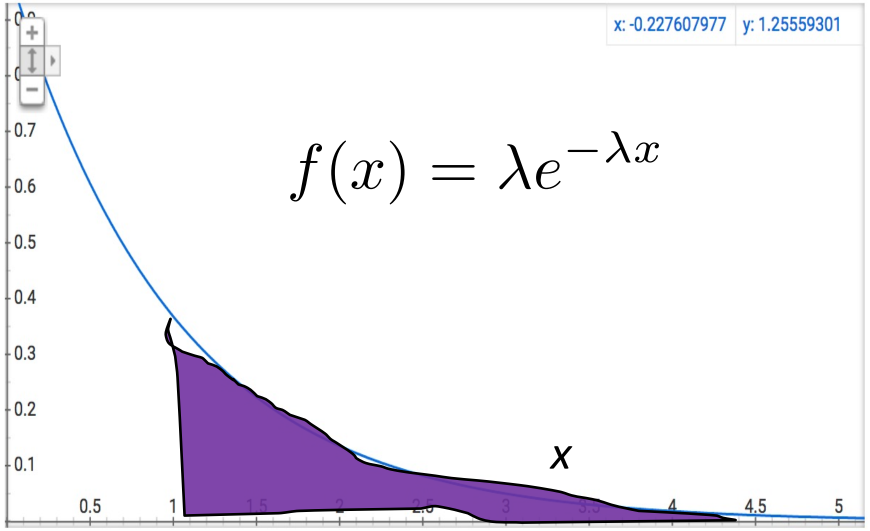
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\ dy$$



Using CDF Example. X is Exp(λ = 1)

Probability
density
function



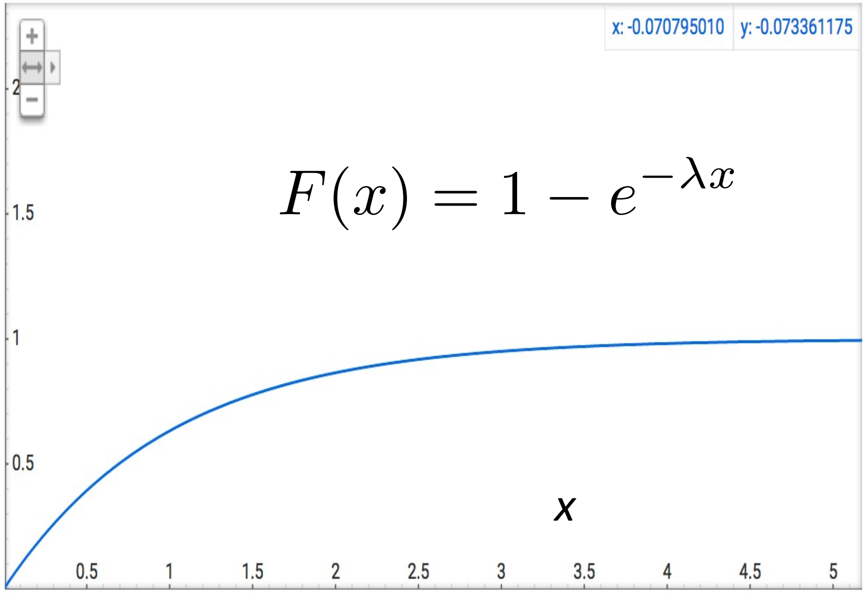
P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\ dx$$

Cumulative
density function

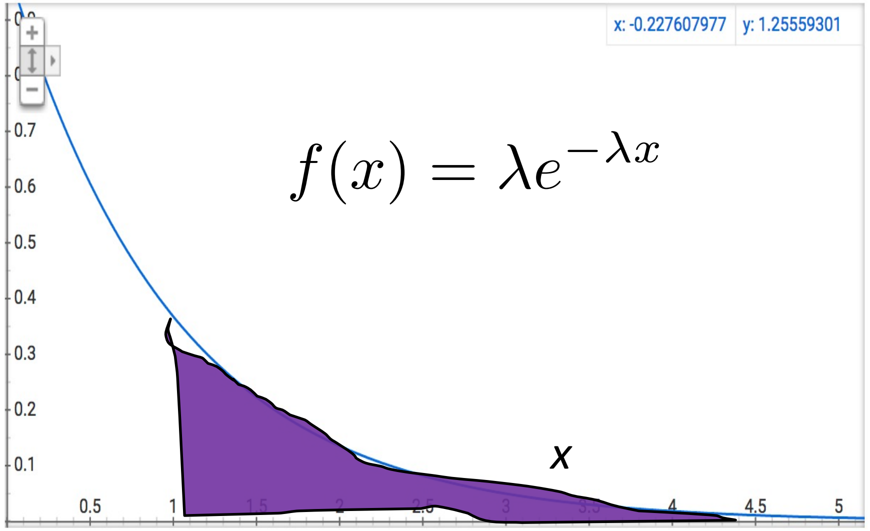
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\ dy$$



Using CDF Example. X is Exp(λ = 1)

Probability
density
function



P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

or

$$= 1 - F(1)$$

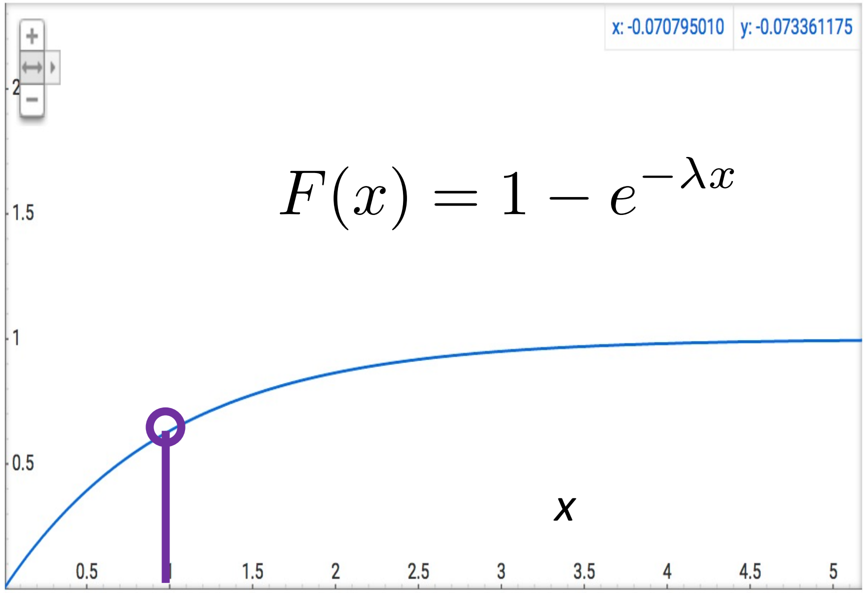
$$= e^{-1}$$

$$\approx 0.37$$

Cumulative
density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability
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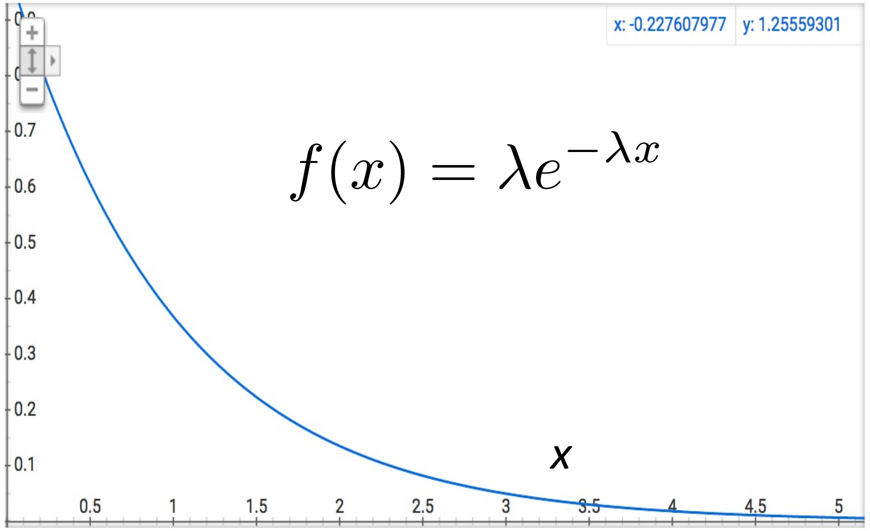

P(1 < X < 2)

Cumulative
density function

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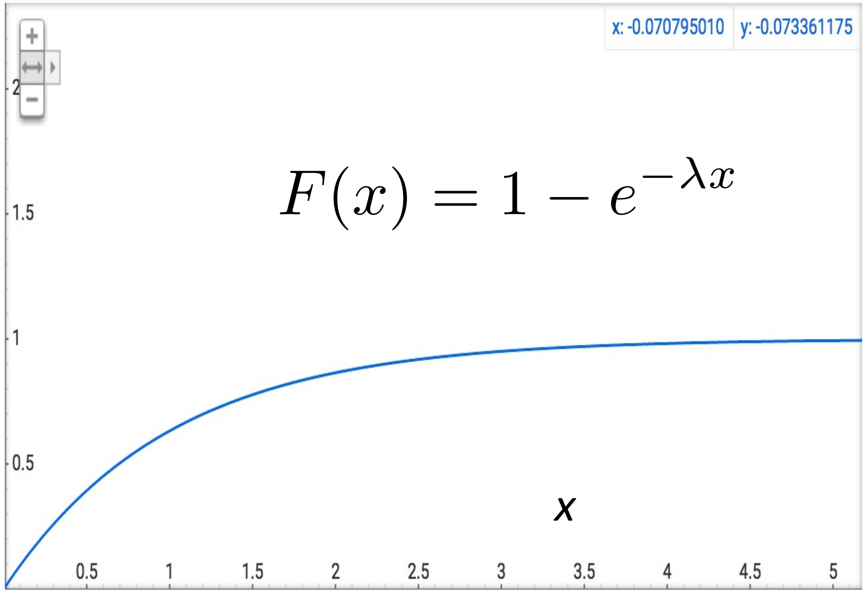
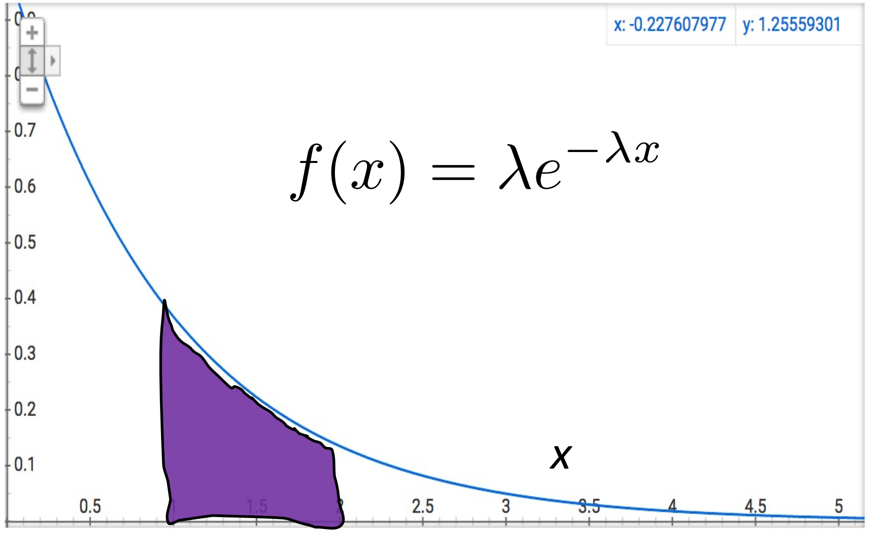


Using CDF Example. X is Exp(λ = 1)

Probability density function



P(1 < X < 2)

$$= \int_{x=1}^{2} f(x)\, dx$$

Cumulative density function

$$F_X(x) = P(X < x)$$

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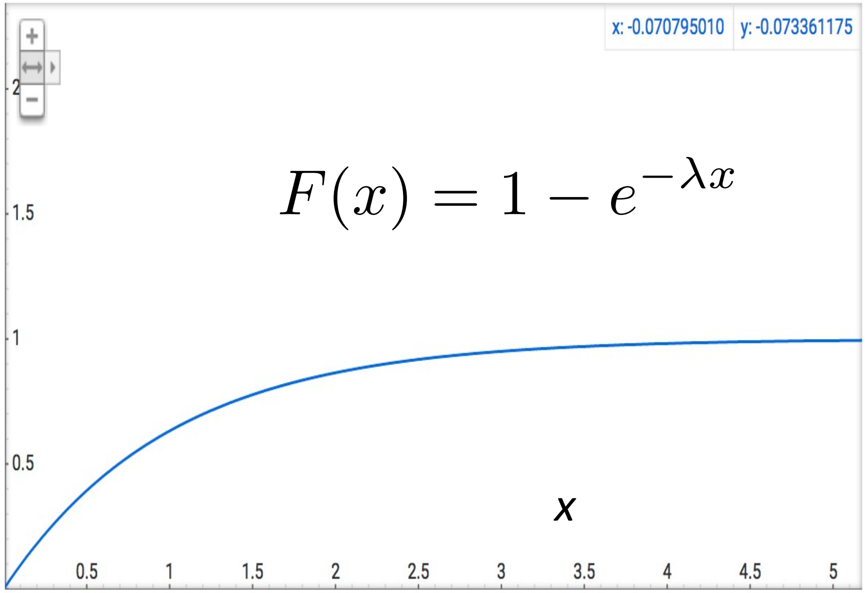

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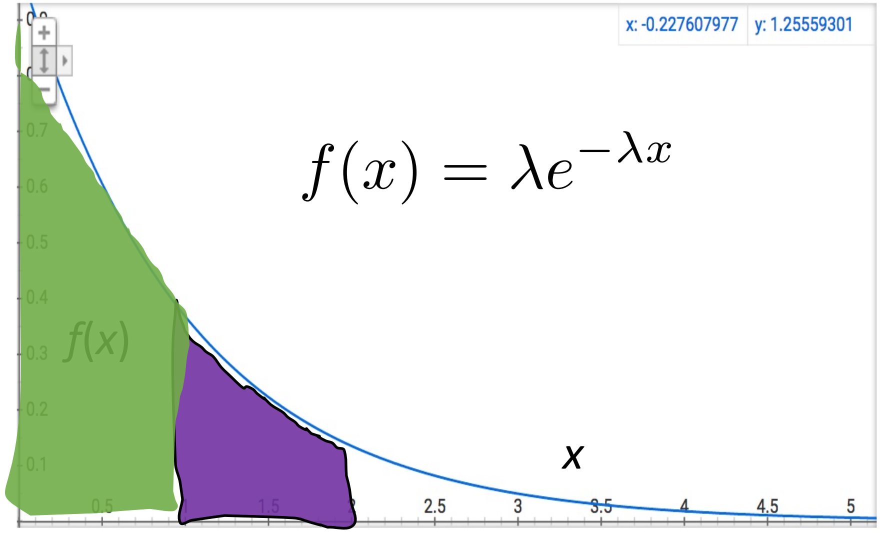

P(1 < X < 2)

$$= \int_{x=1}^{2} f(x)\, dx$$

or

Cumulative density function

$$F_X(x) = P(X < x)$$

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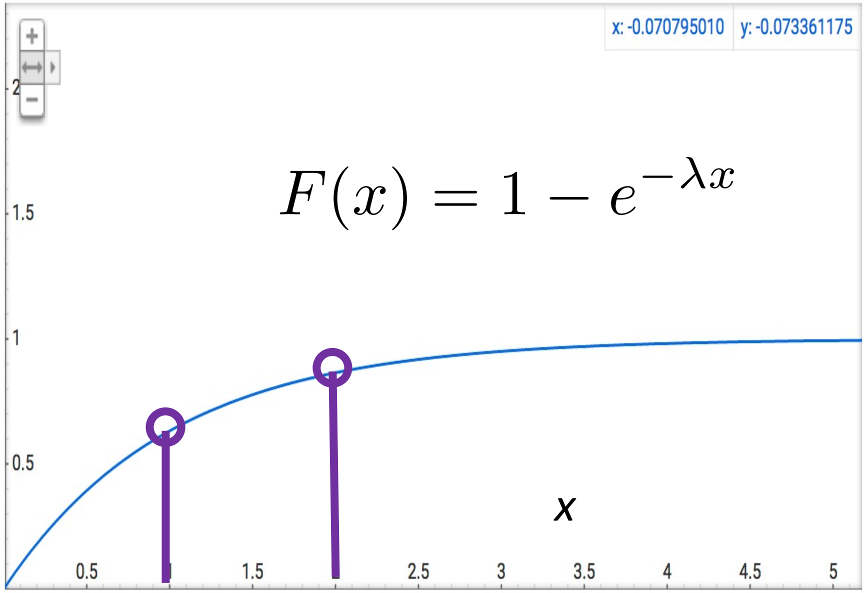


$$= F(2) - F(1)$$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$\approx 0.23$$

Probability of Earthquake in Next 4 Years?

Based on historical data, earthquakes of magnitude 8.0+ happen at a **rate of 0.002** per year*. What is the probability of **an major earthquake in the next 4 years?**

Y = Years until the next earthquake of magnitude 8.0+

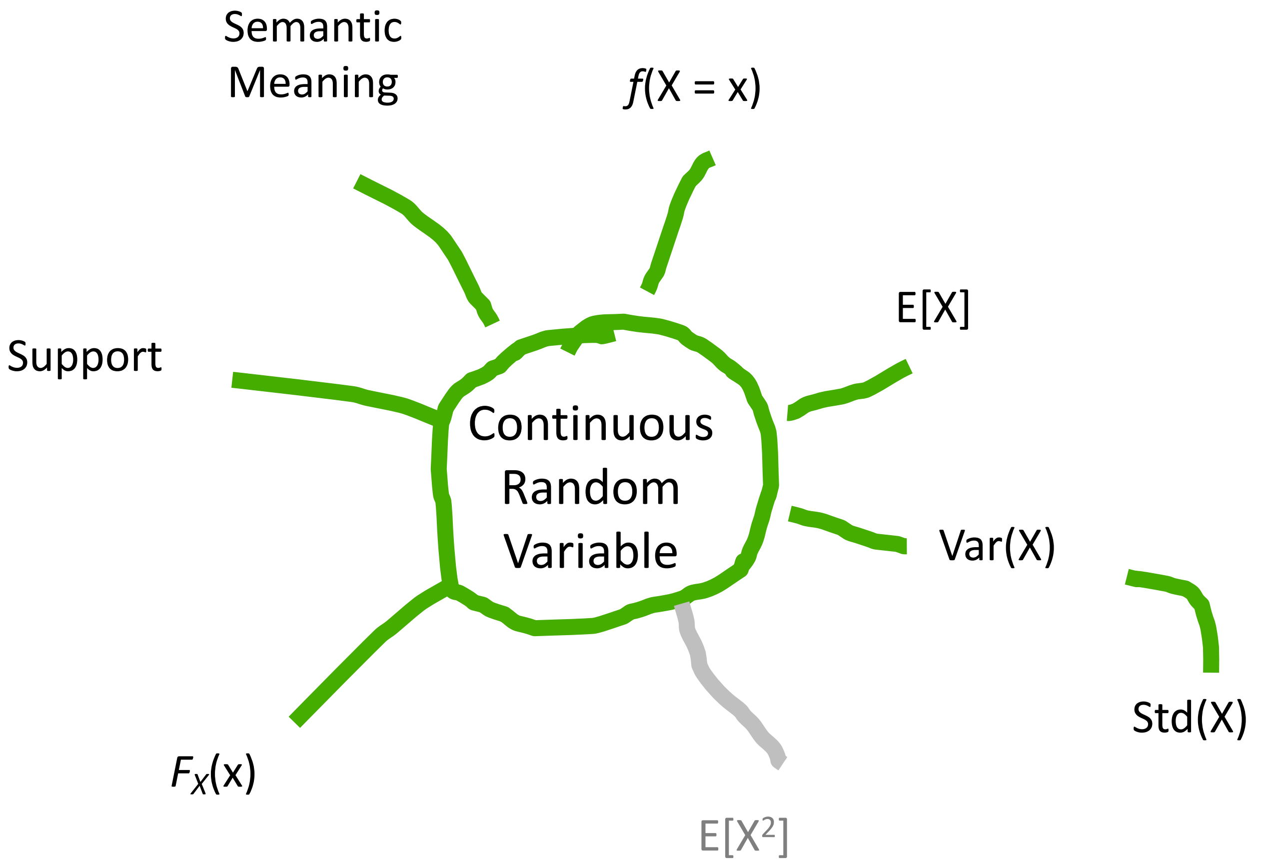
$$Y \sim \text{Exp}(\lambda = 0.002) \qquad F(y) = 1 - e^{-0.002y}$$

$$\begin{aligned} P(Y < 4) &= F(4) \\ &= 1 - e^{-0.002 \cdot 4} \\ &\approx 0.008 \end{aligned}$$

Feeling lucky?

*According to USGS, 2015

Properties for Continuous Random Variable



Here are a few more Random Variables

	number of successes	time to get successes	
One trial	$X \sim \text{Ber}(p)$	$X \sim \text{Geo}(p)$	One success
	$n = 1$	$r = 1$	
Several trials	$X \sim \text{Bin}(n, p)$	$X \sim \text{NegBin}(r, p)$	Several successes
Interval of time	$X \sim \text{Poi}(\lambda)$	$X \sim \text{Exp}(\lambda)$	One success

Extra Problem

Website visits

$X \sim \text{Exp}(\lambda)$ $E[X] = 1/\lambda$
$F(x) = 1 - e^{-\lambda x}$

Suppose a visitor to your website leaves after X minutes.

- On average, visitors leave the site after 5 minutes.
- The length of stay, X, is exponentially distributed.

1. $P(X > 10)$?

2. $P(10 < X < 20)$?

Website visits

$X \sim \text{Exp}(\lambda)$ $E[X] = 1/\lambda$ $F(x) = 1 - e^{-\lambda x}$

Suppose a visitor to your website leaves after X minutes.

- On average, visitors leave the site after 5 minutes.
- The length of stay, X, is exponentially distributed.

1. $P(X > 10)$?

Define

X: when visitor leaves

$X \sim \text{Exp}(\lambda = 1/5 = 0.2)$

Solve

$$P(X > 10) = 1 - F(10)$$

$$= 1 - \left(1 - e^{-10/5}\right) = e^{-2} \approx 0.1353$$

2. $P(10 < X < 20)$?

Solve

$$P(10 < X < 20) = F(20) - F(10)$$

$$= (1 - e^{-4}) - (1 - e^{-2}) \approx 0.1170$$

Exponential is Memoryless

X = time until some event occurs

- X ~ Exp(l)
- What is P(X > s + t | X > s)?

$$P(X > s+t \mid X > s) = \frac{P(X > s+t \text{ and } X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)}$$

$$\frac{P(X > s+t)}{P(X > s)} = \frac{1-F(s+t)}{1-F(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = 1-F(t) = P(X > t)$$

So, $P(X > s+t \mid X > s) = P(X > t)$

- After initial period of time *s*, P(X > t | •) for waiting another *t* units of time until event is same as at start
- "Memoryless" = no impact from preceding period *s*

[Advanced] How to Represent Visual Ability?