02: Combinatorics

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March 30, 2022

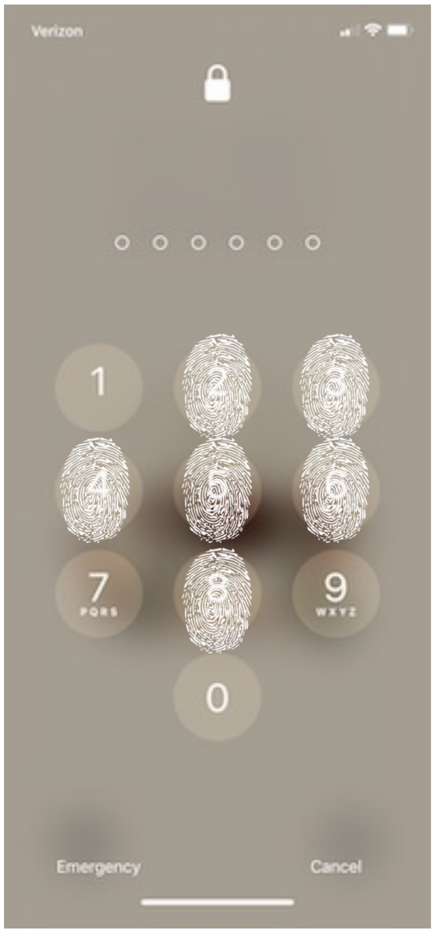
Quick slide reference

Today's discussion thread: https://edstem.org/us/courses/21301/discussion/1324043 *from last time*

Permutations I

Unique 6-digit passcodes with six smudges

from last time



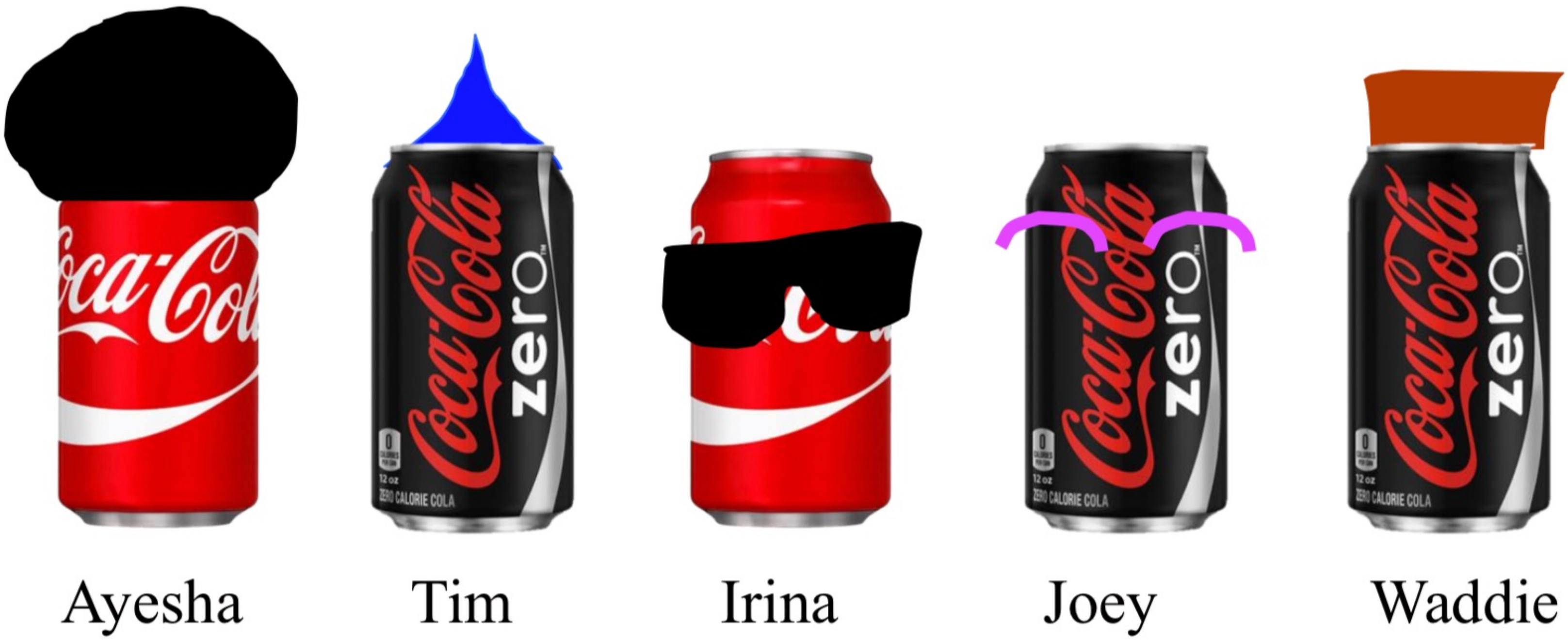
How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

Sort n indistinct objects

from last time

Sort n distinct objects

from last time



Sort n distinct objects

from last time

1st 2nd 3rd 4th 5th

Steps:

1. Choose 1st can — 5 options
2. Choose 2nd can — 4 options

...

5. Choose 5th can — 1 option

Total $= 5 \times 4 \times 3 \times 2 \times 1$

$= 120$

Permutations

from last time

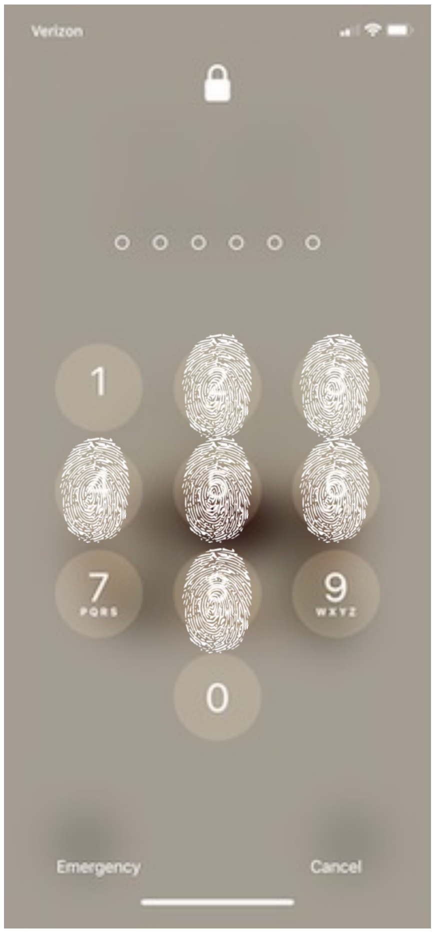
A permutation is an ordered arrangement of objects.

The number of unique orderings (permutations) of n distinct objects is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Unique 6-digit passcodes with six smudges

from last time

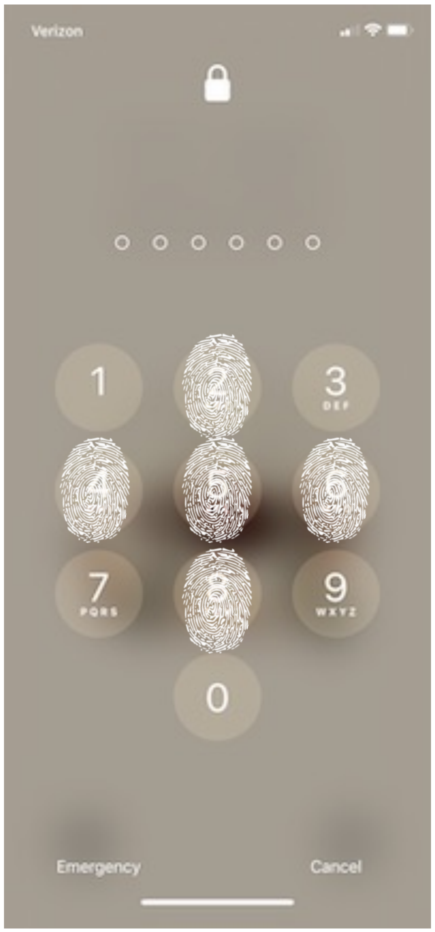


How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6!$$
$$= 720 \text{ passcodes}$$

Unique 6-digit passcodes with five smudges

from last time



How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

Permutations II

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
- Choose k objects (combinations)
- Put objects in r buckets

Sort n distinct objects

# of permutations =

Summary of Combinatorics

Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

Some distinct

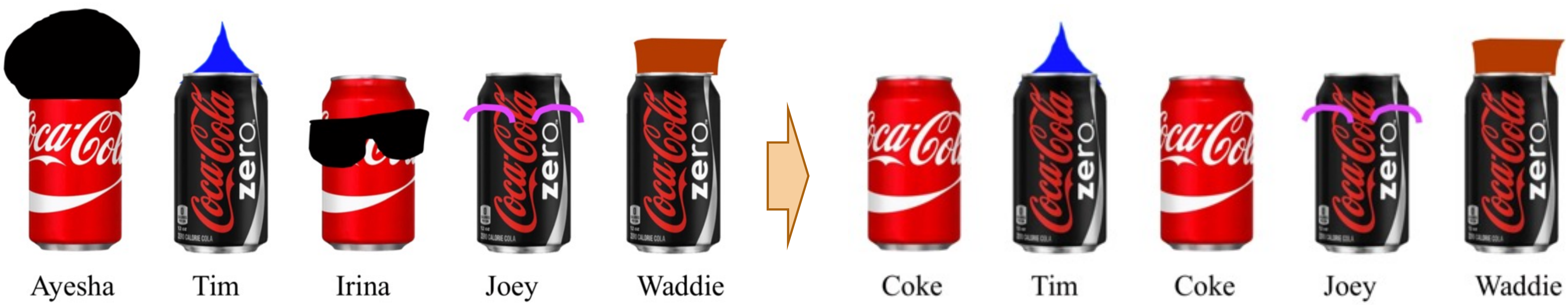
$n!$

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct

Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations of distinct objects = permutations considering some objects are indistinct × Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

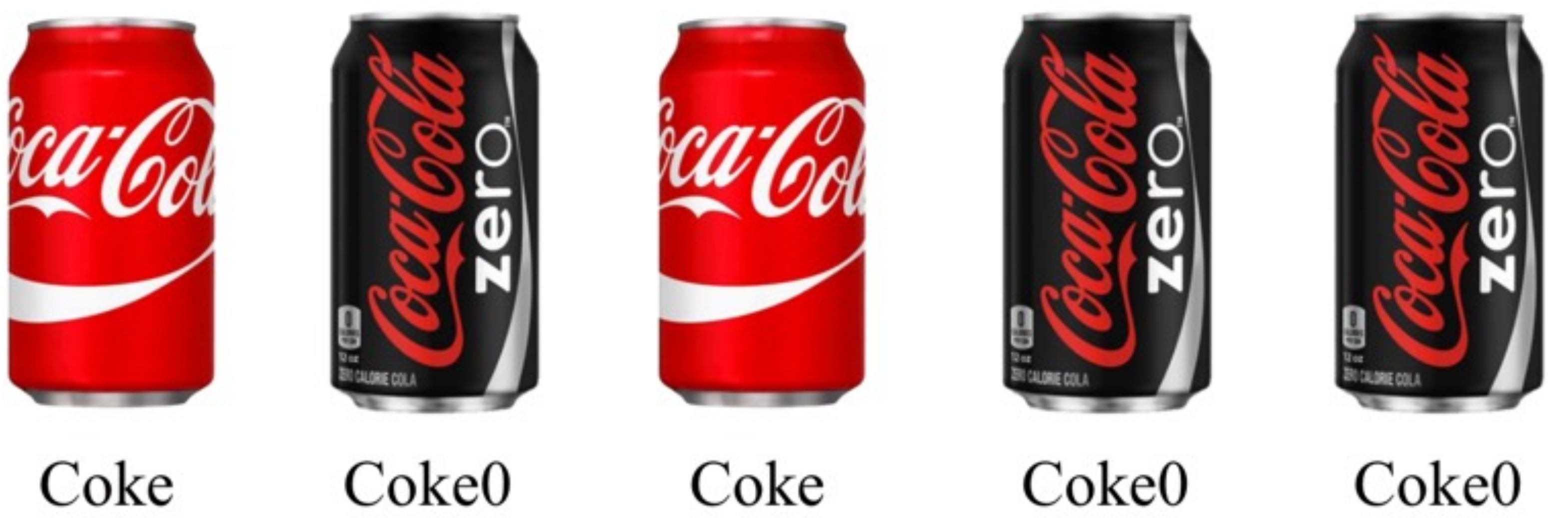
For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects	$\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

$$\frac{5!}{2!\,3!}$$



Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$
- Choose k objects (combinations)
- Put objects in r buckets

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

1. FINALFOUR

$\frac{9!}{2!} \Rightarrow \frac{9!}{2!}$

2. MISSISSIPPI

4 I
4 S
2 P
M 1

$\frac{11!}{4!\,4!\,2!\,1!}$

This is Jerry's dog, Doris. She puts her little Doris paw up to her chin when she's thinking.

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

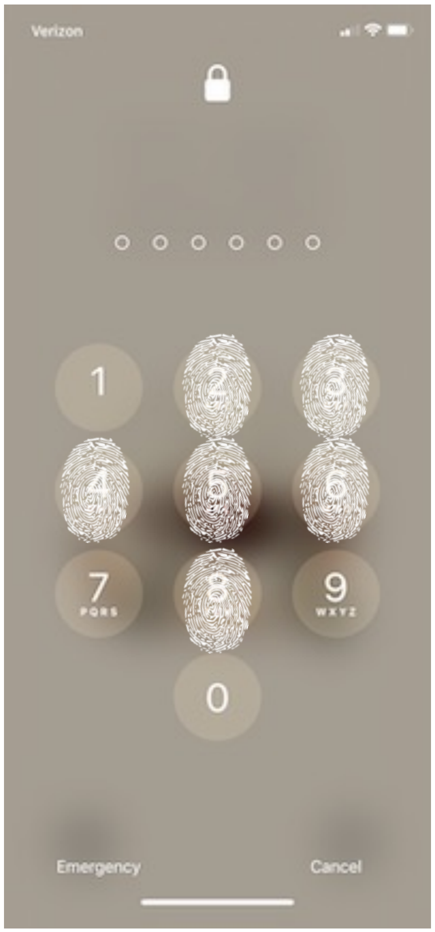
How many letter orderings are possible for the following strings?

1. FINALFOUR $= \frac{9!}{2!} = 181{,}440$

2. MISSISSIPPI $= \frac{11!}{1!4!4!2!} = 34{,}650$

Unique 6-digit passcodes with six smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6!$$
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Unique 6-digit passcodes with five smudges

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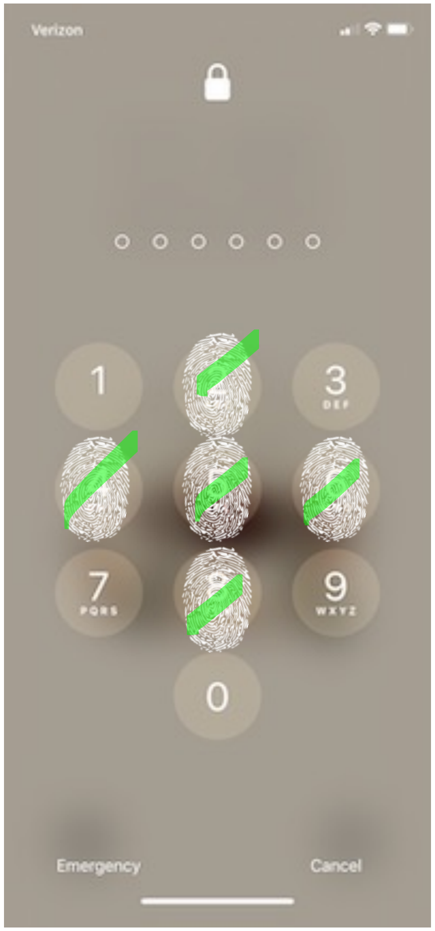

How many unique 6-digit passcodes are possible if a phone password uses each of **five** distinct numbers?

Steps:

1. Choose digit to repeat — 5 outcomes
2. Create passcode — (sort 6 digits: 4 distinct, 2 indistinct)

$$\text{Total} = 5 \times \frac{6!}{2!}$$

$$= 1{,}800 \text{ passcodes}$$

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
- Put objects in r buckets

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
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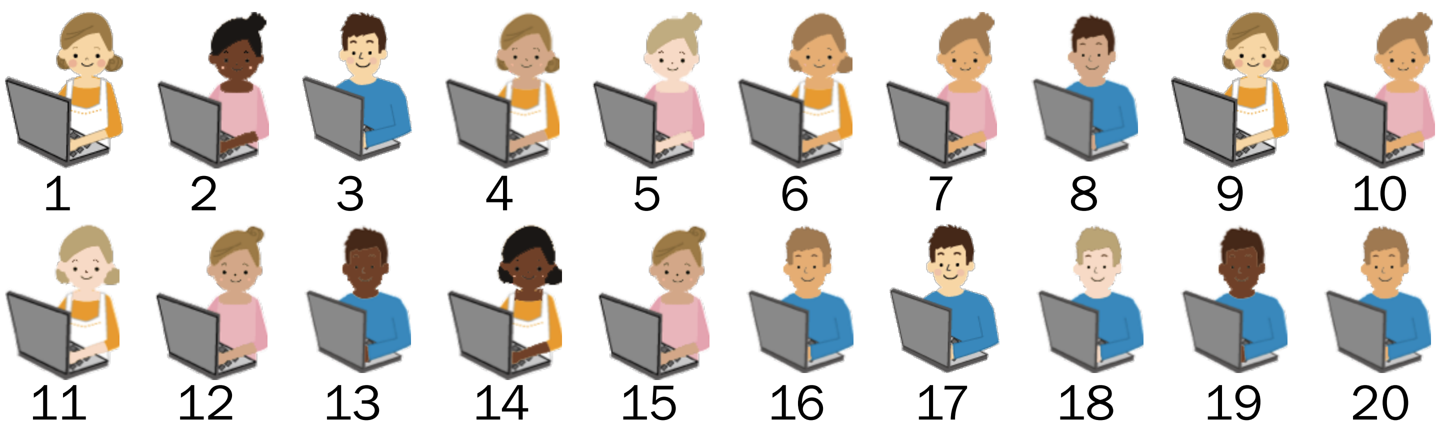
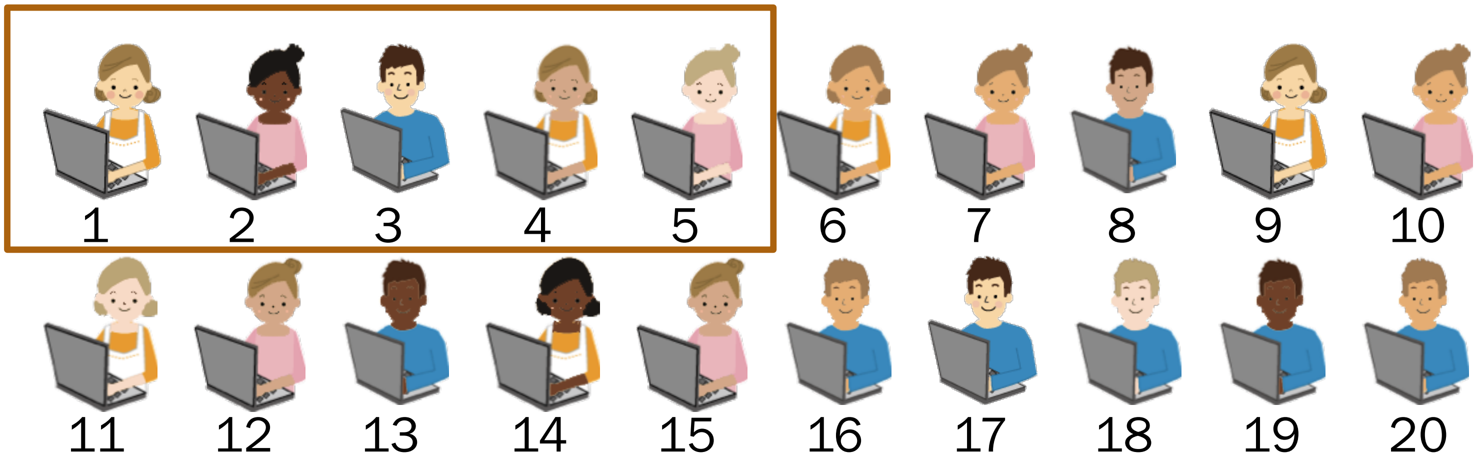


1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line
2. Put first k in cake room

$n!$ ways

1 way

Combinations with cake

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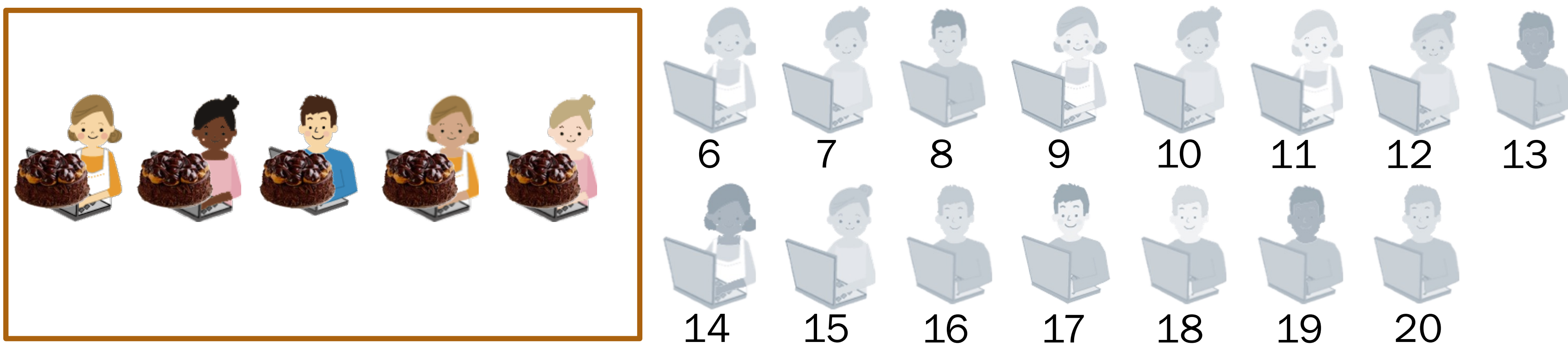

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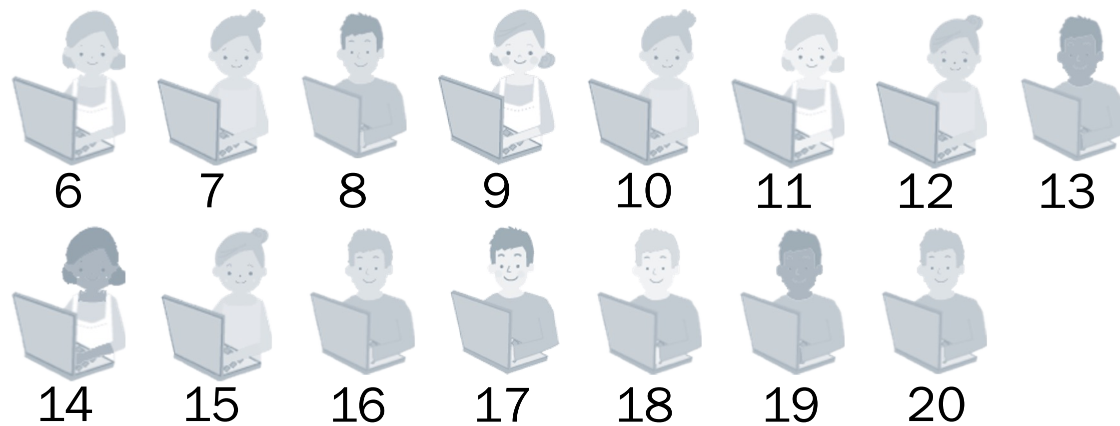

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

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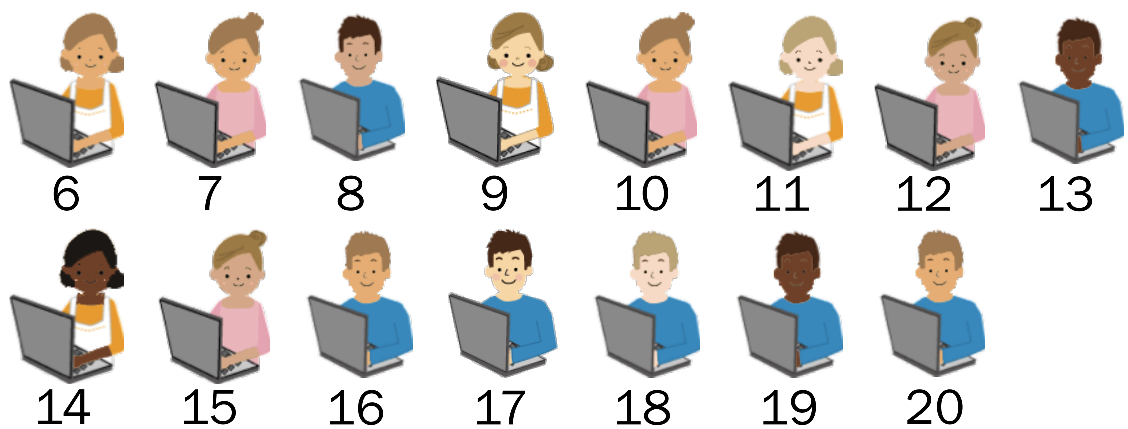


1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line — $n!$ ways
2. Put first k in cake room — 1 way
3. Allow cake group to mingle — $k!$ different permutations lead to the same mingle
4. Allow non-cake group to mingle — $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

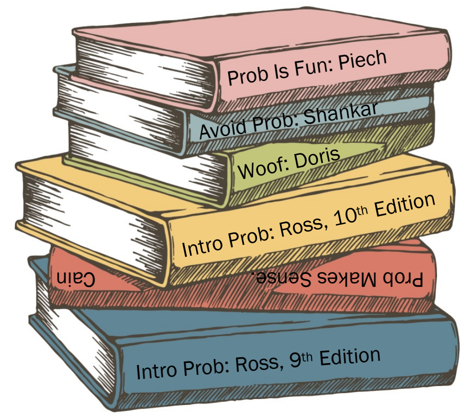
Binomial coefficient

Note: $\binom{n}{n-k} = \binom{n}{k}$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books from a set of 6 distinct books?



$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- **Put objects in r buckets**

General approach to combinations

$\binom{6}{3} = \binom{6}{3,3}$

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \ldots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$\binom{20}{7,7,6}$

$\frac{20!}{7!\,7!\,6!}$

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

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Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

$\frac{13!}{6!\,4!\,3!}$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

$\frac{13!}{6!\,7!} \cdot \frac{7!}{4!\,3!} \cdot \frac{3!}{3!\,0!}$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. Two are by the same author. What if we don't want to choose both?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

B. $\frac{6!}{3!3!2!} = 10$

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

D. $\binom{6}{3} - \binom{4}{1} = 16$

E. Both C and D

F. Something else

Probability textbooks

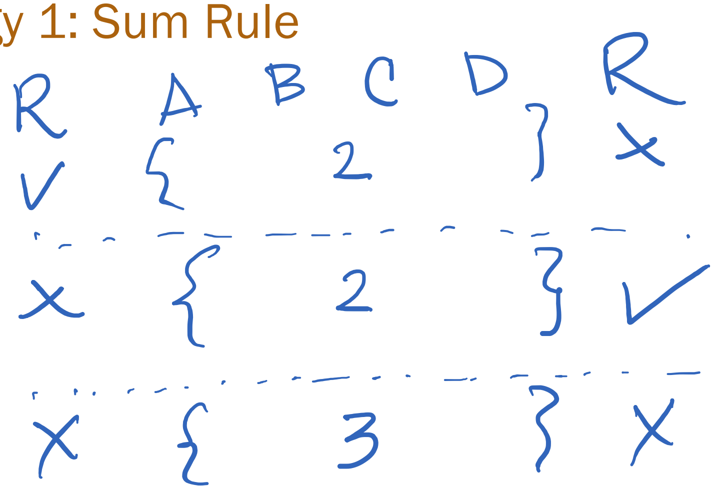
Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. Two are by the same author. What if we don't want to choose both?

Strategy 1: Sum Rule



$$\left(1\cdot\binom{4}{2} + 1\cdot\binom{4}{2} + \binom{4}{3} \right)$$

Probability textbooks

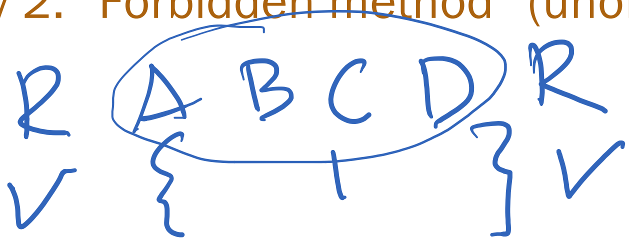
Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. Two are by the same author. What if we don't want to choose both?

Strategy 2: "Forbidden method" (unofficial name)



$$\binom{6}{3} - \binom{4}{1}$$

$$\binom{6}{3} - \binom{4}{1} = 16$$

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.

Mid Slide Deck Stretch

Announcements

Problem Set #1

Out:	today
Due:	Friday 4/8, 3:15pm
Covers:	through Friday 4/1

Section sign-ups

Form released:	already out
Form due:	Saturday 4/2, noon
Results:	latest Monday

Python tutorial

When:	Monday, 4/4 at 7:00pm
Where:	online via Zoom
Recorded?	yes
Notes:	to be posted online

Getting help

Office hours: start tomorrow!
https://cs109.stanford.edu/office-hours

Buckets and The Divider Method

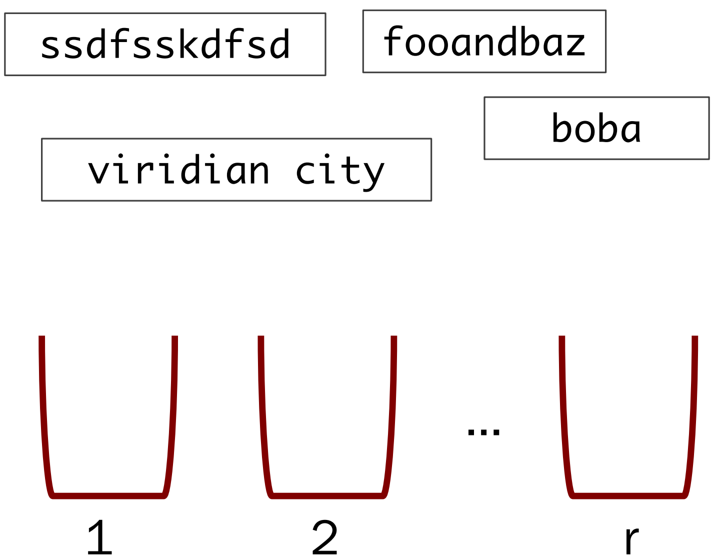
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct
  - Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1st string → r
2. Bucket 2nd string → r

…

n. Bucket nth string → r

r^n outcomes

Summary of Combinatorics

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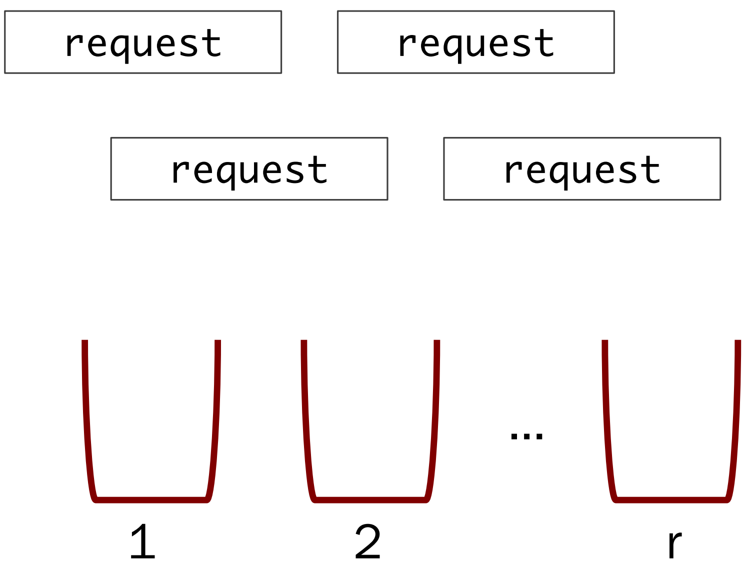
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Servers and indistinct requests

How many ways are there to distribute n indistinct web requests to r servers?



Goal

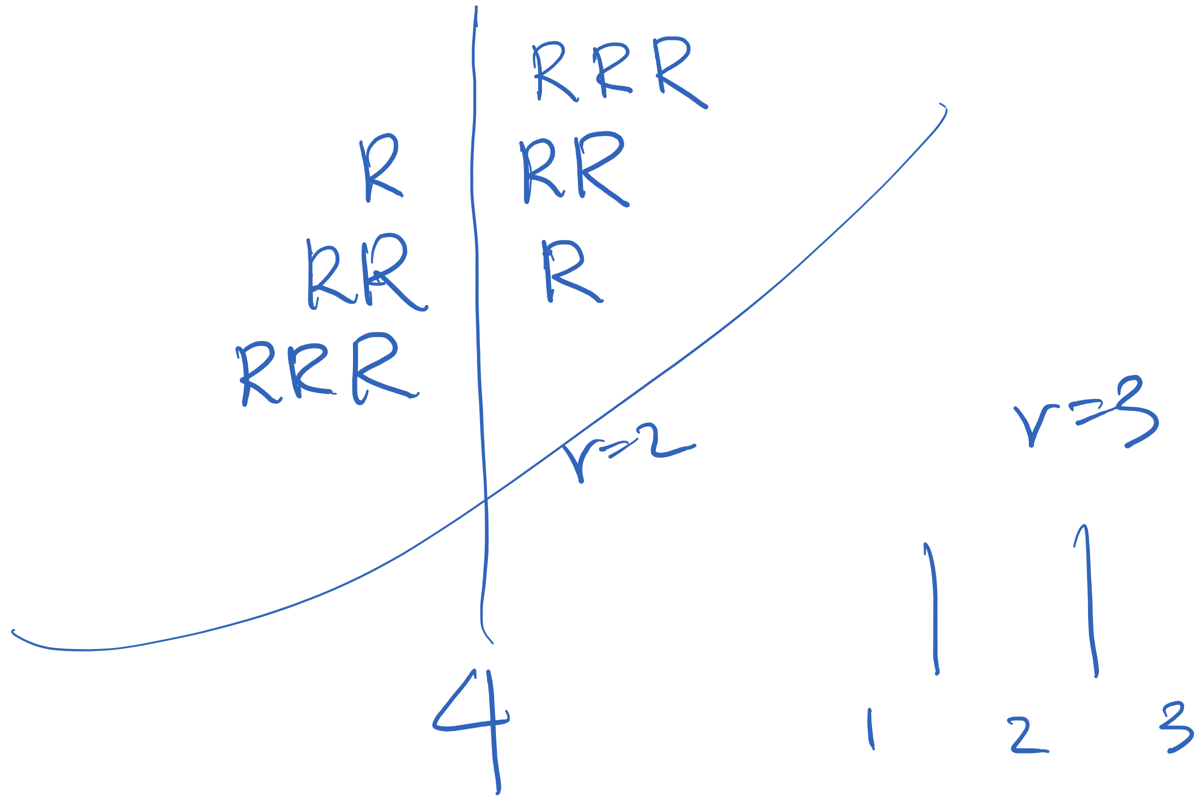
Server 1 has x_1 requests,

Server 2 has x_2 requests,

…

Server r has x_r requests (the rest)

Simple example: $n = 3$ requests and $r = 2$ servers



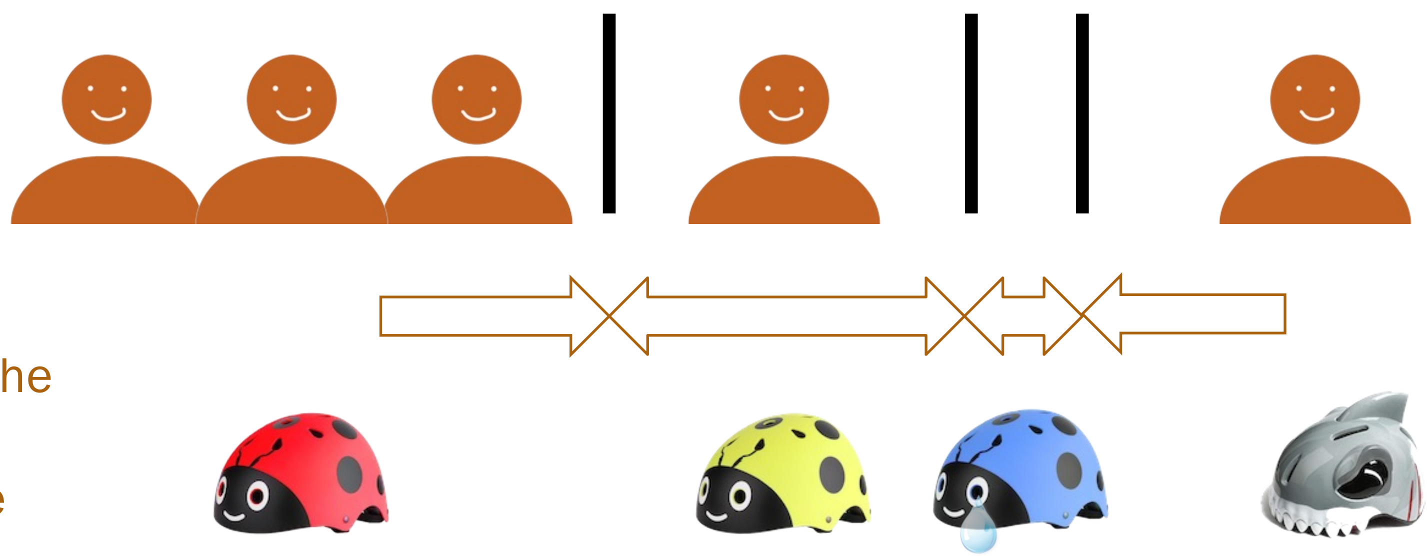
Bicycle helmet sales

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Bicycle helmet sales

1 possible assignment outcome:

Goal Order n indistinct objects and $r-1$ indistinct dividers.



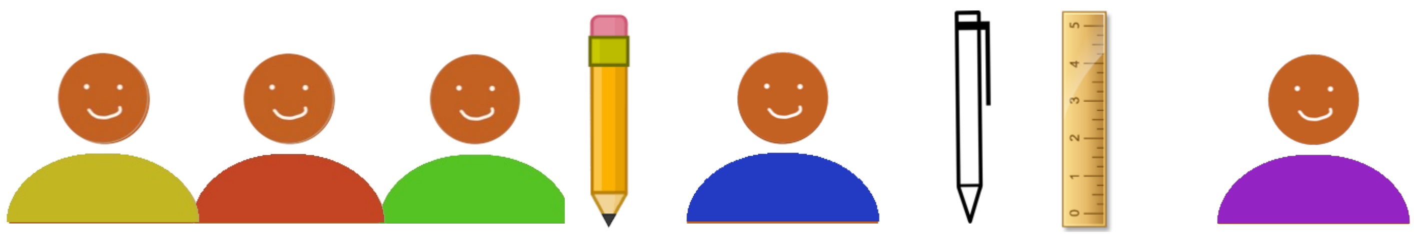
Consider the following generative process...

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



The divider method: A generative proof

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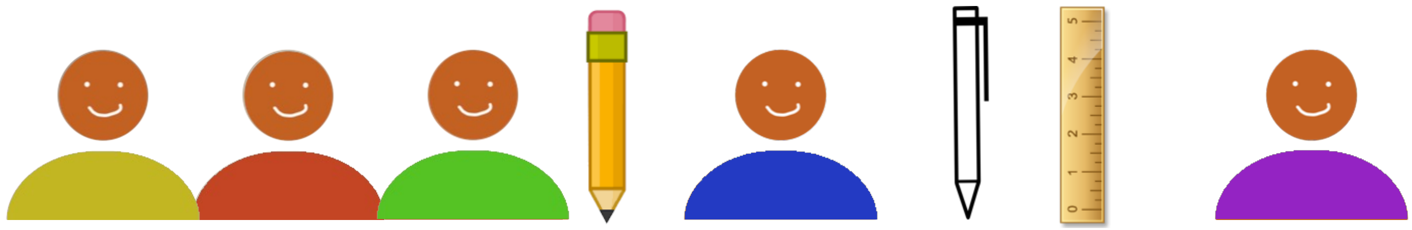

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

The divider method: A generative proof

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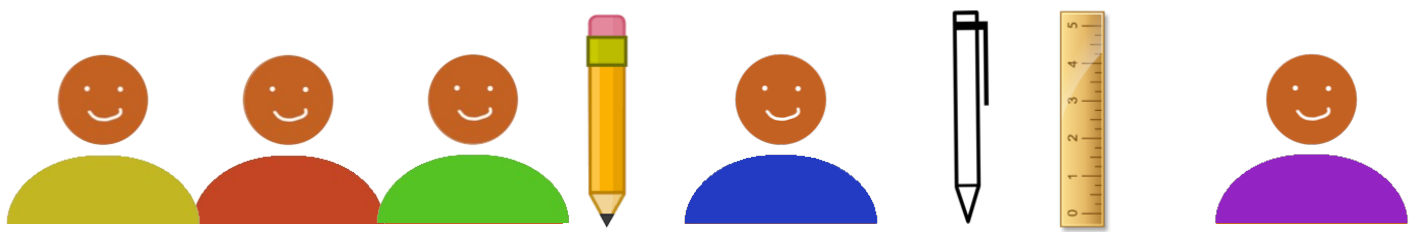

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

The divider method: A generative proof

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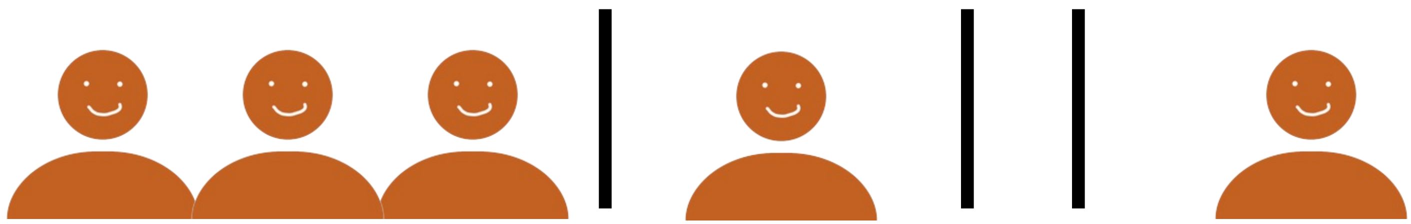


1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$

Integer solutions to equations

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \leq x_i \leq n$?

Positive integer equations can be solved with the divider method.

CS109 Mixer

Introduce yourself to another neighbor!

Then check out the three questions on the next slide (Slide 65).

Discuss how you might use the divider method to solve the following problems. At the very least, discuss the first one. *Hint: what exactly are you dividing?*

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in units of $1 million).

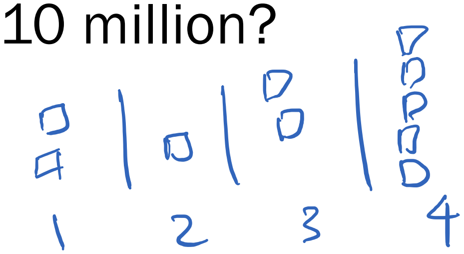
1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don't have to invest all your money?

Venture capitalists. #1

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?



Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

$$\binom{10+4-1}{4-1} = 286$$

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

$x_1 + x_2 + x_3 + x_4 = 7$

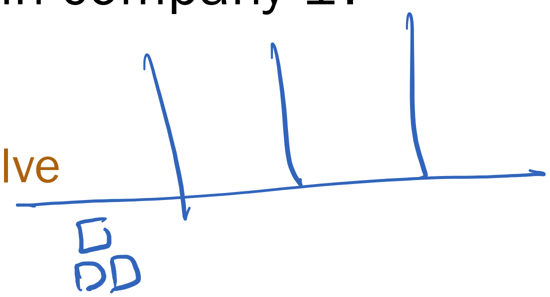
Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$\binom{7+4-1}{4-1} = 120$

Solve



Venture capitalists. #3

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

Set up

$x_1 + x_2 + x_3 + x_4 = 10$

$$x_1 + x_2 + x_3 + x_4 \le 10$$

$x_1 + x_2 + x_3 + x_4 = 9$

x_i: amount invested in company i

$$x_i \ge 0$$

8

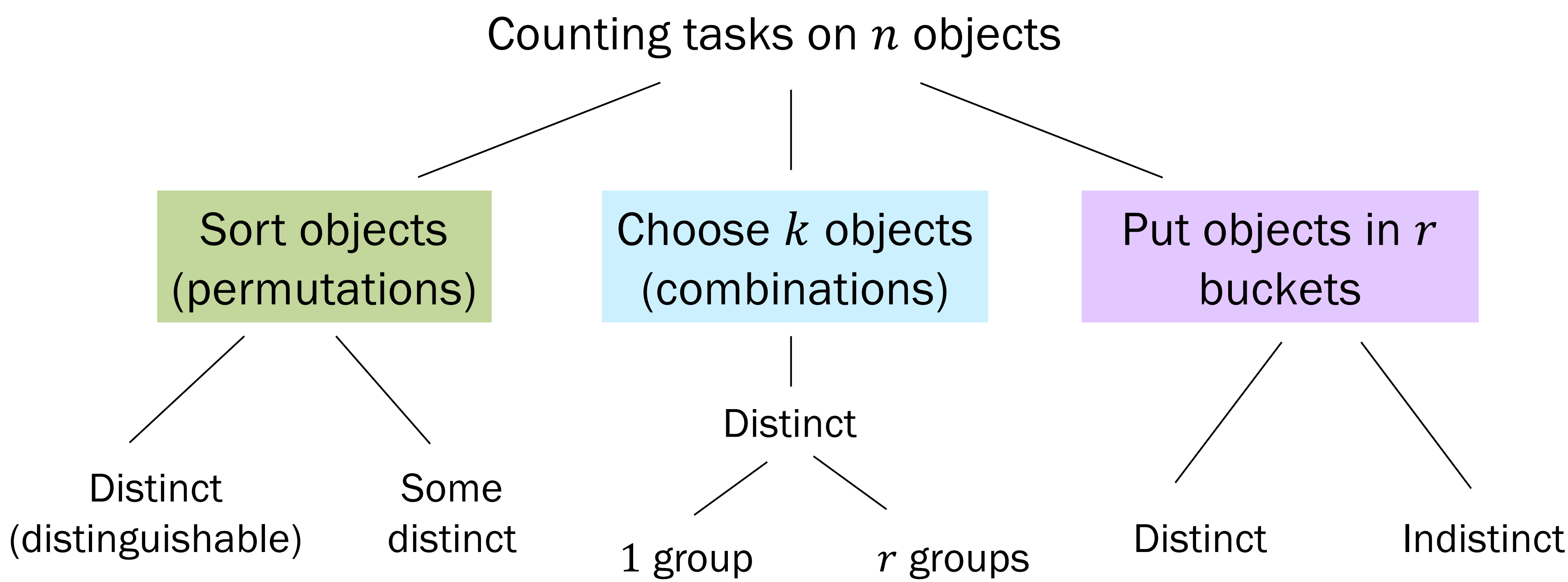
0

Solve

$x_1 + x_2 + x_3 + x_4 + x_5 = 10$

$\binom{10+5-1}{5-1} = 1001$

Summary of Combinatorics



- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases

See you next lecture