04: Conditional Probability and Bayes

Jerry Cain
April 4, 2021
### Quick slide reference

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Conditional Probability
Dice, our misunderstood friends

Roll two 6-sided dice, yielding values $D_1$ and $D_2$.

Let $E$ be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$
$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let $F$ be event: $D_1 = 2$.

What is $P(E, \text{ given } F \text{ already observed})$?
# Conditional Probability

The **conditional probability** of $E$ given $F$ is the probability that $E$ occurs given that $F$ has already occurred. This is known as conditioning on $F$.

Written as: $P(E|F)$

Means: “$P(E,$ given $F$ already observed)”

Sample space $\rightarrow$ all possible outcomes consistent with $F$ (i.e. $S \cap F$)

Event $\rightarrow$ all outcomes in $E$ consistent with $F$ (i.e. $E \cap F$)
Conditional Probability, equally likely outcomes

The conditional probability of $E$ given $F$ is the probability that $E$ occurs given that $F$ has already occurred. This is known as conditioning on $F$.

With equally likely outcomes:

$$P(E|F) = \frac{\# \text{ of outcomes in } E \text{ consistent with } F}{\# \text{ of outcomes in } S \text{ consistent with } F} = \frac{|E \cap F|}{|S \cap F|}$$

$$P(E|F) = \frac{|EF|}{|F|}$$

$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$
Slicing up the spam

24 emails are sent, 6 each to 4 users.
• 10 of the 24 emails are spam.
• All possible outcomes are equally likely.

Let $E = \text{user 1 receives 3 spam emails.}$
What is $P(E)$?

$$P(E) = \frac{\binom{10}{3} \binom{14}{3}}{\binom{24}{6}}$$

Let $F = \text{user 2 receives 6 spam emails.}$
What is $P(E | F)$?

$$P(E | F) = \frac{\binom{4}{3} \binom{14}{3}}{\binom{18}{6}} = \frac{5}{18}$$

Let $G = \text{user 3 receives 5 spam emails.}$
What is $P(G | F)$?

$$P(G | F) = \frac{\binom{4}{5} \binom{14}{1}}{\binom{18}{6}} = 0$$
Slicing up the spam

24 emails are sent, 6 each to 4 users.
- 10 of the 24 emails are spam.
- All possible outcomes are equally likely.

Let $E = \text{user 1 receives 3 spam emails.}$

What is $P(E)$?

$P(E) = \frac{(\binom{10}{3})(\binom{14}{3})}{\binom{24}{6}}$

$\approx 0.3245$

Let $F = \text{user 2 receives 6 spam emails.}$

What is $P(E|F)$?

$P(E|F) = \frac{(\binom{4}{3})(\binom{14}{3})}{\binom{18}{6}}$

$\approx 0.0784$

Let $G = \text{user 3 receives 5 spam emails.}$

What is $P(G|F)$?

$P(G|F) = \frac{(\binom{4}{5})(\binom{14}{1})}{\binom{18}{6}}$

$= 0$

No way to choose 5 spam from 4 remaining spam emails!
Conditional probability in general

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

These properties hold even when outcomes are not equally likely.
Let $E$ = a user watches *Life is Beautiful*. What is $P(E)$?

Equally likely outcomes?

$\times$ Equally likely outcomes? $S = \{\text{watch, not watch}\}$
$E = \{\text{watch}\}$
$P(E) = 1/2$ ?

$\checkmark$ \[
P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\# \text{ people who have watched movie}}{\# \text{ people on Netflix}}
= 10,234,231 / 50,923,123 \approx 0.20
\]
Netflix and Learn

Let $E$ be the event that a user watches the given movie.

\[ P(E|F) = \frac{P(EF)}{P(F)} \]

Definition of Cond. Probability
Netflix and Learn

Let $E$ = a user watches Life is Beautiful.
Let $F$ = a user watches Amelie.

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Definition of Cond. Probability

$$P(E|F) = \frac{\text{# people who have watched both}}{\text{# people on Netflix}} \times \frac{\text{# people who have watched Amelie}}{\text{# people on Netflix}}$$

$$= \frac{\text{# people who have watched both}}{\text{# people who have watched Amelie}}$$

$$\approx 0.42$$
Netflix and Learn

Let $E$ be the event that a user watches the given movie.
Let $F$ be the event that the same user watches Amelie.

$P(E) = 0.19$  $P(E) = 0.32$  $P(E) = 0.20$  $P(E) = 0.09$  $P(E) = 0.20$

$P(E|F) = 0.14$  $P(E|F) = 0.35$  $P(E|F) = 0.20$  $P(E|F) = 0.72$  $P(E|F) = 0.42$

Definition of Cond. Probability

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Law of Total Probability
Today’s tasks

$P(EF)$

Chain rule
(Product rule)

Definition of conditional probability

$P(E|F)$

Law of Total Probability

$P(E)$
Law of Total Probability

**Thm** Let $F$ be an event where $P(F) > 0$. For any event $E$,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

**Proof**

1. $F$ and $F^C$ are disjoint s.t. $F \cup F^C = S$  
   Def. of complement
2. $E = (EF) \cup (EF^C)$  
   (see diagram)
3. $P(E) = P(EF) + P(EF^C)$  
   Additivity axiom
4. $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$  
   Chain rule (product rule)

Note: disjoint sets by definition are mutually exclusive events
General Law of Total Probability

Thm For **mutually exclusive events** $F_1, F_2, \ldots, F_n$
s.t. $F_1 \cup F_2 \cup \cdots \cup F_n = S$,

\[
P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)
\]
Finding $P(E)$ from $P(E|F)$

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is $P$(winning)?

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

Law of Total Probability
Finding $P(E)$ from $P(E|F)$

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is $P($winning$)$?

1. Define events & state goal

   Let: $E$: win, $F$: flip heads
   
   Want: $P($win$) = P(E)$

2. Identify known probabilities

   $P($win$|H) = P(E|F) = 1/6$  
   $P(H) = P(F) = 1/2$  
   $P($win$|T) = P(E|F^C) = 0$  
   $P(T) = P(F^C) = 1 - 1/2$

3. Solve

   
   
   $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$  
   Law of Total Probability

   $P(E) = (1/6)(1/2) + (0)(1/2)$

   $= 1/12 \approx 0.083$
Finding $P(E)$ from $P(E|F)$, an understanding

- Flip a fair coin.
- If heads: roll a fair 6-sided die.
- Else: roll a fair 3-sided die.

You win if you roll a 6. What is $P($winning$)$?

1. Define events & state goal

Let: $E$: win, $F$: flip heads
Want: $P($win$) = P(E)$

"Probability trees" can help connect your understanding of the experiment with the problem statement.
Bayes’ Theorem
Today’s tasks

- Chain rule (Product rule)
- Definition of conditional probability

\[ P(TEF) \]

- Law of Total Probability

\[ P(E) \]

- Bayes’ Theorem

\[ P(F|E) \]

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Thomas Bayes

Rev. Thomas Bayes (~1701-1761):
British mathematician and Presbyterian minister

He looked remarkably similar to Charlie Sheen
(but that’s not important right now)
Detecting spam email

We can easily calculate how many existing spam emails contain “Dear”:

\[ P(E|F) = P\left("\text{Dear}\mid\text{Spam email}\right) \]

But what is the probability that a mystery email containing “Dear” is spam?

\[ P(F|E) = P\left(\text{Spam email}\mid"\text{Dear}\right) \]
(silent drumroll)
Bayes’ Theorem

Thm For any events $E$ and $F$ where $P(E) > 0$ and $P(F) > 0,$

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Proof 2 steps!

Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

Proof 1 more step!
Detecting spam email

- 60% of all email in 2016 is spam.
- 20% of spam has the word “Dear”
- 1% of non-spam (aka ham) has the word “Dear”

You get an email with the word “Dear” in it.

What is the probability that the email is spam?

1. Define events & state goal
   - Want: $P(\text{spam} | \text{“Dear”}) = P(F | E)

2. Identify known probabilities
   - $P(E | F) = 0.20$
   - $P(F) = 0.60$
   - $P(E | F') = 0.01$
   - $P(F') = 0.40$

3. Solve
   - $P(F | E) = \frac{P(E | F)P(F)}{P(E | F)P(F) + P(E | F')P(F')}$
   - $P(F | E) = \frac{(0.20)(0.60)}{(0.20)(0.60) + (0.01)(0.40)} = 0.967$
Detecting spam email, an understanding

- 60% of all email in 2016 is spam.
- 20% of spam has the word “Dear”
- 1% of non-spam (aka ham) has the word “Dear”

You get an email with the word “Dear” in it. What is the probability that the email is spam?

1. Define events & state goal

Let: $E$: “Dear”, $F$: spam

Want: $P(\text{spam}|\text{“Dear”})$  

$= P(F|E)$

Note: You should still know how to use Bayes/ Law of Total Prob., but drawing a probability tree can help you identify which probabilities you have. The branches are determined using the problem setup.
Bayes’ Theorem terminology

- 60% of all email in 2016 is spam.
- 20% of spam has the word “Dear”
- 1% of non-spam (aka ham) has the word “Dear”

You get an email with the word “Dear” in it.
What is the probability that the email is spam?

Want: \( P(F|E) \)

\[
P(F|E) = \frac{P(E|F)P(F)}{P(E)}
\]

normalization constant
This class going forward

Last week
Equally likely events

\[ P(E \cap F) \quad P(E \cup F) \]
(counting, combinatorics)

Today and for most of this course
Not equally likely events

\[ P(E = \text{Evidence} \mid F = \text{Fact}) \]
(collected from data)

Bayes’

\[ P(F = \text{Fact} \mid E = \text{Evidence}) \]
(categorize a new datapoint)
Conditional probability in general

General **definition** of conditional probability:

\[
P(E | F) = \frac{P(EF)}{P(F)}
\]

The **Chain Rule** (aka Product rule):

\[
P(EF) = P(F)P(E | F)
\]

These properties hold even when outcomes are not equally likely.
Then check out the question on the next slide (Slide 34). Brainstorm with your neighbor for a few minutes, then we’ll discuss.
Brainstorm with neighbors

You have a flowering plant.

Let

\[ E = \text{Flowers bloom} \]
\[ F = \text{Plant was watered} \]
\[ G = \text{Plant got sun} \]

1. How would you write
   i. the probability that the plant got sun, given that it was watered and flowers bloomed?
   ii. the probability that the plant got sun and flowers bloomed given that it was watered?

2. Using the Venn diagram, compute the above probabilities.

   i. \[ P(GE) = \underline{P(\text{G|E})} \cdot P(E) \]
   ii. \[ P(GE|F) = P(G|EF) \cdot \underline{P(E|F)} \]
Brainstorm with neighbors

You have a flowering plant.

Let

\( E = \) Flowers bloom
\( F = \) Plant was watered
\( G = \) Plant got sun

1. How would you write
   i. the probability that the plant got sun, given that it was watered and flowers bloomed?
   ii. the probability that the plant got sun and flowers bloomed given that it was watered?

2. Using the Venn diagram, compute the above probabilities.

   i. \( P(GE) = \) \( \) \( \cdot P(E) \)
   ii. \( P(GE|F) = P(G|EF) \cdot \) 

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Bayes’ Theorem II
Why is Bayes’ so important?

👉 It links belief to evidence in probability!
Bayes’ Theorem

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Mathematically:

$$P(E|F) \rightarrow P(F|E)$$

Real-life application:

Given new evidence $E$, update belief of fact $F$

Prior belief $\rightarrow$ Posterior belief

$P(F) \rightarrow P(F|E)$
Zika, an autoimmune disease

A disease spread through mosquito bites. Generally no symptoms, but can cause paralysis in very, very rare cases. During pregnancy: may cause birth defects. Very serious news story in 2015/2016.

If a test returns positive, what is the likelihood you have the disease?

Taking tests: Confusion matrix

<table>
<thead>
<tr>
<th>Evidence</th>
<th>Fact</th>
<th>Evidence, $E$ or $E^C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$, Test +</td>
<td>$F$, disease +</td>
<td>True positive $P(E</td>
</tr>
<tr>
<td>$F^C$, Test -</td>
<td>$F^C$, disease -</td>
<td>False negative $P(E^C</td>
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If a test returns positive, what is the likelihood you have the disease?
Taking tests: Confusion matrix

Take test

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<td>$F$, Test $+$</td>
<td>True positive $P(E</td>
<td>F)$</td>
</tr>
<tr>
<td>$F^C$, Test $-$</td>
<td>False negative $P(E^C</td>
<td>F)$</td>
</tr>
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If a test returns positive, what is the likelihood you have the disease?
Check out the question on the next slide (Slide 43). Strike up chat with a different neighbor, discuss plans for the week, and then brainstorm about the following question.
Zika Testing

- A test is 98% effective at detecting Zika (“true positive”).
- However, the test has a “false positive” rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive? Why would you expect this number?

1. Define events & state goal

Let: $E = \text{you test positive}$
$F = \text{you actually have the disease}$

Want:
$P(\text{disease | test+})$
$= P(F|E)$
Zika Testing

- A test is 98% effective at detecting Zika ("true positive").
- However, the test has a "false positive" rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?
Why would you expect this number?

1. Define events & state goal
   Let: \( E = \) you test positive
   \( F = \) you actually have the disease
   Want: \( P(\text{disease} \mid \text{test+}) = P(F \mid E) \)

2. Identify known probabilities
   \[ P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^C)P(F^C)} \]
   \[ P(F \mid E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(0.995)} \approx 0.330 \]

Bayes' Theorem
Bayes’ Theorem intuition

Original question:
What is the likelihood you have Zika if you test positive for the disease?
Bayes’ Theorem intuition

Original question:
What is the likelihood you have Zika if you test positive for the disease?

Interpretation:
Of the people who test positive, how many actually have Zika?
Bayes’ Theorem intuition

Original question:
What is the likelihood you have Zika if you test positive for the disease?

Interpretation:
Of the people who test positive, how many actually have Zika?

The space of facts, conditioned on a positive test result:

- People who test positive and have Zika
- People who test positive but don’t have Zika
- People who test positive but don’t have Zika
Zika Testing

- A test is 98% effective at detecting Zika (“true positive”).
- However, the test has a “false positive” rate of 1%.
- 0.5% of the US population has Zika.

What is the likelihood you have Zika if you test positive?

Say we have 1000 people:

- 5 have Zika and test positive
- 985 do not have Zika and test negative.
- 10 do not have Zika and test positive.

\[
\approx 0.333
\]
Update your beliefs with Bayes’ Theorem

\[ E = \text{you test positive for Zika} \]
\[ F = \text{you actually have the disease} \]

\[ P(F) \]

I have a 0.5% chance of having Zika.

Take test, results positive

With these test results, I now have a 33% chance of having Zika!!!
Ruminate Solo

Slide 51 is a question to think over by yourself. Take a minute to breathe, relax, clear your mind, and then think about the following question.
Why it’s still good to get tested

- A test is 98% effective at detecting Zika (“true positive”).
- However, the test has a “false positive” rate of 1%.
- 0.5% of the US population has Zika.

Let: \( E \) = you test positive
\( F \) = you actually have the disease

Let: \( E^C \) = you test negative for Zika with this test.

What is \( P(F|E^C) \)?

\[
P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}
\]

Bayes’ Theorem

\[
P(F|E) = 0.380
\]

\[
P(F^C|E^C) = 1 - P(F|E)
\]

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<td>( E ), Test +</td>
<td>True positive ( P(E</td>
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(ruminating)
Why it’s still good to get tested

- A test is 98% effective at detecting Zika (“true positive”).
- However, the test has a “false positive” rate of 1%.
- 0.5% of the US population has Zika.

Let:

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</tr>
<tr>
<td>$P(E</td>
<td>F) = 0.98$</td>
<td>$P(E</td>
</tr>
</tbody>
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What is $P(F|E^C)$?
Why it’s still good to get tested

- A test is 98% effective at detecting Zika (“true positive”).
- However, the test has a “false positive” rate of 1%.
- 0.5% of the US population has Zika.

Let: 
- \(E\) = you test positive
- \(F\) = you actually have the disease

Let: 
- \(E^C\) = you test negative for Zika with this test.

What is \(P(F|E^C)\)?

\[
P(F|E^C) = \frac{P(E^C|F)P(F)}{P(E^C|F)P(F) + P(E^C|F^C)P(F^C)}
\]

Bayes’ Theorem

- \(F\), disease +
  - True positive
  - \(P(E|F) = 0.98\)

- \(F^C\), disease –
  - False positive
  - \(P(E|F^C) = 0.01\)

- \(E^C\), Test –
  - False negative
  - \(P(E^C|F) = 0.02\)

  - True negative
  - \(P(E^C|F^C) = 0.99\)

\[
P(F|E^C) = \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(0.995)} = 0.001
\]
Why it’s still good to get tested

\[ E = \text{you test positive for Zika} \]
\[ F = \text{you actually have the disease} \]
\[ E^C = \text{you test negative for Zika} \]

I have a \(0.5\%\) chance of having Zika disease.

\[ P(F) \]

Take test, results positive

With these test results, I now have a \(33\%\) chance of having Zika!!!

\[ P(F|E) \]

Take test, results negative

With these test results, I now have a \(0.01\%\) chance of having Zika disease!!!

\[ P(F|E^C) \]
Topical probability news: Bayes for COVID-19 testing

- Antibody tests (blood samples) have higher false negative, false positive rates than RT-PCR tests (nasal swab). However, they help explain/identify our body’s reaction to the virus.

- The real world has many more “givens” (current symptoms, existing medical conditions) that improve our belief prior to testing.

- Most importantly, testing gives us a noisy signal of the spread of a disease.

Why test if there are errors?
Monty Hall Problem
Monty Hall Problem

and Wayne Brady
Monty Hall Problem aka Let’s Make a Deal

Behind one door is a prize (equally likely to be any door). Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Should we switch?

Note: If we don’t switch, \( P(\text{win}) = \frac{1}{3} \) (random)

We are comparing \( P(\text{win}) \) and \( P(\text{win} | \text{switch}) \).
If we switch

Without loss of generality, say we pick A (out of Doors A, B, C).

A = prize
- Host opens B or C
- We switch
- We always lose

\[ P(\text{win} \mid A \text{ prize}, \text{picked A}, \text{switched}) = 0 \]

B = prize
- Host must open C
- We switch to B
- We always win

\[ P(\text{win} \mid B \text{ prize}, \text{picked A}, \text{switched}) = 1 \]

C = prize
- Host must open B
- We switch to C
- We always win

\[ P(\text{win} \mid C \text{ prize}, \text{picked A}, \text{switched}) = 1 \]

\[ P(\text{win} \mid \text{picked A, switched}) = 1/3 * 0 + 1/3 * 1 + 1/3 * 1 = 2/3 \]

You should switch.
Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

\[
\begin{align*}
\frac{1}{1000} &= P(\text{envelope is prize}) \\
\frac{999}{1000} &= P(\text{other 999 envelopes have prize})
\end{align*}
\]

2. I open 998 of remaining 999 (showing they are empty).

\[
\begin{align*}
\frac{999}{1000} &= P(998 \text{ empty envelopes had prize}) \\
&\quad + \ P(\text{last other envelope has prize}) \\
&\quad = P(\text{last other envelope has prize}) \\
&\quad = P(\text{envelope is prize})
\end{align*}
\]

3. Should you switch?

No: \( P(\text{win without switching}) = \frac{1}{\text{original \# envelopes}} \)

Yes: \( P(\text{win with new knowledge}) = \frac{1}{\text{original \# envelopes} - 1} \)