05: Independence

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Quick slide reference

Independence I

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

Otherwise E and F are called dependent events.

If E and F are independent, then:

$$P(E|F) = P(E)$$

Intuition through proof

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Definition of conditional probability

$$= \frac{P(E)P\cancel{(F)}}{P\cancel{(F)}}$$

Independence of E and F

$$= P(E)$$

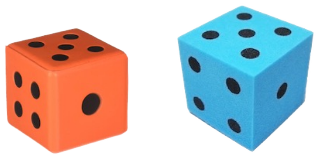
Taking the bus to cancellation city

Knowing that F happened does not change our belief that E happened.

Dice, our misunderstood friends

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$



$G = \{(1,4),(2,3),(3,2),(4,1)\}$

$|G| = 4$

1. Are E and F independent?

$EF = \{(1,6)\}$

$P(E) = 1/6$
$P(F) = 1/6$
$P(EF) = 1/36$

$\frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$

✅ independent

2. Are E and G independent?

$P(E) = 1/6$
$P(G) = 4/36 = 1/9$
$P(EG) = 1/36 \neq P(E)P(G)$

❌ dependent $\neq 1/6 \cdot 1/9$

Generalizing independence

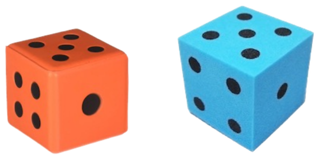
Three events E, F, and G are independent if:

$$\left\{\begin{array}{l} P(EFG) = P(E)P(F)P(G), \text{ and} \\ \left.\begin{array}{l} P(EF) = P(E)P(F), \text{ and} \\ P(EG) = P(E)P(G), \text{ and} \\ P(FG) = P(F)P(G) \end{array}\right\} \text{this is just pairwise independence} \end{array}\right.$$

n events $E_1, E_2, \ldots, E_n$ are independent if:

$$\left\{\begin{array}{l} \text{for } r = 1, \ldots, n: \\ \quad \text{for every subset } E_1, E_2, \ldots, E_r: \\ \qquad P(E_1 E_2 \ldots E_r) = P(E_1)P(E_2) \cdots P(E_r) \end{array}\right.$$

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$

$G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

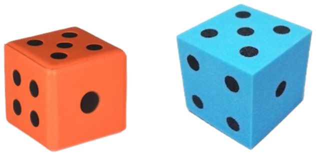
1. Are E and F ✅ independent?
2. Are E and G independent?
3. Are F and G independent?
4. Are E, F, G independent?

$EF = \{(1,6)\}$

$P(EF) = 1/36$

$= 1/6 \cdot 1/6$

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$

$G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

1. Are E and F ✅ independent?

$P(EF) = 1/36$

2. Are E and G ✅ independent?

$P(EG) = P(E)P(G)$

$EG = \{(1,6)\}$

$P(EG) = 1/36 = 1/6 \cdot 1/6$

3. Are F and G ✅ independent?

$P(FG) = P(F)P(G)$

$FG = \{(1,6)\}$

$= P(FG) = 1/36 = 1/6 \cdot 1/6$

4. Are E, F, G ❌ independent?

$EFG = \{(1,6)\}$

$P(EFG) = 1/36$

$\neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$

Pairwise independence is not sufficient to prove independence of >2 events!

Independence II

Independent trials

We often are interested in experiments consisting of n **independent trials**.

- n trials, each with the same set of possible outcomes
- n-way independence: an event in one subset of trials is independent of events in other subsets of trials

Examples:

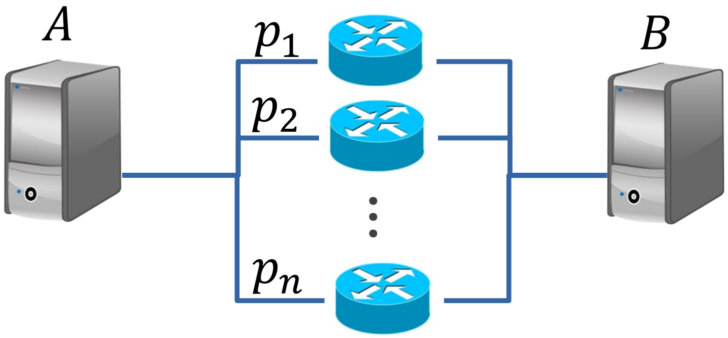
- Flip a coin n times
- Roll a die n times
- Send a multiple-choice survey to n people
- Send n web requests to k different servers

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

What is $P(E)$?



Network reliability

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- n independent routers, each with probability p_i of functioning (where $1 \le i \le n$)
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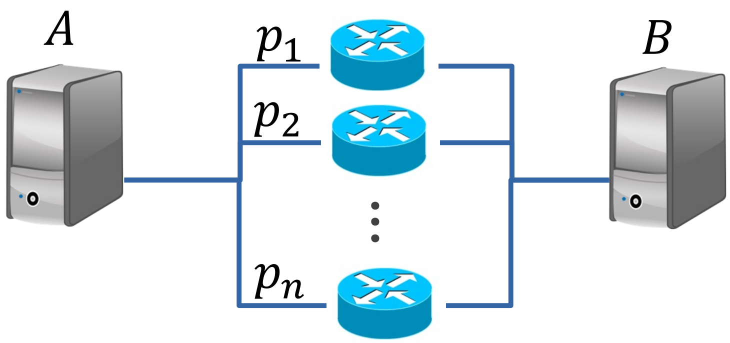
What is $P(E)$?



router i functioning: p_i

not functioning: $(1-p_i)$

$$P(E) = P(\geq 1 \text{ one router works})$$

$$= 1 - P(\text{all routers fail}) = 1 - P(R_1 \text{ fails} \cap R_2 \text{ fails} \cap \cdots \cap R_n \text{ fails})$$

$$= 1 - (1-p_1)(1-p_2)\cdots(1-p_n)$$

$$= 1 - \prod_{i=1}^{n}(1-p_i)$$

$\geq$ 1 with independent trials: take complement

Exercises

Independence?

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

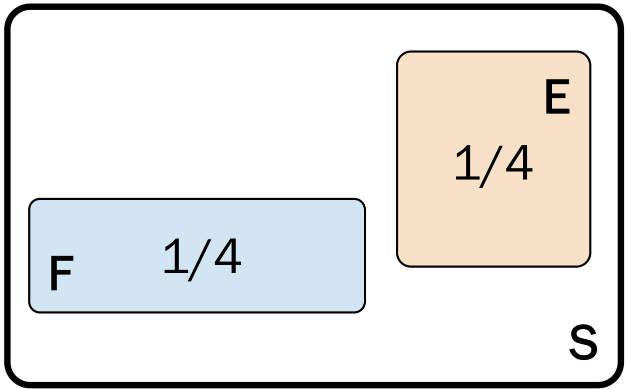
1. True or False? Two events E and F are independent if: Assuming $P(E) > 0$

~~A.~~ Knowing that F happens means that E can't happen. $P(E|F) = 0 \neq P(E)$

[✓ B.] Knowing that F happens doesn't change probability that E happened. $P(E|F) = P(E)$

2. Are E and F independent in the following pictures?

A.

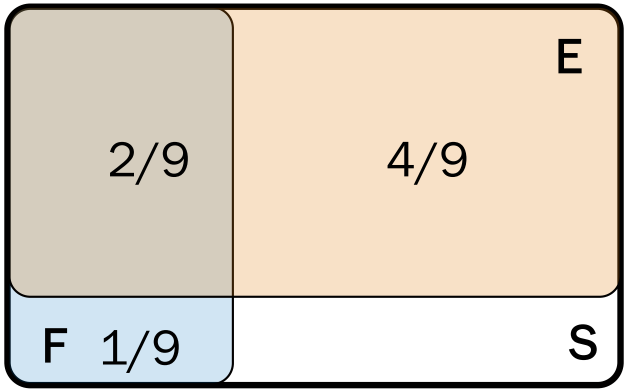


$P(E) = 1/4$, $P(F) = 1/4$

$EF = \emptyset$ mutually exclusive

$P(EF) = 0 \neq \frac{1}{4} \cdot \frac{1}{4}$

B.



$P(E) = 2/3$

$P(F) = 1/3$

$P(EF) = 2/9$

$\overset{?\ \text{yes!}}{=} P(E)P(F)$

$= \frac{2}{3} \cdot \frac{1}{3}$

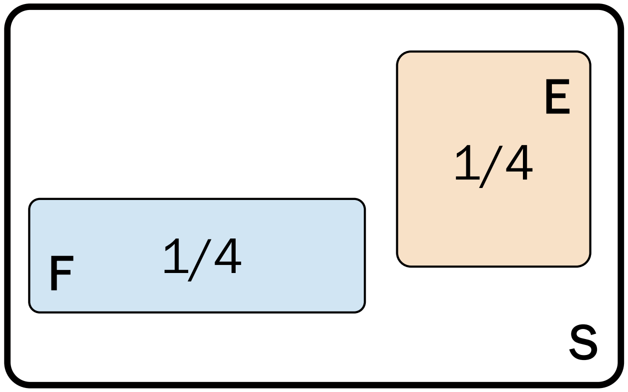
(by yourself)

Independence?

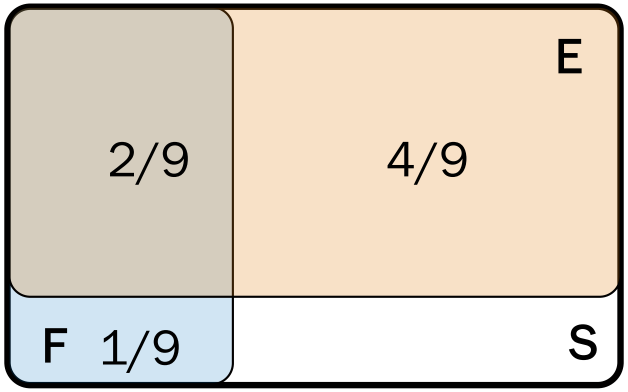
Independent events E and F ⟷ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

1. True or False? Two events E and F are independent if:
   - A. Knowing that F happens means that E can't happen.
   - B. Knowing that F happens doesn't change probability that E happened.
2. Are E and F independent in the following pictures?

A.



B.



Be careful:
- Independence is NOT mutual exclusion.
- Independence is difficult to visualize graphically.

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

For independent events E and F,

- $P(E|F) = P(E)$
- E and F^C are independent.

new

Independence of complements

Statement:

If E and F are independent, then E and F^C are independent.

Proof:

$P(EF^C) = P(E) - P(EF)$	Intersection
$= P(E) - P(E)P(F)$	Independence of E and F
$= P(E)[1 - P(F)]$	Factoring
$= P(E)P(F^C)$	Complement
E and F^C are independent	Definition of independence

$P(E|F^c) = P(E)$

$P(E|F) = P(E)$

Knowing that F **did or didn't happen** does not change our belief that E happened.

(biased) Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. $P(n$ heads on n coin flips$)$
2. $P(n$ tails on n coin flips$)$
3. $P($first k heads, then $n-k$ tails$)$
4. $P($exactly k heads on n coin flips$)$

(biased) Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

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4. $P($exactly k heads on n coin flips$)$

$\underbrace{H}_{p}\underbrace{H}_{p}\underbrace{H}_{p}\underbrace{H}_{p}$ p^n

$\longrightarrow (1-p)^n$

$\underbrace{HHH\cdots H}_{k}\ \underbrace{TT\cdots T}_{n-k}$ $p^k(1-p)^{n-k}$

$\Downarrow$

$HHH\,T\cdots T \longleftarrow p^k(1-p)^{n-k}$

$THHHTH\cdots T \longleftarrow$ same probability

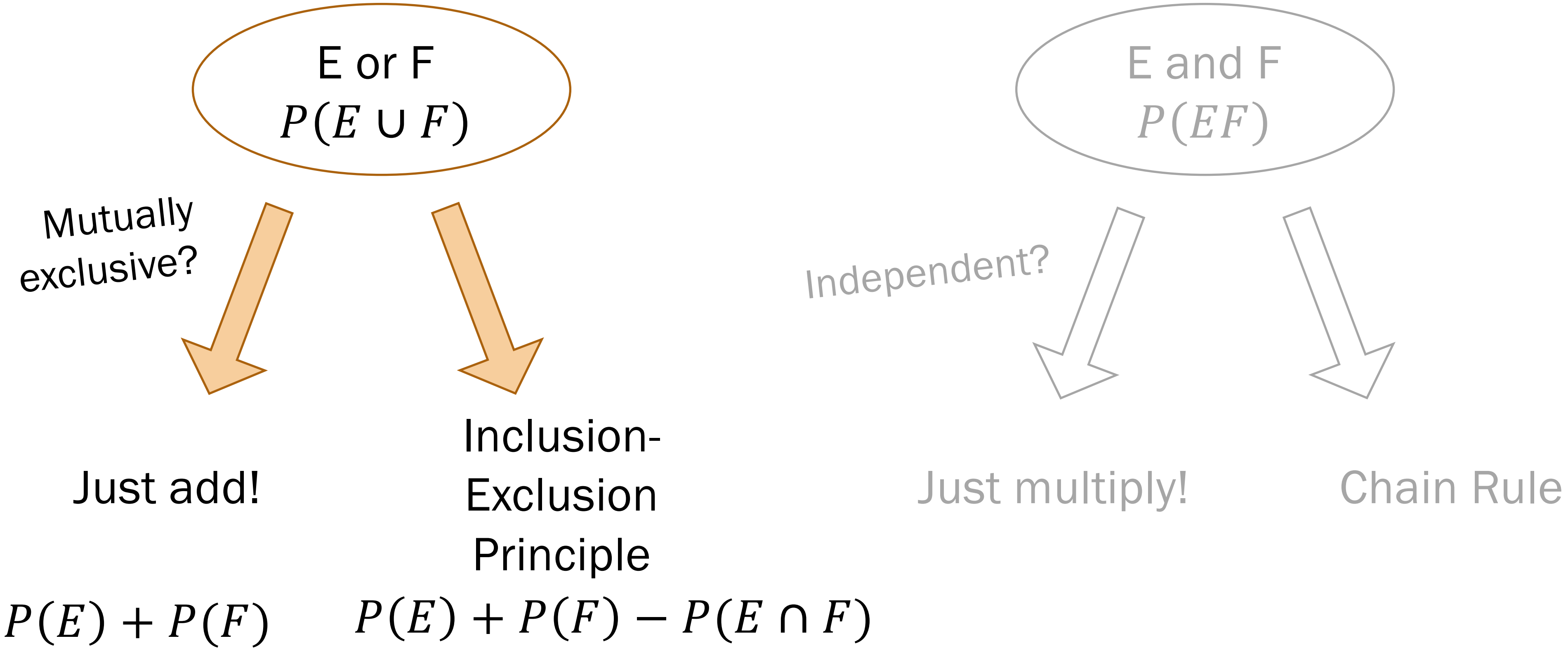
$$\binom{n}{k}\; p^k(1-p)^{n-k}$$

# of mutually exclusive outcomes

$P($a particular outcome's k heads on n coin flips$)$

Make sure you understand #4! It will come up again.

Probability of events



E or F

$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F

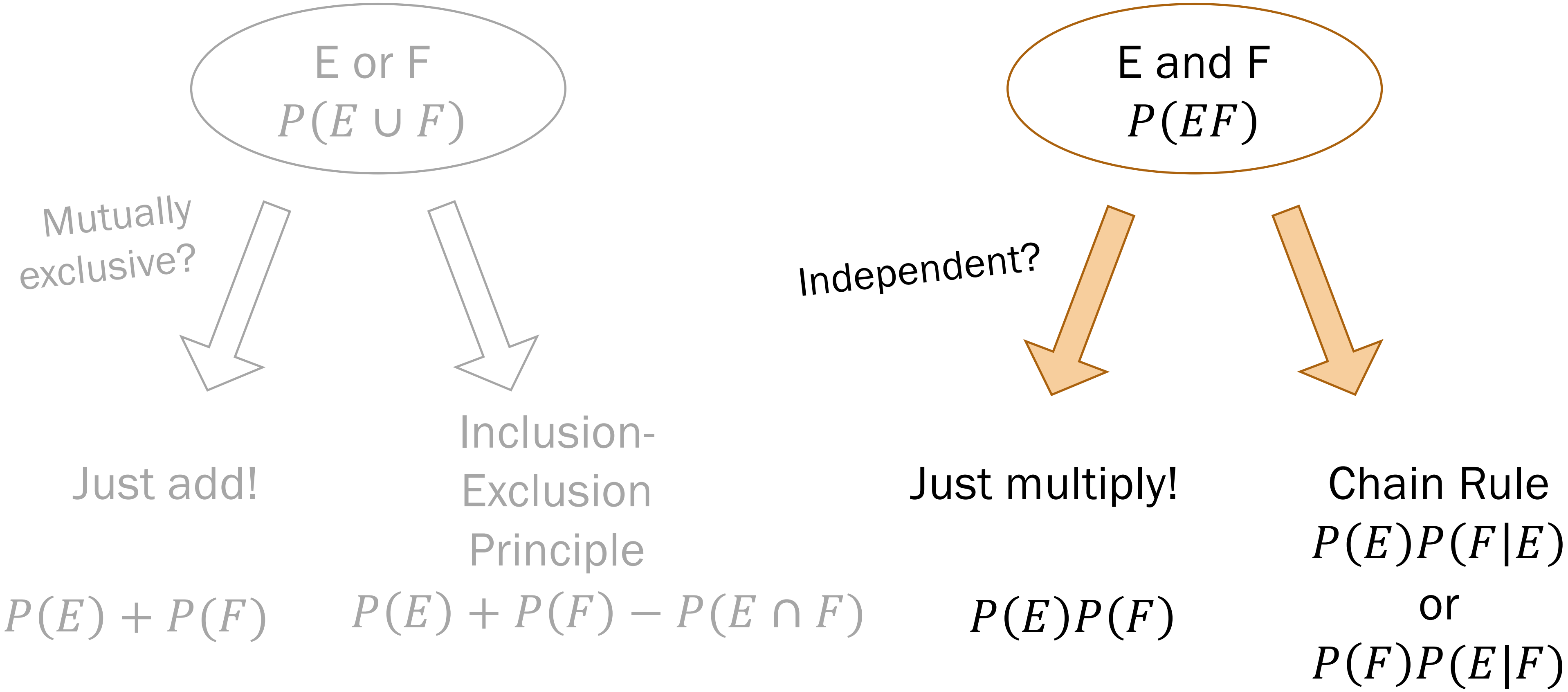
$P(EF)$

Independent?

Just multiply!

Chain Rule

Probability of events



E or F

$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F

$P(EF)$

Independent?

Just multiply!

$P(E)P(F)$

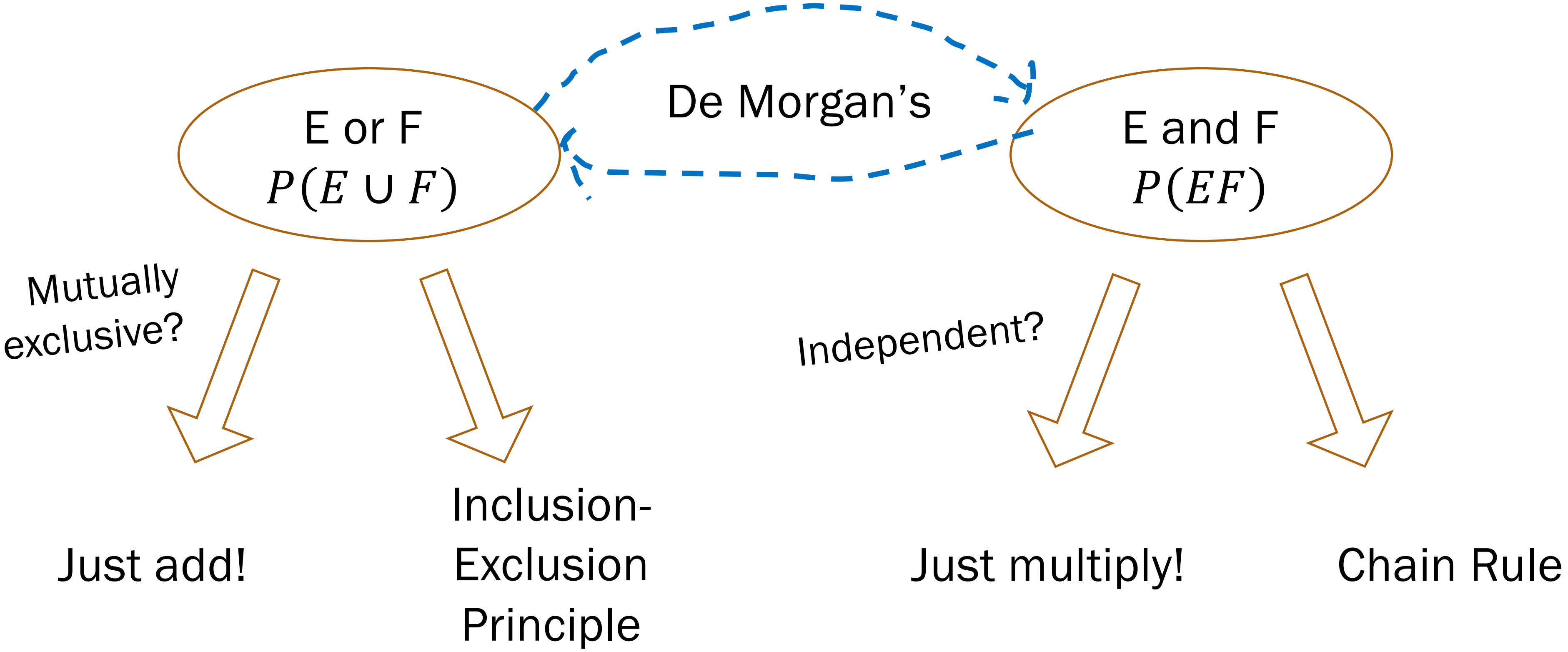
Chain Rule

$P(E)P(F|E)$

or

$P(F)P(E|F)$

Probability of events



E or F

$P(E \cup F)$

De Morgan's

E and F

$P(EF)$

Mutually exclusive?

Just add!

Inclusion-Exclusion Principle

Independent?

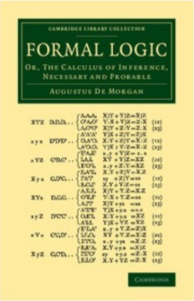
Just multiply!

Chain Rule

Augustus De Morgan

Augustus De Morgan (1806–1871):

British mathematician who wrote the book *Formal Logic* (1847).



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FORMAL LOGIC

Or, The Calculus of Inference, Necessary and Probable

Augustus De Morgan

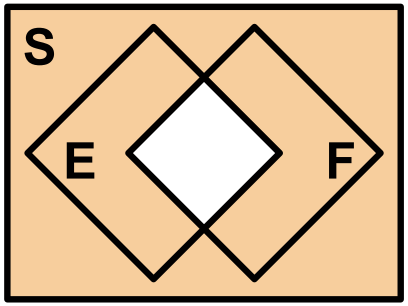
Cambridge

He looked remarkably similar to Jason Alexander (George from Seinfeld)

(but that's not important right now)

De Morgan's Laws

DeMorgan's lets you switch between AND and OR.



$$(E \cap F)^C = E^C \cup F^C$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$$

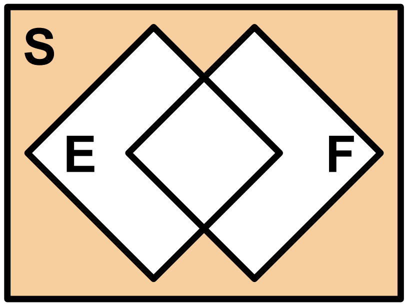
In probability:

$$P(E_1 E_2 \cdots E_n)$$

$$= 1 - P\left((E_1 E_2 \cdots E_n)^C\right)$$

$$= 1 - P\left(E_1^C \cup E_2^c \cup \cdots \cup E_n^c\right)$$

Great if E_i^C mutually exclusive!



$$(E \cup F)^C = E^C \cap F^C$$

$$\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$$

In probability:

$$P(E_1 \cup E_2 \cup \cdots \cup E_n)$$

$$= 1 - P\left((E_1 \cup E_2 \cup \cdots \cup E_n)^C\right)$$

$$= 1 - P(E_1^c E_2^c \cdots E_n^c)$$

Great if E_i independent!

CS109 Mixer

Check out the questions on the next slide (Slide 27). **These are challenging problems.**

Speak with your neighbor about how to best approach these problems.

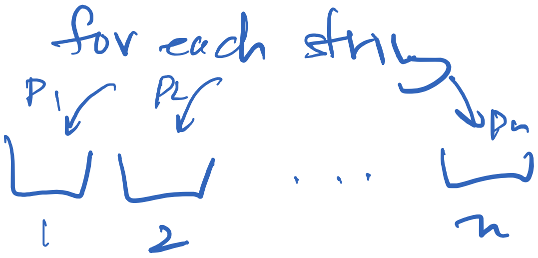
Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

$\Sigma p_i = 1$

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it?



2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it?

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$$P(S_i) = p_1$$
$$P(S_i^C) = 1 - p_1$$

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

WTF (not-real acronym for Want To Find):

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

S_i independent

$P(S_i) = p_1$
$P(S_i^C) = 1 - p_1$

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$

$$= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right) \qquad \text{Complement}$$

$$= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right) \qquad \text{De Morgan's Law}$$

$$= 1 - P\left(S_1^C\right)P\left(S_2^C\right) \cdots P\left(S_m^C\right) = 1 - \left(P\left(S_1^C\right)\right)^m \qquad S_i \text{ independent trials}$$

$$= 1 - (1 - p_1)^m$$

More hash table fun: Possible approach?

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$

$$= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right)$$

$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$

$$? = 1 - P\left(F_1^C\right)P\left(F_2^C\right) \cdots P(F_k^C)$$

⚠ F_i bucket events are *dependent*!

So we cannot approach with complement.

More hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$
\begin{aligned}
P(E) &= P(F_1 \cup F_2 \cup \cdots \cup F_k) \\
&= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right) \\
&= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)
\end{aligned}
$$

$$
\begin{aligned}
&= P(\text{buckets 1 to } k \textbf{ all denied strings}) \\
&= \left(P(\text{each string hashes to } k+1 \text{ or higher})\right)^m \\
&= (1 - p_1 - p_2 \ldots - p_k)^m
\end{aligned}
$$

$$= 1 - (1 - p_1 - p_2 \ldots - p_k)^m$$

The fun never stops with hash tables

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it? ✅
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it? ✅

Looking for a challenge? ☺

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

3. E = each of buckets 1 to k has $\geq$ 1 string hashed into it?

Hint: Use Part 2's event definition:

Define F_i = bucket i has at least one string in it