08: Poisson and More

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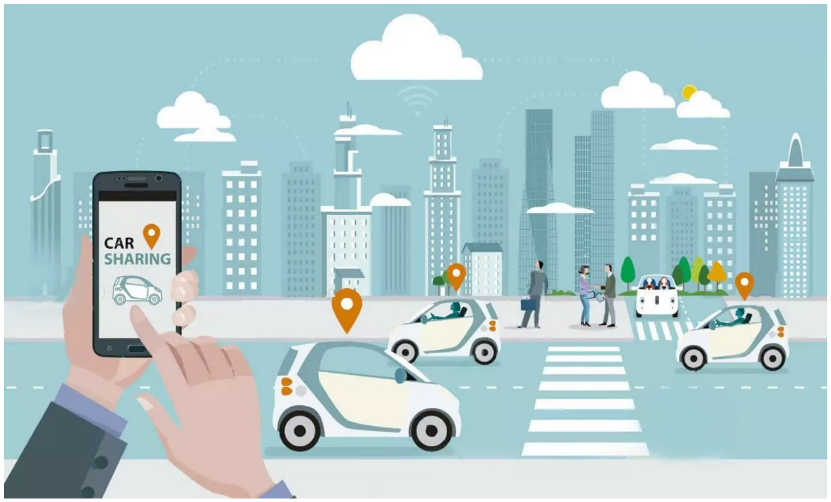


Probability of k requests from this area in the next 1 minute?

Suppose we know: On average, $\lambda = 5$ requests per minute

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60 seconds:

1	2	3	4	5						60
0	0	1	0	1	...	0	0	0	0	1

At each second:

- Independent trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$X \sim \text{Bin}(n = 60,\ p = 5/60)$

$$P(X = k) = \binom{60}{k} \left(\frac{5}{60}\right)^k \left(1 - \frac{5}{60}\right)^{n-k}$$

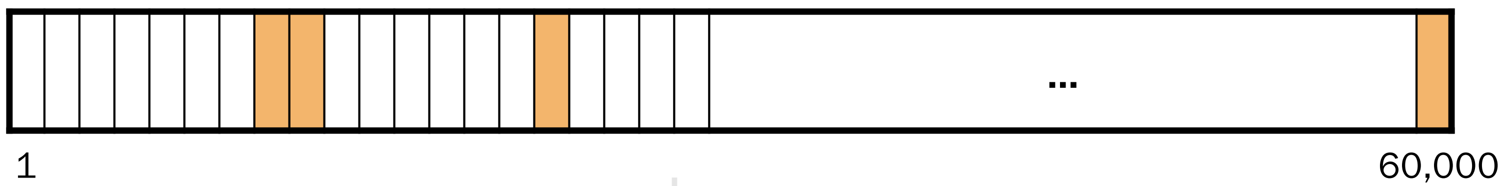
But what if there are *two* requests in the same second?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60,000 milliseconds:



At each millisecond:

- Independent trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n = 60000,\ p = \lambda/n)$$

$$P(X = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

But what if there are *two* requests in the same millisecond?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into infinitely small buckets:

omg so small

1 ∞

For each time bucket:

- Independent trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$X \sim \text{Bin}(n,\ p = \lambda/n)$

$$P(X = k) = \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Who wants to see some cool math?

Binomial in the limit

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$p = \lambda / n$

$$P(X=k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

Expand

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Rearrange

$$= \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Def natural exponent

$$= \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(n-k)!}{(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

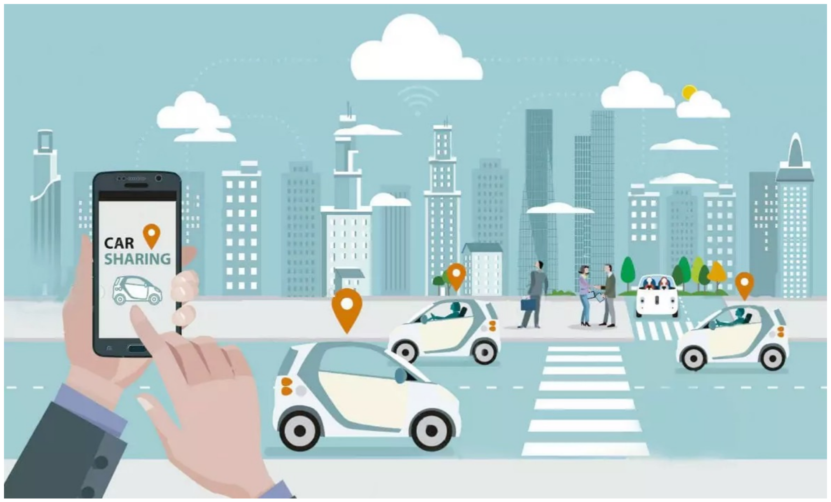
Limit analysis + cancel

$$= \lim_{n\to\infty} \frac{n^k}{n^k} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{1}$$

Simplify

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

Algorithmic ride sharing



Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

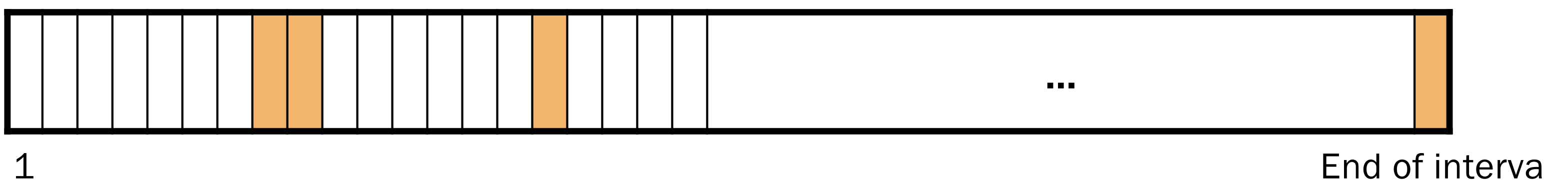
Poisson distribution

Poisson, continued

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

<u>def</u> A Poisson random variable X is the number of successes over the experiment duration, assuming **the time that each success occurs is independent** and the average # of requests over time is constant. λ



Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of successes over the experiment duration, assuming the time that each success occurs is independent and the average # of requests over time is constant.

$$X \sim \text{Poi}(\lambda)$$

Support: $\{0, 1, 2, \ldots\}$

PMF $$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Expectation $E[X] = \lambda$

Variance $\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Yes, expectation == variance for Poisson RV! More later.

Earthquakes

$X \sim \text{Poi}(\lambda)$
$E[X] = \lambda$

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

There are an average of 2.79 major earthquakes in the world each year, and major earthquakes occur independently.

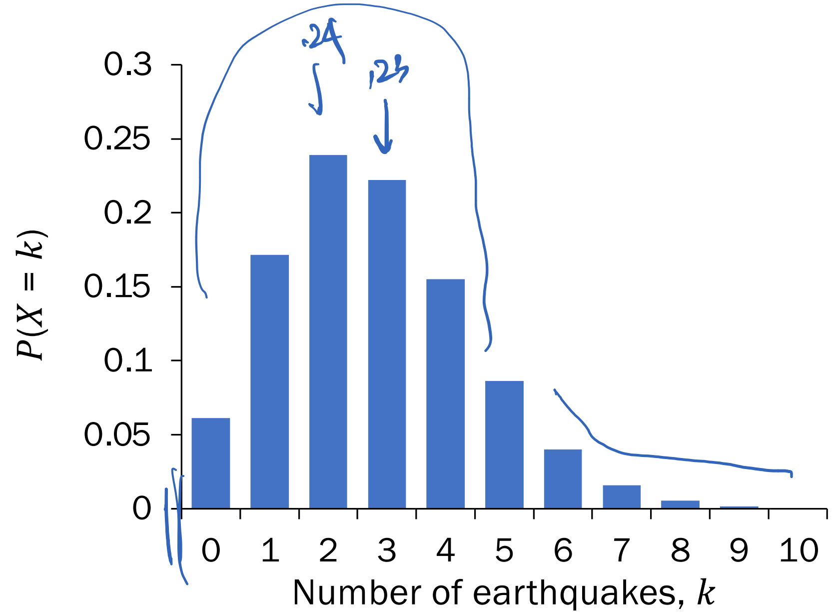
What is the probability of 3 major earthquakes happening next year?

1. Define RVs

$X \sim \text{Poi}(\lambda = 2.79)$

2. Solve

$P(X=3) = e^{-2.79}\frac{(2.79)^3}{3!} \approx .23$



Other Discrete RVs

Grid of random variables

	Number of successes	Time until success	
One trial	$\text{Ber}(p)$		One success
Several trials	$\text{Bin}(n, p)$ ($n = 1$ ↑)		Several successes
Interval of time	$\text{Poi}(\lambda)$	(Friday)	Interval of time to first success

Focus on understanding <u>how</u> and <u>when</u> to use RVs, not on memorizing PMFs.

Geometric RV

H : 0.5 — 1
TH : 0.25 — 2
TTH : 0.125 — 3
k-1 fails + 1 yay — k

r=1 successes needed

Consider an experiment: independent trials of Ber(p) random variables.

def A **Geometric** random variable X is the # of trials until the first success.

$$X \sim \text{Geo}(p)$$

Support: $\{1, 2, \ldots\}$

PMF	$P(X = k) = (1-p)^{k-1}p$
Expectation	$E[X] = \frac{1}{p}$
Variance	$\text{Var}(X) = \frac{1-p}{p^2}$

Examples:

- Flipping a coin ($P(\text{heads}) = p$) until first heads appears
- Generate bits with $P(\text{bit} = 1) = p$ until first 1 generated

Negative Binomial RV

Consider an an experiment: independent trials of Ber(p) random variables.

def A **Negative Binomial** random variable X is the # of trials until r successes.

$$X \sim \text{NegBin}(r, p)$$

Support: $\{r, r+1, \ldots\}$

PMF	$P(X = k) = \binom{k-1}{r-1}(1-p)^{k-r} p^r$
Expectation	$E[X] = \frac{r}{p}$
Variance	$\text{Var}(X) = \frac{r(1-p)}{p^2}$

Examples:

- Flipping a coin until r^{th} heads appears
- # of strings to hash into table until bucket 1 has r entries

$\text{Geo}(p) = \text{NegBin}(1, p)$

Grid of random variables

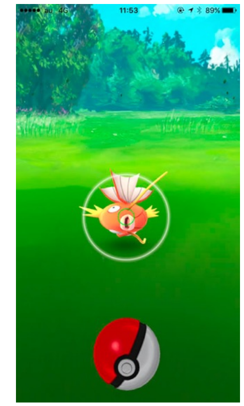
	Number of successes	Time until success	
One trial	$\text{Ber}(p)$	$\text{Geo}(p)$	One success
	↑ $n = 1$	↑ $r = 1$	
Several trials	$\text{Bin}(n, p)$	$\text{NegBin}(r, p)$	Several successes
Interval of time	$\text{Poi}(\lambda)$	(Friday)	Interval of time to first success

Catching Pokemon

Wild Pokemon are captured by throwing Pokeballs at them.

- Each ball has probability p = 0.1 of capturing the Pokemon.
- Each ball is an independent trial.

What is the probability that you catch the Pokemon on the 5th try?



1. Define events/ RVs & state goal

   $X \sim$ some distribution

   Want: $P(X = 5)$

2. Solve

   A. $X \sim \text{Bin}(5, 0.1)$
   B. $X \sim \text{Poi}(0.5)$
   C. $X \sim \text{NegBin}(5, 0.1)$
   D. $X \sim \text{NegBin}(1, 0.1)$
   E. $X \sim \text{Geo}(0.1)$
   F. None/other

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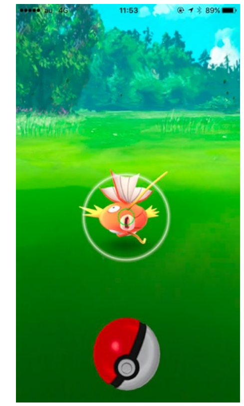

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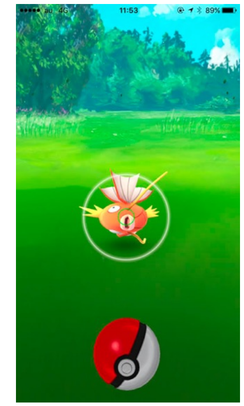
   D. $X \sim \text{NegBin}(1, 0.1)$
   E. $X \sim \text{Geo}(0.1)$
   F. None/other

Catching Pokemon

$X \sim \text{Geo}(p) \quad p(k) = (1-p)^{k-1} p$



Wild Pokemon are captured by throwing Pokeballs at them.

- Each ball has probability p = 0.1 of capturing the Pokemon.
- Each ball is an independent trial.

What is the probability that you catch the Pokemon on the 5th try?

1. Define events/ RVs & state goal
2. Solve

$X \sim \text{Geo}(0.1)$

Want: $P(X = 5)$

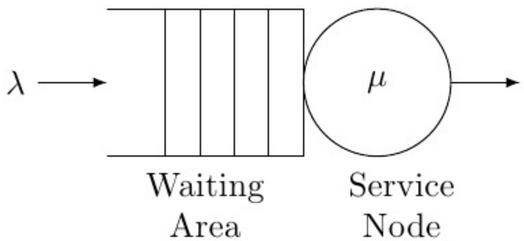
Exercises

The hardest part of problem-solving is determining what is a random variable .

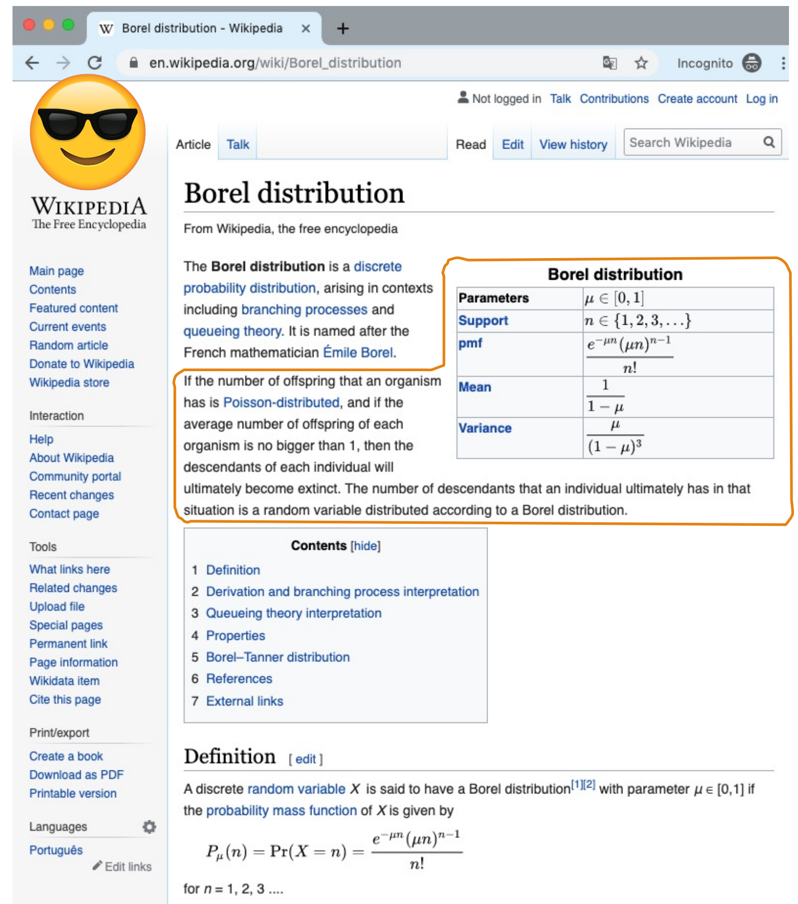
CS109 Learning Goal: Use new RVs

Let's say you are learning about servers/networks.

You read about the M/D/1 queue:



"The service time busy period is distributed as a Borel with parameter $\mu = 0.2$."



Goal: You can recognize terminology and understand experiment setup.

Kickboxing with RVs

Choose from:
A. $\text{Ber}(p)$
B. $\text{Bin}(n,p)$
C. $\text{Poi}(\lambda)$
D. $\text{Geo}(p)$
E. $\text{NegBin}(r,p)$

How would you model the following?

1. # of snapchats you receive in a day
2. # of children until the first one with brown eyes (same parents)
3. If stock went up (1) or down (0) in a day
4. # of probability problems you try until you get 5 correct (if you are randomly correct)
5. # of years in some decade with more than 6 Atlantic hurricanes

Kickboxing with RVs

Choose from:
A. $\text{Ber}(p)$
B. $\text{Bin}(n, p)$
C. $\text{Poi}(\lambda)$
D. $\text{Geo}(p)$
E. $\text{NegBin}(r, p)$

How would you model the following?

1. # of snapchats you receive in a day — C. $\text{Poi}(\lambda)$
2. # of children until the first one with brown eyes (same parents) — D. $\text{Geo}(p)$ or E. $\text{NegBin}(1, p)$
3. If stock went up (1) or down (0) in a day — A. $\text{Ber}(p)$ or B. $\text{Bin}(1, p)$
4. # of probability problems you try until you get 5 correct (if you are randomly correct) — E. $\text{NegBin}(r = 5, p)$
5. # of years in some decade with more than 6 Atlantic hurricanes — B. $\text{Bin}(n = 10, p)$, where $p = P(\geq 6 \text{ hurricanes in a year})$ calculated from C. $\text{Poi}(\lambda)$

Note: These exercises are designed to build intuition; in a problem statement, you will generally have more clues.

Poisson Random Variable

Review

$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0, 1, 2, \ldots\}$	Variance	$\text{Var}(X) = \lambda$

In CS109, a Poisson RV $X \sim \text{Poi}(\lambda)$ most often models

1. # of successes in a fixed interval of time, where successes are independent
   $\lambda = E[X]$, average success/interval

1. Web server load

$X \sim \text{Poi}(\lambda)$ $\quad p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$
$E[X] = \lambda$

Consider requests to a web server in 1 second.

- In the past, server load averages 2 hits/second, where hits arrive independently.
- Let X = # hits the server receives in a second.

What is $P(X < 5)$?

Define RVs

Solve

Poisson Random Variable

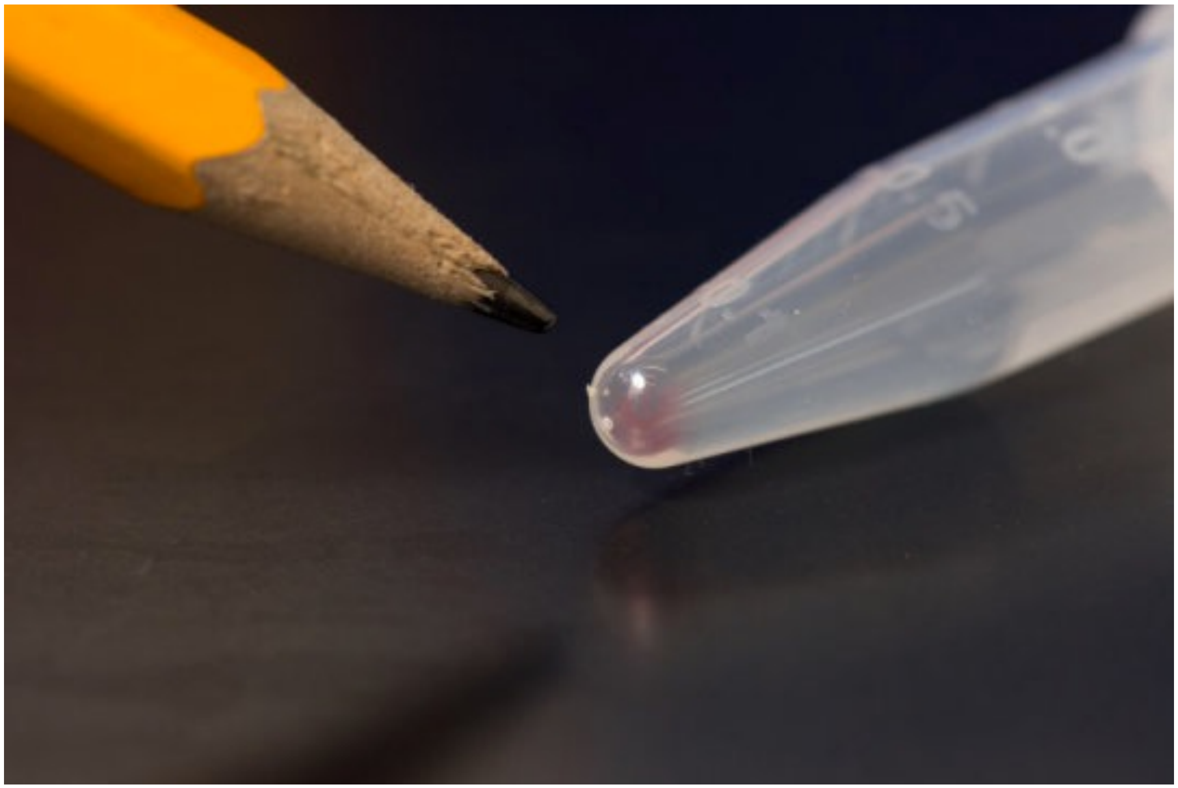
$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0,1,2,\ldots\}$	Variance	$\text{Var}(X) = \lambda$

In CS109, a Poisson RV $X \sim \text{Poi}(\lambda)$ most often models

1. # of successes in a fixed time interval, where successes are independent
   $\lambda = E[X]$, average success/interval
2. Approximation of $Y \sim \text{Bin}(n, p)$ where n is large and p is small.
   $\lambda = E[Y] = np$

   Approximation works well even when trials not entirely independent.

2. DNA



All the movies, images, emails and other digital data from more than 600 smartphones (10,000 GB) can be stored in the faint pink smear of DNA at the end of this test tube.

What is the probability that DNA storage stays uncorrupted?

2. DNA

What is the probability that DNA storage stays uncorrupted?

- In DNA (and real networks), we store large strings.
- Let string length be long, e.g., $n \approx 10^4$
- Probability of corruption of each base pair is very small, e.g., $p = 10^{-6}$
- Let $X =$ # of corruptions.

What is P(DNA storage is uncorrupted) = $P(X = 0)$?

1. Approach 1:

$$X \sim \text{Bin}(n = 10^4, p = 10^{-6})$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

unwieldy! ⚠️ $= \binom{10^4}{0} 10^{-6 \cdot 0} (1 - 10^{-6})^{10^4 - 0}$

≈ 0.990049829

2. Approach 2:

$$X \sim \text{Poi}(\lambda = 10^4 \cdot 10^{-6} = 0.01)$$

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} = e^{-0.01} \frac{0.01^0}{0!}$$

$$= e^{-0.01}$$

≈ 0.990049834 a good ✅ approximation!

When is a Poisson approximation appropriate?

$$P(X=k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \cdots$$

Def natural exponent

$$= \lim_{n\to\infty} \frac{n!}{n^k (n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(n-k)!}{(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

Limit analysis

$$= \lim_{n\to\infty} \frac{n^k}{n^k} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{1}$$

Simplify

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

Under which conditions will $X \sim \text{Bin}(n, p)$ behave like $\text{Poi}(\lambda)$, where $\lambda = np$?

A. Large n, large p
B. Small n, small p
C. Large n, small p
D. Small n, large p
E. Other

Poisson approximation

$X \sim \text{Poi}(\lambda)$ $\quad Y \sim \text{Bin}(n, p)$
$E[X] = \lambda$ $\quad E[Y] = np$

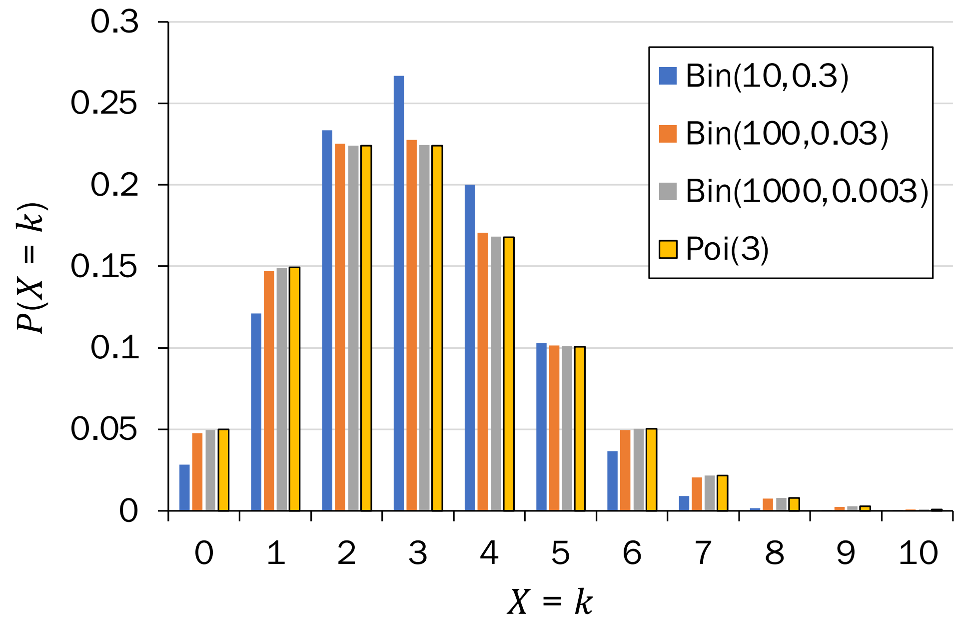
Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate."

Different interpretations of "moderate":

- $n > 20$ and $p < 0.05$
- $n > 100$ and $p < 0.1$

Poisson is Binomial in the limit:

- $\lambda = np$, where $n \to \infty, p \to 0$



Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of occurrences over the experiment duration.

$X \sim \text{Poi}(\lambda)$

Support: $\{0, 1, 2, \ldots\}$

PMF $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$

Expectation $E[X] = \lambda$

Variance $\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Time to show intuition for why expectation == variance!

Properties of Poi(λ) with the Poisson paradigm

Recall the Binomial:

$Y \sim \text{Bin}(n, p)$	Expectation	$E[Y] = np$
	Variance	$\text{Var}(Y) = np(1-p)$

Consider $X \sim \text{Poi}(\lambda)$, where $\lambda = np$ $(n \to \infty, p \to 0)$:

$X \sim \text{Poi}(\lambda)$	Expectation	$E[X] = \lambda$
	Variance	$\text{Var}(X) = \lambda$

Proof:

$$E[X] = np = \lambda$$
$$\text{Var}(X) = np(1-p) \to \lambda(1-0) = \lambda$$

Poisson Approximation, approximately

Poisson can still provide a good approximation of the Binomial, even when assumptions are "mildly" violated.

You can apply the Poisson approximation when:

- "Successes" in trials are not entirely independent
  e.g.: # entries in each bucket in large hash table.
- Probability of "Success" in each trial varies (slightly), like a small relative change in a very small p
  e.g. Average # requests to web server/sec may fluctuate slightly due to load on network

We won't explore this too much, but I want you to know it exists.

Can these Binomial RVs be approximated?

Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate."

Different interpretations of "moderate":

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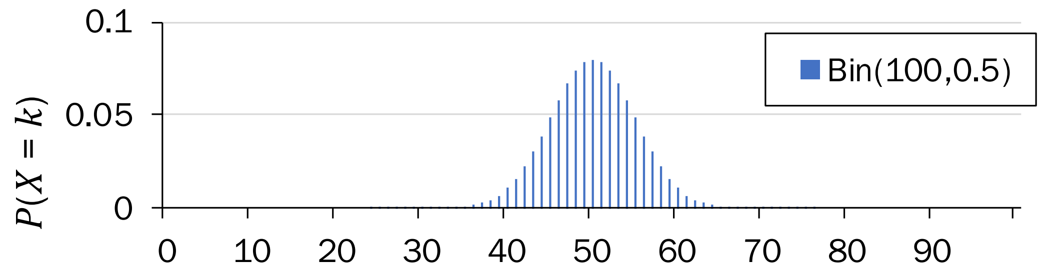

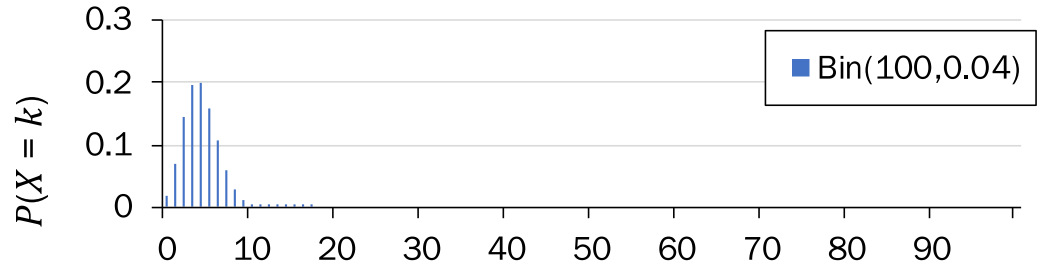

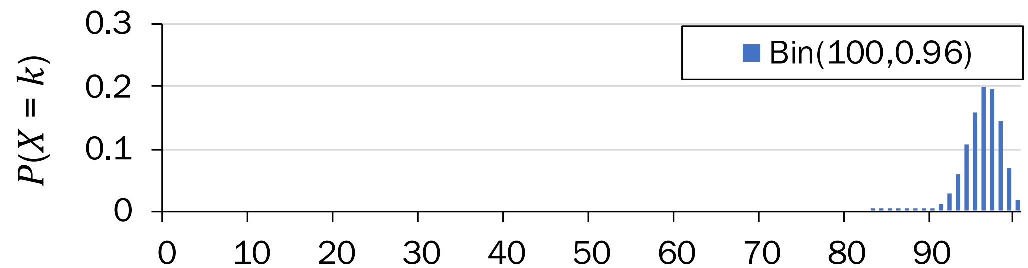

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