

10: Normal Distributions

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Normal RV

Normal Random Variable

def A **Normal** random variable X is defined as follows:

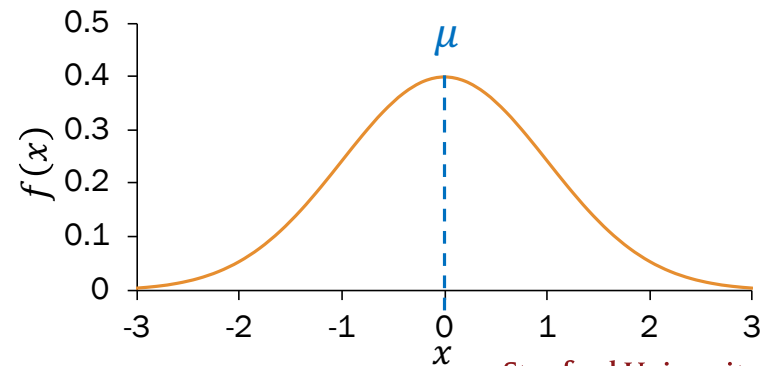
$X \sim \mathcal{N}(\mu, \sigma^2)$	PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Support: $(-\infty, \infty)$	Expectation	$E[X] = \mu$
	Variance	$\text{Var}(X) = \sigma^2$

Handwritten notes: "expected average value" with an arrow pointing to μ ; "variance" with an arrow pointing to σ^2 .

Other names: **Gaussian** random variable

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Handwritten notes: "mean" with an arrow pointing to μ ; "variance" with an arrow pointing to σ^2 .



Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.



Johann Carl Friedrich Gauss ([/ɡaʊs/](#); **German:** *Gauß* [\[ɡaʊs\]](#) (listen); **Latin:** *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including [algebra](#), [analysis](#), [astronomy](#), [differential geometry](#), [electrostatics](#), [geodesy](#), [geophysics](#), [magnetic fields](#), [matrix theory](#), [mechanics](#), [number theory](#), [optics](#) and [statistics](#).

} just wow!

Sometimes referred to as the *Princeps mathematicorum*^[1] (Latin for "the foremost of mathematicians") and "the greatest mathematician since antiquity". Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.^[2]

Did *not* invent Normal distribution but rather popularized it.

Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

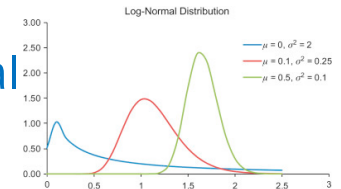
That's what they
want you to believe...



Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

Actually log-normal



Just an assumption

Only if equally weighted

(okay this one is true, we'll see this in 3 weeks)

Okay, so why the Normal?

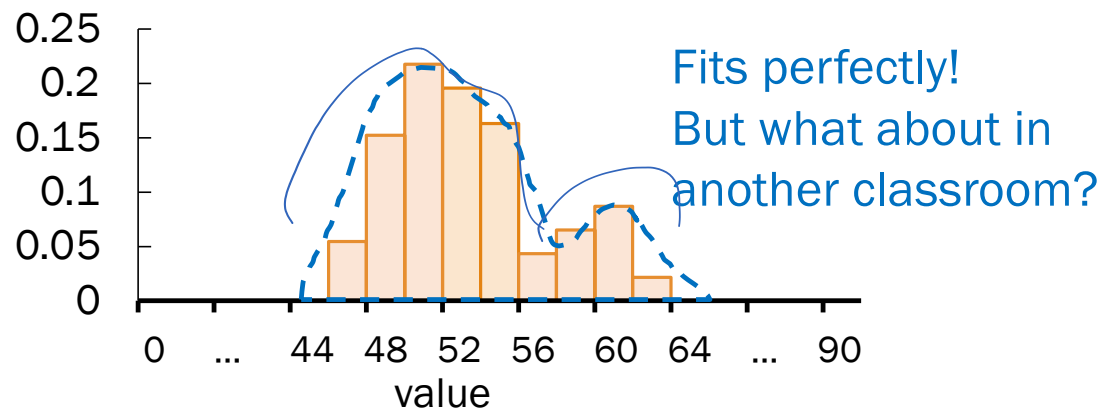
Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: **model real life situations** with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.



Okay, so why the Normal?

μ, σ^2



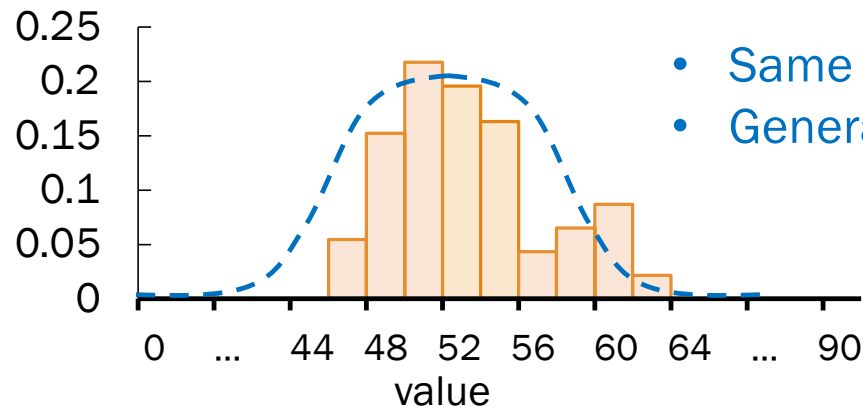
Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: **model real life situations** with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.



- Same mean/var
- Generalizes well

Occam's Razor:

"Non sunt multiplicanda entia sine necessitate."

Entities should not be multiplied without necessity.

A Gaussian maximizes **entropy** for a given mean and variance.

Why the Normal?

- Common for natural phenomena: height, weight, etc.

Actually log-normal

- Most noise in the world is Normal

assumption

- Often results from sum of random variables

Only if equally weighted

- Sample means are distributed normally

(okay this one is true, we'll see this in 3 weeks)

because it's easy to use

Stay critical of how to model real-world phenomena.

Anatomy of a beautiful equation

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of X is defined as:

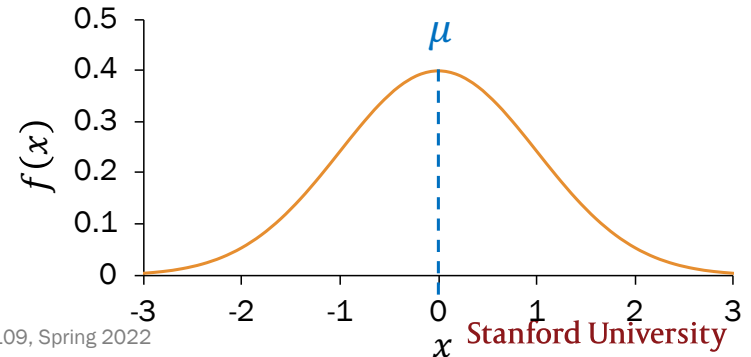
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

normalizing constant

exponential tail

symmetric around μ

variance σ^2 manages spread

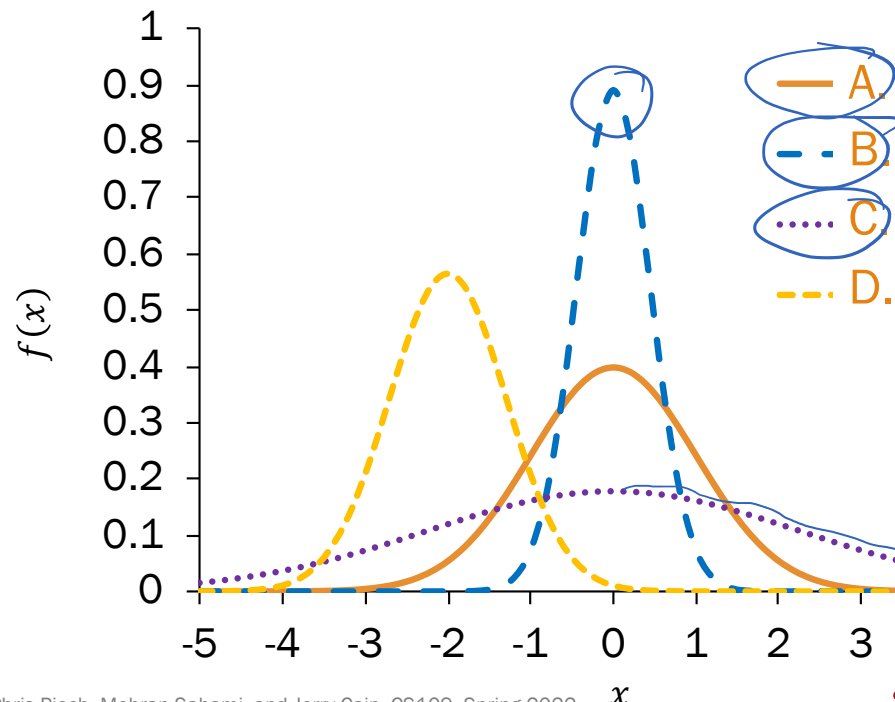


Normal Random Variable

$$X \sim \mathcal{N}(\overset{\text{mean}}{\mu}, \overset{\text{variance}}{\sigma^2})$$

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$ A
2. $\mathcal{N}(\underline{-2}, 0.5)$ D
3. $\mathcal{N}(0, 5)$ C
4. $\mathcal{N}(0, 0.2)$ B

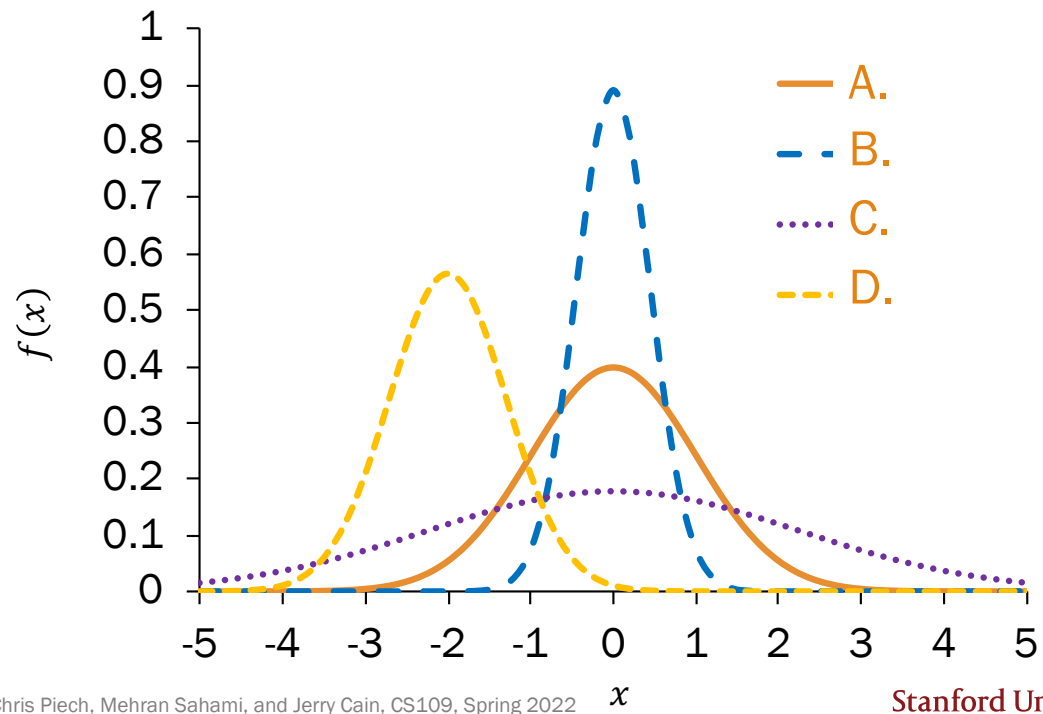


Normal Random Variable

$$X \sim \mathcal{N}(\overset{\text{mean}}{\mu}, \overset{\text{variance}}{\sigma^2})$$

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$
2. $\mathcal{N}(-2, 0.5)$
3. $\mathcal{N}(0, 5)$
4. $\mathcal{N}(0, 0.2)$



Getting to class

You spend some minutes, X , traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes²

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^{\infty} f(x) dx = \int_6^{\infty} \frac{1}{2\sqrt{\pi}} e^{-\frac{(x-4)^2}{4}} dx$$

(call Jerry if you solve this analytically)



Loving, not scary
...except this time

Computing probabilities with Normal RVs

For a Normal RV $X \sim \mathcal{N}(\mu, \sigma^2)$, its CDF has no closed form.

$$P(X \leq x) = F(x) = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

! Cannot be solved analytically

However, we can solve for probabilities numerically using a function Φ :

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$

A function that has been solved numerically

To get here, we'll first need to know some properties of Normal RVs.



Normal RV: Properties

Properties of Normal RVs

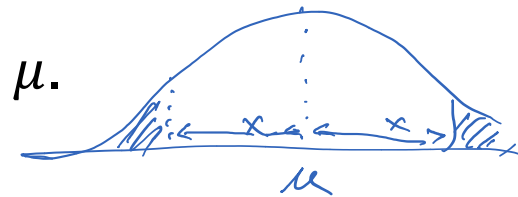
Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

1. Linear transformations of Normal RVs are also Normal RVs.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

2. The PDF of a Normal RV is symmetric about the mean μ .

$$F(\mu - x) = 1 - F(\mu + x)$$



1. Linear transformations of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

Linear transformations of X are also Normal.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof:

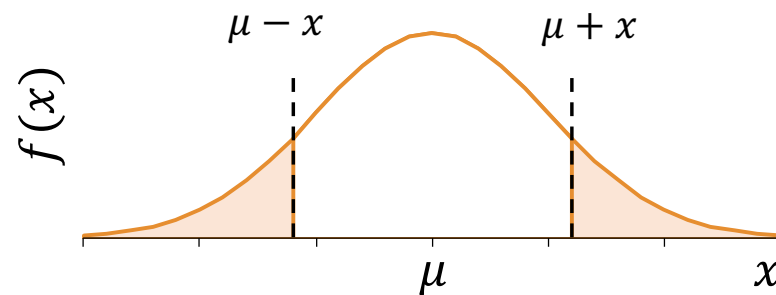
- $E[Y] = E[aX + b] = aE[X] + b = a\mu + b$ Linearity of Expectation
 - $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$ $\text{Var}(aX + b) = a^2\text{Var}(X)$
 - Y is also Normal
- Proof in Ross,
10th ed (Section 5.4)

2. Symmetry of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

The PDF of a Normal RV is symmetric about the mean μ .

$$F(\mu - x) = 1 - F(\mu + x)$$



Using symmetry of the Normal RV

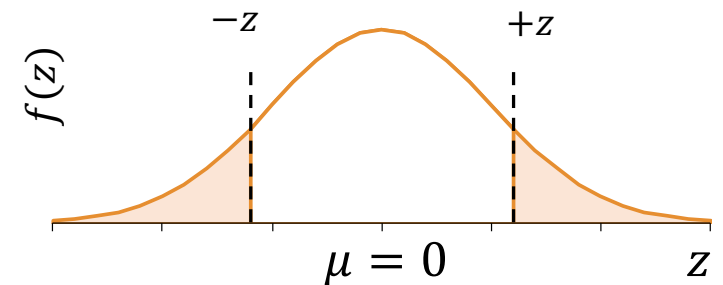
Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $z, y \geq 0$.

How do we compute the following probabilities?

1. $P(Z \leq z) = F(z)$
2. $P(Z < z)$
3. $P(Z \geq z)$
4. $P(Z \leq -z)$
5. $P(Z \geq -z)$
6. $P(y < Z < z)$

$$F(\mu - x) = 1 - F(\mu + x)$$



- A. $F(z)$
- B. $1 - F(z)$
- C. $F(z) - F(y)$



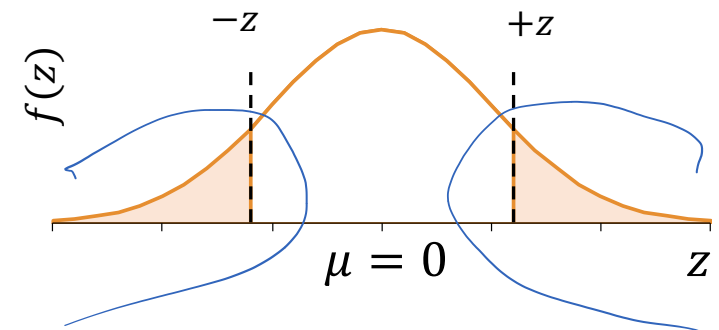
Using symmetry of the Normal RV

$$F(\mu - x) = 1 - F(\mu + x)$$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $z, y \geq 0$.

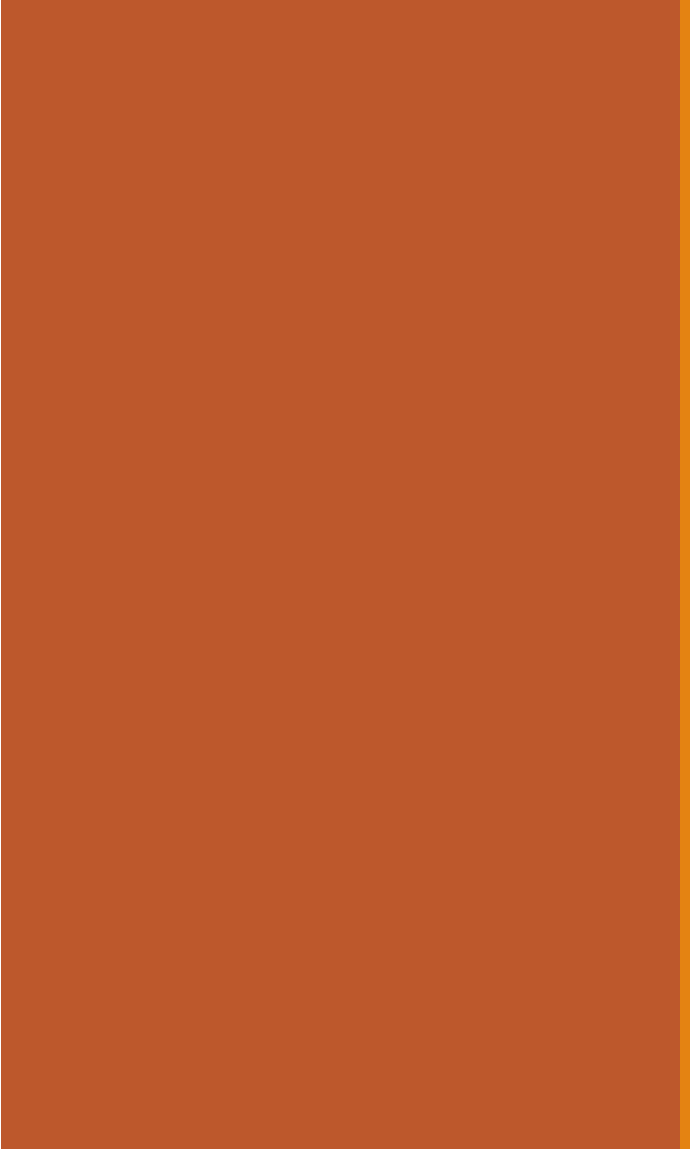
How do we compute the following probabilities?



1. $P(Z \leq z) = F(z)$
2. $P(Z < z) = F(z)$
3. $P(Z \geq z) = 1 - F(z)$
4. $P(Z \leq -z) = 1 - F(z)$
5. $P(Z \geq -z) = F(z)$
6. $P(y < Z < z) = F(z) - F(y)$

- A. $F(z)$
- B. $1 - F(z)$
- C. $F(z) - F(y)$

Symmetry is particularly useful when computing probabilities of zero-mean Normal RVs.



Normal RV: Computing probability

Computing probabilities with Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

To compute the CDF, $P(X \leq x) = F(x)$:

- We cannot analytically solve the integral (it has no closed form)
- ...but we **can** solve numerically using a function Φ :

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of the Standard Normal, Z

Standard Normal RV, Z

The **Standard Normal** random variable Z is defined as follows:

$$Z \sim \mathcal{N}(0, 1)$$

Expectation $E[Z] = \mu = 0$

Variance $\text{Var}(Z) = \sigma^2 = 1$

Note: not a new distribution; just a special case of the Normal

Other names: **Unit Normal**

CDF of Z defined as: $P(Z \leq z) = \Phi(z)$

Φ has been numerically computed

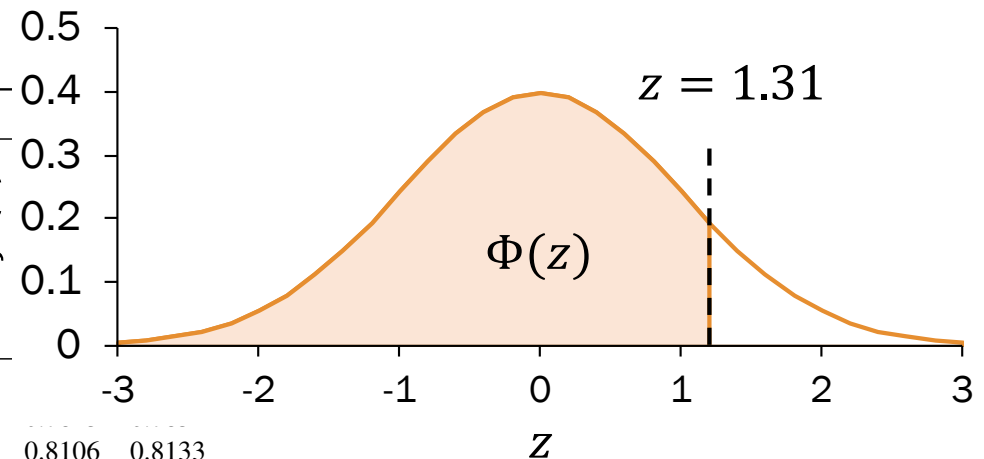
Standard Normal Table

An entry in the table is the area under the curve to the left of z , $P(Z \leq z) = \Phi(z)$.



$$P(Z \leq 1.31) = \Phi(1.31)$$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0	$f(z)$
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0	
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0	
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0	
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429



Standard Normal Table only has probabilities $\Phi(z)$ for $z \geq 0$.

History fact: Standard Normal Table

T A B L E S

S E R V A N T

AU CALCUL DES RÉFRACTIONS

APPROCHANTES DE L'HORIZON.

TABLE PREMIÈRE.

Intégrales de $e^{-t^2} dt$, depuis une valeur quelconque de t jusqu'à t infinie.

t	Intégrale.	Diff. prem.	Diff. II.	Diff. III.
0,00	0,88622692	999968	201	199
0,01	0,87622724	999767	400	199
0,02	0,86622057	999367	599	200
0,03	0,85623500	998768	799	199
0,04	0,84624822	997969	998	197
0,05	0,83626853	996971	1195	199
0,06	0,82629882	995776	1394	196

The first Standard Normal Table was computed by Christian Kramp, French astronomer (1760–1826), in *Analyse des Réfractions Astronomiques et Terrestres*, 1799

Used a Taylor series expansion to the third power

integral from $x = 0.03$ to infinity of e^{-x^2}

 Extended Keyboard  Upload

Definite integral:

$$\int_{0.03}^{\infty} e^{-x^2} dx = 0.856236$$

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use Φ , the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Proof:

$$\begin{aligned}
 F(x) &= P(X \leq x) \\
 &= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\
 &= P\left(Z \leq \frac{x - \mu}{\sigma}\right) \\
 &= \Phi\left(\frac{x - \mu}{\sigma}\right)
 \end{aligned}$$

Definition of CDF
Algebra + $\sigma > 0$

- $\frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X .
- This is distributed as $\mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0, 1)$
- In other words, $\frac{X - \mu}{\sigma} = Z \sim \mathcal{N}(0, 1)$ with CDF Φ .

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use Φ , the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Proof:

$$\begin{aligned} F(x) &= P(X \leq x) && \text{Definition of CDF} \\ &= P(X - \mu \leq x - \mu) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) && \text{Algebra + } \sigma > 0 \\ &= P\left(Z \leq \frac{x - \mu}{\sigma}\right) \left\{ \begin{array}{l} \bullet \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \text{ is a linear transform of } X. \\ \bullet \text{ This is a linear transform of } X \text{ with } a = \frac{1}{\sigma} \text{ and } b = -\frac{\mu}{\sigma}. \end{array} \right. \\ &= \Phi\left(\frac{x - \mu}{\sigma}\right) \end{aligned}$$

1. Compute $z = (x - \mu)/\sigma$.
2. Look up $\Phi(z)$ in Standard Normal table.

Campus bikes

You spend some minutes, X , traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes²

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?



$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2) \quad \times \quad P(X \geq 6) = \int_6^{\infty} f(x) dx \quad (\text{no analytic solution})$$

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$\begin{aligned} P(X \geq 6) &= 1 - F_x(6) \sqrt{2} \\ &= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right) \\ &\approx 1 - \Phi(1.41) \end{aligned}$$

2. Look up $\Phi(z)$ in table

$$\begin{aligned} &1 - \Phi(1.41) \\ &\approx 1 - 0.9207 \\ &= 0.0793 \end{aligned}$$

Is there an easier way? (yes)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?

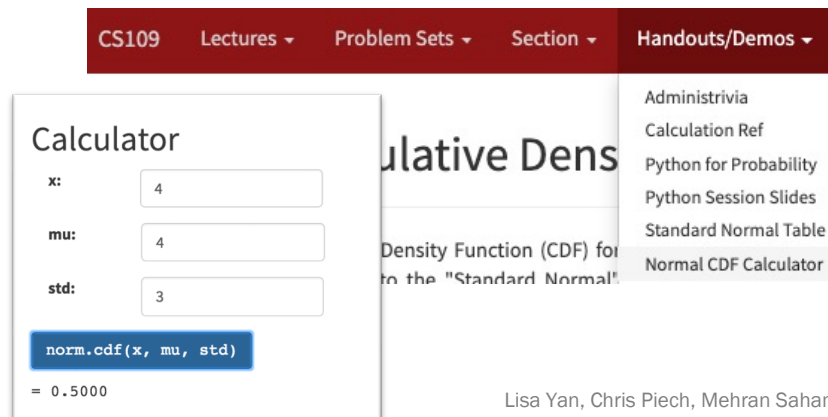
- Use Python

```
from scipy import stats
X = stats.norm(mu, std)
X.cdf(x)
```

SciPy reference:

<https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html>

- Use website tool



The screenshot shows a web interface for a Normal CDF calculator. At the top, there is a navigation bar with 'CS109' and several dropdown menus: 'Lectures', 'Problem Sets', 'Section', and 'Handouts/Demos'. Below the navigation bar, there is a 'Calculator' section with three input fields: 'x:' with the value '4', 'mu:' with the value '4', and 'std:' with the value '3'. Below these fields is a blue button labeled 'norm.cdf(x, mu, std)'. The result of the calculation is displayed as '= 0.5000'. To the right of the calculator, there is a dropdown menu with the following options: 'Administrivia', 'Calculation Ref', 'Python for Probability', 'Python Session Slides', 'Standard Normal Table', and 'Normal CDF Calculator'.

Website tool:

<https://web.stanford.edu/class/cs109/handouts/normalCDF.html>



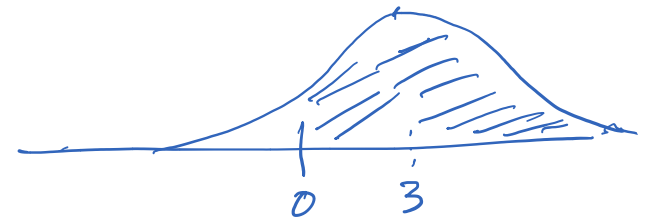
Exercises

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

$$\begin{aligned} 1. \quad P(X > 0) &= 1 - F(0) \\ &= 1 - \Phi\left(\frac{0-3}{4}\right) \\ &= 1 - \Phi\left(-\frac{3}{4}\right) \\ &= 1 - \left(1 - \Phi\left(\frac{3}{4}\right)\right) \\ &= \Phi\left(\frac{3}{4}\right) \\ &= 0.7734 \end{aligned}$$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$



Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.

Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, Φ ?

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$



Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$

2. $P(2 < X < 5) = F(5) - F(2)$

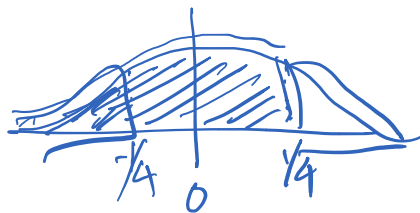
$$= \Phi\left(\frac{5-3}{4}\right) - \Phi\left(\frac{2-3}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right)$$

$$= \Phi\left(\frac{1}{2}\right) + \Phi\left(\frac{1}{4}\right) - 1$$

$$= 0.2902$$



- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$\begin{aligned} P(X < -3) + P(X > 9) \\ &= F(-3) + (1 - F(9)) \\ &= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right) \end{aligned}$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

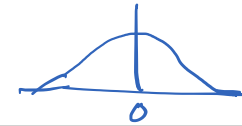
$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$\approx 0.1337$$

Noisy Wires

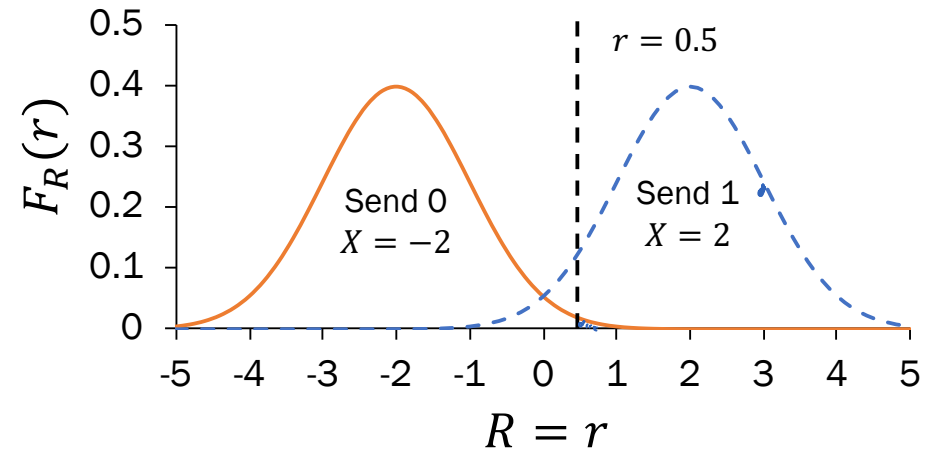
$$\mu = 0$$
$$\sigma^2 = 1$$
$$N(\mu, \sigma^2)$$
$$z(0, 1)$$



Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: $\left. \begin{array}{l} 1 \text{ if } R \geq 0.5 \\ 0 \text{ otherwise.} \end{array} \right\}$



1. What is $P(\text{decoding error} \mid \text{original bit is 1})$?
i.e., we sent 1, but we decoded as 0?
2. What is $P(\text{decoding error} \mid \text{original bit is 0})$?

These probabilities are unequal. Why might this be useful?

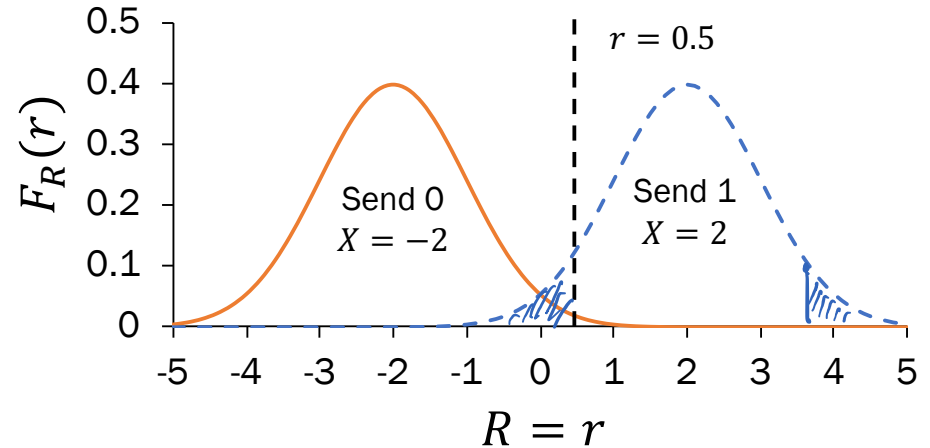


Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
 0 otherwise.



1. What is $P(\text{decoding error} \mid \text{original bit is 1})$?
 i.e., we sent 1, but we decoded as 0?



$$P(R < 0.5 \mid X = 2) = P(2 + Y < 0.5) = P(Y < -1.5)$$

Y is Standard Normal

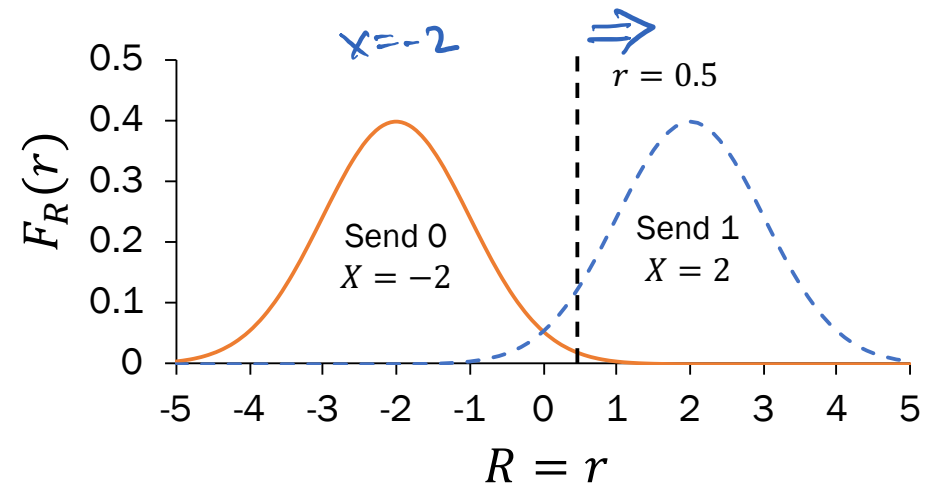
$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
 0 otherwise.



1. What is $P(\text{decoding error} \mid \text{original bit is 1})$?
 i.e., we sent 1, but we decoded as 0?

0.0668

2. What is $P(\text{decoding error} \mid \text{original bit is 0})$?

$$P(R \geq 0.5 \mid X = -2) = P(-2 + Y \geq 0.5) = P(Y \geq 2.5) \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.



Challenge: Sampling with the Normal RV

ELO ratings

Basketball == Stats



What is the probability that the Warriors win?
How do you model zero-sum games?

ELO ratings

$$S_1 = 1487$$
$$S_2 = 1200$$

Each team has an ELO score S , calculated based on its past performance.



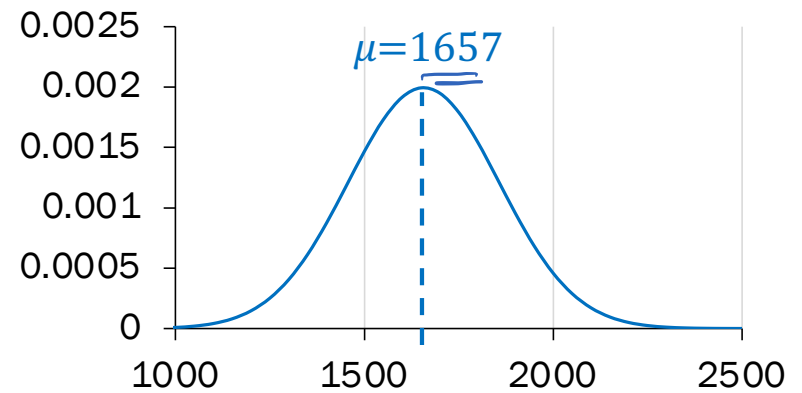
Arpad Elo

- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.

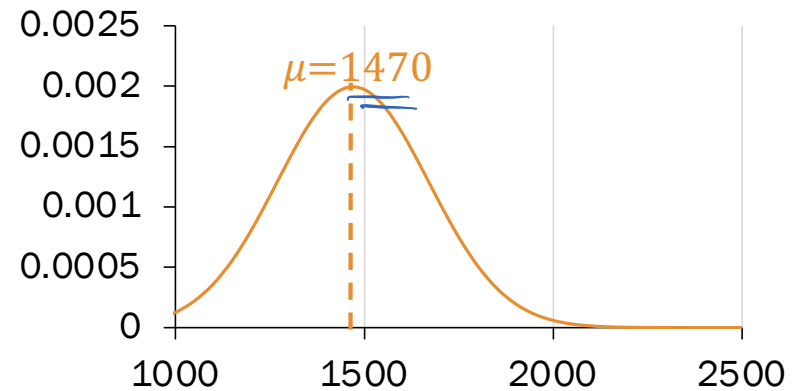
What is the probability that Warriors win this game?

$$\text{Want: } P(\text{Warriors win}) = P(A_W > A_O)$$

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



ELO ratings

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

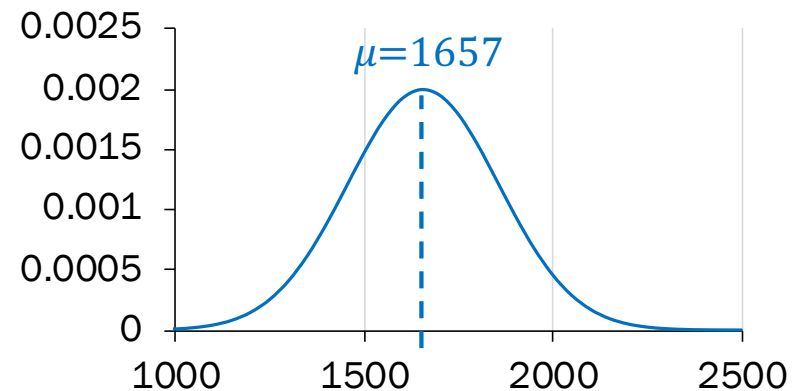
```
from scipy import stats
WARRIORS_ELO = 1657
OPPONENT_ELO = 1470
STDEV = 200
NTRIALS = 10000

nSuccess = 0
for i in range(NTRIALS):
    w = stats.norm.rvs(WARRIORS_ELO, STDEV)
    o = stats.norm.rvs(OPPONENT_ELO, STDEV)
    if w > o: nSuccess += 1

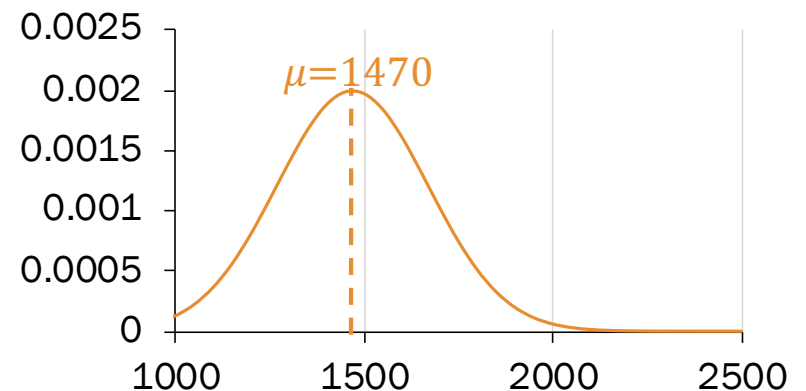
print("Warriors sampled win fraction",
      float(nSuccess) / NTRIALS)
```

≈ 0.7488, calculated by sampling

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



Is there a better way?

$$P(A_W > A_O)$$

- This is a probability of an event involving *two continuous random variables!*
- We'll solve this problem analytically in two weeks' time.

Big goal for next time: Events involving *two discrete random variables.*

Stay tuned!