

17: Continuous Joint Distributions (II)

Jerry Cain
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Convolution: Sum of independent Uniform RVs

Today's lecture

Take what we've seen in **discrete** joint distributions...

...and translate them to **continuous** joint distributions!

For the most part, this is easy. For example:

$$\begin{array}{l} \text{Marginal} \\ \text{distributions} \end{array} \quad p_X(a) = \sum_y p_{X,Y}(a, y) \quad f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy$$

$$\text{Independent RVs} \quad p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

But some concepts, while mathematically accessible, are difficult to implement in practice.

We'll focus on some of these today.

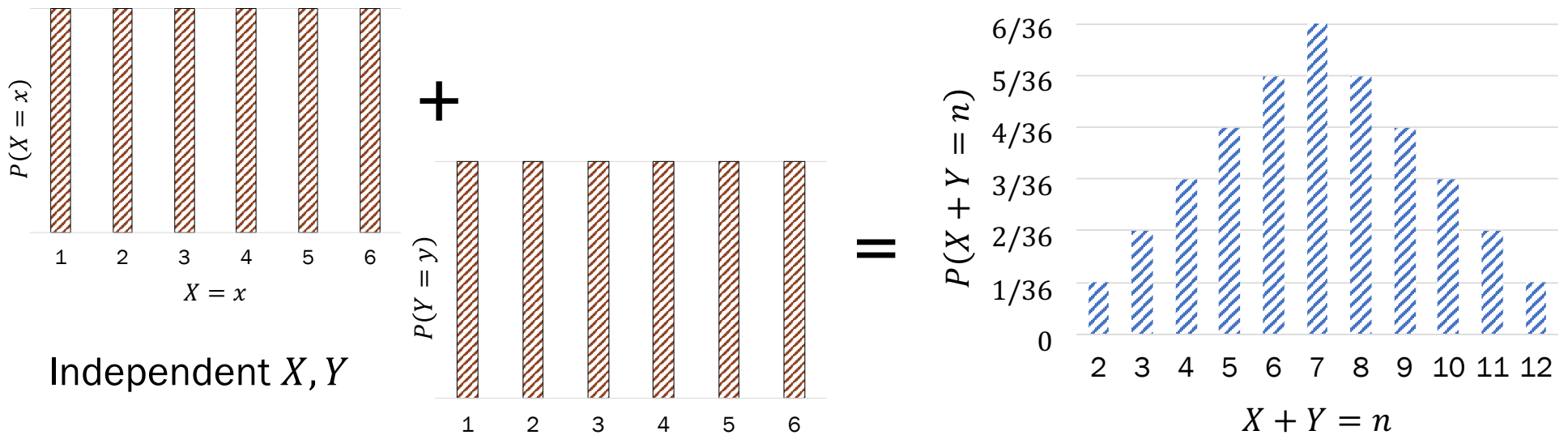
Goal of CS109 continuous joint distributions unit:
build mathematical maturity

Dance, Dance, Convolution

Recall that for independent discrete random variables X and Y :

$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

the **convolution** of p_X and p_Y



Independent X, Y

Dance, Dance, Convolution

Recall that for independent discrete random variables X and Y :

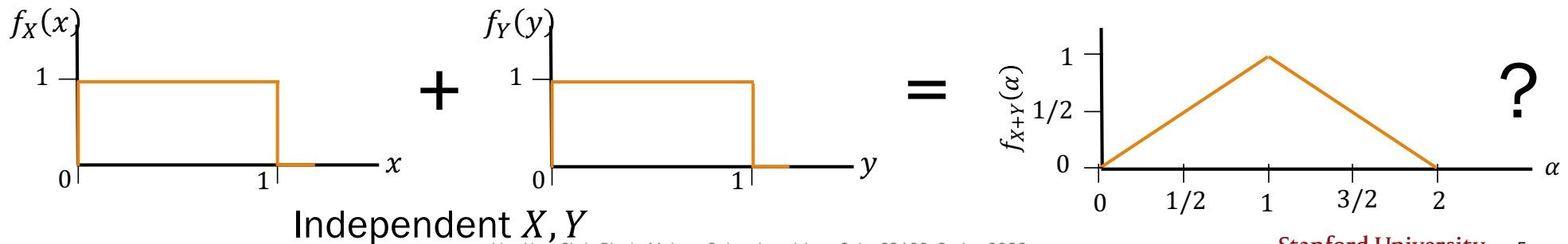
$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

the convolution of p_X and p_Y

For independent continuous random variables X and Y :

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x)f_Y(\alpha - x)dx$$

the convolution of f_X and f_Y



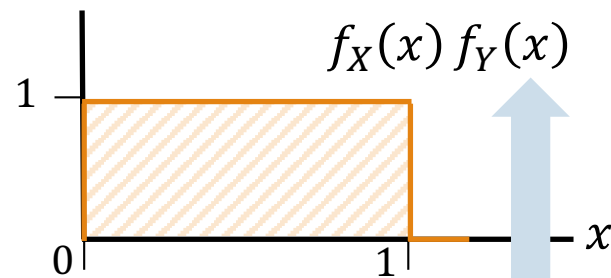
Independent X, Y

Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.

What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$



Isn't this just one??



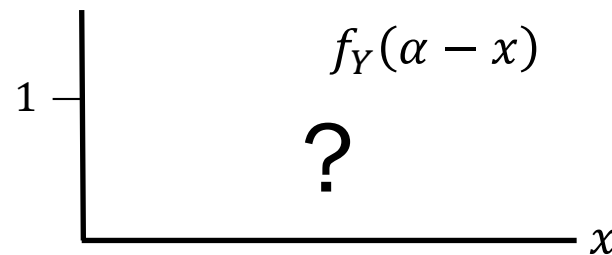
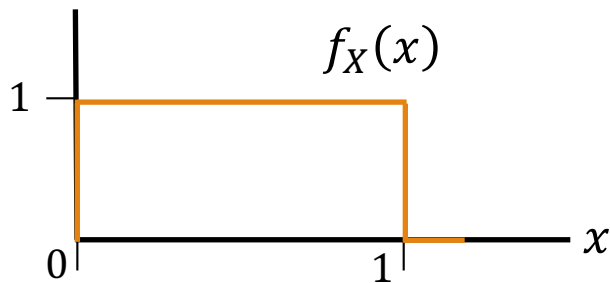
Not so fast...

Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.

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$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$



$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(\alpha - x) = \begin{cases} 1 & \text{if } 0 \leq \alpha - x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } \alpha - 1 \leq x \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

α is a constant
in the integral
w.r.t. x .

Sum of independent Uniforms

X and Y
independent
+ continuous

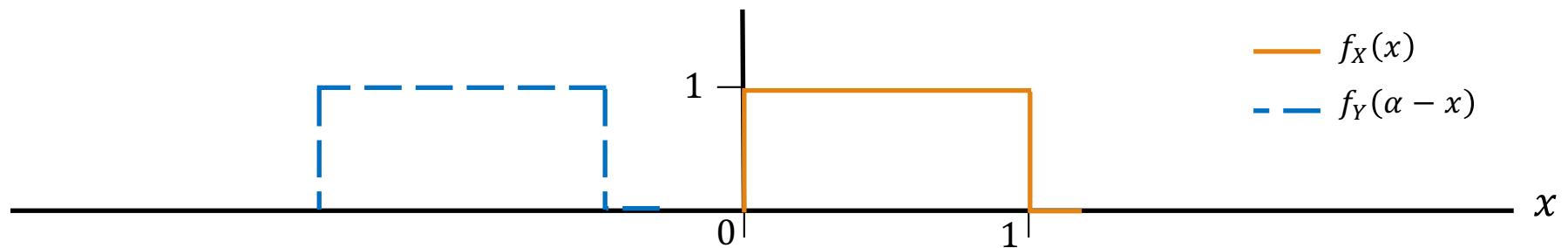
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1. $\alpha \leq 0$ 0



Sum of independent Uniforms

X and Y
independent
+ continuous

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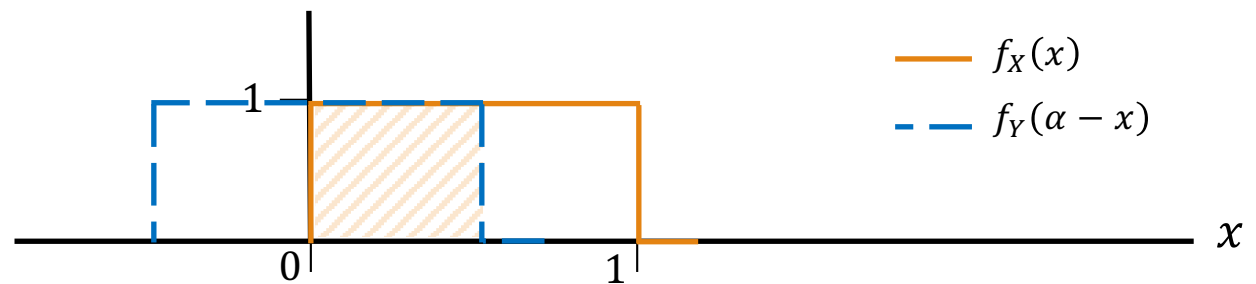
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1. $\alpha \leq 0$ 0

2. $\alpha = 1/2$ 1/2



Integral = area under the curve
This curve = product of 2 functions of x

Sum of independent Uniforms

X and Y
independent
+ continuous

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1. $\alpha \leq 0$ 0

2. $\alpha = 1/2$ 1/2

3. $\alpha = 1$

4. $\alpha = 3/2$

5. $\alpha \geq 2$



Sum of independent Uniforms

X and Y
independent
+ continuous

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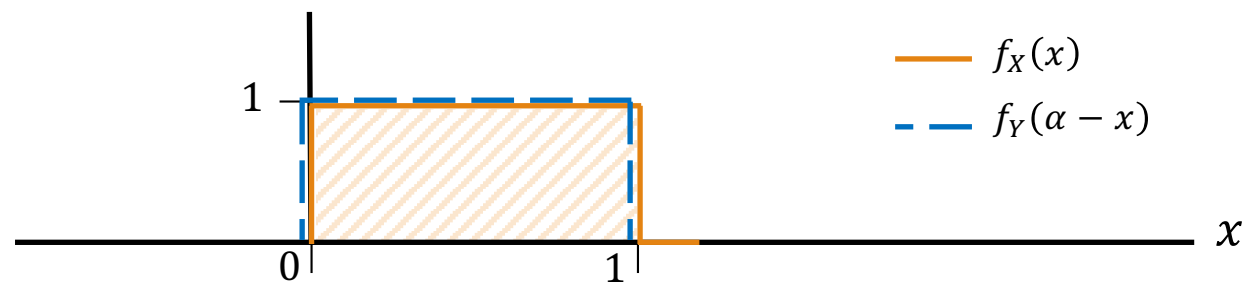
1. $\alpha \leq 0$ **0**

2. $\alpha = 1/2$ **1/2**

3. $\alpha = 1$ **1**

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Sum of independent Uniforms

X and Y
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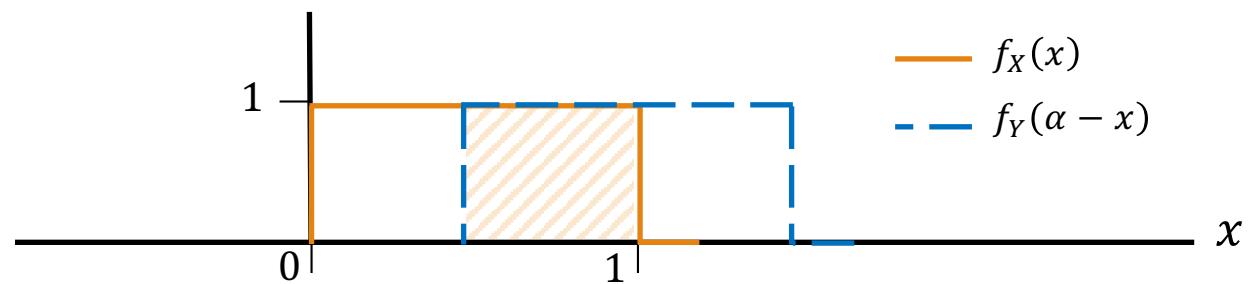
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2. $\alpha = 1/2$ **1/2**

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5. $\alpha \geq 2$



Sum of independent Uniforms

X and Y
independent
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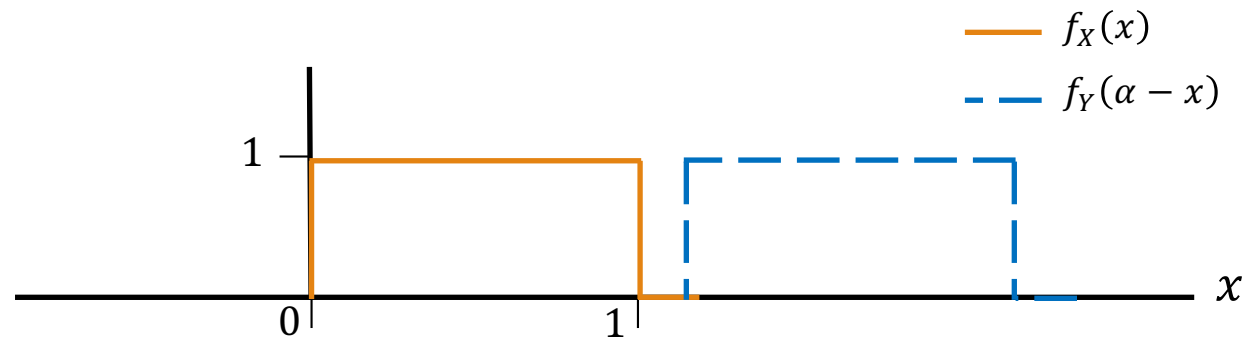
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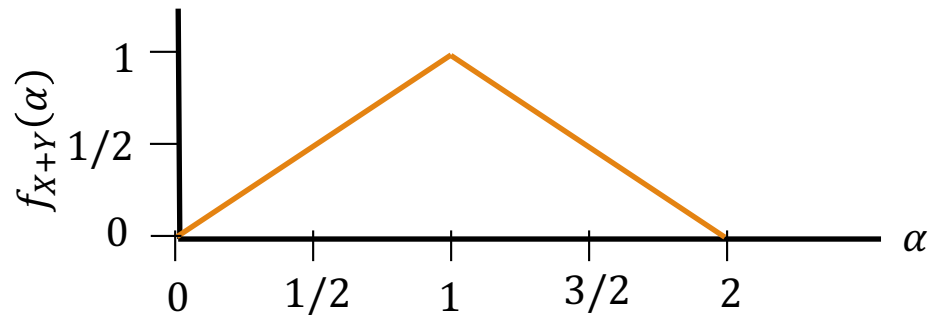
Sum of independent Uniforms

X and Y
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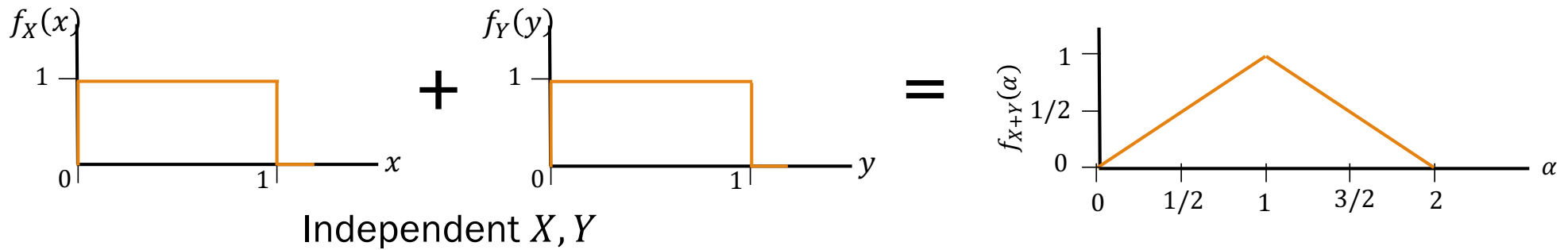
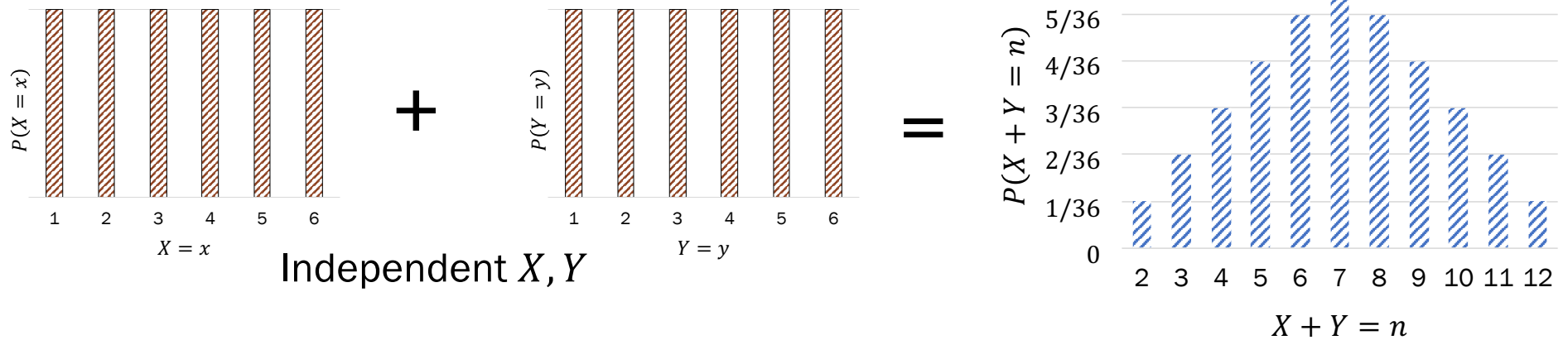
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1. $\alpha \leq 0$ 0
2. $\alpha = 1/2$ 1/2
3. $\alpha = 1$ 1
4. $\alpha = 3/2$ 1/2
5. $\alpha \geq 2$ 0



$$f_{X+Y}(\alpha) = \begin{cases} \alpha & 0 \leq \alpha \leq 1 \\ 2 - \alpha & 1 \leq \alpha \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Dance, Dance, Convolution Extreme





Sums of independent Normal RVs

Sum of independent Normals

$$\begin{array}{l} X \sim \mathcal{N}(\mu_1, \sigma_1^2), \\ Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \\ X, Y \text{ independent} \end{array} \quad \Rightarrow \quad X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

(proof left to [Wikipedia](#))

Holds in general case:

$$\begin{array}{l} X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \\ X_i \text{ independent for } i = 1, \dots, n \end{array} \quad \Rightarrow \quad \sum_{i=1}^n X_i \sim \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$$

Back for another playoffs game



What is the probability that the Warriors win?
How do you model zero-sum games?

$$P(A_W > A_B)$$

This is a probability of an event involving *two* random variables!

We will compute:

$$P(\underbrace{A_W - A_B}_{\text{A sum of Normals!}} > 0)$$

Motivating idea: Zero sum games



Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

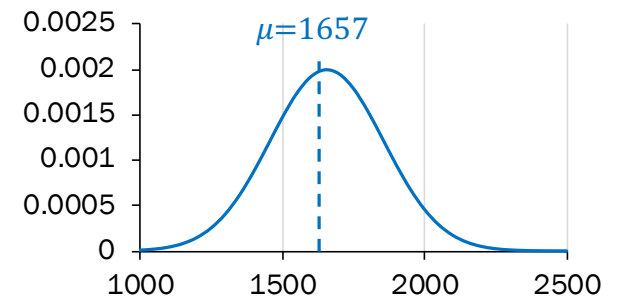
Assume A_W, A_B are independent.

Let $D = A_W - A_B$.

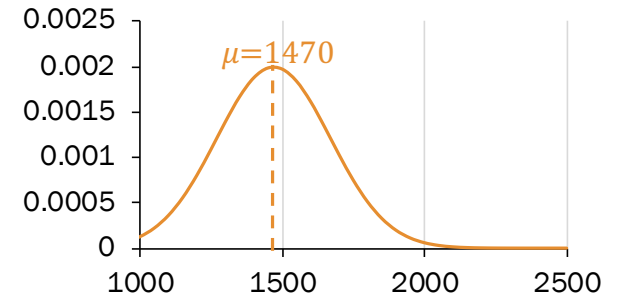
What is the distribution of D ?

- A. $D \sim \mathcal{N}(1657 - 1470, 200^2 - 200^2)$
- B. $D \sim \mathcal{N}(1657 - 1470, 200^2 + 200^2)$
- C. $D \sim \mathcal{N}(1657 + 1470, 200^2 + 200^2)$
- D. Dance, Dance, Convolution
- E. None/other

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$



Motivating idea: Zero sum games



Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

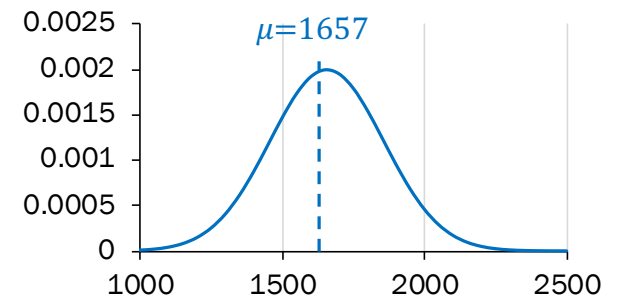
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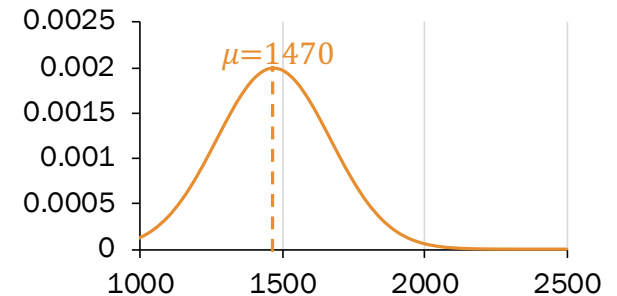
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- C. $D \sim \mathcal{N}(1657 + 1470, 200^2 + 200^2)$
- D. Dance, Dance, Convolution
- E. None/other

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$



If $X \sim \mathcal{N}(\mu_1, \sigma^2)$,
then $(-X) \sim \mathcal{N}(-\mu, (-1)^2 \sigma^2 = \sigma^2)$.

Motivating idea: Zero sum games



Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

Assume A_W, A_B are independent.

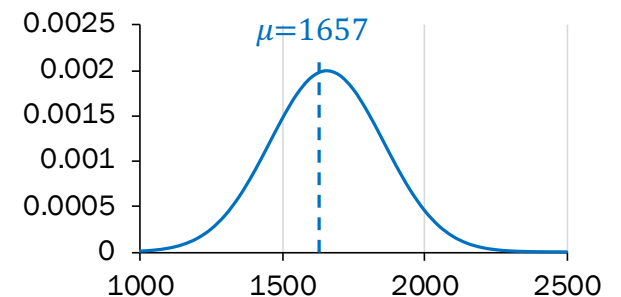
Let $D = A_W - A_B$.

$$\begin{aligned} D &\sim \mathcal{N}(1657 - 1470, 200^2 + 200^2) \\ &\sim \mathcal{N}(187, 2 \cdot 200^2) \quad \sigma \approx 283 \end{aligned}$$

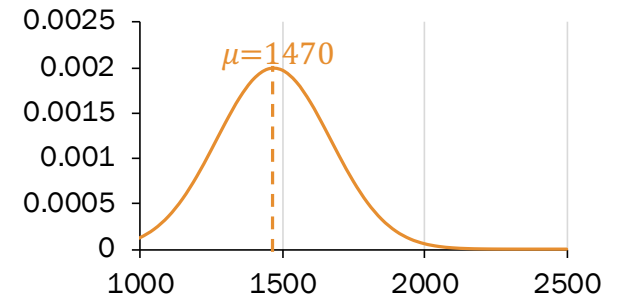
$$\begin{aligned} P(D > 0) &= 1 - F_D(0) = 1 - \Phi\left(\frac{0 - 187}{283}\right) \\ &\approx 0.7454 \end{aligned}$$

Compare with **0.7488**, calculated by sampling!

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$





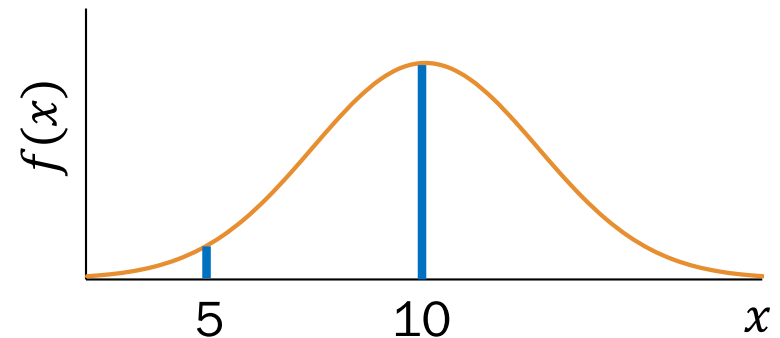
Ratio of PDFs

Relative probabilities of continuous random variables

Let X = time to finish problem set 4.

Suppose $X \sim \mathcal{N}(10, 2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?



$$\frac{P(X = 10)}{P(X = 5)} =$$

- A. $0/0 = \text{undefined}$
- B. $\frac{f(10)}{f(5)}$
- C. stay healthy



Relative probabilities of continuous random variables

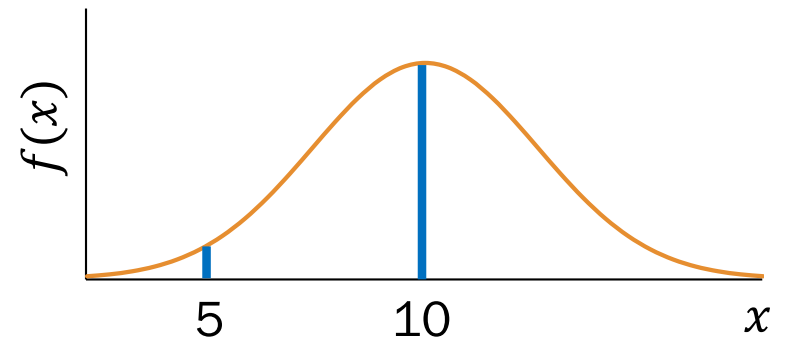
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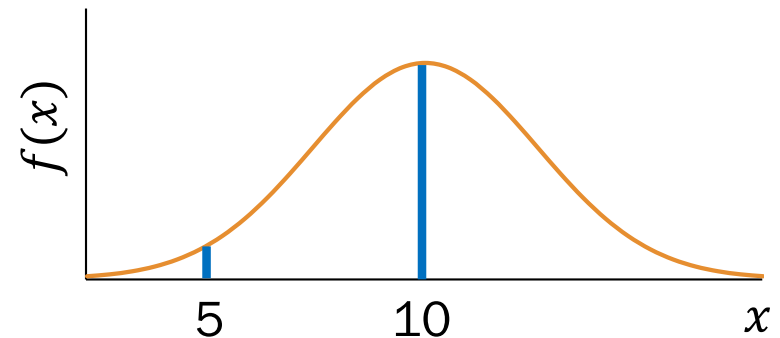


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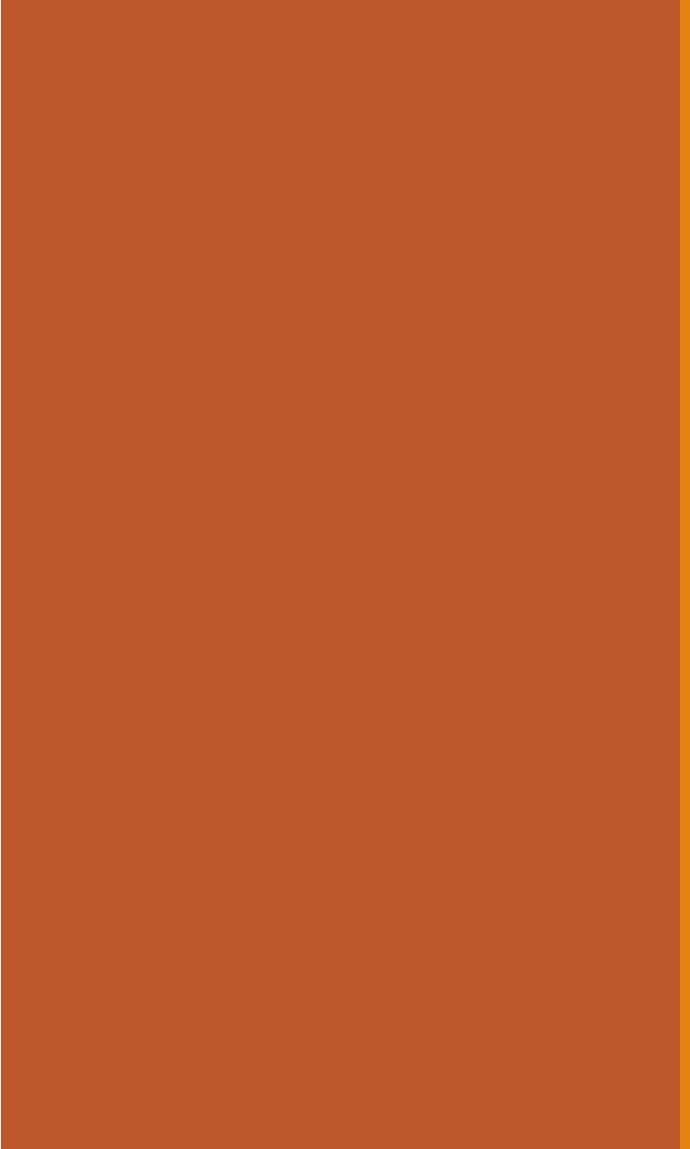
$$\frac{P(X = 10)}{P(X = 5)} = \frac{f(10)}{f(5)}$$

$$P(X = a) = P\left(a - \frac{\varepsilon}{2} \leq X \leq a + \frac{\varepsilon}{2}\right) = \int_{a - \frac{\varepsilon}{2}}^{a + \frac{\varepsilon}{2}} f(x) dx \approx \varepsilon f(a)$$

Therefore $\frac{P(X = a)}{P(X = b)} = \frac{\varepsilon f(a)}{\varepsilon f(b)} = \frac{f(a)}{f(b)}$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(10 - \mu)^2}{2\sigma^2}}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(5 - \mu)^2}{2\sigma^2}}} = \frac{e^{-\frac{(10 - 10)^2}{2 \cdot 2}}}{e^{-\frac{(5 - 10)^2}{2 \cdot 2}}} = \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

Ratios of PDFs
are meaningful!



Continuous conditional distributions

Continuous conditional distributions

For continuous RVs X and Y , the **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{where } f_Y(y) > 0$$

Intuition: $P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} \iff f_{X|Y}(x|y)\varepsilon_X = \frac{f_{X,Y}(x,y)\varepsilon_X\varepsilon_Y}{f_Y(y)\varepsilon_Y}$

Note that conditional PDF $f_{X|Y}$ is a "true" density:

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)} dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

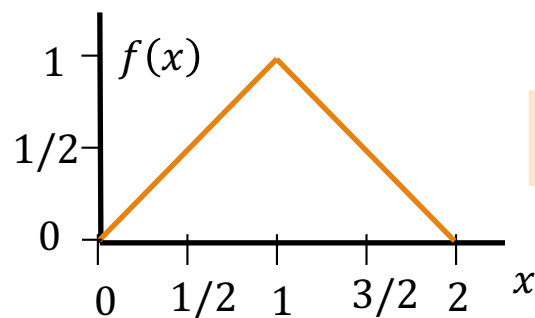
Why sums of random variables?

Sometimes modeling and understanding a complex RV, X , is difficult.

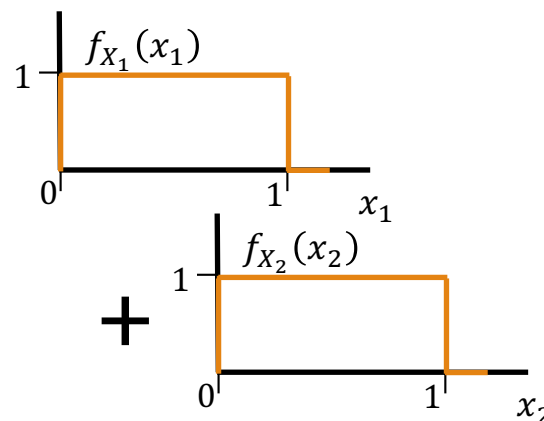
But if we can decompose X into the **sum of independent simpler** RVs,

- We can then compute distributions on X .
- We can then to understand how X changes as its parts change.

What can we model
with a triangular PDF?



Sum of uniforms!



We're covering the
reverse direction for
now; the forward
direction will come
on Friday

Everything* in probability is a sum or a product (or both)

*except conditional probability (a ratio)

Sum of values that can be considered separately (possibly weighted by prob. of happening)

$$E[X] = \sum_x x \underbrace{p(x)}_{\text{weight}}$$

$$E[X|Y = y] = \int_{-\infty}^{\infty} x \underbrace{f_{X|Y}(x|y)}_{\text{weight}} dx$$

$$P(E) = \sum_{i=1}^n P(E|F_i) \underbrace{P(F_i)}_{\text{weight}}$$

Law of Total Probability

$$P(E) = \sum_{i=1}^n P(E_i)$$

Axiom 3, $E = E_1 \cup \dots \cup E_n$

Product of values that can each be considered in sequence

$$P(E \cap F \cap G) = P(E)P(F|E)P(G|EF)$$

Chain Rule

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Independent cont. RVs

$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

Sum of indep. discrete RVs (convolution)

Multiple Bayes' Theorems



with
events

$$P(F|E) = \frac{P(F)P(E|F)}{P(E)}$$



with
discrete RVs

$$p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)}$$



with
continuous RVs

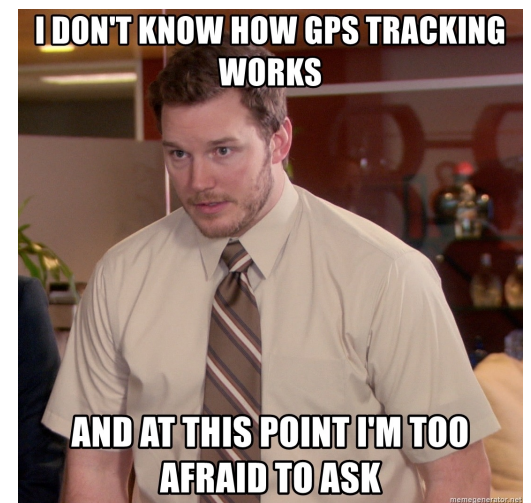
You are given
this value...

$$f_{Y|X}(y|x) = \frac{f_Y(y)f_{X|Y}(x|y)}{f_X(x)}$$

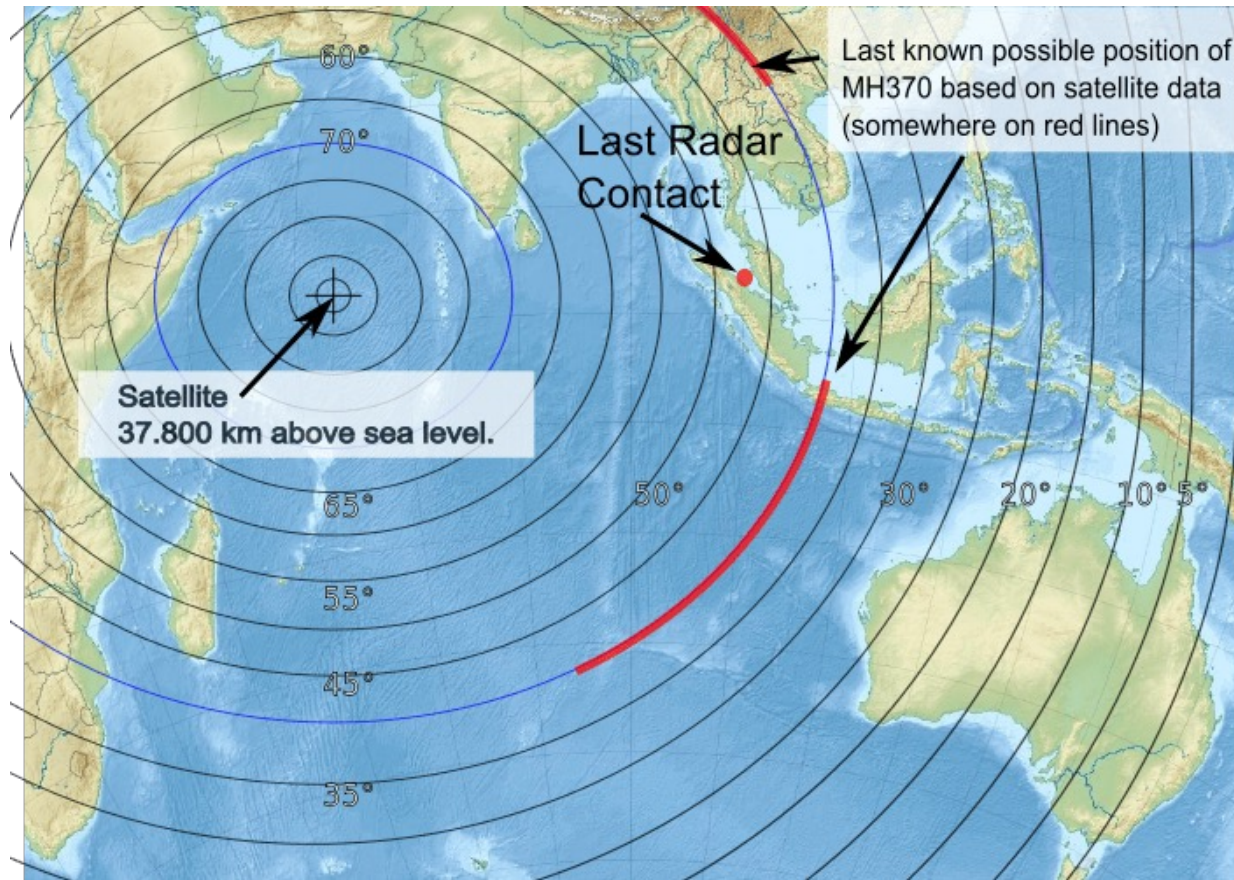
...so this is just a scalar

Really all the
same idea!

Intense Exercise



Tracking in 2-D space



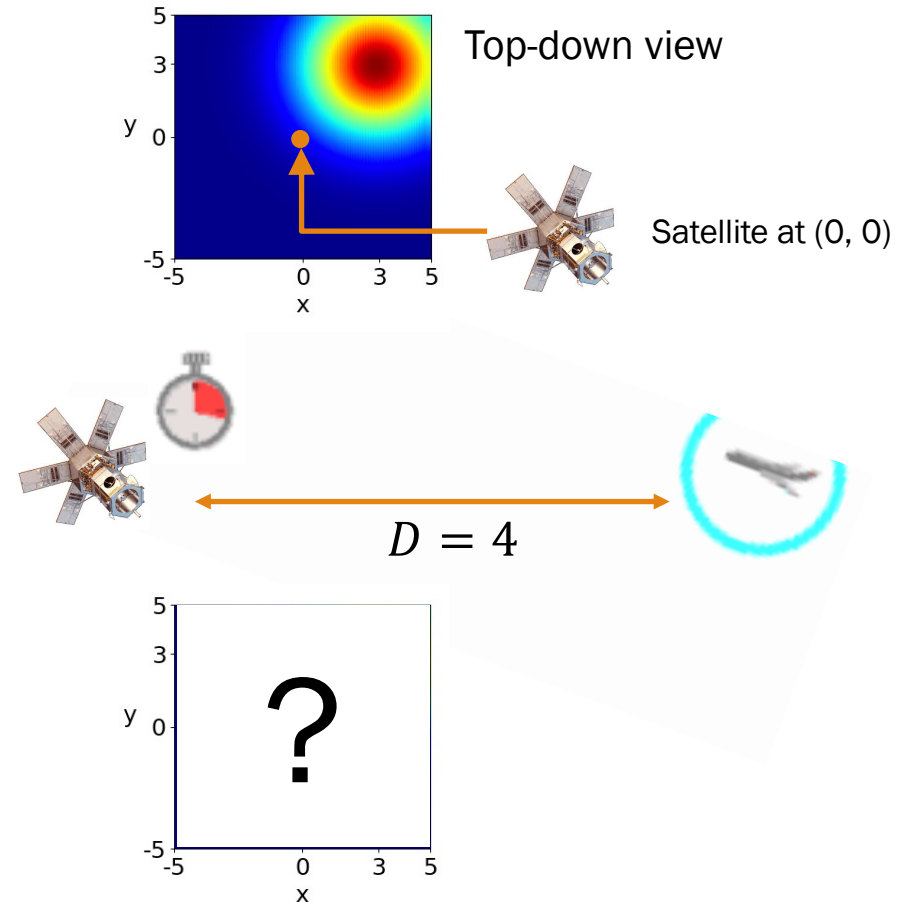
You want to know the 2-D location of an object.

Your satellite ping gives you a noisy 1-D measurement of the distance of the object from the satellite (0,0).

Using the satellite measurement, where is the object?

Tracking in 2-D space

- Before measuring, we have some **prior belief** about the 2-D location of an object, (X, Y) .
- We observe some noisy **measurement** $D = 4$, the Euclidean distance of the object to a satellite.
- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?



Tracking in 2-D space

- You have a **prior belief** about the 2-D location of an object, (X, Y) .
- You observe a **noisy distance measurement**, $D = 4$.
- What is your **updated (posterior) belief** of the 2-D location of the object after observing the measurement?

Recall Bayes terminology:

$$f_{X,Y|D}(x, y|d) = \frac{\begin{array}{l} \text{likelihood} \\ \text{(of evidence)} \end{array} f_{D|X,Y}(d|x, y) \begin{array}{l} \text{prior} \\ \text{belief} \end{array} f_{X,Y}(x, y)}{\begin{array}{l} \text{normalization constant} \\ f_D(d) \end{array}}$$

1. Define prior

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

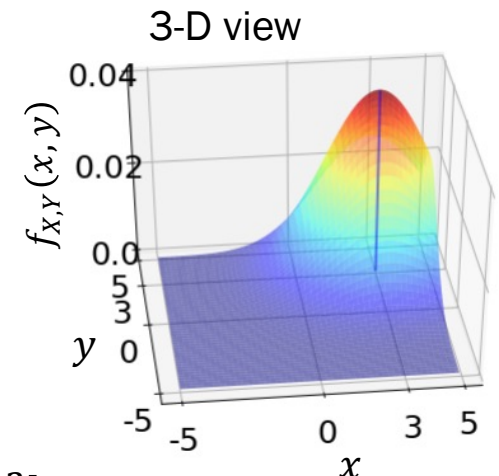
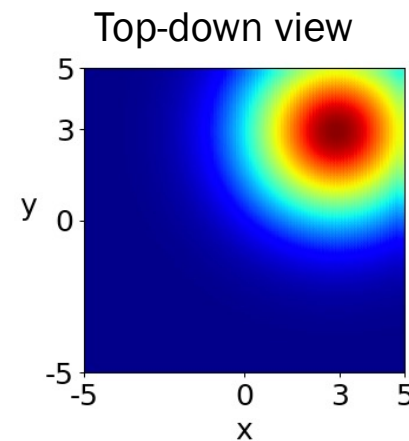
You have a **prior belief** about the 2-D location of an object, (X, Y) .

Let (X, Y) = object's 2-D location, assuming satellite is at $(0,0)$

Suppose the prior distribution is a symmetric bivariate normal distribution:

$$f_{X,Y}(x, y) = \frac{1}{2\pi 2^2} e^{-\frac{[(x-3)^2 + (y-3)^2]}{2(2^2)}} = K_1 \cdot e^{-\frac{[(x-3)^2 + (y-3)^2]}{8}}$$

normalizing constant



2. Define likelihood

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

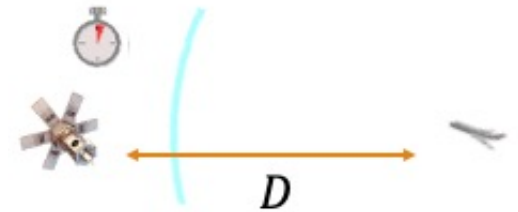
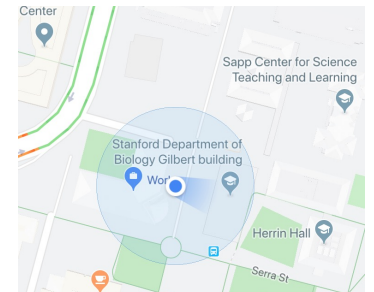
You observe a **noisy distance measurement**, $D = 4$.

If you knew your actual location (x, y) , you could say **how likely** a measurement $D = 4$ is:

Let D = distance from the satellite (radially).

Suppose you knew your actual position: (x, y) .

- D is still noisy! Suppose noise is **standard normal**.
- On average, D is your true Euclidean distance: $\sqrt{x^2 + y^2}$



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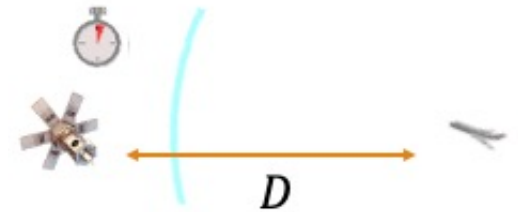
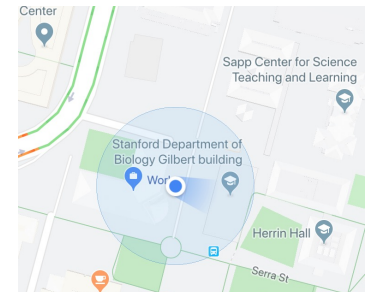
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$$D|X, Y \sim N(\mu = (A), \sigma^2 = (B))$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{(C) \sqrt{2\pi}} e^{\{ (D) \}}$$



2. Define likelihood

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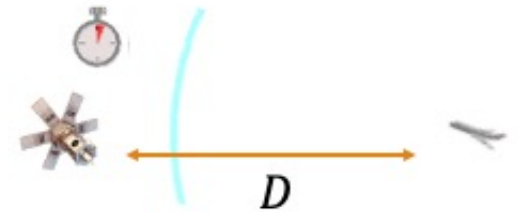
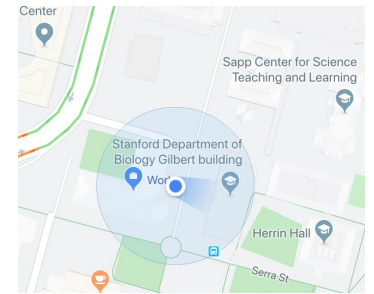
Suppose you knew your actual position: (x, y) .

- D is still noisy! Suppose noise is **standard normal**.
- On average, D is your true Euclidean distance: $\sqrt{x^2 + y^2}$

$$D|X, Y \sim N\left(\mu = \sqrt{x^2 + y^2}, \sigma^2 = 1\right)$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d - \sqrt{x^2 + y^2})^2}{2}} = K_2 \cdot e^{-\frac{(d - \sqrt{x^2 + y^2})^2}{2}}$$

normalizing constant



3. Compute posterior

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

What is your **updated (posterior) belief** of the 2-D location of the object after observing the measurement?

Compute:

Posterior
belief

$$f_{X,Y|D}(x, y|4) = f_{X,Y|D}(X = x, Y = y|D = 4)$$

3. Compute posterior

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$$f_{X,Y|D}(x, y|4) = f_{X,Y|D}(X = x, Y = y|D = 4)$$

Know:

Prior
belief $f_{X,Y}(x, y) = K_1 \cdot e^{-\frac{[(x-3)^2 + (y-3)^2]}{8}}$

Observation
likelihood $f_{D|X,Y}(d|x, y) = K_2 \cdot e^{-\frac{(d - \sqrt{x^2 + y^2})^2}{2}}$

Tips

- Use Bayes' Theorem!
- $f_D(4)$ is just a scaling constant. Why?
- How can we approximate the final scaling constant with a computer?



Tracking in 2-D space

What is your **updated (posterior) belief** of the 2-D location of the object after observing the measurement?

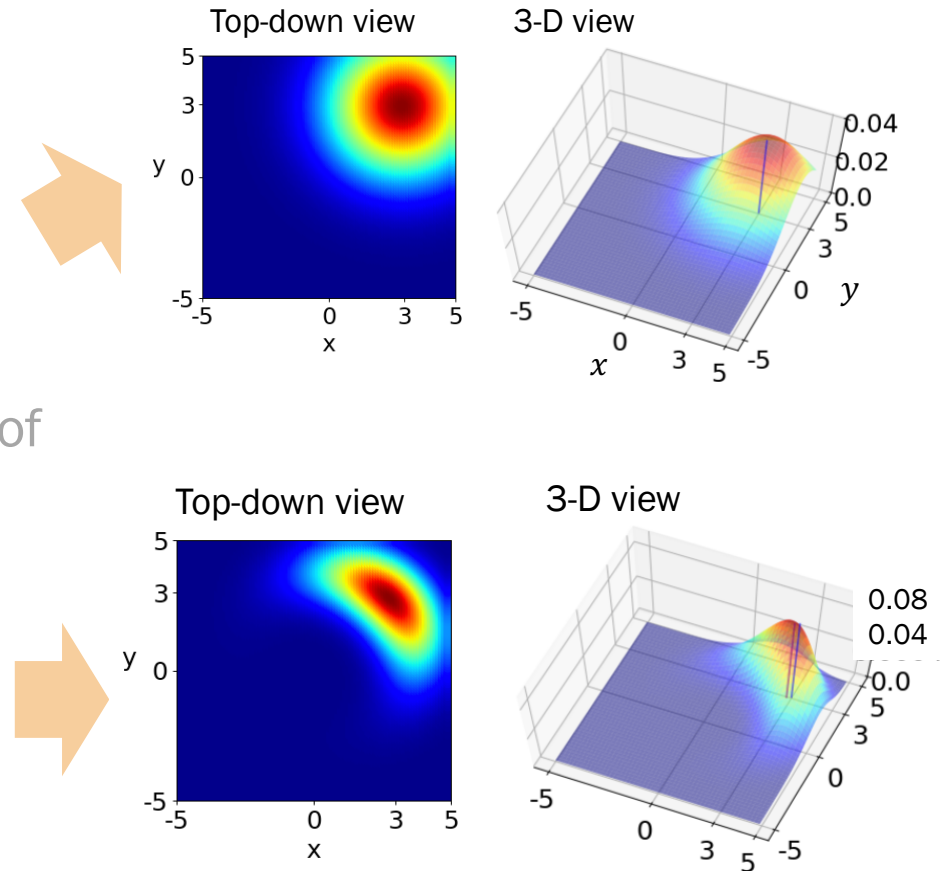
$$\begin{aligned} f_{X,Y|D}(X = x, Y = y | D = 4) &= \frac{\overset{\text{likelihood of } D = 4}{f_{D|X,Y}(D = 4 | X = x, Y = y)} \overset{\text{prior belief}}{f_{X,Y}(x, y)}}{f(D = 4)} \quad \text{Bayes' Theorem} \\ &= \frac{K_2 \cdot e^{-\frac{(4 - \sqrt{x^2 + y^2})^2}{2}} \cdot K_1 \cdot e^{-\frac{[(x-3)^2 + (y-3)^2]}{8}}}{f(D = 4)} \\ &= \frac{K_3 \cdot e^{-\left[\frac{(4 - \sqrt{x^2 + y^2})^2}{2} + \frac{[(x-3)^2 + (y-3)^2]}{8}\right]}}{f(D = 4)} \\ &= K_4 \cdot e^{-\left[\frac{(4 - \sqrt{x^2 + y^2})^2}{2} + \frac{[(x-3)^2 + (y-3)^2]}{8}\right]} \end{aligned}$$

Key: Once we know the part dependent on x, y , we can computationally approximate K_4 such that $f_{X,Y|D}$ is a valid PDF.

Tracking in 2-D space

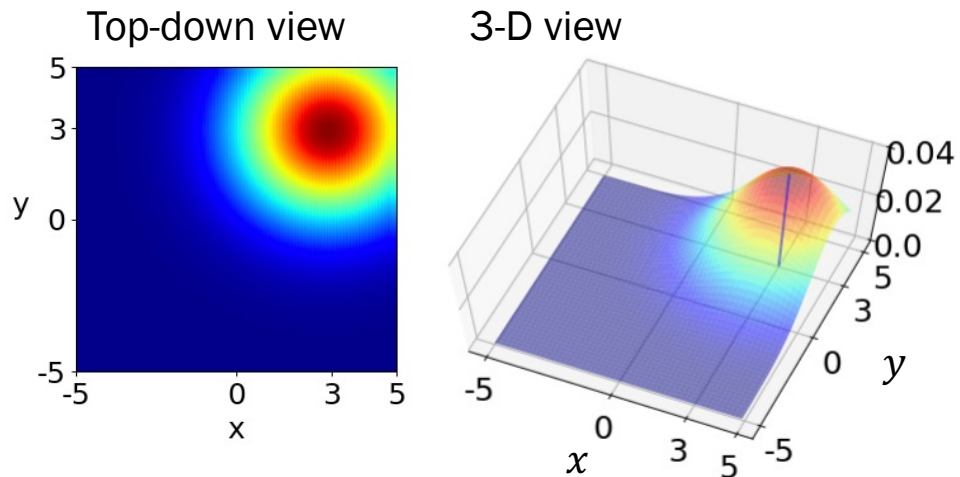
With this continuous version of Bayes' theorem, we can explore new domains.

- Before measuring, we have some **prior belief** about the 2-D location of an object, (X, Y) .
- We observe some noisy measurement of the distance of the object to a satellite.
- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?



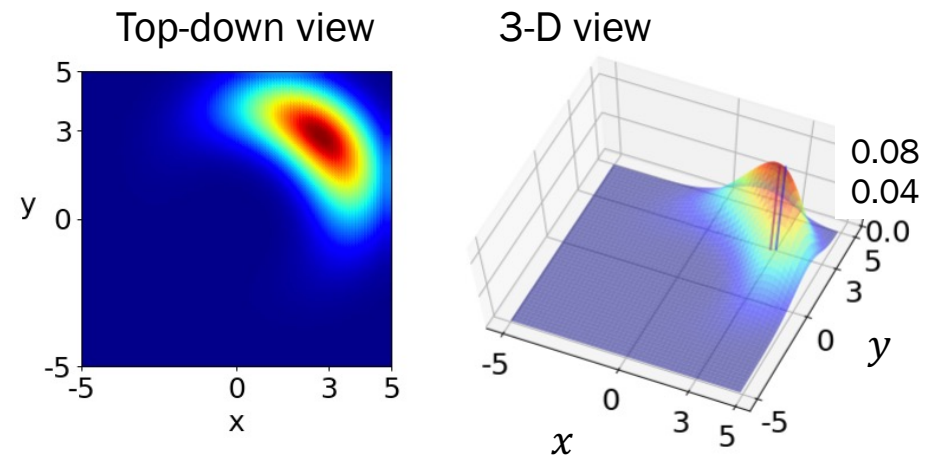
Tracking in 2-D space: Posterior belief

Prior belief



$$f_{X,Y}(x, y) = K_1 \cdot e^{-\frac{[(x-3)^2 + (y-3)^2]}{8}}$$

Posterior belief



$$f_{X,Y|D}(x, y|4) = K_4 \cdot e^{-\left[\frac{(4 - \sqrt{x^2 + y^2})^2}{2} + \frac{[(x-3)^2 + (y-3)^2]}{8}\right]}$$

How'd you compute that K_4 ?

To be a valid conditional PDF, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y|D}(x, y|4) dx dy = 1$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_4 \cdot e^{-\left[\frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]} dx dy = 1$$

➔ $\frac{1}{K_4} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left[\frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]} dx dy$ (pull out K_4 , divide)

Approximate:

$$\frac{1}{K_4} \approx \sum_y \sum_x e^{-\left[\frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]} \Delta x \Delta y$$
 Use a computer!