17: Continuous Joint Distributions (II)

Jerry Cain
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Convolution: Sum of independent Uniform RVs
Today’s lecture

Take what we’ve seen in **discrete** joint distributions...

...and translate them to **continuous** joint distributions!

For the most part, this is easy. For example:

\[
p_X(a) = \sum_y p_{X,Y}(a, y) \quad f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy
\]

\[
p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad f_{X,Y}(x, y) = f_X(x)f_Y(y)
\]

But some concepts, while mathematically accessible, are difficult to implement in practice.

We’ll focus on some of these today.

---

Goal of CS109 continuous joint distributions unit:
**build mathematical maturity**
Dance, Dance, Convolution

Recall that for independent discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_{k} P(X = k)P(Y = n - k)$$

the convolution of $p_X$ and $p_Y$
Dance, Dance, Convolution

Recall that for independent discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

the convolution of $p_X$ and $p_Y$

For independent continuous random variables $X$ and $Y$:

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x)f_Y(\alpha - x)dx$$

the convolution of $f_X$ and $f_Y$
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$

Isn’t this just one??

Not so fast...
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.

What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$

$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(\alpha - x) = \begin{cases} 1 & \text{if } 0 \leq \alpha - x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$\alpha$ is a constant in the integral w.r.t. $x$. 

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2022
Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0$  

$X$ and $Y$ independent + continuous

$f_{X+Y}(\alpha) = \int_{-\infty}^{\alpha} f_X(x) f_Y(\alpha - x) \, dx$

$f_X(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}
$

$f_Y(\alpha - x) = \begin{cases} 
1 & \text{if } \alpha - 1 \leq x \leq \alpha \\
0 & \text{otherwise}
\end{cases}
$
## Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0$ \hspace{1cm} 0

2. $\alpha = 1/2$ \hspace{1cm} 1/2

\[
X \text{ and } Y \text{ independent and continuous}
\]

\[
f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) \, dx
\]

\[
f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
f_Y(\alpha - x) = \begin{cases} 1 & \text{if } \alpha - 1 \leq x \leq \alpha \\ 0 & \text{otherwise} \end{cases}
\]

Integral = area under the curve
This curve = product of 2 functions of $x$
Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0$  
   $0$

2. $\alpha = 1/2$  
   $1/2$

3. $\alpha = 1$

4. $\alpha = 3/2$

5. $\alpha \geq 2$
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0$  
   0

2. $\alpha = 1/2$  
   1/2

3. $\alpha = 1$  
   1

4. $\alpha = 3/2$

5. $\alpha \geq 2$

\[
X \text{ and } Y \text{ independent and continuous} \quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) \, dx
\]

\[
f_X(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
f_Y(\alpha - x) = \begin{cases} 
1 & \text{if } \alpha - 1 \leq x \leq \alpha \\
0 & \text{otherwise}
\end{cases}
\]
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0$ \hspace{1cm} 0

2. $\alpha = 1/2$ \hspace{1cm} 1/2

3. $\alpha = 1$ \hspace{1cm} 1

4. $\alpha = 3/2$ \hspace{1cm} 1/2

5. $\alpha \geq 2$
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0 \quad 0$

2. $\alpha = 1/2 \quad 1/2$

3. $\alpha = 1 \quad 1$

4. $\alpha = 3/2 \quad 1/2$

5. $\alpha \geq 2 \quad 0$

$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(\alpha - x) = \begin{cases} 1 & \text{if } \alpha - 1 \leq x \leq \alpha \\ 0 & \text{otherwise} \end{cases}$

$X$ and $Y$ independent + continuous $f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x)f_Y(\alpha - x) \, dx$
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs. What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

1. $\alpha \leq 0$ 0
2. $\alpha = 1/2$ 1/2
3. $\alpha = 1$ 1
4. $\alpha = 3/2$ 1/2
5. $\alpha \geq 2$ 0

Let $X$ and $Y$ be independent and continuous RVs.

$$f_{X+Y}(\alpha) = \int_{-\infty}^\infty f_X(x) f_Y(\alpha - x) \, dx$$

$$f_{X+Y}(\alpha) = \begin{cases} \alpha & 0 \leq \alpha \leq 1 \\ 2 - \alpha & 1 \leq \alpha \leq 2 \\ 0 & \text{otherwise} \end{cases}$$
Dance, Dance, Convolution Extreme

Independent $X, Y$

$p(X = x)$

$p(Y = y)$

$\begin{align*}
\begin{array}{cccccccc}
X & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
p(x) & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\end{align*}$

$\begin{align*}
\begin{array}{cccccccc}
Y & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
p(y) & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\end{align*}$

$p(X + Y = n)$

$\begin{align*}
\begin{array}{cccccccc}
X + Y & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
\end{array}
\end{align*}$

Independent $X, Y$

$f_X(x)$

$f_Y(y)$

$\begin{align*}
\begin{array}{cccccccc}
x & 0 & 1 & 1 \\
\hline
f(x) & 0 & 1 & 0 \\
\end{array}
\end{align*}$

$\begin{align*}
\begin{array}{cccccccc}
y & 0 & 1 & 1 \\
\hline
f(y) & 0 & 1 & 0 \\
\end{array}
\end{align*}$

$\begin{align*}
\begin{array}{cccccccc}
\alpha & 0 & 1/2 & 1 & 3/2 & 2 \\
\hline
f_{X+Y}(\alpha) & 0 & 1/2 & 1 & 1/2 & 0 \\
\end{array}
\end{align*}$

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Sums of independent Normal RVs
Sum of independent Normals

\[ X \sim \mathcal{N}(\mu_1, \sigma_1^2), \]
\[ Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \]
\[ X, Y \text{ independent} \]

\[ X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \]

(proof left to [Wikipedia](https://en.wikipedia.org/))

Holds in general case:

\[ X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \]
\[ X_i \text{ independent for } i = 1, \ldots, n \]

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N} \left( \sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2 \right) \]
Back for another playoffs game

What is the probability that the Warriors win? How do you model zero-sum games?

\[ P(A_W > A_B) \]

This is a probability of an event involving \textit{two} random variables!

We will compute:

\[ P(A_W - A_B > 0) \]

A sum of Normals!
Motivating idea: Zero sum games

Want: \( P(\text{Warriors win}) = P(A_W - A_B > 0) \)

Assume \( A_W, A_B \) are independent.
Let \( D = A_W - A_B \).

What is the distribution of \( D \)?

A. \( D \sim \mathcal{N}(1657 - 1470, 200^2 - 200^2) \)
B. \( D \sim \mathcal{N}(1657 - 1470, 200^2 + 200^2) \)
C. \( D \sim \mathcal{N}(1657 + 1470, 200^2 + 200^2) \)
D. Dance, Dance, Convolution
E. None/other
Motivating idea: Zero sum games

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C. \( D \sim \mathcal{N}(1657 + 1470, 200^2 + 200^2) \)
D. Dance, Dance, Convolution
E. None/other

If \( X \sim \mathcal{N}(\mu_1, \sigma^2) \),
then \((-X) \sim \mathcal{N}(-\mu, (-1)^2 \sigma^2 = \sigma^2)\).
Motivating idea: Zero sum games

Want: \( P(\text{Warriors win}) = P(A_W - A_B > 0) \)

Assume \( A_W, A_B \) are independent.
Let \( D = A_W - A_B \).

\[
D \sim \mathcal{N}(1657 - 1470, 200^2 + 200^2) \\
\sim \mathcal{N}(187, 2 \cdot 200^2) \quad \sigma \approx 283
\]

\[
P(D > 0) = 1 - F_D(0) = 1 - \Phi\left(\frac{0 - 187}{283}\right) \\
\approx 0.7454
\]

Compare with 0.7488, calculated by sampling!
Ratio of PDFs
Relative probabilities of continuous random variables

Let $X = \text{time to finish problem set 4}$. 
Suppose $X \sim \mathcal{N}(10,2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?

\[
\frac{P(X = 10)}{P(X = 5)} = \quad \text{?}
\]

- **A.** $0/0 = \text{undefined}$
- **B.** \( \frac{f(10)}{f(5)} \)  
  \text{ratio of densities ??}
- **C.** stay healthy
Relative probabilities of continuous random variables

Let $X =$ time to finish problem set 4.
Suppose $X \sim \mathcal{N}(10,2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?

\[
\frac{P(X = 10)}{P(X = 5)} = \frac{f(10)}{f(5)}
\]

A. $0/0 = \text{undefined}$
B. $\frac{f(10)}{f(5)}$
C. stay healthy
Relative probabilities of continuous random variables

Let $X = \text{time to finish problem set 4}$. Suppose $X \sim \mathcal{N}(10,2)$. How much more likely are you to complete in 10 hours than 5 hours?

\[
\frac{P(X = 10)}{P(X = 5)} = \frac{f(10)}{f(5)}
\]

Therefore

\[
\frac{P(X = a)}{P(X = b)} = \frac{\varepsilon f(a)}{\varepsilon f(b)} = \frac{f(a)}{f(b)}
\]

\[
\begin{align*}
\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(10 - \mu)^2}{2\sigma^2}} &= \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(5 - \mu)^2}{2\sigma^2}} = \frac{e^0}{e^{-\frac{25}{4}}} = 518
\end{align*}
\]

Ratios of PDFs are meaningful!
Continuous conditional distributions
Continuous conditional distributions

For continuous RVs $X$ and $Y$, the conditional PDF of $X$ given $Y$ is

\[ f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \]

where $f_Y(y) > 0$

Intuition: $P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} \quad \leftrightarrow \quad f_{X|Y}(x|y) \epsilon_x = \frac{f_{X,Y}(x, y) \epsilon_x \epsilon_Y}{f_Y(y) \epsilon_Y}$

Note that conditional PDF $f_{X|Y}$ is a "true" density:

\[
\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x, y)}{f_Y(y)}dx = \frac{f_Y(y)}{f_Y(y)} = 1
\]
Why sums of random variables?

Sometimes modeling and understanding a complex RV, $X$, is difficult. But if we can decompose $X$ into the sum of independent simpler RVs,

- We can then compute distributions on $X$.
- We can then understand how $X$ changes as its parts change.

What can we model with a triangular PDF?

We’re covering the reverse direction for now; the forward direction will come on Friday.
Everything* in probability is a sum or a product (or both)

*except conditional probability (a ratio)

**Sum** of values that can be considered separately (possibly weighted by prob. of happening)

\[ E[X] = \sum_{x} x p(x) \]

\[ E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \]

\[ P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i) \]

Law of Total Probability

\[ P(E) = \sum_{i=1}^{n} P(E_i) \]

Axiom 3, \( E = E_1 \cup \cdots \cup E_n \)

**Product** of values that can each be considered in sequence

\[ P(E \cap F \cap G) = P(E)P(F|E)P(G|EF) \]

Chain Rule

\[ f_{X,Y}(x,y) = f_X(x)f_Y(y) \]

Independent cont. RVs

\[ P(X + Y = n) = \sum_{k} P(X = k)P(Y = n - k) \]

Sum of indep. discrete RVs (convolution)
Conditional probability and Bayes’ Theorem

Definition

\[ P(F|E) = \frac{P(E \cap F)}{P(E)} \]

**Scaling** to the correct sample space

Bayes’ Theorem

\[ P(F|E) = \frac{P(F)P(E|F)}{P(E)} \]

**Prior**: some prob. of event \( F \)

**Likelihood**

**Posterior**: prob. of \( F \) knowing that \( E \) happened

**Scaling** to the correct sample space

Independence

\( E, F \) independent

\[ P(F|E) = P(F) \]

Sample space doesn’t need to be scaled
Multiple Bayes’ Theorems

\[
P(F|E) = \frac{P(F)P(E|F)}{P(E)}
\]

\[
p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)}
\]

\[
f_{Y|X}(y|x) = \frac{f_Y(y)f_{X|Y}(x|y)}{f_X(x)}
\]

Really all the same idea!

You are given this value...

...so this is just a scalar
Intense Exercise
You want to know the 2-D location of an object.

Your satellite ping gives you a noisy 1-D measurement of the distance of the object from the satellite (0,0).

Using the satellite measurement, where is the object?
Tracking in 2-D space

• Before measuring, we have some **prior belief** about the 2-D location of an object, \((X, Y)\).

• We observe some noisy **measurement** \(D = 4\), the Euclidean distance of the object to a satellite.

• After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?
Tracking in 2-D space

- You have a **prior belief** about the 2-D location of an object, \((X, Y)\).
- You observe a **noisy distance measurement**, \(D = 4\).
- What is your **updated (posterior) belief** of the 2-D location of the object after observing the measurement?

Recall Bayes terminology:

\[
    f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y)f_{X,Y}(x, y)}{f_D(d)}
\]

- **likelihood (of evidence)**
- **prior belief**
- **normalization constant**
1. Define prior

You have a **prior belief** about the 2-D location of an object, \((X, Y)\).

Let \((X, Y) = \text{object’s 2-D location, assuming satellite is at (0,0)}\)

Suppose the prior distribution is a symmetric bivariate normal distribution:

\[
f_{X,Y}(x, y) = \frac{1}{2\pi \sigma^2} e^{-\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2\sigma^2}} = K_1 \cdot e^{-\frac{(x-3)^2 + (y-3)^2}{8}}
\]

(normalizing constant)

\[
f_{X,Y\mid D}(x, y \mid d) = \frac{f_{D\mid X,Y}(d \mid x, y) f_{X,Y}(x, y)}{f_D(d)}
\]
2. Define likelihood

You observe a noisy distance measurement, \( D = 4 \).

If you knew your actual location \((x, y)\), you could say how likely a measurement \( D = 4 \) is:

Let \( D \) = distance from the satellite (radially). Suppose you knew your actual position: \((x, y)\).

- \( D \) is still noisy! Suppose noise is standard normal.
- On average, \( D \) is your true Euclidean distance: \( \sqrt{x^2 + y^2} \)
2. Define likelihood

You observe a **noisy distance measurement**, \( D = 4 \).

If you knew your actual location \((x, y)\), you could say **how likely** a measurement \( D = 4 \) is:

Let \( D \) = distance from the satellite (radially). Suppose you knew your actual position: \((x, y)\).

- \( D \) is still noisy! Suppose noise is **standard normal**.
- On average, \( D \) is your true Euclidean distance: \( \sqrt{x^2 + y^2} \)

\[
D | X, Y \sim \mathcal{N}(\mu = (A), \sigma^2 = (B))
\]

\[
f_{D|X,Y}(D = d | X = x, Y = y) = \frac{1}{(C)} \sqrt{2\pi} e^{\{ (D) \}}
\]
2. Define likelihood

You observe a noisy distance measurement, \( D = 4 \).

If you knew your actual location \((x, y)\), you could say how likely a measurement \( D = 4 \) is:

Let \( D \) = distance from the satellite (radially).

Suppose you knew your actual position: \((x, y)\).

- \( D \) is still noisy! Suppose noise is **standard normal**.
- On average, \( D \) is your true Euclidean distance: \( \sqrt{x^2 + y^2} \)

\[
D \mid X, Y \sim N \left( \mu = \sqrt{x^2 + y^2}, \sigma^2 = 1 \right)
\]

\[
f_{D \mid X, Y}(D = d \mid X = x, Y = y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d-\sqrt{x^2+y^2})^2}{2}} = K_2 \cdot e^{-\frac{(d-\sqrt{x^2+y^2})^2}{2}}
\]

normalizing constant
3. Compute posterior

What is your **updated (posterior) belief** of the 2-D location of the object after observing the measurement?

Compute:

**Posterior belief**

\[ f_{X,Y|D}(x, y|d) = f_{X,Y|D}(X = x, Y = y|D = 4) \]
3. Compute posterior

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Compute:

\[ f_{X,Y|D}(x, y|4) = f_{X,Y|D}(X = x, Y = y|D = 4) \]

Know:

Prior belief
\[ f_{X,Y}(x, y) = K_1 \cdot e^{-\frac{[(x-3)^2 + (y-3)^2]}{8}} \]

Observation likelihood
\[ f_{D|X,Y}(d|x, y) = K_2 \cdot e^{\frac{-(d-\sqrt{x^2+y^2})^2}{2}} \]

Tips
• Use Bayes’ Theorem!
• \( f_D(4) \) is just a scaling constant. Why?
• How can we approximate the final scaling constant with a computer?
Tracking in 2-D space

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

\[
f_{X,Y|D}(X = x, Y = y|D = 4) = \frac{f_{D|X,Y}(D = 4|X = x, Y = y)f_{X,Y}(x, y)}{f(D = 4)}
\]

Bayes' Theorem

\[
= K_2 \cdot e^{-\frac{(4-\sqrt{x^2+y^2})^2}{2}} \cdot K_1 \cdot e^{-\frac{(x-3)^2+(y-3)^2}{8}}
\]

\[
= K_3 \cdot e^{-\frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{(x-3)^2+(y-3)^2}{8}}
\]

\[
= K_4 \left( e^{-\frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{(x-3)^2+(y-3)^2}{8}} \right)
\]

Key: Once we know the part dependent on \(x, y\), we can computationally approximate \(K_4\) such that \(f_{X,Y|D}\) is a valid PDF.
Tracking in 2-D space

With this continuous version of Bayes’ theorem, we can explore new domains.

• Before measuring, we have some prior belief about the 2-D location of an object, \((X, Y)\).

• We observe some noisy measurement of the distance of the object to a satellite.

• After the measurement, what is our updated (posterior) belief of the 2-D location of the object?
Tracking in 2-D space: Posterior belief

Prior belief

Top-down view

3-D view

$$f_{X,Y}(x, y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

Posterior belief

Top-down view

3-D view

$$f_{X,Y|D}(x, y|4) = K_4 \cdot e^{-\left[\frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$
How'd you compute that $K_4$?

To be a valid conditional PDF, \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y|D}(x, y |4) \, dx \, dy = 1 \)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_4 \cdot e^{-\left[ \frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{(x-3)^2 + (y-3)^2}{8} \right]} \, dx \, dy = 1
\]

\[
\frac{1}{K_4} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left[ \frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{(x-3)^2 + (y-3)^2}{8} \right]} \, dx \, dy \tag{pull out $K_4$, divide}
\]

Approximate:

\[
\frac{1}{K_4} \approx \sum_x \sum_y e^{-\left[ \frac{(4-\sqrt{x^2+y^2})^2}{2} + \frac{(x-3)^2 + (y-3)^2}{8} \right]} \Delta x \Delta y \tag{Use a computer!}
\]