# 17: Continuous Joint Distributions (II)

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satellite

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# Convolution: Sum of independent Uniform RVs

# Today's lecture

Take what we've seen in discrete joint distributions…

…and translate them to continuous joint distributions!

For the most part, this is easy. For example:

| | | |
|---|---|---|
| Marginal distributions | $p_X(a) = \sum_y p_{X,Y}(a, y)$ | $f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy$ |
| Independent RVs | $p_{X,Y}(x, y) = p_X(x) p_Y(y)$ | $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ |

But some concepts, while mathematically accessible, are difficult to implement in practice.

We'll focus on some of these today.

Goal of CS109 continuous joint distributions unit:
**build mathematical maturity**

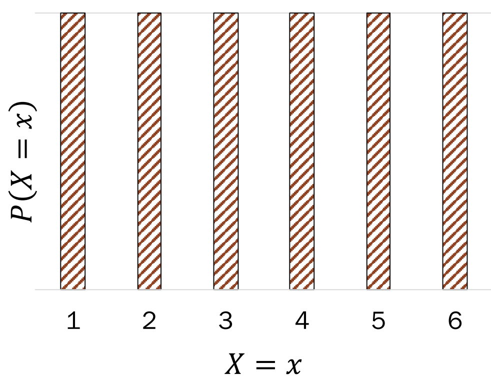
# Dance, Dance, Convolution

Review

Recall that for independent discrete random variables $X$ and $Y$:

$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of $p_X$ and $p_Y$

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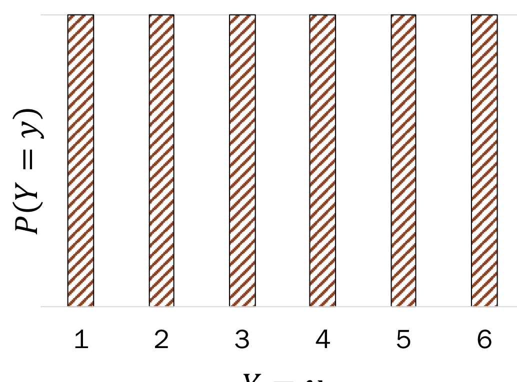


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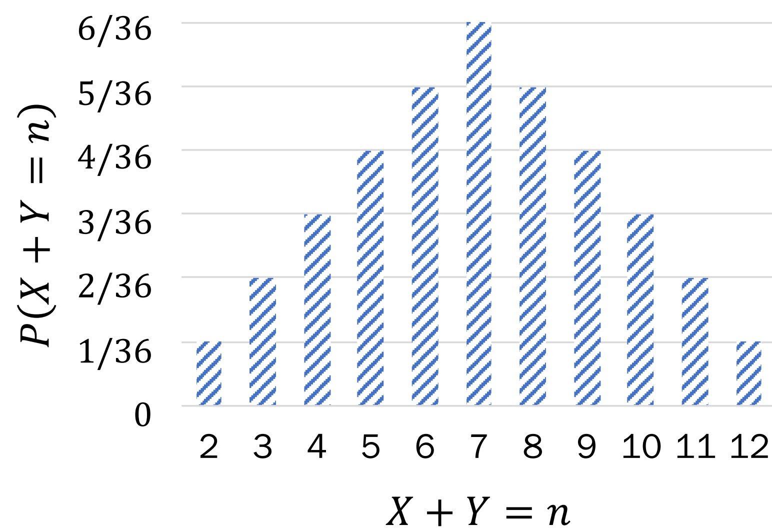



Independent $X, Y$

# Dance, Dance, Convolution

Recall that for independent discrete random variables $X$ and $Y$:

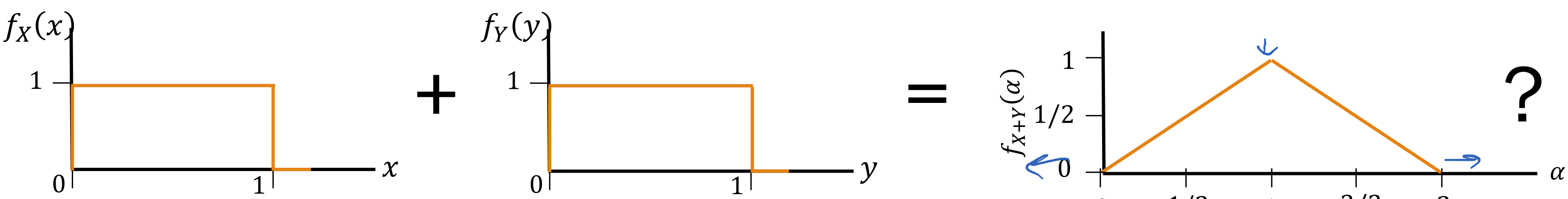
$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of $p_X$ and $p_Y$

For independent continuous random variables $X$ and $Y$:

$$f_{X+Y}(\alpha)=\int_{-\infty}^{\infty} f_X(x)f_Y(\alpha-x)dx$$

the convolution of $f_X$ and $f_Y$

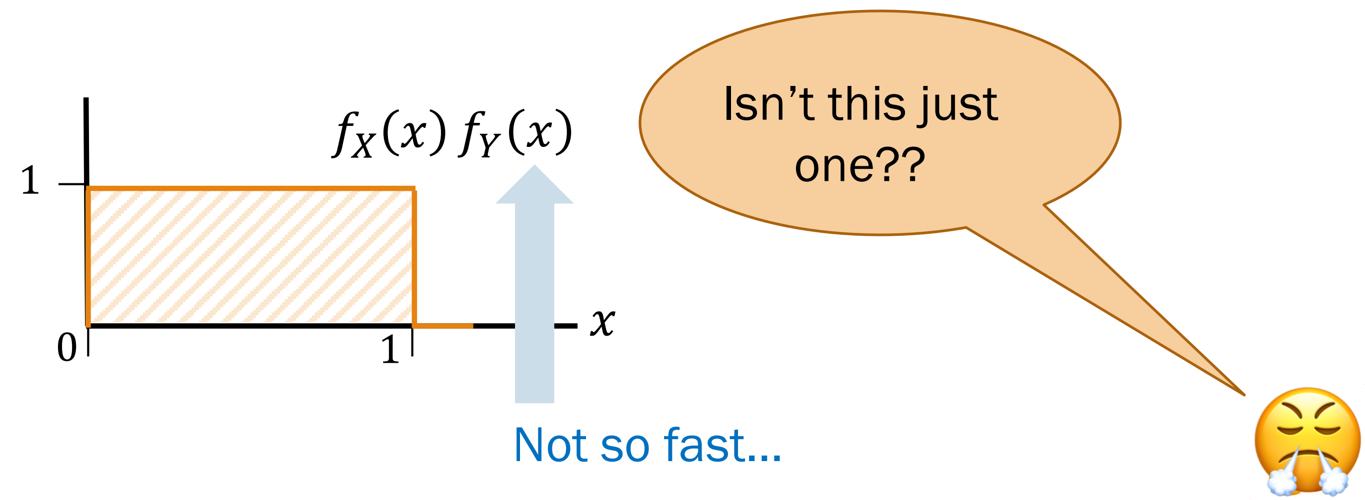



Independent $X, Y$

# Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.
What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$



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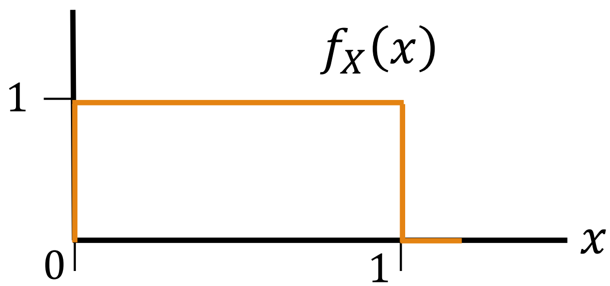


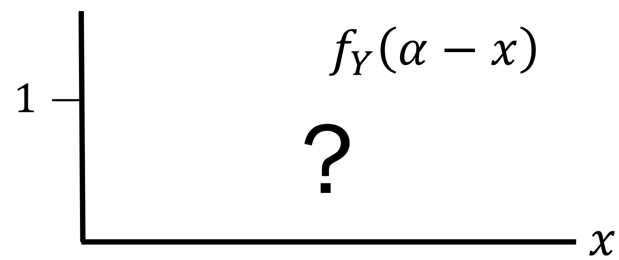


$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(\alpha - x) = \begin{cases} 1 & \text{if } 0 \le \alpha - x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \alpha - 1 \le x \le \alpha \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(\alpha - x)$

$\alpha$ is a constant in the integral w.r.t. $x$.

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$X$ and $Y$ independent + continuous $f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

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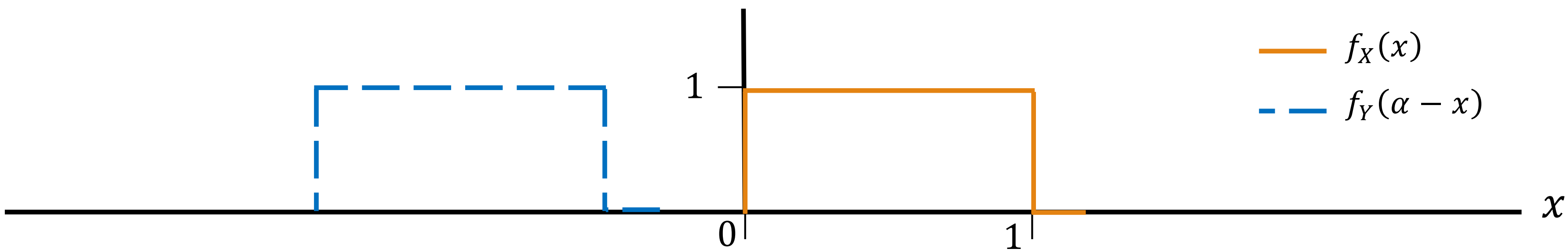
1. $\alpha \le 0$     0



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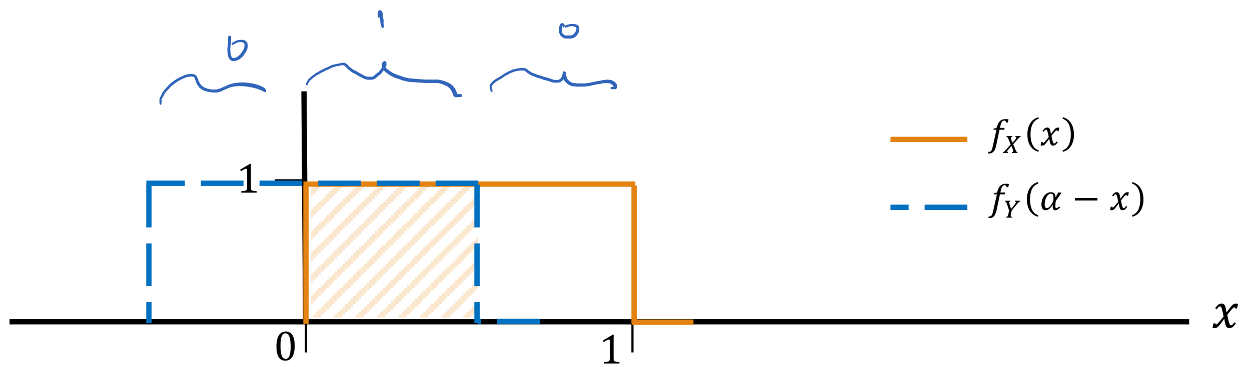


Integral = area under the curve
This curve = product of 2 functions of $x$

# Sum of independent Uniforms

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3. $\alpha = 1$
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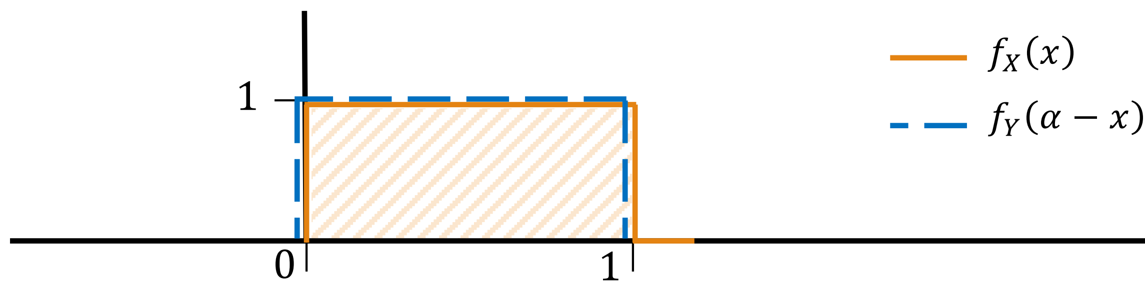

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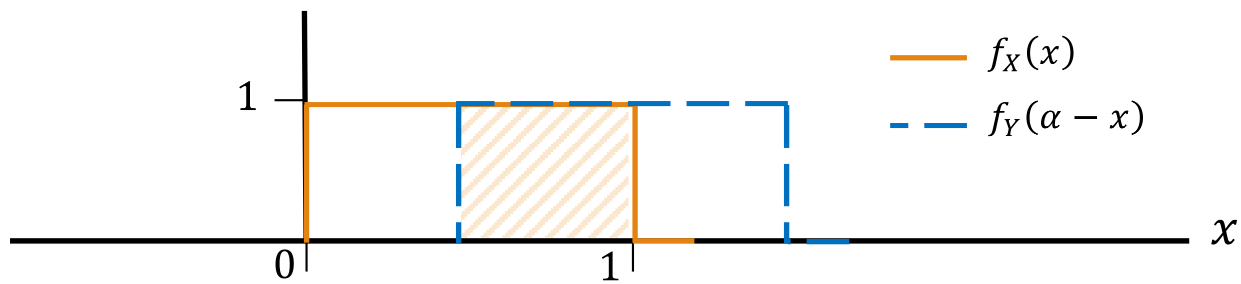

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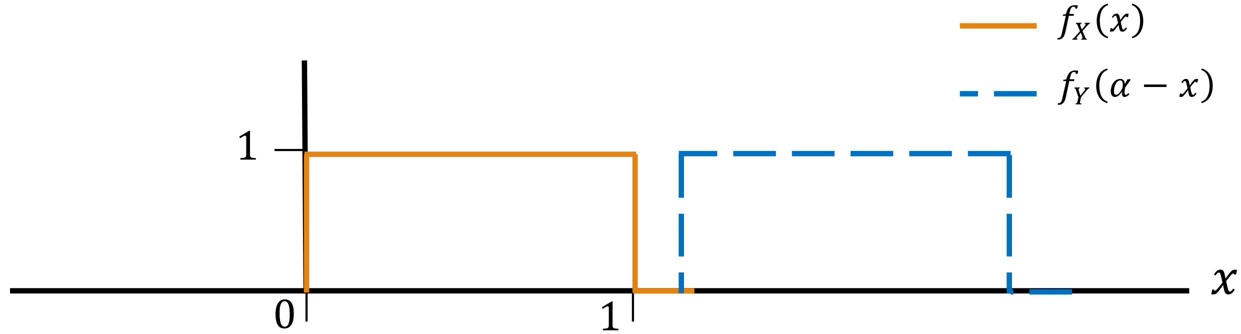

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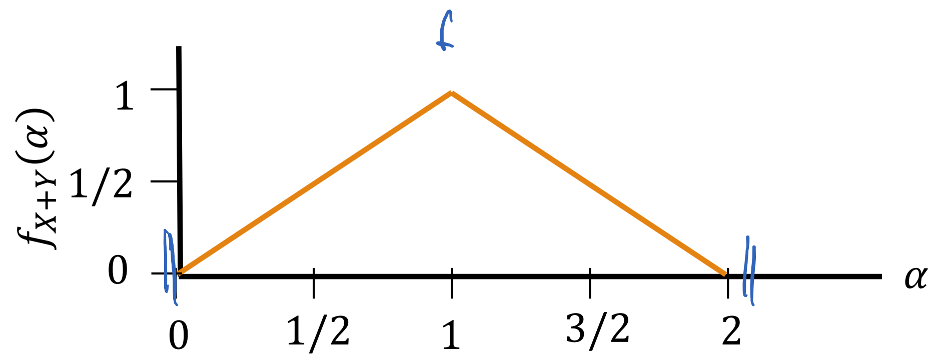
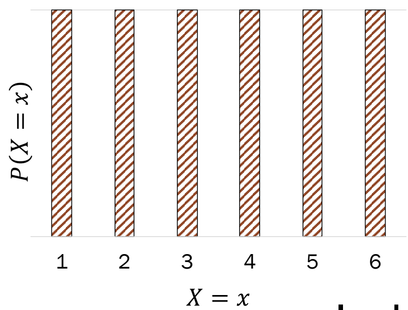



$$f_{X+Y}(\alpha) = \begin{cases} \alpha & 0 \leq \alpha \leq 1 \\ 2 - \alpha & 1 \leq \alpha \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

# Dance, Dance, Convolution Extreme

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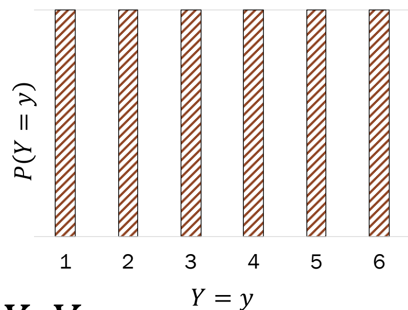


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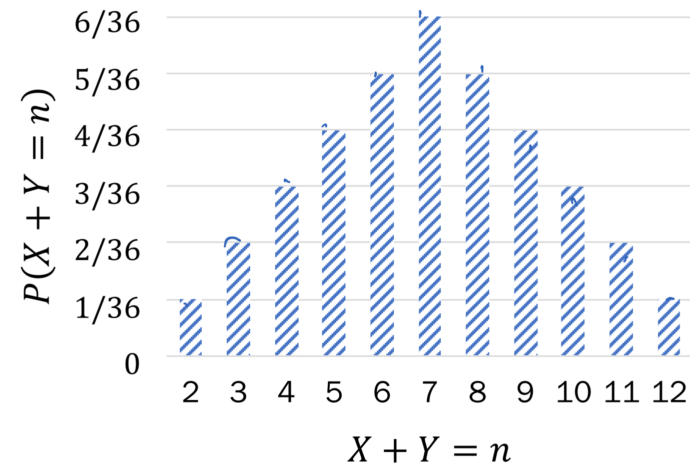
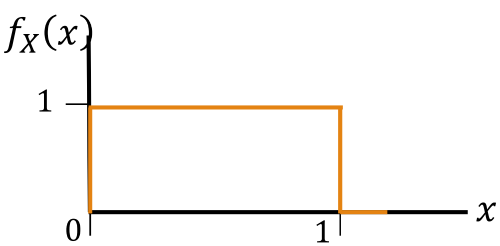



Independent $X, Y$

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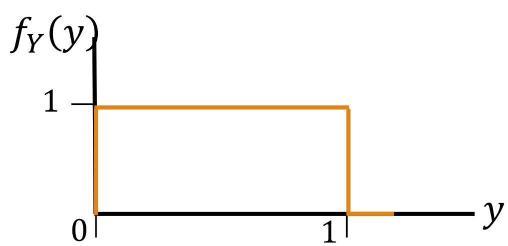


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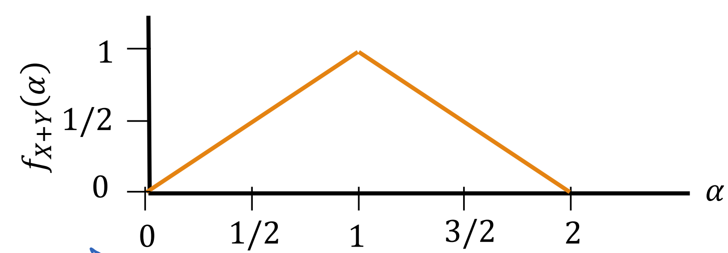



Independent $X, Y$

# Sums of independent Normal RVs

# Sum of independent Normals

$X \sim \mathcal{N}(\mu_1, \sigma_1^2)$,
$Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$
$X, Y$ independent

$$\Rightarrow \quad \underbrace{X + Y}_{\text{RV}} \sim \mathcal{N}(\underbrace{\mu_1 + \mu_2}_{\mu}, \underbrace{\sigma_1^2 + \sigma_2^2}_{\sigma^2})$$

(proof left to Wikipedia)

## Holds in general case:

$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$
$X_i$ independent for $i = 1, \ldots, n$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

# Back for another playoffs game


What is the probability that the Warriors win?
How do you model zero-sum games?

$$P(A_W > A_B)$$

This is a probability of an event involving ***two*** random variables!

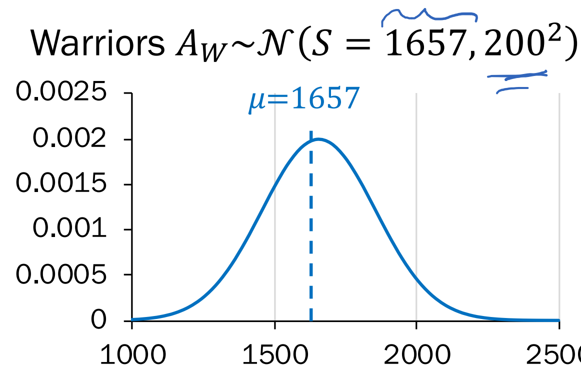
We will compute:

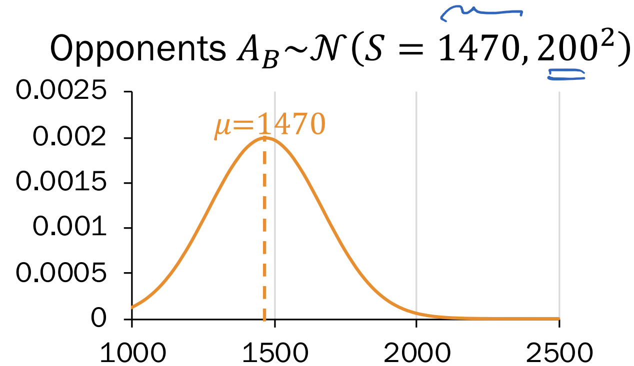
$$P(\underbrace{A_W - A_B}_{\text{A sum of Normals!}} > 0)$$

# Motivating idea: Zero sum games

Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

Assume $A_W, A_B$ are independent.
Let $D = A_W - A_B$.




What is the distribution of $D$?

A. $D \sim \mathcal{N}(1657 - 1470,\ 200^2 - 200^2)$
B. $D \sim \mathcal{N}(1657 - 1470,\ 200^2 + 200^2)$
C. $D \sim \mathcal{N}(1657 + 1470,\ 200^2 + 200^2)$
D. Dance, Dance, Convolution
E. None/other

# Motivating idea: Zero sum games

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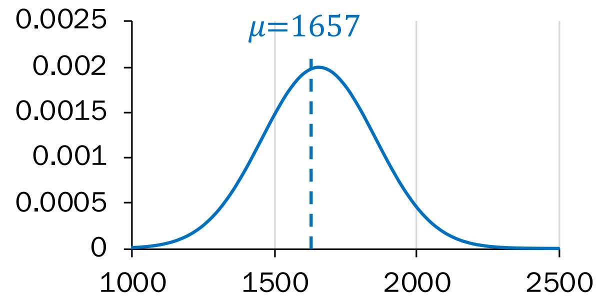
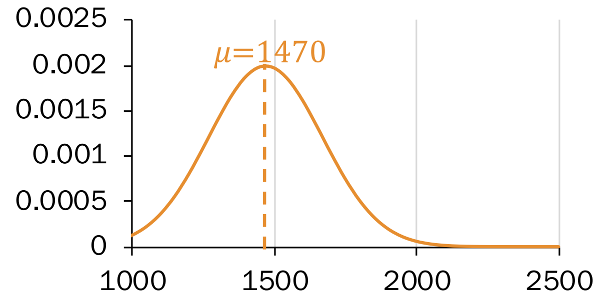
Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$




Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$




If $X \sim \mathcal{N}(\mu_1, \sigma^2)$,
then $(-X) \sim \mathcal{N}(-\mu, (-1)^2\sigma^2 = \sigma^2)$.

# Motivating idea: Zero sum games

Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

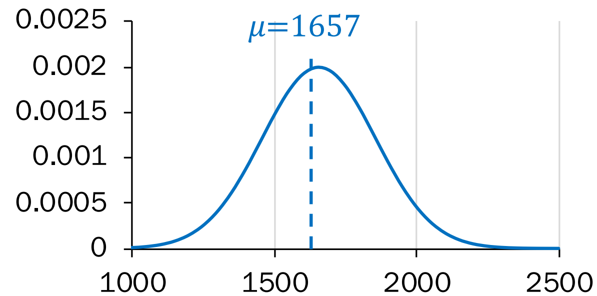
Assume $A_W, A_B$ are independent.
Let $D = A_W - A_B$.

$$D \sim \mathcal{N}(1657 - 1470,\ \ 200^2 + 200^2)$$
$$\sim \mathcal{N}(187, 2 \cdot 200^2) \quad \sigma \approx 283$$

$$P(D > 0) = 1 - F_D(0) = 1 - \Phi\left(\frac{0 - 187}{283}\right)$$
$$\approx 0.7454$$

Compare with 0.7488, calculated by sampling!

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$

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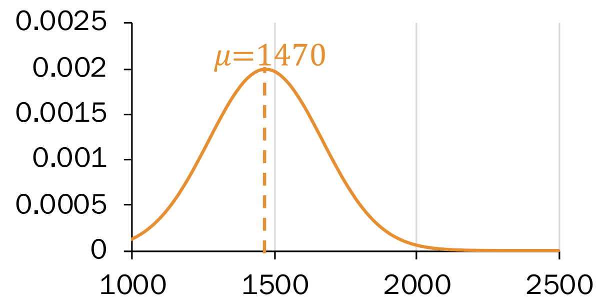


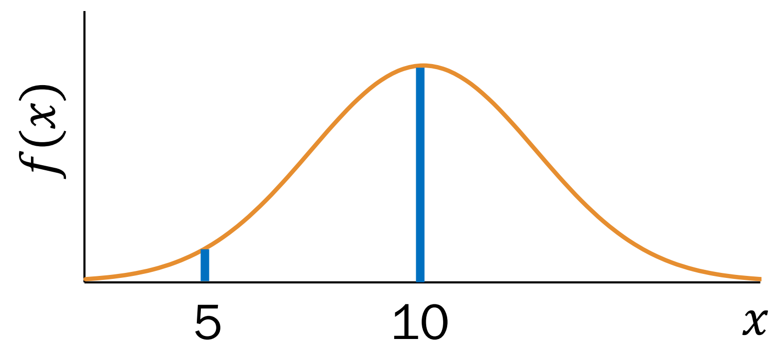
# Ratio of PDFs

# Relative probabilities of continuous random variables

Let $X$ = time to finish problem set 4.

Suppose $X \sim \mathcal{N}(10, 2)$.

How much *more likely* are you to complete in 10 hours than 5 hours?




$$\frac{P(X = 10)}{P(X = 5)} =$$

A. 0/0 = undefined

B. $\dfrac{f(10)}{f(5)}$ ratio of densities ??

C. stay healthy

# Relative probabilities of continuous random variables

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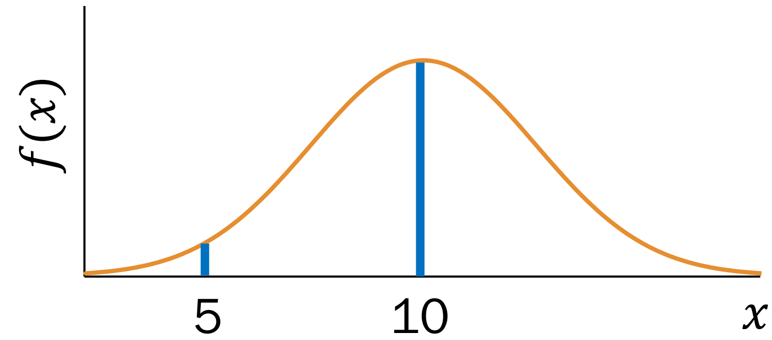


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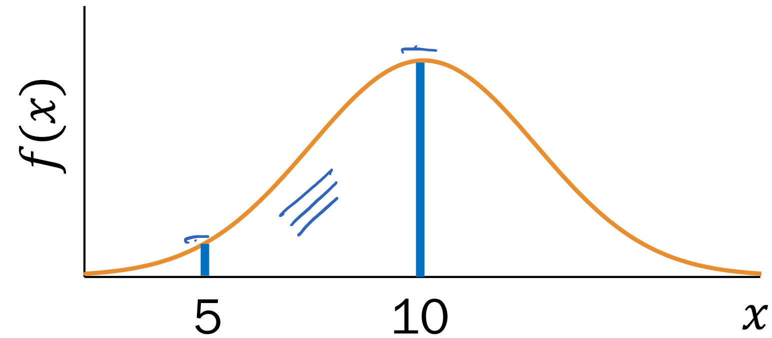



$$\frac{P(X=10)}{P(X=5)} = \frac{f(10)}{f(5)}$$

$$P(X=a) = P\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \approx \varepsilon f(a)$$

Therefore $\frac{P(X=a)}{P(X=b)} = \frac{\varepsilon f(a)}{\varepsilon f(b)} = \frac{f(a)}{f(b)}$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}} = \frac{e^{-\frac{(10-10)^2}{2\cdot 2}}}{e^{-\frac{(5-10)^2}{2\cdot 2}}} = \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

Ratios of PDFs are meaningful!

# Continuous conditional distributions

# Continuous conditional distributions

For continuous RVs $X$ and $Y$, the **conditional PDF** of $X$ given $Y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

where $f_Y(y) > 0$

Intuition: $P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$ ⟺ $f_{X|Y}(x|y)\varepsilon_X = \frac{f_{X,Y}(x,y)\varepsilon_X\varepsilon_Y}{f_Y(y)\varepsilon_Y}$

Note that conditional PDF $f_{X|Y}$ is a "true" density:

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

# Why sums of random variables?

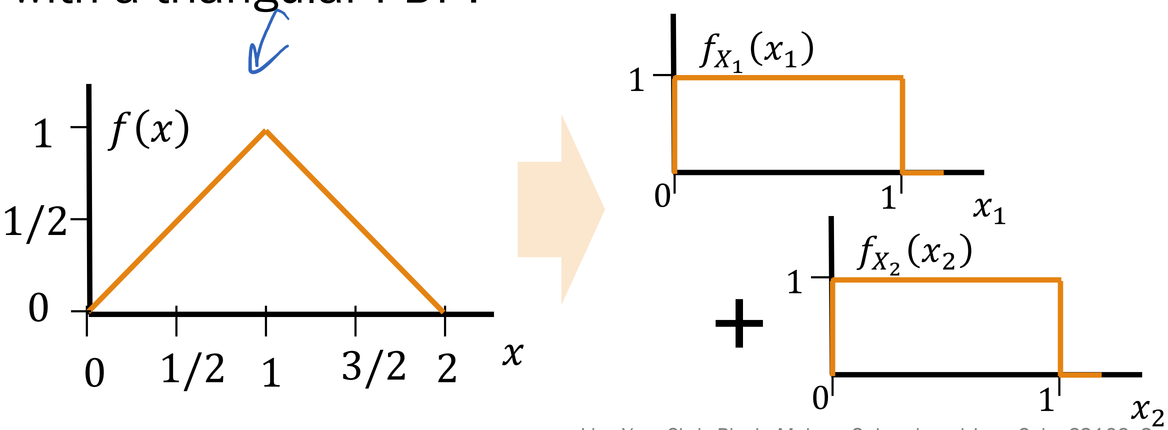
Sometimes modeling and understanding a complex RV, $X$, is difficult.

But if we can decompose $X$ into the **sum of independent simpler** RVs,

- We can then compute distributions on $X$.
- We can then to understand how $X$ changes as its parts change.

What can we model with a triangular PDF?

Sum of uniforms!




We're covering the reverse direction for now; the forward direction will come on Friday

# Everything* in probability is a sum or a product (or both)

*except conditional probability (a ratio)

**Sum** of values that can be considered separately (possibly weighted by prob. of happening)

$$E[X] = \sum_x x\underbrace{p(x)}_{\text{weight}}$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \underbrace{f_{X|Y}(x|y)}_{\text{weight}} dx$$

$$P(E) = \sum_{i=1}^{n} P(E|F_i)\underbrace{P(F_i)}_{\text{weight}}$$

Law of Total Probability

$$P(E) = \sum_{i=1}^{n} P(E_i)$$

Axiom 3, $E = E_1 \cup \cdots \cup E_n$

**Product** of values that can each be considered in sequence

$$P(E \cap F \cap G) = P(E)P(F|E)P(G|EF)$$

Chain Rule

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Independent cont. RVs

$$P(X+Y=n) = \sum_k P(X=k)P(Y=n-k)$$

Sum of indep. discrete RVs (convolution)

# Conditional probability and Bayes' Theorem

Definition

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

**Scaling** to the correct sample space

Independence

$E, F$ independent

$$P(F|E) = P(F)$$

Sample space doesn't need to be scaled

Bayes' Theorem

$$P(F|E) = \frac{P(F)P(E|F)}{P(E)}$$

**Prior**: some prob. of event $F$

**Likelihood**

**Posterior**: prob. of $F$ knowing that $E$ happened

**Scaling** to the correct sample space

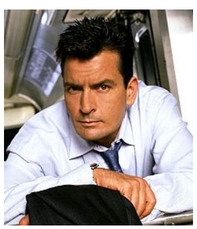
# Multiple Bayes' Theorems

with
events

$$P(F|E) = \frac{P(F)P(E|F)}{P(E)}$$

with
discrete RVs

$$p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)}$$



with
continuous RVs

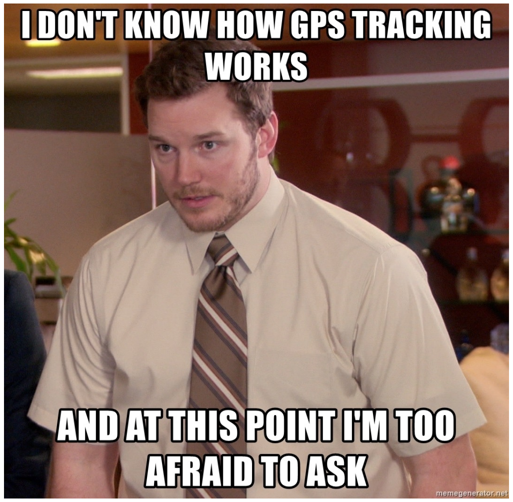
You are given
this value...

$$f_{Y|X}(y|x) = \frac{f_Y(y)f_{X|Y}(x|y)}{f_X(x)}$$

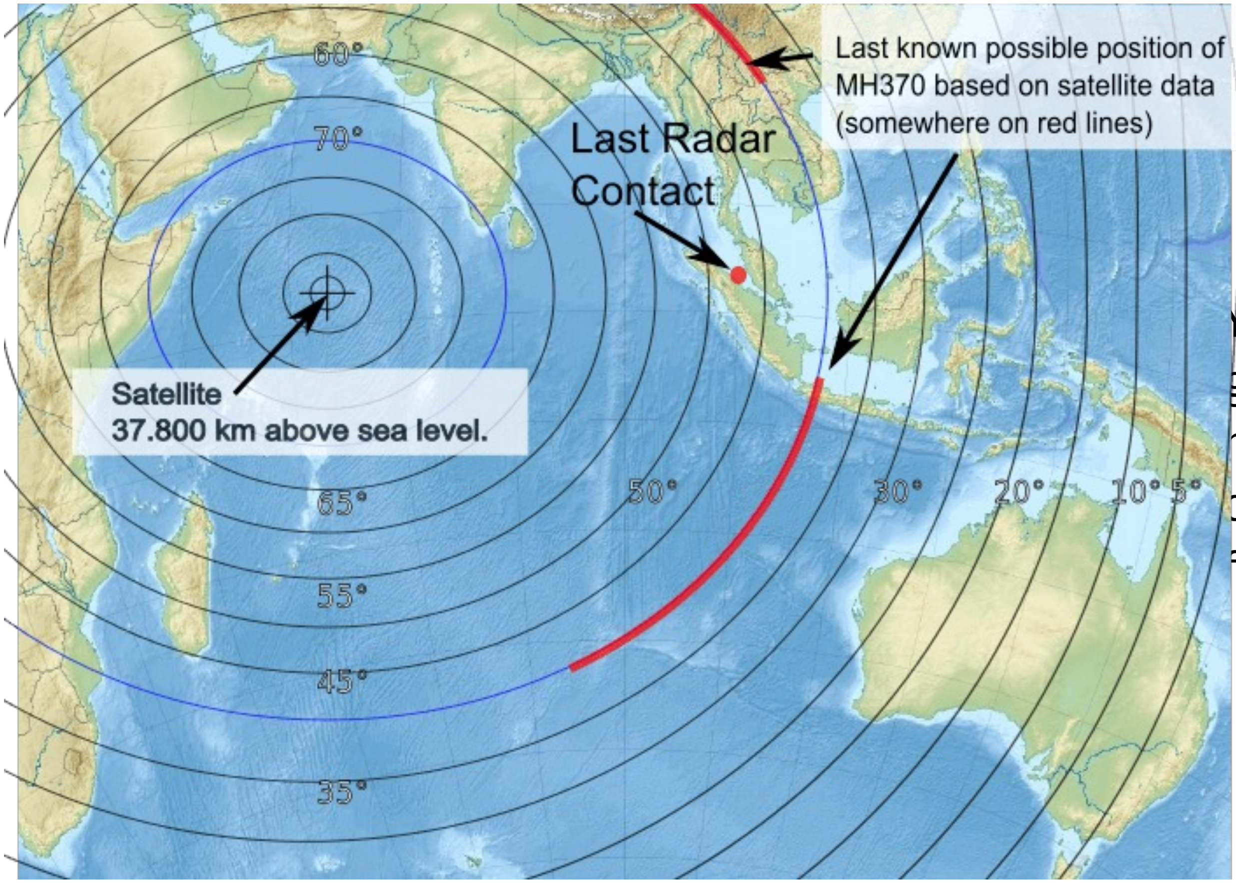
...so this is just a scalar

Really all the same idea!

# Intense Exercise

# Tracking in 2-D space



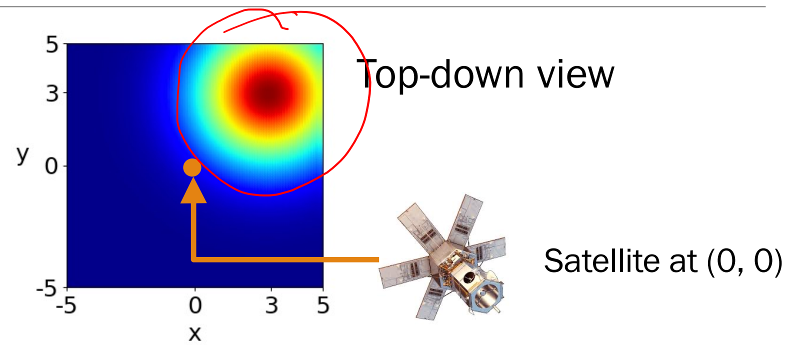

You want to know the 2-D location of an object.

Your satellite ping gives you a noisy 1-D measurement of the distance of the object from the satellite (0,0).

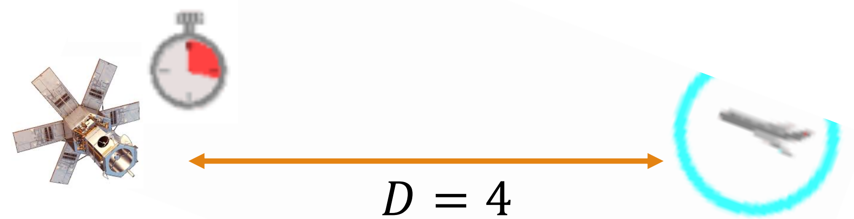
Using the satellite measurement, where is the object?

# Tracking in 2-D space

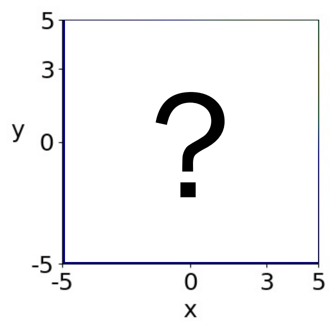
- Before measuring, we have some **prior belief** about the 2-D location of an object, $(X, Y)$.




- We observe some noisy **measurement** $D = 4$, the Euclidean distance of the object to a satellite.




- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?



# Tracking in 2-D space

- You have a prior belief about the 2-D location of an object, $(X, Y)$.
- You observe a noisy distance measurement, $D = 4$.
- What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Recall Bayes terminology:

posterior belief; likelihood (of evidence); prior belief

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

normalization constant

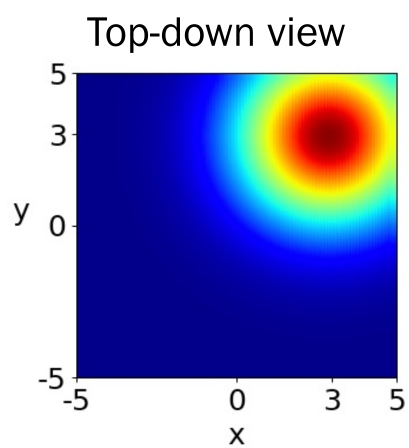
# 1. Define prior

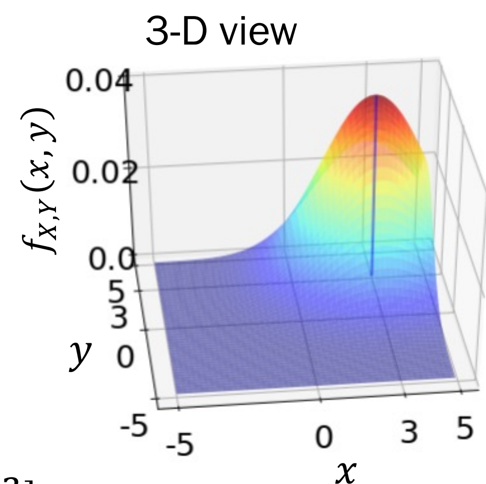
$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\, f_{X,Y}(x,y)}{f_D(d)}$$

You have a prior belief about the 2-D location of an object, $(X, Y)$.

Let $(X, Y)$ = object's 2-D location, assuming satellite is at (0,0)

Suppose the prior distribution is a symmetric bivariate normal distribution:



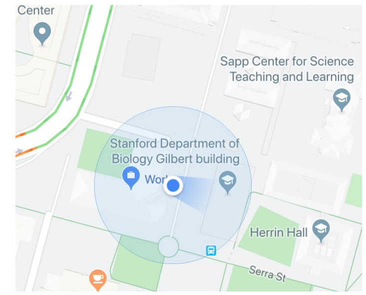

$$f_{X,Y}(x,y) = \frac{1}{2\pi 2^2} e^{-\frac{[(x-3)^2+(y-3)^2]}{2(2^2)}} = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

normalizing constant

# 2. Define likelihood

f(d|x,y)

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\, f_{X,Y}(x,y)}{f_D(d)}$$

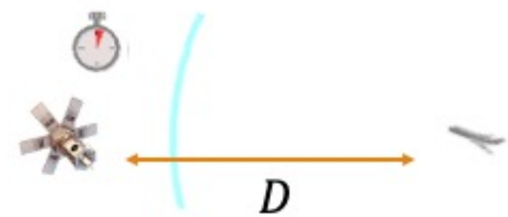



You observe a noisy distance measurement, $D = 4$.

If you knew your actual location $(x, y)$, you could say how likely a measurement $D = 4$ is:

Let $D$ = distance from the satellite (radially).
Suppose you knew your actual position: $(x, y)$.

- $D$ is still noisy! Suppose noise is **standard normal.**
- On average, $D$ is your true Euclidean distance: $\sqrt{x^2 + y^2}$



# 2. Define likelihood

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\, f_{X,Y}(x,y)}{f_D(d)}$$

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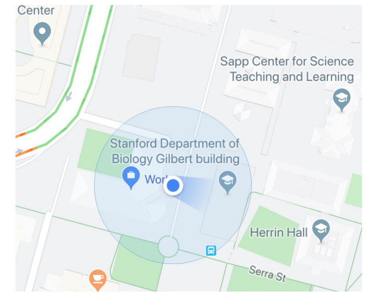


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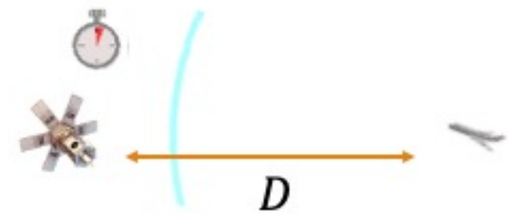
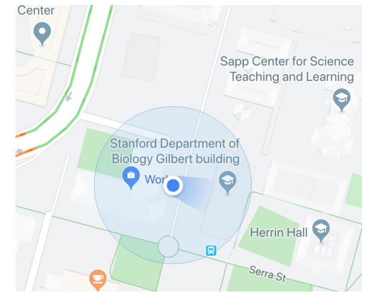



$$D|X,Y \sim N(\mu = \ (A)\ , \sigma^2 = \ (B)\ )$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{(C)\ \sqrt{2\pi}} e^{\{\quad (D) \quad\}}$$

# 2. Define likelihood

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\, f_{X,Y}(x,y)}{f_D(d)}$$




You observe a noisy distance measurement, $D = 4$.

If you knew your actual location $(x, y)$, you could say how likely a measurement $D = 4$ is:

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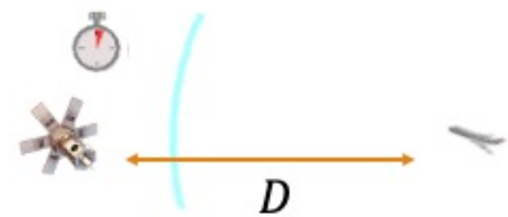
- On average, $D$ is your true Euclidean distance: $\sqrt{x^2 + y^2}$




$$D|X,Y \sim N\left(\mu = \sqrt{x^2+y^2}, \sigma^2 = 1\right)$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}} = K_2 \cdot e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}}$$

normalizing constant

# 3. Compute posterior

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\ f_{X,Y}(x,y)}{f_D(d)}$$

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

<u>Compute:</u>

Posterior belief

$$f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x, Y=y|D=4)$$

# 3. Compute posterior

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\; f_{X,Y}(x,y)}{f_D(d)}$$

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Compute:

Posterior belief

$$f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x, Y=y|D=4)$$

Know:

Prior belief

$$f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

Observation likelihood

$$f_{D|X,Y}(d|x,y) = K_2 \cdot e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}}$$

Tips
- Use Bayes' Theorem!
- $f_D(4)$ is just a scaling constant. Why?
- How can we approximate the final scaling constant with a computer?

# Tracking in 2-D space

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

likelihood of $D = 4$ | prior belief

$$f_{X,Y|D}(X = x, Y = y|D = 4) = \frac{f_{D|X,Y}(D = 4|X = x, Y = y) f_{X,Y}(x, y)}{f(D = 4)}$$ Bayes' Theorem

1-D | 2D

$$= \frac{K_2 \cdot e^{-\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}} \cdot K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}}{f(D = 4)}$$

$$= \frac{K_3 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}}{f(D = 4)}$$

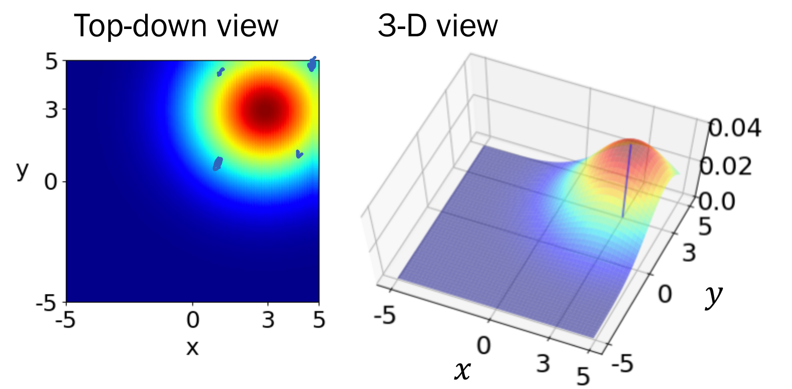
$$= K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

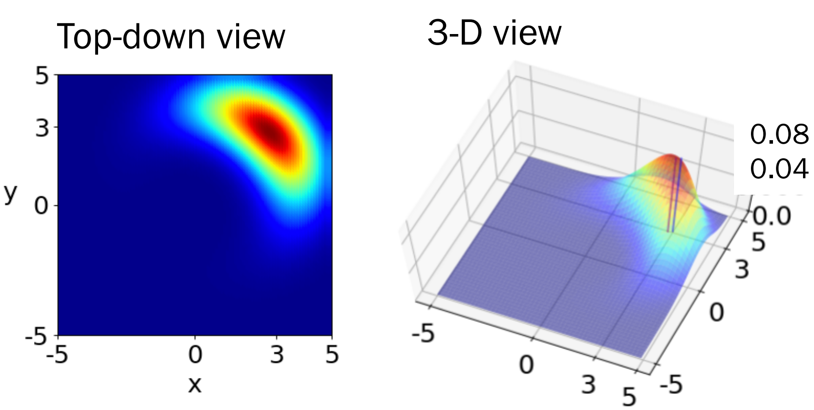
Key: Once we know the part dependent on $x, y$, we can computationally approximate $K_4$ such that $f_{X,Y|D}$ is a valid PDF.

# Tracking in 2-D space

With this continuous version of Bayes' theorem, we can explore new domains.

- Before measuring, we have some **prior belief** about the 2-D location of an object, $(X, Y)$.
- We observe some noisy **measurement** of the distance of the object to a satellite.
- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?

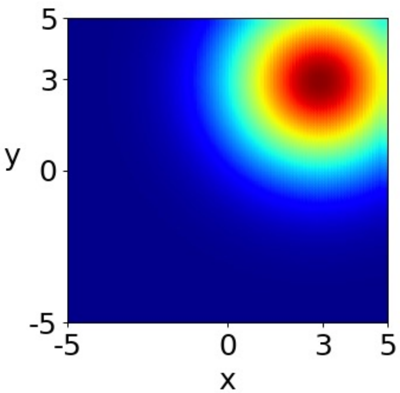


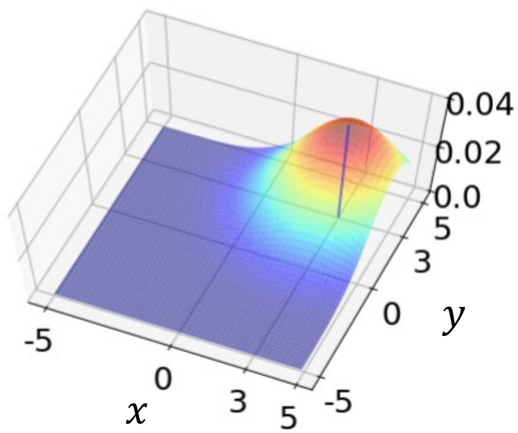
# Tracking in 2-D space: Posterior belief

## Prior belief

Top-down view



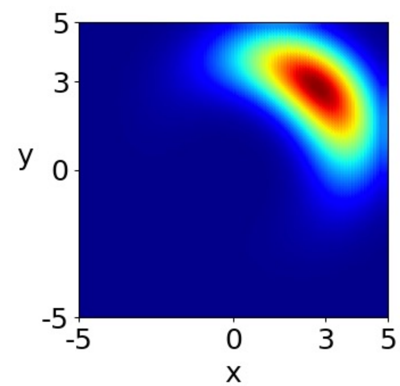

3-D view



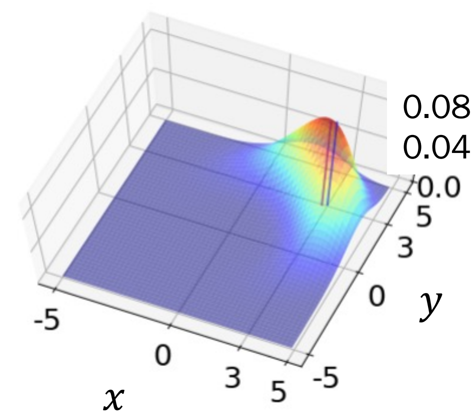

$$f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

## Posterior belief

Top-down view




3-D view




$$f_{X,Y|D}(x,y|4) = K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

# How'd you compute that $K_4$?

To be a valid conditional PDF, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y|D}(x, y|4)\, dx\, dy = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{\left[(x-3)^2+(y-3)^2\right]}{8}\right]} dx\, dy = 1$$

$$\frac{1}{K_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{\left[(x-3)^2+(y-3)^2\right]}{8}\right]} dx\, dy$$

(pull out $K_4$, divide)

Approximate:

$$\frac{1}{K_4} \approx \sum_y \sum_x e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{\left[(x-3)^2+(y-3)^2\right]}{8}\right]} \Delta x \Delta y$$

Use a computer!