18: Central Limit Theorem

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iid Random Variables

i.i.d. random variables

Consider n variables $X_1, X_2, \ldots, X_n$.

$X_1, X_2, \ldots, X_n$ are independent and identically distributed if

- $X_1, X_2, \ldots, X_n$ are independent, and
- all have the same (discrete) PMF or (continuous) PDF.

$\Rightarrow E[X_i] = \mu$ for $i = 1, \ldots, n$

$\Rightarrow \text{Var}(X_i) = \sigma^2$ for $i = 1, \ldots, n$

Synonyms: i.i.d. iid IID

Quick check

Are $X_1, X_2, \ldots, X_n$ i.i.d. with the following distributions?

1. $X_i \sim \text{Exp}(\lambda)$, X_i independent ✅
2. $X_i \sim \text{Exp}(\lambda_i)$, X_i independent ❌ (unless λ_i equal)
3. $X_i \sim \text{Exp}(\lambda)$, $X_1 = X_2 = \cdots = X_n$ ❌ dependent: $X_1 = X_2 = \cdots = X_n$
4. $X_i \sim \text{Bin}(n_i, p)$, X_i independent ❌ (unless n_i equal)
   Note underlying Bernoulli RVs are i.i.d.!

Central Limit Theorem

Central Limit Theorem

Consider n **independent and identically distributed (i.i.d.)** variables $X_1, X_2, \ldots, X_n$ with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$.

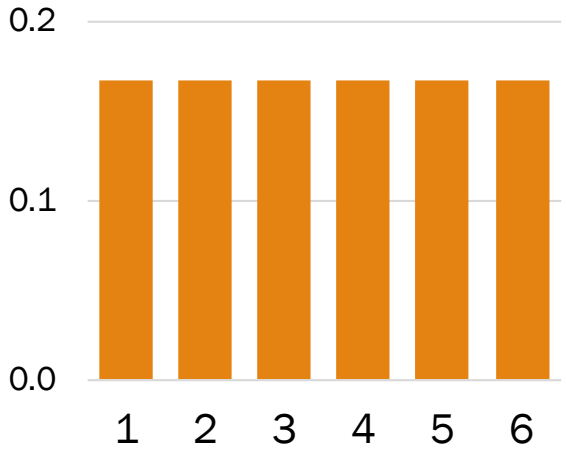
As $n \to \infty$

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

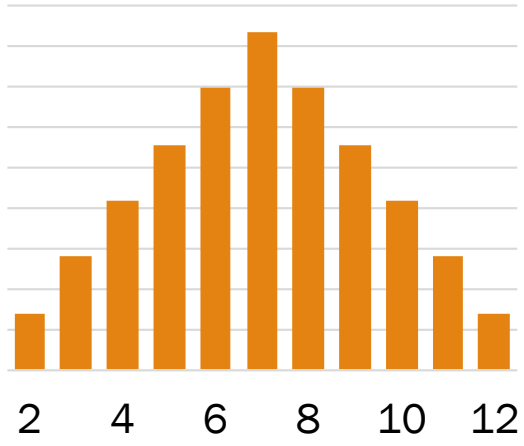
The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

Sum of dice rolls

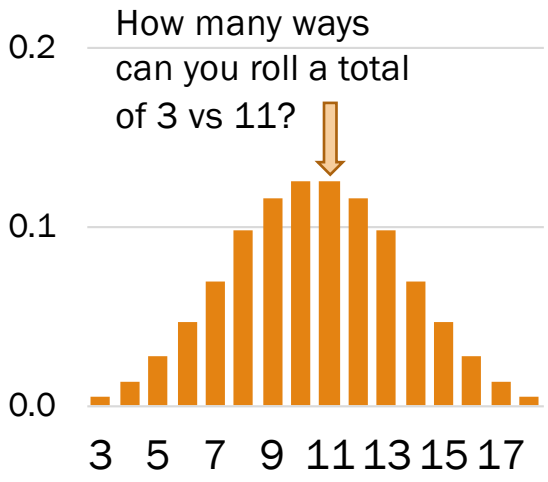
Roll n independent dice. Let X_i be the outcome of roll i. X_i are i.i.d.



$\sum_{i=1}^{1} X_i$ Sum of 1 die roll



$\sum_{i=1}^{2} X_i$ Sum of 2 dice rolls

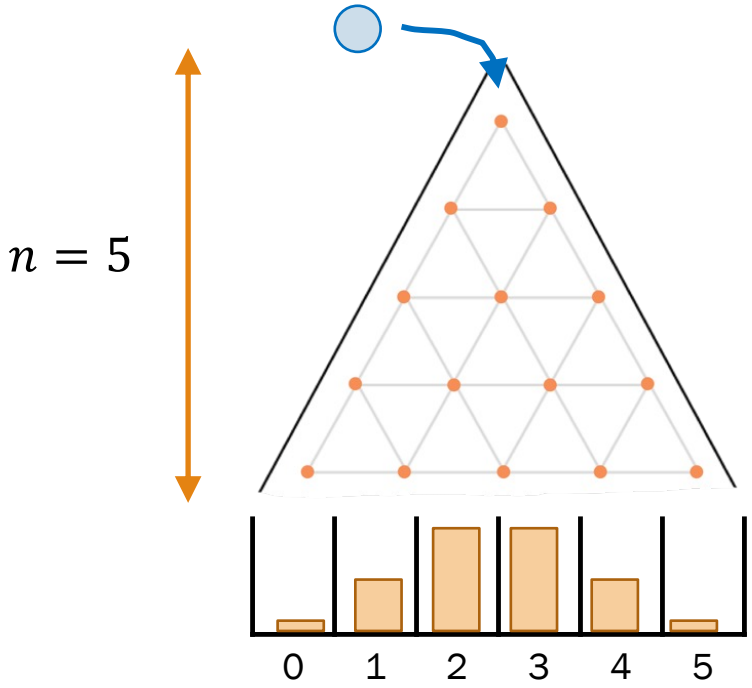


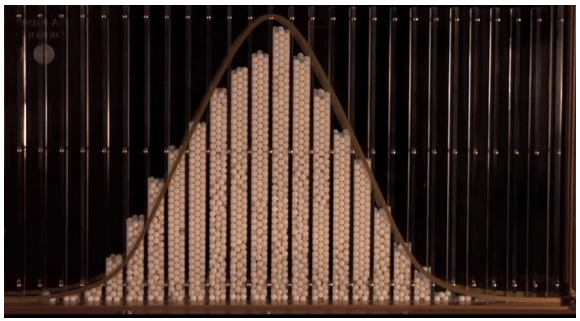
$\sum_{i=1}^{3} X_i$ Sum of 3 dice rolls

CLT explains a lot

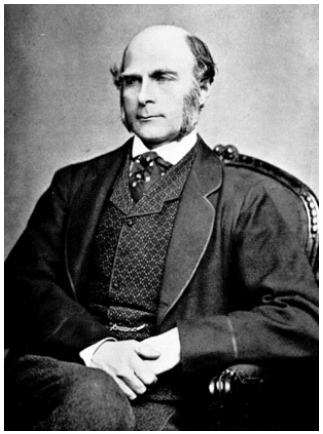
$$\text{As } n \to \infty \qquad \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.





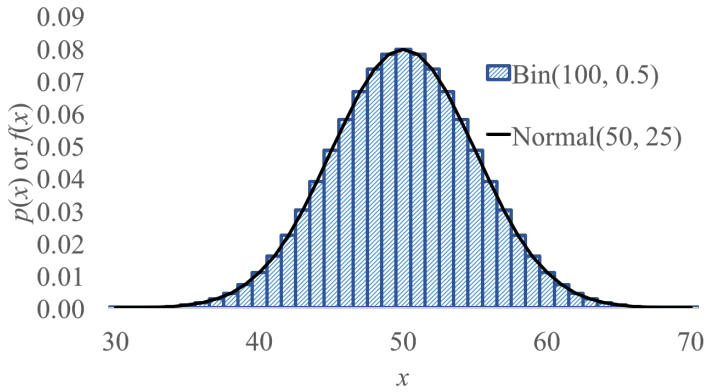
Galton Board, by Sir Francis Galton (1822-1911)



CLT explains a lot

$$\text{As } n \to \infty \qquad \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.



Normal approximation of Binomial

Sum of i.i.d. Bernoulli RVs ≈ Normal

Proof:

Let $X_i \sim \text{Ber}(p)$ for $i = 1, \ldots, n$, where X_i are i.i.d.
$E[X_i] = p$, $\text{Var}(X_i) = p(1-p)$

$$X = \sum_{i=1}^{n} X_i \qquad (X \sim \text{Bin}(n, p))$$

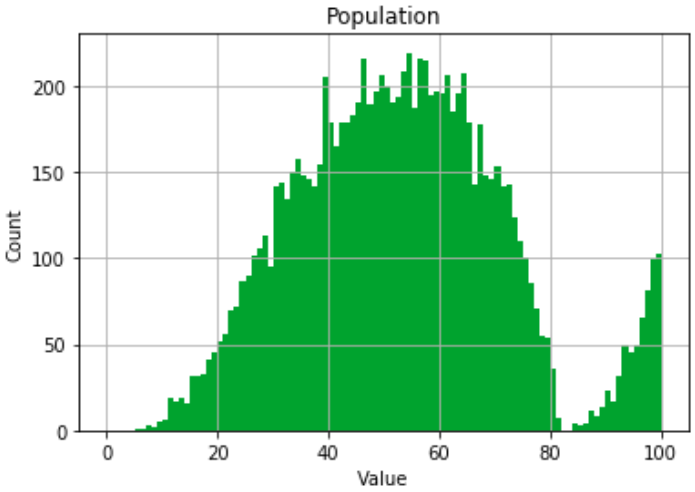
$$X \sim \mathcal{N}(n\mu, n\sigma^2) \qquad (\text{CLT, as } n \to \infty)$$

$$X \sim \mathcal{N}(np, np(1-p)) \qquad (\text{substitute mean, variance of Bernoulli})$$

CLT explains a lot

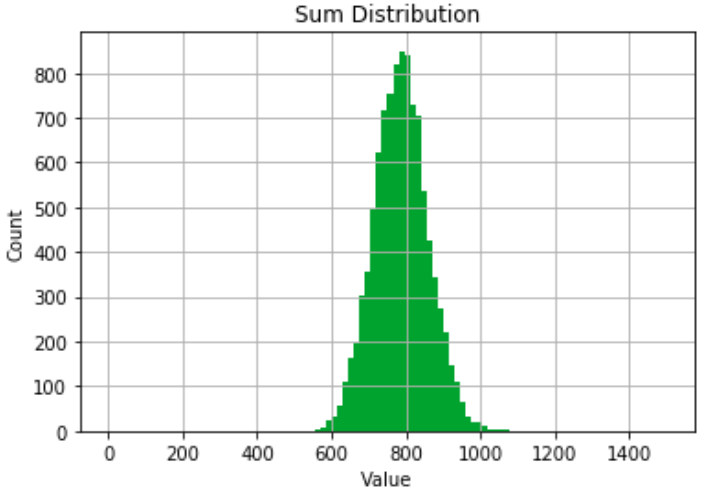
$$\text{As } n \to \infty \qquad \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.



Distribution of X_i

Sample of size 15, **sum** values

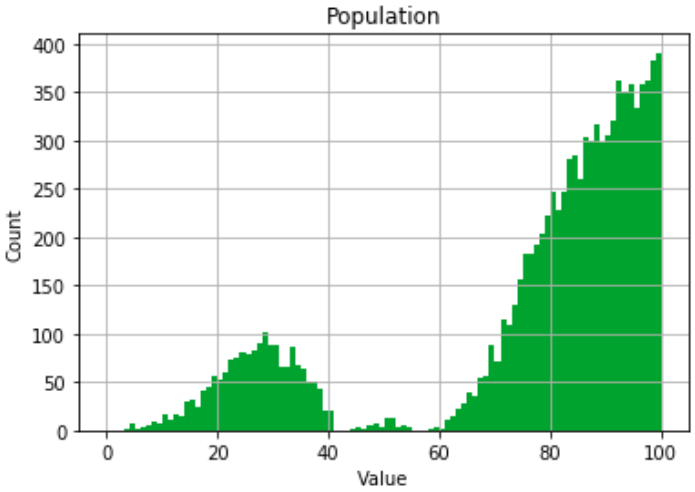


Distribution of $\sum_{i=1}^{15} X_i$

CLT explains a lot

$$\text{As } n \to \infty \quad \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

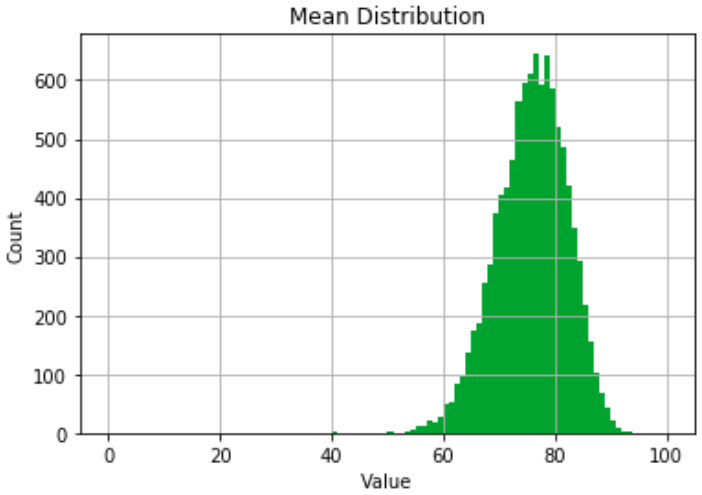
The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.



Distribution of X_i

Sample of size 15, **average** values

(sample mean)



Distribution of $\frac{1}{15}\sum_{i=1}^{15} X_i$

Proof of CLT

$$\text{As } n \to \infty \qquad \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

Proof:

- The Fourier Transform of a PDF is called a **characteristic function**.
- Take the characteristic function of the probability mass of the sample distance from the mean, divided by standard deviation
- Show that this approaches an exponential function in the limit as $n \to \infty$: $f(x) = e^{-\frac{x^2}{2}}$
- This function is in turn the characteristic function of the Standard Normal, $Z \sim \mathcal{N}(0,1)$.

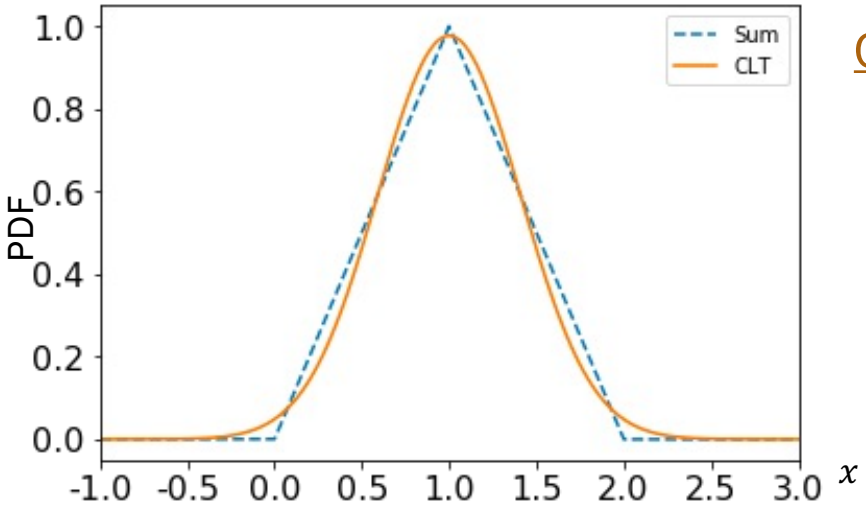
(proof is beyond the scope of CS109, but only because we don't teach Fourier transforms)

Sum of n independent Uniform RVs

Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$. $\mu = E[X_i] = 1/2$, $\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \leq n/3)$?

$n = 2$:



Exact $\quad P(X \leq 2/3) \approx 0.2222$

CLT approximation

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2) \implies Y \sim \mathcal{N}(1, 1/6)$$

$$P(X \leq 2/3) \approx P(Y \leq 2/3)$$

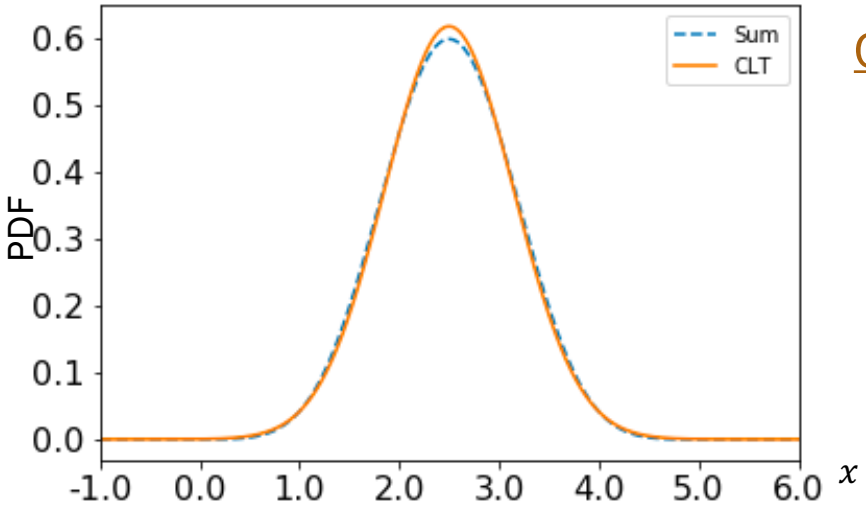
$$= \Phi\left(\frac{2/3 - 1}{\sqrt{1/6}}\right) \approx 0.2071$$

Sum of n independent Uniform RVs

Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$. $\mu = E[X_i] = 1/2$, $\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \leq n/3)$?

$n = 5$:



Exact $\quad P(X \leq 5/3) \approx 0.1017$

CLT approximation

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2) \implies Y \sim \mathcal{N}(5/2, 5/12)$$

$$P(X \leq 5/3) \approx P(Y \leq 5/3)$$

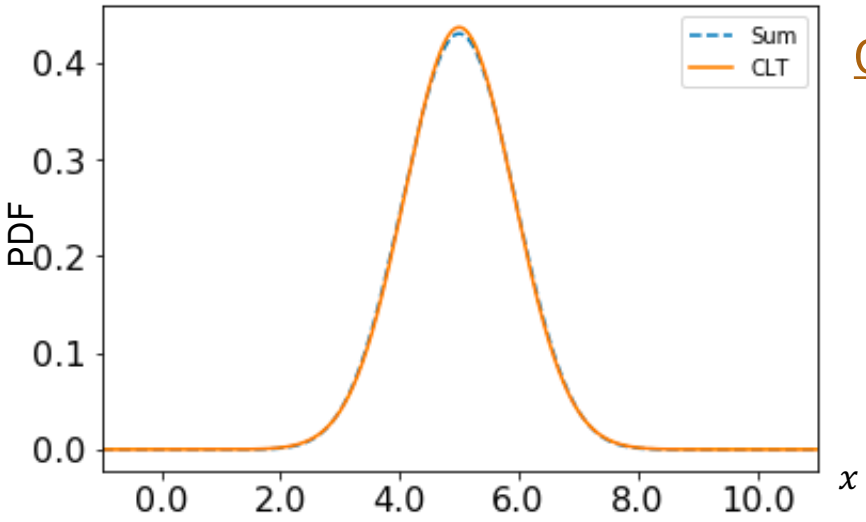
$$= \Phi\left(\frac{5/3 - 5/2}{\sqrt{5/12}}\right) \approx 0.0984$$

Sum of n independent Uniform RVs

Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$. $\mu = E[X_i] = 1/2$, $\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \leq n/3)$?

$n = 10$:



Exact

$$P(X \leq 10/3) \approx 0.0337$$

CLT approximation

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2) \implies Y \sim \mathcal{N}(5, 5/6)$$

$$P(X \leq 10/3) \approx P(Y \leq 10/3)$$

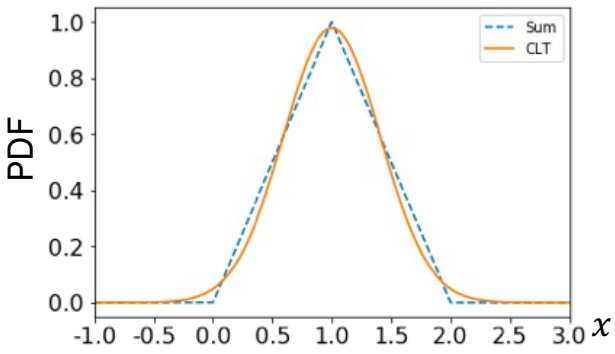
$$= \Phi\left(\frac{10/3 - 5}{\sqrt{5/6}}\right) \approx 0.0339$$

Sum of n independent Uniform RVs

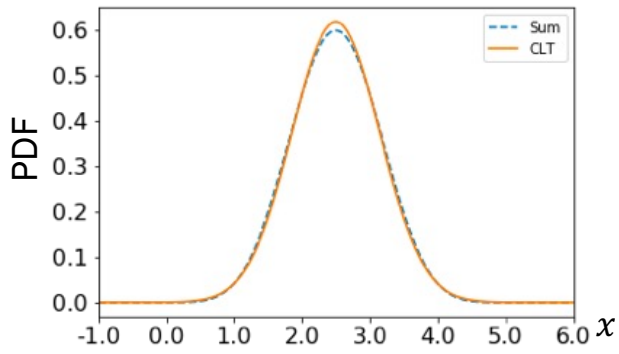
Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$. $\mu = E[X_i] = 1/2$, $\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \leq n/3)$?

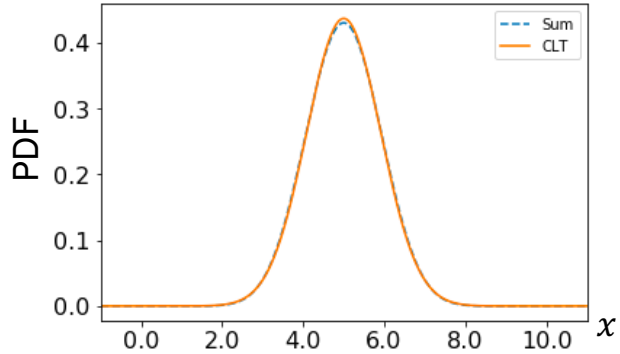
$n = 2$:



$n = 5$:



$n = 10$:



Most books will tell you that CLT holds if $n \geq 30$, but it can hold for smaller n depending on the distribution of your i.i.d. X_i's.

Sample Statistics

What about other functions?

Let $X_1, X_2, \ldots, X_n$ be i.i.d., where $E[X_i] = \mu, \text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$	Sum of i.i.d. RVs
?	Average of i.i.d. RVs (sample mean)
?	Max of i.i.d. RVs

What about other functions?

Let $X_1, X_2, \ldots, X_n$ be i.i.d., where $E[X_i] = \mu, \text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$	Sum of i.i.d. RVs
?	Average of i.i.d. RVs (sample mean)
?	Max of i.i.d. RVs

Distribution of sample mean

Let $X_1, X_2, \ldots, X_n$ be i.i.d., where $E[X_i] = \mu, \text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

Define: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ (sample mean) $Y = \sum_{i=1}^{n} X_i$ (sum)

$Y \sim \mathcal{N}(n\mu, n\sigma^2)$ (CLT, as $n \to \infty$)

$\bar{X} = \frac{1}{n} Y$

$\bar{X} \sim \mathcal{N}(\,?\,,\,?\,)$ (Linear transform of a Normal)

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

The average of **i.i.d.** random variables (i.e., sample mean) is normally distributed with mean μ and variance σ^2/n.

Demo: http://onlinestatbook.com/stat_sim/sampling_dist/

What about other functions?

Let $X_1, X_2, \ldots, X_n$ be i.i.d., where $E[X_i] = \mu, \text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

Sum of i.i.d. RVs

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Average of i.i.d. RVs
(sample mean)

Gumbel

Max of i.i.d. RVs

(see Fisher-Tippett Gnedenko Theorem)

Exercises

Dice game

As $n \to \infty$: $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$

You will roll 10 6-sided dice $(X_1, X_2, \ldots, X_{10})$.

- Let $X = X_1 + X_2 + \cdots + X_{10}$, the total value of all 10 rolls.
- You win if $X \leq 25$ or $X \geq 45$.

To the demo!

Dice game

As $n \to \infty$: $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$

You will roll 10 6-sided dice $(X_1, X_2, \ldots, X_{10})$.

- Let $X = X_1 + X_2 + \cdots + X_{10}$, the total value of all 10 rolls.
- You win if $X \leq 25$ or $X \geq 45$.

And now the truth (according to the CLT)…

1. Define RVs and state goal.

$E[X_i] = 3.5,$
$\text{Var}(X_i) = 35/12$

Want: $P(X \leq 25 \text{ or } X \geq 45)$

Approximate: ?

2. Solve.

Dice game

As $n \to \infty$: $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$

You will roll 10 6-sided dice $(X_1, X_2, \ldots, X_{10})$.

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And now the truth (according to the CLT)…

1. Define RVs and state goal.

$E[X_i] = 3.5,$
$\text{Var}(X_i) = 35/12$

Want: $P(X \leq 25 \text{ or } X \geq 45)$

Approximate:

$X \approx Y \sim \mathcal{N}(10(3.5), 10(35/12))$

2. Solve.

$P(Y \leq 25.5) + P(Y \geq 44.5)$ or $1 - P(25.5 \leq Y \leq 44.5)$

continuity correction

Dice game

As $n \to \infty$: $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$

You will roll 10 6-sided dice $(X_1, X_2, \ldots, X_{10})$.

- Let $X = X_1 + X_2 + \cdots + X_{10}$, the total value of all 10 rolls.
- You win if $X \leq 25$ or $X \geq 45$.

And now the truth (according to the CLT)…

1. Define RVs and state goal.

$E[X_i] = 3.5$, $\text{Var}(X_i) = 35/12$

Want: $P(X \leq 25 \text{ or } X \geq 45)$

Approximate:

$$X \approx Y \sim \mathcal{N}(10(3.5), 10(35/12))$$

2. Solve.

$$P(Y \leq 25.5) + P(Y \geq 44.5) = \Phi\left(\frac{25.5 - 35}{\sqrt{10(35/12)}}\right) + \left(1 - \Phi\left(\frac{44.5 - 35}{\sqrt{10(35/12)}}\right)\right)$$

$$\approx \Phi(-1.76) + \big(1 - \Phi(1.76)\big) \approx (1 - 0.9608) + (1 - 0.9608) = 0.0784$$

Dice game

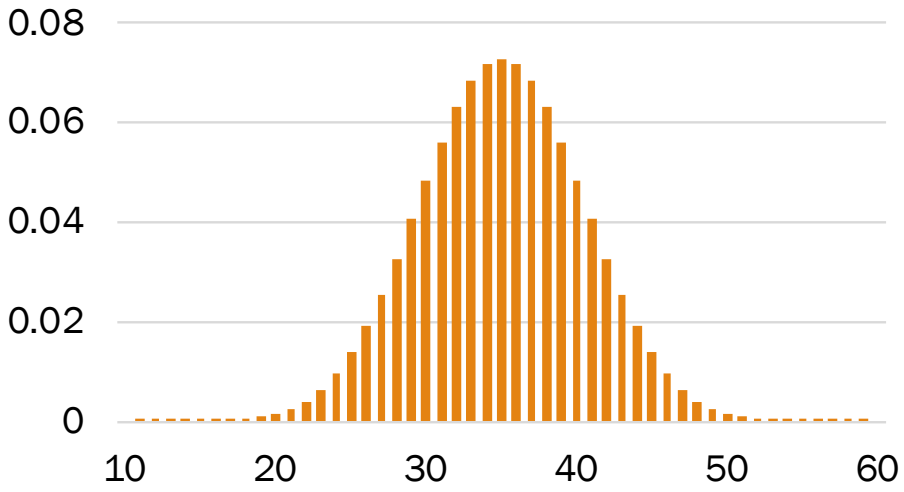
As $n \to \infty$: $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$

You will roll 10 6-sided dice $(X_1, X_2, \ldots, X_{10})$.

- Let $X = X_1 + X_2 + \cdots + X_{10}$, the total value of all 10 rolls.
- You win if $X \leq 25$ or $X \geq 45$.

And now the truth (according to the CLT)...

Check out the code!



(by CLT)

$$\approx P(Y \leq 25.5) + P(Y \geq 44.5)$$
$$\approx 0.0786$$

(exact, by computer)

$$P(X \leq 25 \text{ or } X \geq 45) = 0.0780$$

(exact, by computer)

$$P(X \leq 25 \text{ or } X \geq 45) \approx 0.0776$$

Summary: Working with the CLT

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu, \text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

Sum of i.i.d. RVs

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Average of i.i.d. RVs
(sample mean)

If X_i is discrete:

Use the **continuity correction** on Y!

Crashing website

- Let X = number of visitors to a website, where $X \sim \text{Poi}(100)$.
- The server crashes if there are ≥ 120 requests/minute.

What is $P(\text{server crashes in next minute})$?

Strategy:
Poisson (exact)

$$P(X \geq 120) = \sum_{k=120}^{\infty} \frac{(100)^k e^{-100}}{k!} \approx 0.0282$$

Strategy:
CLT
(approx.)

How would we involve CLT here?

(Hint: Is there a way to represent X as a sum of i.i.d. RVs?)

Crashing website

- Let X = number of visitors to a website, where $X \sim \text{Poi}(100)$.
- The server crashes if there are ≥ 120 requests/minute.

What is $P(\text{server crashes in next minute})$?

Strategy: Poisson (exact)

$$P(X \geq 120) = \sum_{k=120}^{\infty} \frac{(100)^k e^{-100}}{k!} \approx 0.0282$$

Strategy: CLT (approx.)

State approx. goal

$$\text{Poi}(100) \sim \sum_{i=1}^{n} \text{Poi}(100/n)$$

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2)$$

$$P(X \geq 120) \approx P(Y \geq 119.5)$$

Check out the code!

Solve

$$P(Y \geq 119.5) = 1 - \Phi\left(\frac{119.5 - 100}{\sqrt{100}}\right) = 1 - \Phi(1.95) \approx 0.0256$$

Next time

Central Limit Theorem:

- Sample mean $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$
- If we know μ and σ^2, we can compute probabilities on sample mean $\bar{X}$ of a given sample size n

In real life:

- Yes, the CLT still holds….
- But we often don't know μ or σ^2 of our original distribution
- However, we can collect data (a sample of size n)!
- How can we estimate the values μ and σ^2 from our sample?

…until next time!