24: Naïve Bayes

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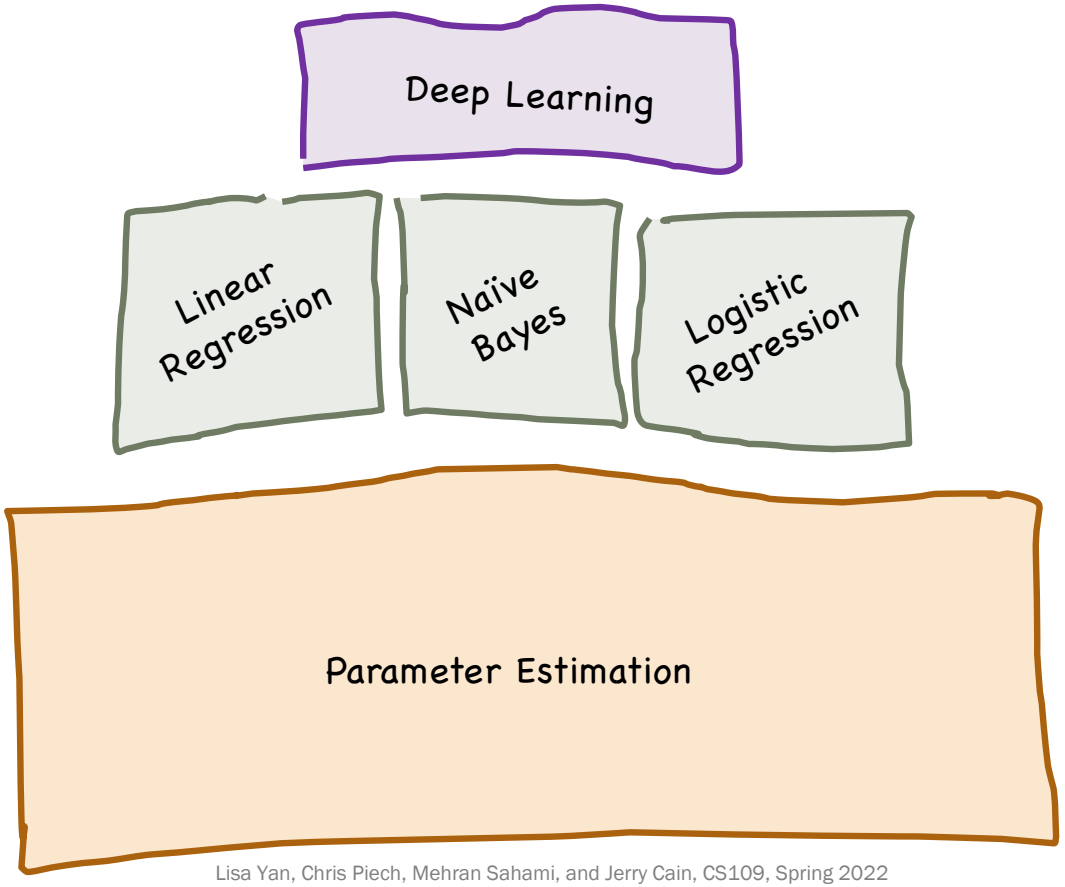
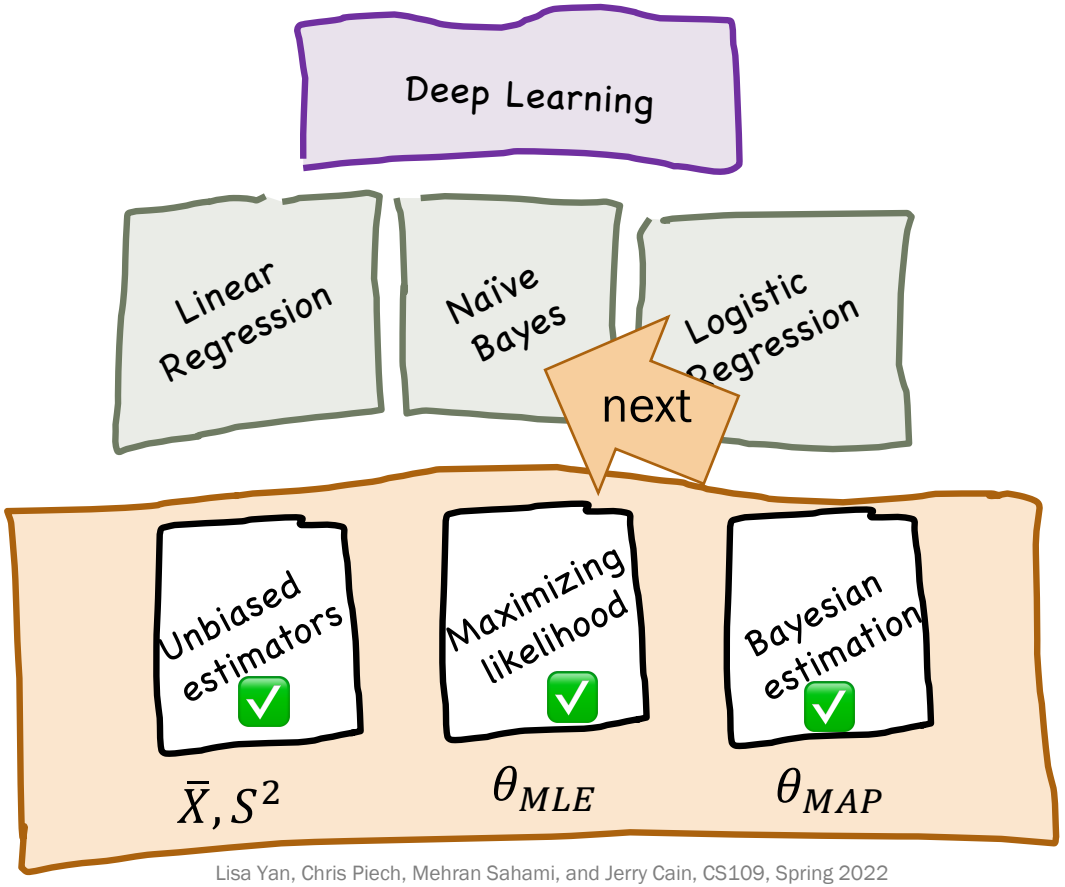
Table of Contents

Intro: Machine Learning

Our path from here



Our path from here



Machine Learning (formally)

Many different forms of Machine Learning

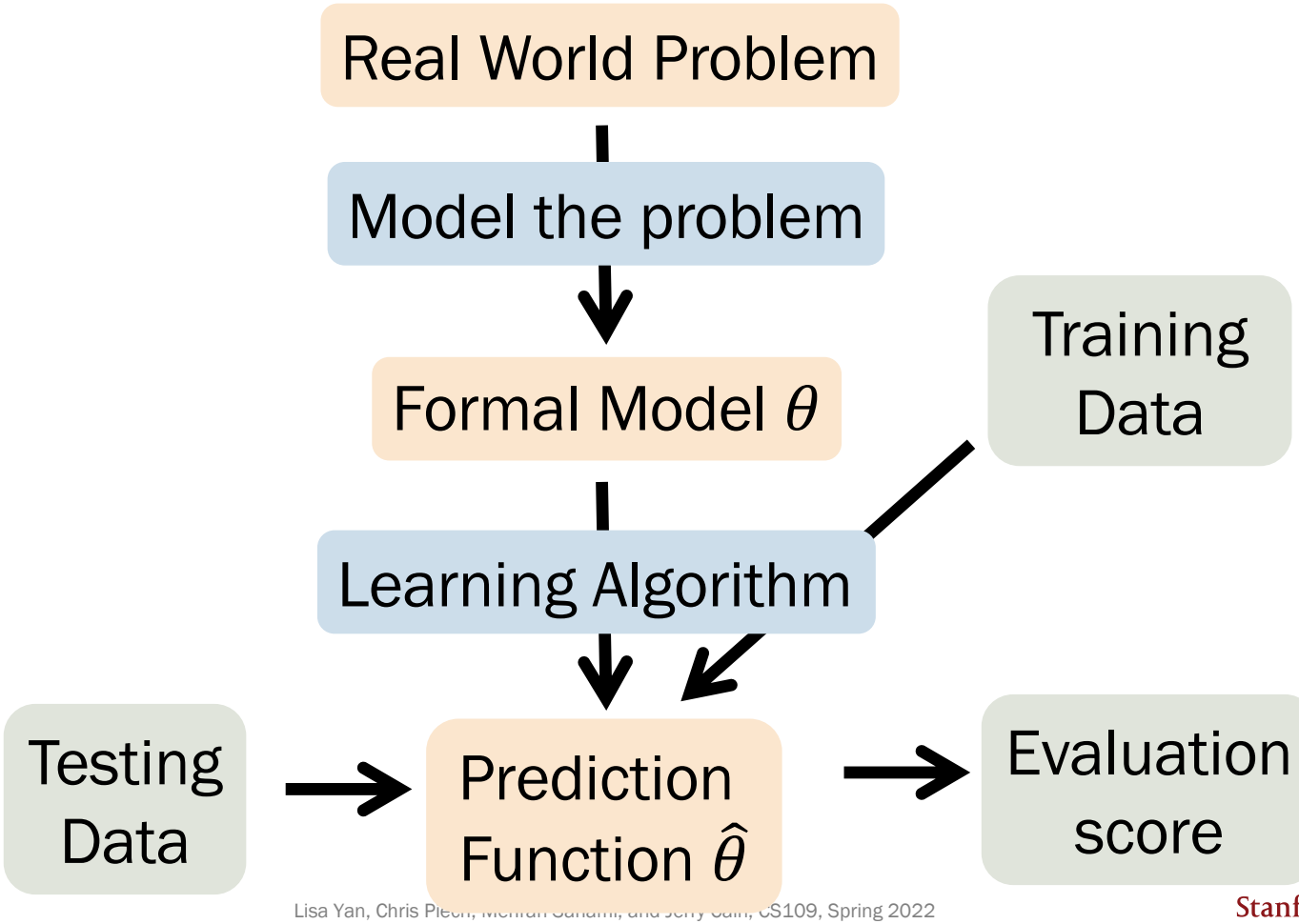
- We focus on the problem of **prediction** given prior observations.

Machine Learning uses a lot of data.

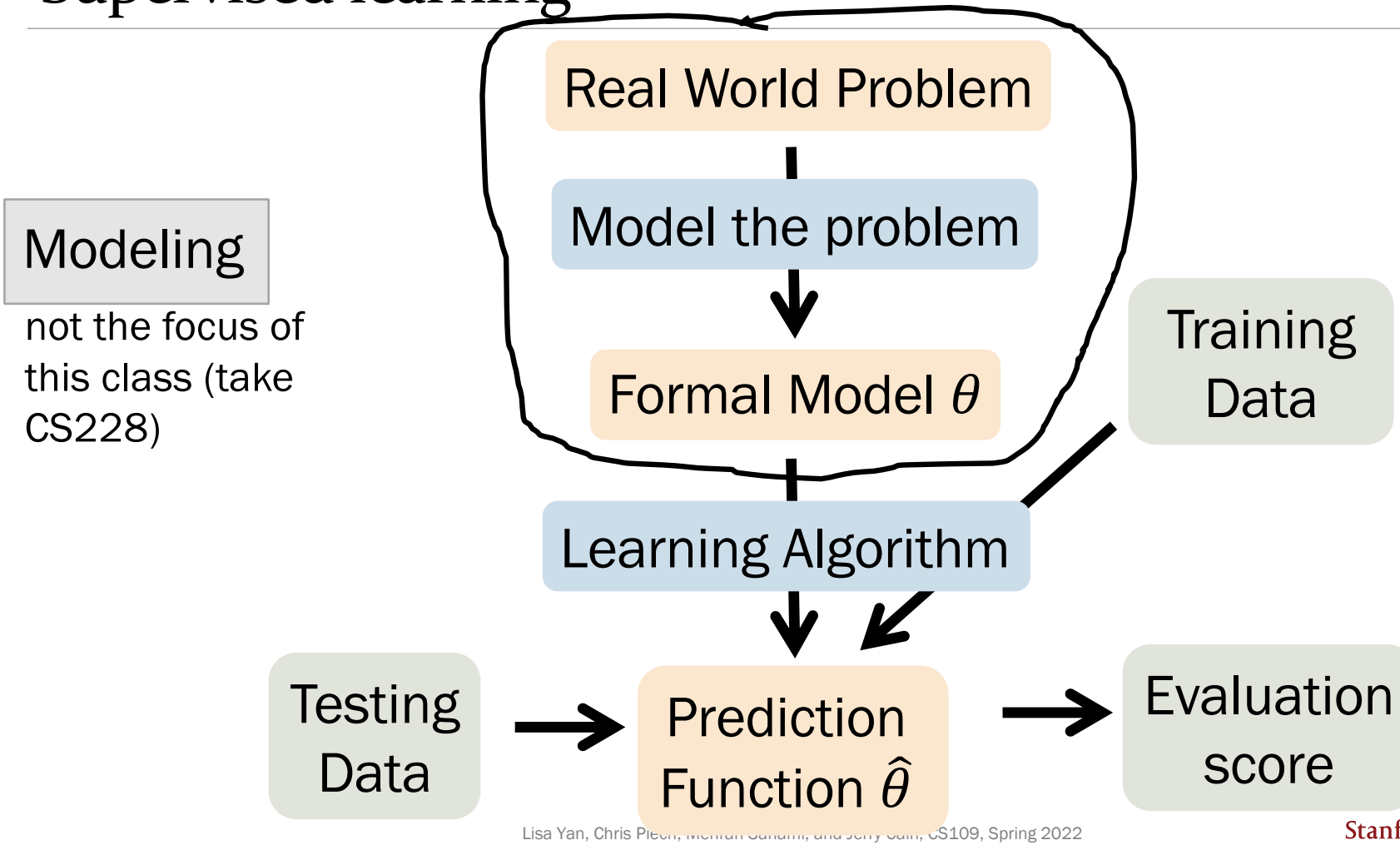
Task: Identify the chair
Data: All the chairs ever

Supervised learning: A category of machine learning where you have labeled data for the problem you are solving.

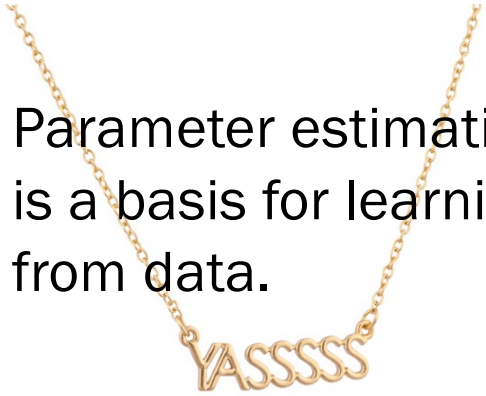
Supervised learning



Supervised learning



Supervised learning



Parameter estimation is a basis for learning from data.

Training

Model and dataset

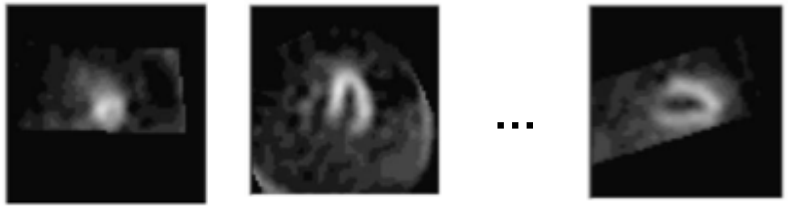
Many different forms of machine learning

- We focus on a specific type of problem: **prediction** from observations.

Goal	Based on observed $\boldsymbol{X}$, predict some unknown Y
• Features	Vector $\boldsymbol{X}$ of m observations (new term: **feature vector**) $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$
• Output	Variable Y (also called **class label** if discrete)
Model	$\hat{Y} = g(\boldsymbol{X})$, a function on $\boldsymbol{X}$

Training data

$$\boldsymbol{X} = (X_1, X_2, X_3, \ldots, X_{300})$$



	Feature 1	Feature 2	...	Feature 300	Output
Patient 1	1	0	...	1	1
Patient 2	1	1	...	0	0
...			⋮		⋮
Patient n	0	0	...	1	1

Training data notation

$$\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$$

n datapoints, assumed to be iid

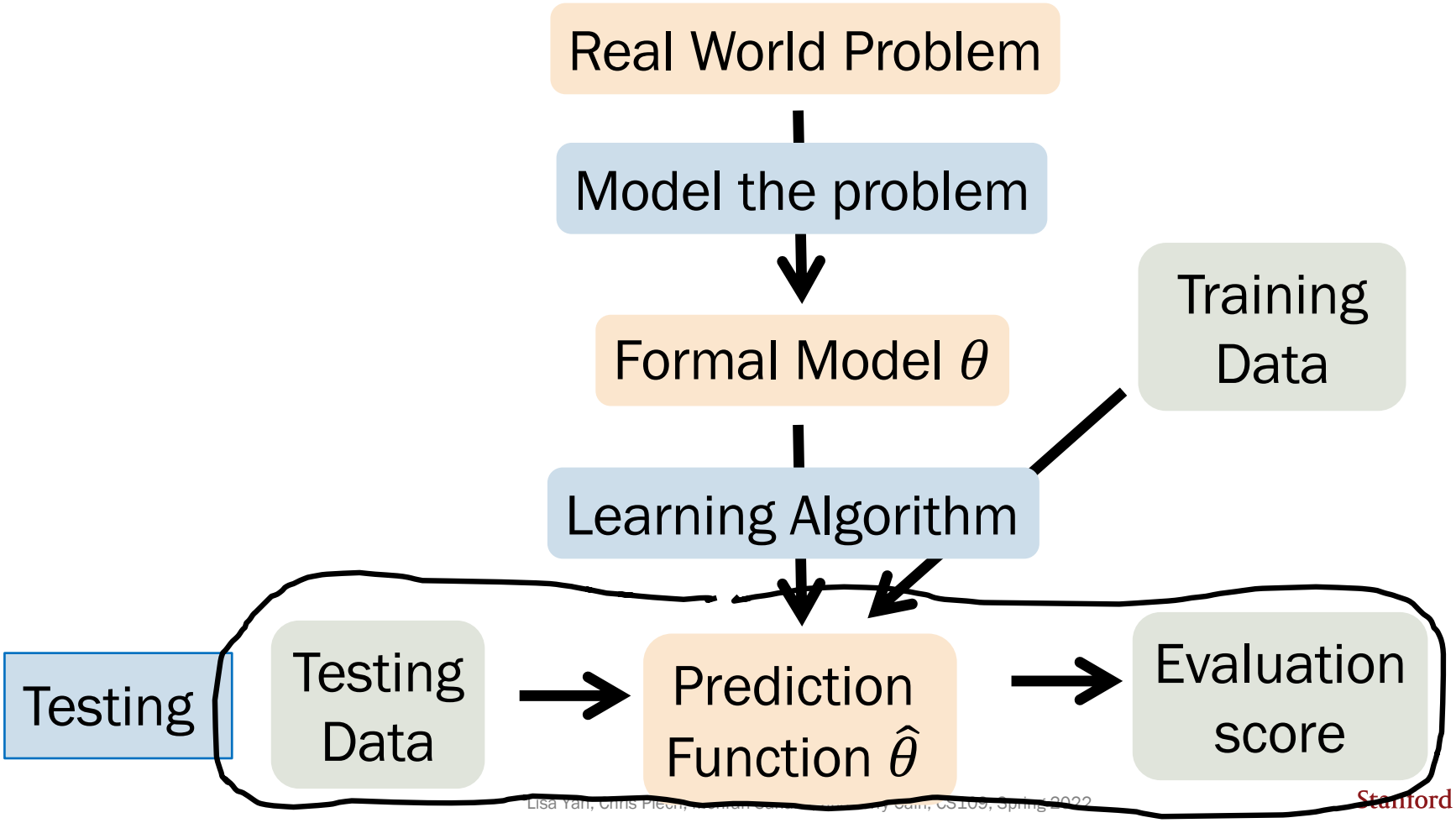
i–th datapoint $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$:

- m features: $\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}\right)$
- A single output $y^{(i)}$
- Independent of all other datapoints

Training Goal:

Use these n datapoints to learn a model $\hat{Y} = g(\boldsymbol{X})$ that predicts Y

Supervised learning



Testing data notation

$$\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$$

n_{test} **other** datapoints, assumed to be iid

i–th datapoint $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$:

- Has the same structure as your training data

Testing Goal:

Leveraging the model $\hat{Y} = g(\boldsymbol{X})$ that you trained, see how well you can predict Y on known data

Two tasks we will focus on

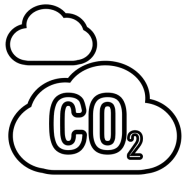
Many different forms of "Machine Learning"

- We focus on the problem of **prediction** based on observations.

Goal	Based on observed $\boldsymbol{X}$, predict some unknown Y
• Features	Vector $\boldsymbol{X}$ of m observations (new term: feature vector) $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$
• Output	Variable Y (also called **class label** if discrete)
Model	$\hat{Y} = g(\boldsymbol{X})$, a function on $\boldsymbol{X}$
• **Regression**	prediction when Y is continuous
• **Classification**	prediction when Y is discrete

Regression: Predicting real numbers

Training data: $(\boldsymbol{x}^{(1)}, y^{(1)}), (\boldsymbol{x}^{(2)}, y^{(2)}), \ldots, (\boldsymbol{x}^{(n)}, y^{(n)})$



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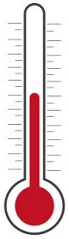
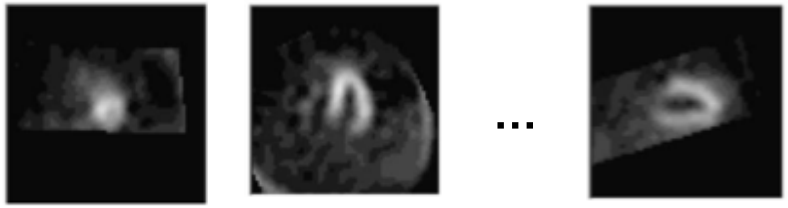


Global Land-Ocean temperature

	CO2 levels	Sea level	...	Feature m	Output
Year 1	338.8	0	...	1	0.26
Year 2	340.0	1	...	0	0.32
...			⋮		⋮
Year n	340.76	0	...	1	0.14

Classification: Predicting class labels

$$\boldsymbol{X} = (X_1, X_2, X_3, \ldots, X_{300})$$



	Feature 1	Feature 2		Feature 300	Output
Patient 1	1	0	…	1	1
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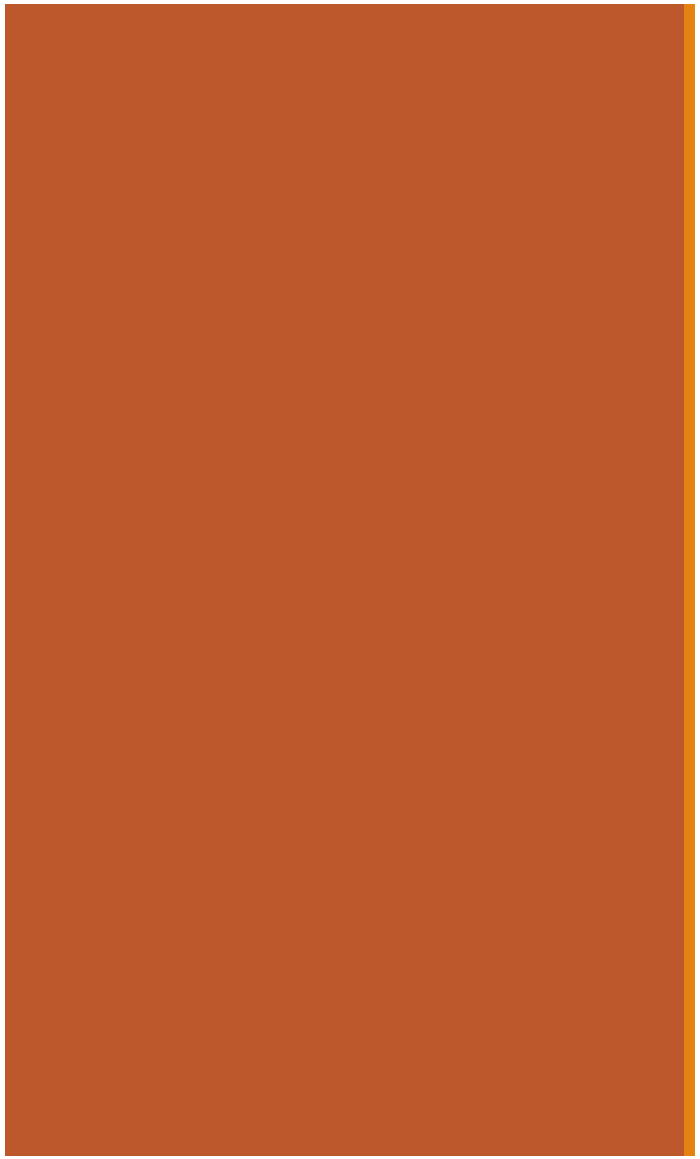
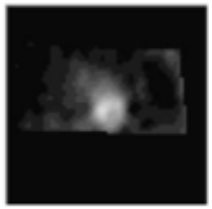
…			⋮		⋮
Patient n	0	0	…	1	1



Brute Force Bayes

Classification: Having a healthy heart

$$\boldsymbol{X} = (X_1)$$



	Feature 1	Output
Patient 1	1	0
Patient 2	1	1
	⋮	⋮
Patient n	0	1

Single feature: Region of Interest (ROI) is healthy (1) or unhealthy (0)

How can we predict whether heart is healthy (1) or unhealthy (0)?

The following strategy is **not used in practice** but helps us understand how to approach classification.

Classification: Brute Force Bayes

$$\hat{Y} = g(\boldsymbol{X})$$

Our prediction for Y is a function of $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} P(Y \mid \boldsymbol{X})$$

Proposed model: Choose the Y that is more or most likely given $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} \frac{P(\boldsymbol{X}|Y)P(Y)}{P(\boldsymbol{X})}$$

(Bayes' Theorem)

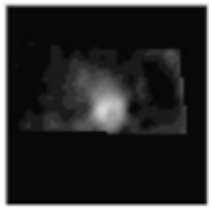
$$= \arg\max_{y=\{0,1\}} P(\boldsymbol{X}|Y)P(Y)$$

($1/P(\boldsymbol{X})$ is constant w.r.t. y)

If we estimate $P(\boldsymbol{X}|Y)$ and $P(Y)$, we can classify datapoints!

Training: Estimate parameters

$$\boldsymbol{X} = (X_1)$$



	Feature 1	Output
Patient 1	1	0
Patient 2	1	1
	⋮	⋮
Patient n	0	1

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Conditional probability tables $\hat{P}(\boldsymbol{X}|Y)$

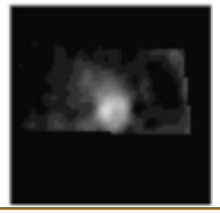
	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	θ_1	θ_3
$X_1 = 1$	θ_2	θ_4

Marginal probability table $\hat{P}(Y)$

	$\hat{P}(Y)$
$Y = 0$	θ_5
$Y = 1$	θ_6

Training Goal: Use n datapoints to learn $2 \cdot 2 + 2 = 6$ parameters.

Training: Estimate parameters $\hat{P}(\boldsymbol{X}|Y)$



Count:	# datapoints
X_1 = 0, Y = 0:	4
X_1 = 1, Y = 0:	6
X_1 = 0, Y = 1:	0
X_1 = 1, Y = 1:	100
Total:	110

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Patient n 0 1

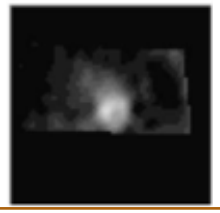
	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	θ_1	θ_3
$X_1 = 1$	θ_2	θ_4

$\boldsymbol{X}|Y=0$ and $\boldsymbol{X}|Y=1$
are each multinomials with 2 outcomes!

(MAP with Laplace smoothing)
Laplace

Use MLE or Laplace (MAP) estimate $\hat{P}(\boldsymbol{X}|Y)$ and $\hat{P}(Y)$ as parameters.

Training: MLE estimates, $\hat{P}(\boldsymbol{X}|Y)$



Count:	# datapoints
$X_1 = 0, Y = 0$:	4
$X_1 = 1, Y = 0$:	6
$X_1 = 0, Y = 1$:	0
$X_1 = 1, Y = 1$:	100
Total:	110

Patient n 0 1

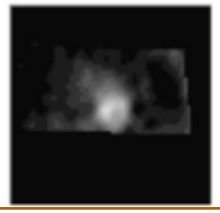
MLE

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	$0.4 = \frac{4}{10}$	$0.0 = \frac{0}{100}$
$X_1 = 1$	$0.6 = \frac{6}{10}$	$1.0 = \frac{100}{100}$

MLE of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y)}{\#(Y = y)}$

Just count!

Training: Laplace (MAP) estimates, $\hat{P}(\boldsymbol{X}|Y)$



Count:	# datapoints
X_1 = 0, Y = 0:	4
X_1 = 1, Y = 0:	6
X_1 = 0, Y = 1:	0
X_1 = 1, Y = 1:	100
Total:	110

Patient n 0 1

MLE

MLE of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y)}{\#(Y = y)}$

Just count!

with Laplace smoothing

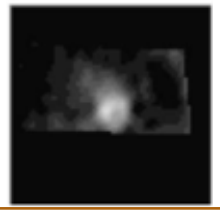
MAP

	$\hat{P}(\boldsymbol{X}\|Y = 0)$	$\hat{P}(\boldsymbol{X}\|Y = 1)$
$X_1 = 0$	0.4	0.0
$X_1 = 1$	0.6	1.0

Laplace of $\hat{P}(X_1 = x|Y = y) = \ ?$

Just count + add imaginary trials!

Training: Laplace (MAP) estimates, $\hat{P}(\boldsymbol{X}|Y)$



Count:	# datapoints
$X_1 = 0, Y = 0$:	4
$X_1 = 1, Y = 0$:	6
$X_1 = 0, Y = 1$:	0
$X_1 = 1, Y = 1$:	100
Total:	110

Patient n 0 1

$P(X_1=1, Y=1) = 1 + \text{observed}$

$P(X_1=1, Y=0) = 1 + \text{observed}$

$P(X_1=0, Y=1) = 1 + \text{observed}$

$P(X_1=0, Y=0) = 1 + \text{observed}$

MLE

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.4	0.0
$X_1 = 1$	0.6	1.0

MLE of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y)}{\#(Y = y)}$

Just count!

MAP

	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.42 $= \frac{4+1}{10+2}$	0.01 $= \frac{1}{102}$
$X_1 = 1$	0.58 $\frac{6+1}{10+2}$	0.99 $= \frac{100}{102}$

Laplace of $\hat{P}(X_1 = x|Y = y) = \dfrac{\#(X_1 = x, Y = y) + 1}{\#(Y = y) + 2}$

Just count + add imaginary trials!

Testing

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

(MAP)	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.42	0.01
$X_1 = 1$	0.58	0.99

(MLE)	$\hat{P}(Y)$
$Y = 0$	0.09
$Y = 1$	0.91

New patient has a healthy ROI ($X_1 = 1$). What is your prediction, $\hat{Y}$?

$$\hat{P}(X_1 = 1|Y = 0)\hat{P}(Y = 0) = 0.58 \cdot 0.09 \approx 0.052$$
$$\hat{P}(X_1 = 1|Y = 1)\hat{P}(Y = 1) = 0.99 \cdot 0.91 \approx 0.901$$

A. $0.052 < 0.5 \Rightarrow \hat{Y} = 1$
B. $0.901 > 0.5 \Rightarrow \hat{Y} = 1$
C. $0.052 < 0.901 \Rightarrow \hat{Y} = 1$

Sanity check: Why don't these sum to 1?

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

(MAP)	$\hat{P}(\boldsymbol{X}\|Y=0)$	$\hat{P}(\boldsymbol{X}\|Y=1)$
$X_1 = 0$	0.42	0.01
$X_1 = 1$	0.58	0.99

(MLE)	$\hat{P}(Y)$
$Y = 0$	0.09
$Y = 1$	0.91

New patient has a healthy ROI ($X_1 = 1$). What is your prediction, $\hat{Y}$?

$$\hat{P}(X_1 = 1|Y = 0)\hat{P}(Y = 0) = 0.58 \cdot 0.09 \approx 0.052 \quad P(X_1=1 \cap Y=0)$$
$$\hat{P}(X_1 = 1|Y = 1)\hat{P}(Y = 1) = 0.99 \cdot 0.91 \approx 0.901 \quad P(X_1=1 \cap Y=1)$$

A. $0.052 < 0.5 \Rightarrow \hat{Y} = 1$
B. $0.901 > 0.5 \Rightarrow \hat{Y} = 1$
C. $0.052 < 0.901 \Rightarrow \hat{Y} = 1$

Sanity check: Why don't these sum to 1?

Brute Force Bayes classifier

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

($\hat{P}(Y)$ is an estimate of $P(Y)$,
$\hat{P}(\boldsymbol{X}|Y)$ is an estimate of $P(\boldsymbol{X}|Y)$)

Training

Estimate these probabilities—i.e., learn these parameters using MLE or Laplace (MAP)

$$\hat{P}(X_1, X_2, \ldots, X_m|Y = 1)$$
$$\hat{P}(X_1, X_2, \ldots, X_m|Y = 0)$$
$$\hat{P}(Y = 1) \qquad \hat{P}(Y = 0)$$

Testing

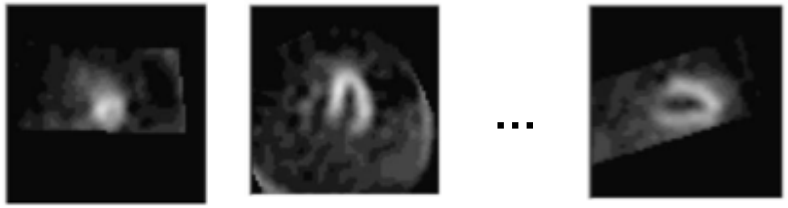
Given an observation $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$, predict

$$\hat{Y} = \underset{y=\{0,1\}}{\arg\max} \left(\hat{P}(X_1, X_2, \ldots, X_m|Y)\hat{P}(Y) \right)$$

Naïve Bayes

Brute Force Bayes: $m = 300$ (# features)

$$\boldsymbol{X} = (X_1, X_2, X_3, \ldots, X_{300})$$

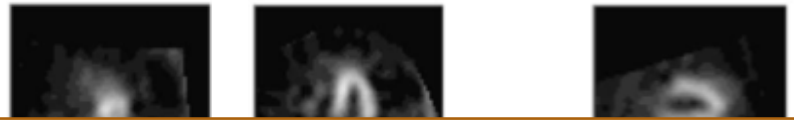


	Feature 1	Feature 2	...	Feature 300	Output
Patient 1	1	0	...	1	1
Patient 2	1	1	...	0	0
...			⋮		⋮
Patient n	0	0	...	1	1

This won't be too bad, right?

Brute Force Bayes: $m = 300$ (# features)

$$\boldsymbol{X} = (X_1, X_2, X_3, \ldots, X_{300})$$



Count:	# datapoints
$X_1 = 0, X_2 = 0, \ldots, X_{299} = 0, X_{300} = 0, Y = 0$:	0
$X_1 = 0, X_2 = 0, \ldots, X_{299} = 0, X_{300} = 1, Y = 0$:	0
$X_1 = 0, X_2 = 0, \ldots, X_{299} = 1, X_{300} = 0, Y = 0$:	1
…	
$X_1 = 0, X_2 = 0, \ldots, X_{299} = 0, X_{300} = 0, Y = 1$:	2
$X_1 = 0, X_2 = 0, \ldots, X_{299} = 0, X_{300} = 1, Y = 1$:	1
$X_1 = 0, X_2 = 0, \ldots, X_{299} = 1, X_{300} = 0, Y = 1$:	1

Pat

Pat

Patient n 0 0 … 1 1

This won't be too bad, right?

Brute Force Bayes: $m = 300$ (# features)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Learn parameters through MLE or MAP

- $\hat{P}(Y = 1 \mid \boldsymbol{x})$: estimated probability a heart is healthy given $\boldsymbol{x}$
- $\boldsymbol{X} = (X_1, X_2, \ldots, X_{300})$: whether 300 regions of interest (ROI) are healthy (1) or unhealthy (0)

How many parameters do we have to learn?

	$\hat{P}(\boldsymbol{X}\|Y)$		$\hat{P}(Y)$	
A.	$2 \cdot 2$	$+ 2$		$= 6$
B.	$2 \cdot 300$	$+ 2$		$= 602$
C.	$2 \cdot 2^{300}$	$+ 2$		$=$ a lot

Brute Force Bayes: $m = 300$ (# features)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Learn parameters through MLE or MAP

This approach requires you to learn $O(2^m)$ parameters.

- $\hat{P}(Y = 1 \mid \boldsymbol{x})$: estimated probability a heart is healthy given $\boldsymbol{x}$
- $\boldsymbol{X} = (X_1, X_2, \ldots, X_{300})$: whether 300 regions of interest (ROI) are healthy (1) or unhealthy (0)

How many parameters do we have to learn?

	$\hat{P}(\boldsymbol{X}\|Y)$		$\hat{P}(Y)$	
A.	$2 \cdot 2$	$+$	2	$= 6$
B.	$2 \cdot 300$	$+$	2	$= 602$
C.	$2 \cdot 2^{300}$	$+$	2	$=$ a lot

2^{272}

$x_1 = 0/1, x_2 = 0/1, x_3 = 0/1 \ldots x_n = 0/1, y = 0/1$

The problem with our current classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

Choose the Y that is most likely given $\boldsymbol{X}$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

(Bayes' Theorem)

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

($1/P(\boldsymbol{X})$ is constant w.r.t. y)

$$\hat{P}(X_1, X_2, \ldots, X_m|Y)$$

Estimating this joint conditional distribution is intractable.

What if we could make a simplifying assumption—even if incredibly naïve—to make our parameter estimation effort computationally tractable?

The Naïve Bayes assumption

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(Y \mid \boldsymbol{X})$$

$$= \arg\max_{y=\{0,1\}} \frac{\hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)}{\hat{P}(\boldsymbol{X})}$$

$$= \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

$$= \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Assumption:

$X_1, \ldots, X_m$ are all **conditionally independent** given Y.

Naïve Bayes Assumption

Naïve Bayes Classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

$j=1 \ldots m$

$P(X_j=0|Y=0)$
$P(X_j=1|Y=0)$
$P(X_j=0|Y=1)$
$P(X_j=1|Y=1)$

$4m+2$

Training

What is the Big-O of # of parameters we need to learn?

A. $O(m)$
B. $O(2^m)$
C. other

Naïve Bayes Classifier

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

for $j = 1, \ldots, m$:

$\hat{P}(X_j = 1|Y = 0)$,
$\hat{P}(X_j = 1|Y = 1)$

$\hat{P}(Y = 1)$

Use MLE or Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\log \hat{P}(Y) + \sum_{j=1}^{m} \log \hat{P}(X_j|Y) \right)$$

log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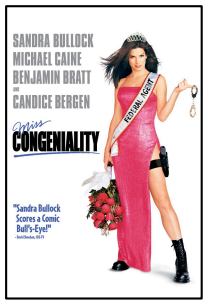
Classification terminology check

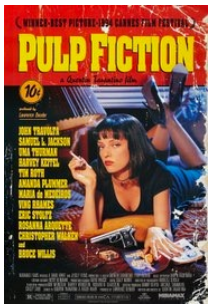
Training data: $(\boldsymbol{x}^{(1)}, y^{(1)}), (\boldsymbol{x}^{(2)}, y^{(2)}), \ldots, (\boldsymbol{x}^{(n)}, y^{(n)})$

Movie 1



Movie 2

...



Movie m

Output

User 1	1.	1	0	...	1	2.	1
User 2	3.	1	1	...	0		0
...				⋮			⋮
User n		0	4. 0	...	1		1

A. $\boldsymbol{x}^{(i)}$
B. $y^{(i)}$
C. $(\boldsymbol{x}^{(i)}, y^{(i)})$
D. $x_j^{(i)}$

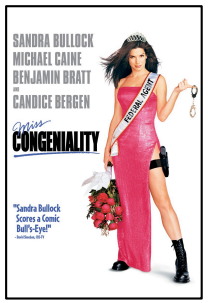
1: like movie
0: dislike movie

1 → A: $\boldsymbol{x}^{(1)}$
2 → B: $y^{(2)}$
3 - C: $(\boldsymbol{x}^{(2)}, y^{(2)})$
4 - D: $x_2^{(n)}$

Classification terminology check

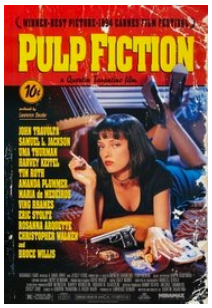
Training data: $(\boldsymbol{x}^{(1)}, y^{(1)}), (\boldsymbol{x}^{(2)}, y^{(2)}), \ldots, (\boldsymbol{x}^{(n)}, y^{(n)})$

Movie 1



Movie 2

...



Movie m

Output

User 1	1.	1	0	...	1	2.	1
User 2	3.	1	1	...	0		0
...				⋮			⋮
User n		0	4. 0	...	1		1

A. $\boldsymbol{x}^{(i)}$
B. $y^{(i)}$
C. $\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)$
D. $x_j^{(i)}$

1: like movie
0: dislike movie

i: i-th user
j: movie j

Predicting user TV preferences

Will a user like the Pokémon TV series?

Observe indicator variables $\boldsymbol{X} = (X_1, X_2)$:

$X_1 = 1$:
"likes Star Wars"

$X_2 = 1$:
"likes Harry Potter"

Output Y indicator:

$Y = 1$:
"likes Pokémon"

$$\text{Predict } \hat{Y} = \underset{y=\{0,1\}}{\arg\max}\, \hat{P}(Y \mid \boldsymbol{X})$$

Predicting user TV preferences

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Naïve Bayes Assumption $$P(\boldsymbol{X}|Y) = \prod_{j=1}^{m} P(X_j|Y)$$

Which probabilities do you need to estimate? How many are there?

- Brute Force Bayes
  (strawman, without NB assumption)

#probs needed $= 2 \cdot 2^2 + 2 = 10$ $O(2^m)$

- Naïve Bayes

During training, how to estimate the prob $\hat{P}(X_1 = 1, X_2 = 1|Y = 0)$ with MLE? with Laplace?

- Brute Force Bayes

MLE → $\frac{\#(X_1=1 \cap X_2=1 \cap Y=0)}{\#(Y=0)}$

- ~~Naïve Bayes~~

MAP = $\frac{\#(X_1=1 \cap X_2=1 \cap Y=0) + 1}{\#(Y=0) + 4}$

Predicting user TV preferences

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y)$$

Naïve Bayes Assumption $$P(\boldsymbol{X}|Y) = \prod_{j=1}^{m} P(X_j|Y)$$

Which probabilities do you need to estimate? How many are there?

- Brute Force Bayes (strawman, without NB assumption)
- Naïve Bayes

$P(X_1=1 \mid Y=0)$
$P(X_1=1 \mid Y=1)$
$P(X_2=0 \mid Y=0)$
$P(X_2=0 \mid Y=1)$

4
$X_1 = X$

$X_1 \to X_2$
4
$X_2 = X$

$P(Y=0)$
$P(Y=1)$

$4 \cdot m + 2$

1D
$O(m)$

During training, how to estimate the prob $\hat{P}(X_1 = 1, X_2 = 1|Y = 0)$ with MLE? with Laplace?

- Brute Force Bayes

$\hat{P}(X_1=1 \mid Y=0)\,\hat{P}(X_2=1 \mid Y=0)$

MLE → $\frac{P(X_1=1 \wedge Y=0)}{P(Y=1)} \cdot \frac{P(X_2=1 \wedge Y=0)}{P(Y=0)}$

- Naïve Bayes

MAP → $\frac{\#(X_1=1 \wedge Y=0)+1}{\#(Y=0)+2}$

(Strawman Brute Force) Multinomial MLE and MAP

Model: Multinomial, m outcomes:
p_j probability of outcome j

Observe: $n_j = \#$ of trials with outcome j
Total of $\sum_{j=1}^{m} n_j$ trials

MLE $\quad \hat{p}_j = \dfrac{n_j}{\sum_{j=1}^{m} n_j}$

Laplace estimate (MAP w/Laplace smoothing) $\quad \hat{p}_j = \dfrac{n_j + 1}{\sum_{j=1}^{m} n_j + m}$

X_1	X_2	Y
0	1	1
1	1	0
1	0	1
...	...	...
1	1	1

training data

$\hat{P}(X_1 = 1\, X_2 = 1 | Y = 0)$

(Naïve Bayes) Multinomial MLE and MAP

Model: Multinomial, m outcomes:
p_j probability of outcome j

Observe: $n_j = \#$ of trials with outcome j
Total of $\sum_{j=1}^{m} n_j$ trials

MLE

$$\hat{p}_j = \frac{n_j}{\sum_{j=1}^{m} n_j}$$

Laplace estimate (MAP w/Laplace smoothing)

$$\hat{p}_j = \frac{n_j + 1}{\sum_{j=1}^{m} n_j + m}$$

X_1	X_2	Y
0	1	1
1	1	0
1	0	1
...	...	...
1	1	1

training data

$$\hat{P}(X_1 = 1\, X_2 = 1 | Y = 0)$$

Ex 1. Naïve Bayes Classifier (MLE)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$\forall i$: $\hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$
$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$
$\hat{P}(Y = 1), \hat{P}(Y = 0)$

Use **MLE** or Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training: Naïve Bayes for TV shows (MLE)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Training data counts

1. How many datapoints (n) are in our training data?
2. Compute MLE estimates for $\hat{P}(X_1|Y)$:

Y \ X_1	0	1
0	$\hat{P}(X_1 = 0\|Y = 0)$	$\hat{P}(X_1 = 1\|Y = 0)$
1	$\hat{P}(X_1 = 0\|Y = 1)$	$\hat{P}(X_1 = 1\|Y = 1)$

Training: Naïve Bayes for TV shows (MLE)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
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Y \ X_1	0	1
0	3	10
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Y \ X_2	0	1
0	5	8
1	7	10

Training data counts

1. How many datapoints (n) are in our training data?
2. Compute MLE estimates for $\hat{P}(X_1|Y)$:

Y \ X_1	0	1
0		
1		

Training: Naïve Bayes for TV shows (MLE)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Y	
0	13
1	17

Training data counts

Y \ X_1	0	1
0	0.23	0.77
1	0.24	0.76

Y \ X_2	0	1
0	$5/13 \approx 0.38$	$8/13 \approx 0.62$
1	$7/17 \approx 0.41$	$10/17 \approx 0.59$

Y	
0	$13/30 \approx 0.43$
1	$17/30 \approx 0.57$

(from last slide)

Training : Naïve Bayes for TV shows (MLE)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

Y \ X_1	0	1
0	0.23	0.77
1	0.24	0.76

Y \ X_2	0	1
0	0.38	0.62
1	0.41	0.59

Y	
0	0.43
1	0.57

Now that we've trained and found parameters,
It's time to classify new users!

Ex 1. Naïve Bayes Classifier (MLE)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$$\forall i: \hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(Y = 1), \hat{P}(Y = 0)$$

Use **MLE** or Laplace (MAP)

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Testing: Naïve Bayes for TV shows (MLE)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

Y \ X_1	0	1
0	0.23	0.77
1	0.24	0.76

Y \ X_2	0	1
0	0.38	0.62
1	0.41	0.59

Y	
0	0.43
1	0.57

Suppose a new person "likes Star Wars" ($X_1 = 1$) but "dislikes Harry Potter" ($X_2 = 0$).

Will they like Pokemon? Need to predict Y:

$$\hat{Y} = \arg\max_{y=\{0,1\}} \hat{P}(\boldsymbol{X}|Y)\hat{P}(Y) \quad = \arg\max_{y=\{0,1\}} \hat{P}(X_1|Y)\hat{P}(X_2|Y)\hat{P}(Y)$$

If $Y = 0$: $\hat{P}(X_1 = 1|Y = 0)\hat{P}(X_2 = 0|Y = 0)\hat{P}(Y = 0) = 0.77 \cdot 0.38 \cdot 0.43 = 0.126$

If $Y = 1$: $\hat{P}(X_1 = 1|Y = 1)\hat{P}(X_2 = 0|Y = 1)\hat{P}(Y = 1) = 0.76 \cdot 0.41 \cdot 0.57 = 0.178$

Since term is greatest when Y = 1, predict $\hat{Y} = 1$

Ex 2. Naïve Bayes Classifier (MAP)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$\forall i$: $\hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$ Use MLE or
$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$ Laplace (MAP)
$\hat{P}(Y = 1), \hat{P}(Y = 0)$

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

(note the same as before)

Training: Naïve Bayes for TV shows (MAP)

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Training data counts

What are our MAP estimates using Laplace smoothing for $\hat{P}(X_j|Y)$?

$\hat{P}(X_j = x|Y = y)$:

A. $\frac{\#(X_j=x, Y=y)}{\#(Y=y)}$

B. $\frac{\#(X_j=x, Y=y)+1}{\#(Y=y)+2}$

C. $\frac{\#(X_j=x, Y=y)+1}{\#(Y=y)+4}$

D. other

Training: Naïve Bayes for TV shows (MAP)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{i=1}^{m} \hat{P}(X_i|Y) \right) \hat{P}(Y)$$

Observe indicator vars. $\boldsymbol{X} = (X_1, X_2)$:

- X_1: "likes Star Wars"
- X_2: "likes Harry Potter"

Predict Y: "likes Pokémon"

Y \ X_1	0	1
0	3	10
1	4	13

Y \ X_2	0	1
0	5	8
1	7	10

Y	
0	13
1	17

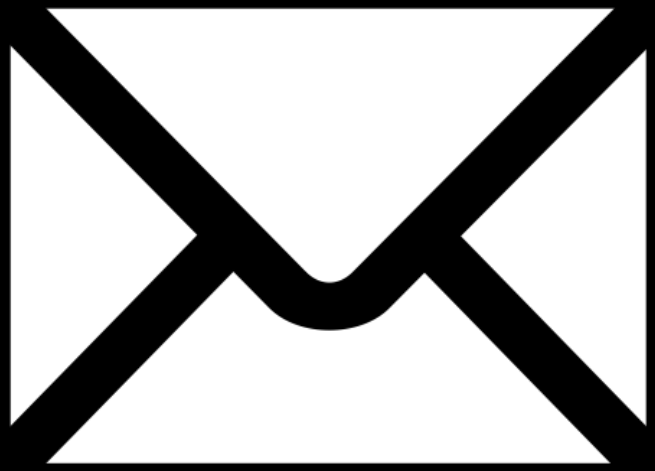
Training data counts

Y \ X_1	0	1
0	0.27	0.73
1	0.26	0.74

Y \ X_2	0	1
0	0.40	0.60
1	0.42	0.58

In practice:

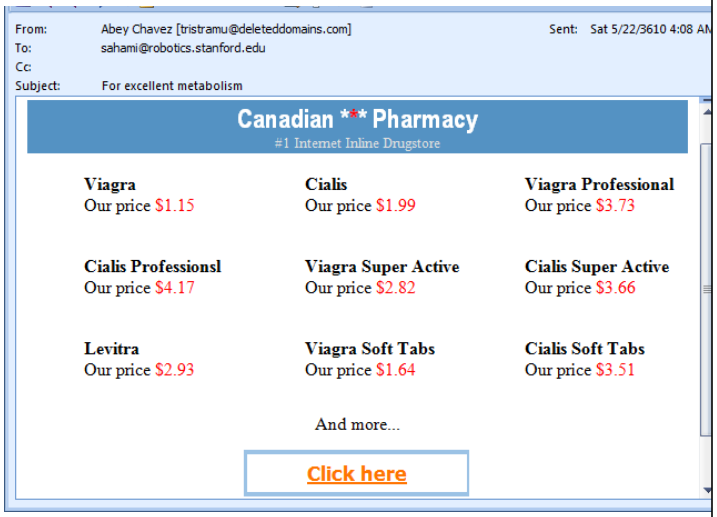
- We use Laplace for $\hat{P}(X_j|Y)$ in case some events $X_j = x_j$ don't appear
- We don't use Laplace for $\hat{P}(Y)$, because all class labels should appear reasonably often



and Learn

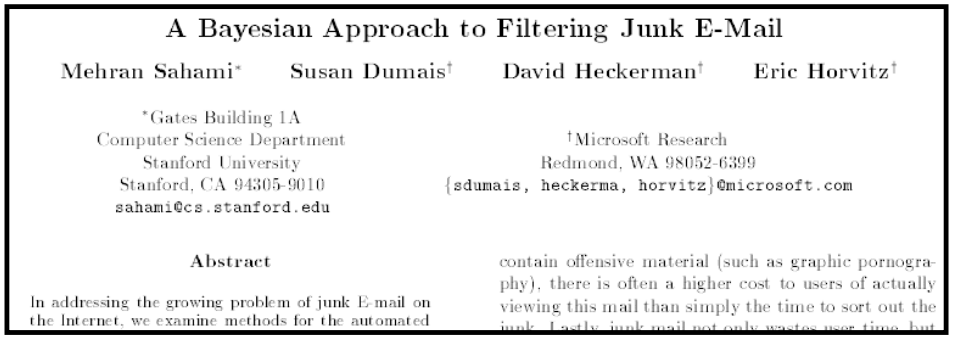
naively

What is Bayes doing in my mail server?



Let's get Bayesian on your spam:

```
Content analysis details:   (49.5 hits, 7.0 required)
 0.9 RCVD_IN_PBL            RBL: Received via a relay in Spamhaus PBL
                            [93.40.189.29 listed in zen.spamhaus.org]
 1.5 URIBL_WS_SURBL         Contains an URL listed in the WS SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_JP_SURBL         Contains an URL listed in the JP SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_OB_SURBL         Contains an URL listed in the OB SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_SC_SURBL         Contains an URL listed in the SC SURBL blocklist
                            [URIs: recragas.cn]
 2.0 URIBL_BLACK            Contains an URL listed in the URIBL blacklist
                            [URIs: recragas.cn]
 8.0 BAYES_99               BODY: Bayesian spam probability is 99 to 100%
                            [score: 1.0000]
```



A Bayesian Approach to Filtering Junk E-Mail

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Abstract

In addressing the growing problem of junk E-mail on the Internet, we examine methods for the automated

contain offensive material (such as graphic pornography), there is often a higher cost to users of actually viewing this mail than simply the time to sort out the

Ex 3. Naïve Bayes Classifier (m, n large)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{i=1}^{m} \hat{P}(X_i|Y) \right) \hat{P}(Y)$$

Training

$\forall i$: $\hat{P}(X_i|Y = 0)$, Use MLE or

$\hat{P}(Y = 0)$, Laplace (MAP)

What changes are necessary?

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{i=1}^{m} \hat{P}(X_i|Y) \right) \hat{P}(Y)$$

Email classification

Goal Based on email content $\boldsymbol{X}$, predict if email is spam or not.

Features Consider a lexicon m words (for English: $m \approx 100{,}000$).

$\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$, m indicator variables

$X_j = 1$ if word j appeared in document

Output $Y = 1$ if email is spam

Note: m **is huge**. Make Naïve Bayes assumption: $P(\boldsymbol{X}|\text{spam}) = \prod_{j=1}^{m} P(X_j|\text{spam})$

Appearances of words in email are conditionally independent given the email is spam or not

Training: Naïve Bayes Email classification

Train set n previous emails $\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$

$\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}\right)$ for each word, whether it appears in email i

$y^{(i)} = 1$ if spam, 0 if not spam

Note: m is huge.

Which estimator should we use for $\hat{P}\left(X_j | Y\right)$?

A. MLE
B. Laplace estimate (MAP)
C. Other MAP estimate
D. Both A and B

Training: Naïve Bayes Email classification

Train set n previous emails $\left(\boldsymbol{x}^{(1)}, y^{(1)}\right), \left(\boldsymbol{x}^{(2)}, y^{(2)}\right), \ldots, \left(\boldsymbol{x}^{(n)}, y^{(n)}\right)$

$\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}\right)$ for each word, whether it appears in email i

$y^{(i)} = 1$ if spam, 0 if not spam

Note: m is huge.

Which estimator should we use for $\hat{P}\left(X_j | Y\right)$?

A. MLE
B. Laplace estimate (MAP)
C. Other MAP estimate
D. Both A and B

Many words are likely to not appear at all in the training set!

Ex 3. Naïve Bayes Classifier (m, n large)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$$\forall j: \hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y = 0),$$
$$\hat{P}(Y = 1), \hat{P}(Y = 0)$$

Use MLE or **Laplace (MAP)**

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=} \right.$$

Laplace (MAP) estimates avoid estimating 0 probabilities for events that don't occur in your training data.

Testing: Naïve Bayes Email classification

For a new email:

- Generate $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$
- Classify as spam or not using Naïve Bayes assumption

Note: m is huge.

Suppose train set size n also huge (many labeled emails).

Can we still use the below prediction?

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Testing: Naïve Bayes Email classification

For a new email:

- Generate $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$
- Classify as spam or not using Naïve Bayes assumption

Note: m is huge.

Suppose train set size n also huge (many labeled emails).

Can we still use the below prediction?

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Will probably lead to underflow!

Ex 3. Naïve Bayes Classifier (m, n large)

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\prod_{j=1}^{m} \hat{P}(X_j|Y) \right) \hat{P}(Y)$$

Training

$$\forall i: \hat{P}(X_j = 1|Y = 0), \hat{P}(X_j = 0|Y =$$
$$\hat{P}(X_j = 1|Y = 1), \hat{P}(X_j = 0|Y =$$
$$\hat{P}(Y = 1), \hat{P}(Y = 0)$$

Use sums of log-probabilities for numerical stability.

Testing

$$\hat{Y} = \arg\max_{y=\{0,1\}} \left(\log \hat{P}(Y) + \sum_{j=1}^{m} \log \hat{P}(X_j|Y) \right)$$

How well does Naïve Bayes perform?

After training, you can test with another set of data, called the **test set**.

- Test set also has known values for Y so we can see how often we were right/wrong in our predictions $\hat{Y}$.

Typical workflow:

- Have a dataset of 1789 emails (1578 spam, 211 ham)
- Train set: First 1538 emails (by time)
- Test set: Next 251 messages

Evaluation criteria on test set:

$$\textbf{precision} = \frac{(\#\text{ correctly predicted class } Y)}{(\#\text{ predicted class } Y)}$$

$$\textbf{recall} = \frac{(\#\text{ correctly predicted class } Y)}{(\#\text{ real class } Y \text{ messages})}$$

	Spam		Non-spam	
	Prec.	Recall	Prec.	Recall
Words only	97.1%	94.3%	87.7%	93.4%
Words + addtl features	100%	98.3%	96.2%	100%

What are precision and recall?

Accuracy (# correct)/(# total) sometimes just doesn't cut it.

Precision: Of the emails you predicted as spam, how many are *truly* spam?

Measure of false positives

Recall: Of the emails that are truly spam, how many did you predict?

Measure of false negatives

More on Wikipedia (https://en.wikipedia.org/wiki/Precision_and_recall)