Problem Set #1 Solutions

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1. a. 10! = 3,628,800. Any arrangement of 10 entities is possible.

   b. 2! · 9! = 725,760. Since A and B have to sit together, you can think of them as "one" person in the arrangement (giving 9!), then A and B can be permuted in 2! ways.

   c. 2 · 5! · 5! = 28,800. The 5 adults must either sit in seats 1, 3, 5, 7, 9 or 2, 4, 6, 8, 10 with the children occupying the remaining 5 seats, giving 2 possibilities. Then the 5 adults can be permuted among their 5 designated seats, and similarly for the 5 children.

   d. 5! · (2!)^5 = 3,840. The 5 couples can each be thought of as one entity, yielding 5! permutations. Now within each of the 5 pairs of couples, the 2 people can be permuted in 2! ways, yielding (2!)^5 possibilities.

2. a. \left(\binom{8}{3}\right)\left(\binom{6}{3}\right) = 1120 possibilities. From the 8 bird species you choose 3. From the 6 reptile species you choose 3.

   b. \left(\binom{6}{3}\right) \cdot \left(\binom{8}{3} + \binom{2}{1} \cdot \binom{6}{2}\right) = 1000 possibilities. From the 6 reptile species you always choose 3. From the 8 bird species, you either choose 3 from the 6 species that have no issues with each other, or you choose 2 from the 6 that have no issues, and then choose 1 of the 2 species that cannot be put together.

   Alternatively, an equivalent answer is \left(\binom{6}{3}\right) \cdot \left[\binom{8}{3} - \binom{6}{3}\right]. We first choose 3 of the 6 reptile species. Then, out of the \left(\binom{8}{3}\right) possible ways to choose 3 of 8 bird species, we subtract the forbidden combination, where we choose the 2 species that have issues and a third bird from the remaining 6.

   c. \left(\binom{5}{3}\right) \cdot \left(\binom{3}{3}\right) + \left(\binom{5}{3}\right) + \left(\binom{5}{3}\right) = 910 possibilities. From the 7 bird species and 5 reptiles who have no issues you can choose 3 from each group. If you select the 1 bird species who cannot be with a particular reptile species, then you can choose 2 other birds from the remaining 7, and choose any 3 of the 5 reptile species who have no issue. If you select the 1 reptile species that cannot be with a particular bird species, then you can choose 3 birds from the 7 that have no issue, and then choose any 2 of the 5 remaining reptiles.

   Alternatively, an equivalent answer is \left(\binom{5}{3}\right) \cdot \left(\binom{6}{3}\right) - \left(\binom{2}{1}\right) \cdot \left(\binom{5}{2}\right) = 910. Using part (a), we subtract out the forbidden combination, which is choosing the 1 bird species and 1 reptile who have issues with each other, then choose 2 other birds and 2 other reptiles.

3. a. \left(\binom{13}{3}\right) = 286. Since you must invest the minimum in all the opportunities, you must invest 1 + 2 + 3 + 4 = $10 million. Then you have $10 million left to invest in the 4 opportunities, which has the same number of possibilities as solutions to \(x_1 + x_2 + \cdots + x_4 = 10\).
b. The following solution follows the interpretation where a minimum investment does not need to be made in all four companies. We will also accept the solution where a minimum investment is first made in all four companies, and then the rest of the money is allocated according to the rules listed in the question.

\[ \binom{13}{1} + \binom{13}{2} + \binom{14}{2} + \binom{15}{2} + \binom{16}{2} = 680. \]

First, you still need to consider all the cases where you invest in all 4 opportunities (same as in part (a)), then you can use a similar analysis from part (a) to consider the number of possibilities if you (i) didn’t invest in company 1, (ii) didn’t invest in company 2, (iii) didn’t invest in company 3, and (iv) didn’t invest in company 4, respectively. Summing all these possibilities give the complete answer.

c. \[ \sum_{j=0}^{k} \binom{j+3}{3}. \]

Since \[ \sum_{i=1}^{4} x_i = j, \] where \( 0 \leq j \leq k \). For any particular value \( j \) million to invest, the number of investment strategies is equivalent to the number of non-negative integer solutions to \( x_1 + x_2 + x_3 + x_4 = j \), or \( \binom{j+3}{3} \). To count all possible strategies, we sum this quantity over all values of \( j \) from 0 to \( k \).

Alternatively, an equivalent answer is: \( \binom{k+4}{4} \). This answer comes from conceptually adding an extra element \( x_5 \) to the vector to represent the remainder for investments less than \( k \), and then counting the number of non-negative integer solutions to \( x_1 + x_2 + x_3 + x_4 + x_5 = k \), where as before, we first invest the minimum in all opportunities. Note that if we were to subtract \( x_5 \) from both sides of that equation, we get \( x_1 + x_2 + x_3 + x_4 + x_5 = k - x_5 \), which is equivalent to \( x_1 + x_2 + x_3 + x_4 \leq k \), since \( x_5 \) is a non-negative integer \( \leq k \).

4. a. \( \binom{4}{1} \left( \binom{13}{5} \right) \left( \binom{52}{s} \right) \). Choose 1 of the 4 suits for the flush, then choose 5 of the 13 cards in that suit.

b. \( \binom{13}{1} \binom{4}{2} \left( \binom{4}{1} \binom{44}{s} \right) \left( \binom{52}{s} \right) \). Choose 2 of the 13 card ranks for the ranks for the two pairs, and choose 2 of the 4 cards of each rank to form each pair. Then choose 1 of the 44 (=52-8) cards that does not share the same rank as one of the 2 chosen pairs to complete the hand.

c. \( \binom{13}{1} \binom{4}{4} \left( \binom{48}{s} \right) \left( \binom{52}{s} \right) \). Choose 1 of the 13 card ranks for the rank of the four of a kind, and choose 4 of the 4 cards of that rank to form the four of a kind. Then choose 1 of the 48 (=52-4) cards that does not share the same rank as the four of a kind to complete the hand.

5. a. \( \binom{6}{2} \binom{6}{3} \left( \binom{6}{3} \right) \left( \binom{52}{s} \right) \left( \binom{6}{s} \right) / 6^6 \). First, select the 2 different numbers (out of 6) that are each rolled three times. Then, in the 6 rolls, there are 2 sets of 3 indistinguishable rolls (i.e., each number is rolled three times). Since the die rolls are distinct, there are \( 6^6 \) total outcomes in the sample space for rolling the die 6 times.

b. \[ \left[ \binom{6}{1} \left( \binom{6}{3} \right) 5^3 - 2 \cdot \binom{6}{2} \left( \binom{6}{3} \right) \right] / 6^6. \] First, we select 1 of the 6 numbers that will appear exactly 3 times (call this number \( x \)). Then, we select the 3 rolls (out of 6) where \( x \) is rolled. The other 3 rolls can be any of the 5 other numbers on the die that are not \( x \). That’s the source of the \( \binom{6}{1} \left( \binom{6}{3} \right) 5^3 \) term.

However, among those \( \binom{6}{1} \left( \binom{6}{3} \right) 5^3 \) sextuplets are several **double** triples, which are specifically forbidden by the problem statement. What’s worse is that each double triple is counted **twice**! Consider, for example, the configuration 1, 1, 1, 2, 2, 2:
• 1, 1, 1, 2, 2, 2 is counted once when 1 is constrained to be the number appearing three times, and the first three positions were chosen to hold those 1’s while the remaining positions were incidentally populated with three 2’s.

• 1, 1, 1, 2, 2, 2 is counted again (!) when 2 is constrained to be the number appearing three times, and the last three positions were chosen to hold those 2’s while the first three positions were incidentally populated with three 1’s.

That means we need to subtract the answer to part (a) not once, but twice, to arrive at an accurate answer, for an overall count of $\binom{6}{1,1,1,2,2,2}$. Tricky!!

Similar to part (a), there are $6^6$ total outcomes in the sample space for rolling the die 6 times, so that’s what goes in the denominator.

An alternative solution to create the event space would be to decompose the task of rolling some number exactly 3 times into two mutually exclusive cases:

• Roll 4 unique numbers. First, choose the 4 unique numbers and select which one will be the triplet. Then, order the 6 rolls, where 3 are the indistinct rolls: $\binom{6}{4} \binom{6}{1} \frac{6!}{3!}$.

• Roll 3 unique numbers. First, choose the 3 unique numbers and then separate them into three groups: the triplet, the pair, and the single. Then, order the 6 rolls with two indistinct groups, the triplet and the pair: $\binom{6}{3} \binom{3}{1,1,1} \frac{6!}{3!2!}$.

Summing up these cases results in the same event space as above.

6. \( \binom{N}{k} \binom{M}{r-k} \) or equivalently \( \binom{r}{k} \binom{M+N-r}{N-k} \binom{M+N}{N} \). The first expression gives the answer by focusing on forming the first r bits of the received message. If we were to order all the N 1’s with integers from 1 to N, we could select k of them to be in the first r bits of the message, yielding \( \binom{N}{k} \) possibilities. Similarly, if we were to order all the M 0’s with integers from 1 to M, we could select \( r-k \) of them to be in the first r bits of the message, yielding \( \binom{M}{r-k} \) possibilities. Multiplying, we get \( \binom{N}{k} \binom{M}{r-k} \) total ways of arranging the first r bits, having exactly k 1’s. The number of ways to form the first r bits of the message (with no constraints), is to think of numbering the \( M+N \) bits with integers from 1 to \( M+N \), and then choosing r of these integers to form the first r bits, yielding \( \binom{M+N}{r} \) for the denominator.

Alternatively, we can get a mathematically equivalent answer to the problem (the second solution given above) by determining how to construct the whole binary string, so as to have k 1’s in the first r bits. Consider the message being sent as have \( M+N \) ‘slots’ to fill with 0’s and 1’s. The denominator is determined by choosing N of the \( M+N \) slots to put 1’s into. The numerator is determined by choosing k of the first r slots to put 1’s in, then choosing \( N-k \) of the remaining \( M+N-r \) slots to put 1’s in. Once all slots for 1’s are determined, the 0’s uniquely fill the remaining unfilled slots.

7. \( \binom{12}{4} \cdot \binom{8}{3} \cdot \frac{20!}{2^2!4^3!} / 12^{20} \). We select 4 of the 12 users to receive exactly 2 emails each, and 3 of the remaining 8 users to receive exactly 4 emails each. Treating all the emails as distinct objects, the 20 emails are now being grouped into 4 sets of size 2 and 3 sets of size 4 (each set representing the emails received by a particular user). This is a multinomial coefficient given by
The total number of ways that the 20 emails could have been distributed to 10 users (again treating the emails as distinct objects for consistency) is $12^{20}$.

8.  
   a. $\frac{1}{n}$. You can think of this as analogous to arranging the passwords linearly, with a numbering from 1 to $n$ on the passwords. We want to determine the probability that the correct password is given the number $k$. This is $\frac{1}{n}$, since the number $k$ is equally likely to be assigned to any of the $n$ passwords.

   b. $\frac{(n-1)^{k-1}}{n^k}$. There are $n^k$ ways of choosing any password for the first $k$ attempts. Out of those, the ways of getting the right password for the first time on the $k$-th attempt are the number of ways of choosing any wrong password for each of the first $k-1$, which is $(n-1)^{k-1}$, times the number of ways of getting the right password on the $k$-th try, which is 1.

   This can also be solved using independence. To successfully log in on exactly the $k$-th try, the hacker would have had to pick one of the wrong passwords (where the probability of picking one of the wrong passwords on a particular attempt is $\frac{n-1}{n}$) on each of the $k-1$ previous tries, yielding $\left(\frac{n-1}{n}\right)^{k-1}$. Then, on the $k$-th try, the hacker would have a $\frac{1}{n}$ probability of selecting the right password, giving the final answer.

9.  
   a. $\frac{2^{n-1}}{n!}$. First, note that there are $n!$ orderings of the values 1 through $n$ we insert into the BST. Now, the only way to produce a completely degenerate BST is to have every successive insertion be either the minimal or maximal value of the values remaining to be inserted. This means we have one of 2 choices for every successive element we insert into the BST (minimal or maximal), except for the last element inserted (since it is both the minimal and maximal remaining element). Since we insert $n$ elements total, we have $2^{n-1}$ insertion orderings that can produce a completely degenerate BST.

   b. $n = 9$ is the smallest value such that $\frac{2^{n-1}}{n!} < 0.001$. Specifically, $\frac{2^8}{9!} \approx 0.000705$. Note how rapidly the probability of producing a degenerate BST decreases with the number of elements inserted.