Section #6: Uncertainty Theory

Based on the work of many CS109 instructors and course staff members.

1 Warmups

1.1 Sum of Random Variables

1. What is the distribution (with name and parameter(s)) of the average of \( n \) i.i.d. random variables, \( X_1, \ldots, X_n \), each with mean \( \mu \) and variance \( \sigma^2 \)?

2. For each \( X \) and \( Y \) below, let \( X \) and \( Y \) be independent.
   
   a. Let \( X \sim N(\mu_1, \sigma_1^2) \) and \( Y \sim N(\mu_2, \sigma_2^2) \). What is \( \mu \) and \( \sigma^2 \) for \( X + Y \sim N(\mu, \sigma^2) \)?
   
   b. Let \( X \sim \text{Uni}(0, 1) \) and \( Y \sim \text{Uni}(0, 1) \). What is the PDF for \( X + Y \)?
   
   c. In general, two random variables \( X \) and \( Y \), what is the PDF \( f \) of \( X + Y \)?

1.2 Sample and Population Mean

Computing the sample mean is similar to the population mean: sum all available points and divide by the number of points. However, sample variance is slightly different from population variance.

1. What is the difference between the population variance, \( \sigma^2 \), and sample variance, \( S^2 \)? What is the difference between the sample variance, \( S^2 \), and variance of the sample mean, \( \text{Var}(\bar{X}) \)?

2. Consider the equation for population variance, and an analogous equation for sample variance.

\[
\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \quad \text{(1)}
\]

\[
S^2_{\text{biased}} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad \text{(2)}
\]

\( S^2_{\text{biased}} \) is a random variable to estimate the constant \( \sigma^2 \). Because it is biased, \( E[S^2_{\text{biased}}] \neq \sigma^2 \). Is \( E[S^2_{\text{biased}}] \) greater or less than \( \sigma^2 \)?

3. Consider an alternative Random Variable, \( S^2_{\text{unbiased}} \) (known simply as \( S^2 \) in class). The technique of unbiasing variance is known as Bessel’s correction. Write the \( S^2_{\text{unbiased}} \) equation.

4. (Optional) Can you think of any examples in which the biased estimator might be preferred over the unbiased estimator?
1.3 Beta

1. Suppose you have a coin where you have no prior belief on its true probability of heads \( p \). How can you model this belief as a Beta distribution?

2. Suppose you have a coin which you believe is fair, with “strength” \( \alpha \). That is, pretend you’ve seen \( \alpha \) heads and \( \alpha \) tails. How can you model this belief as a Beta distribution?

3. Now suppose you take the coin from the previous part and flip it 10 times. You see 8 heads and 2 tails. How can you model your posterior belief of the coin’s probability of heads?

2 Problems

2.1 Beta Sum

What is the distribution of the sum of 100 i.i.d. Betas? Let \( X \) be the sum

\[ X = \sum_{i=1}^{100} X_i \]

where each \( X_i \sim \text{Beta}(a = 3, b = 4) \)

Note the variance of a Beta:

\[ \text{Var}(X_i) = \frac{ab}{(a + b)^2(a + b + 1)} \]

where \( X_i \sim \text{Beta}(a, b) \)

2.2 Food for Thought

Karel the dog eats an unpredictable amount of food. Every day, the dog is equally likely to eat between a continuous amount in the range 100 to 300 grams. How much Karel eats is independent of all other days. You only have 6.5kg of food for the next 30 days. What is the probability that 6.5kg will be enough for the next 30 days?