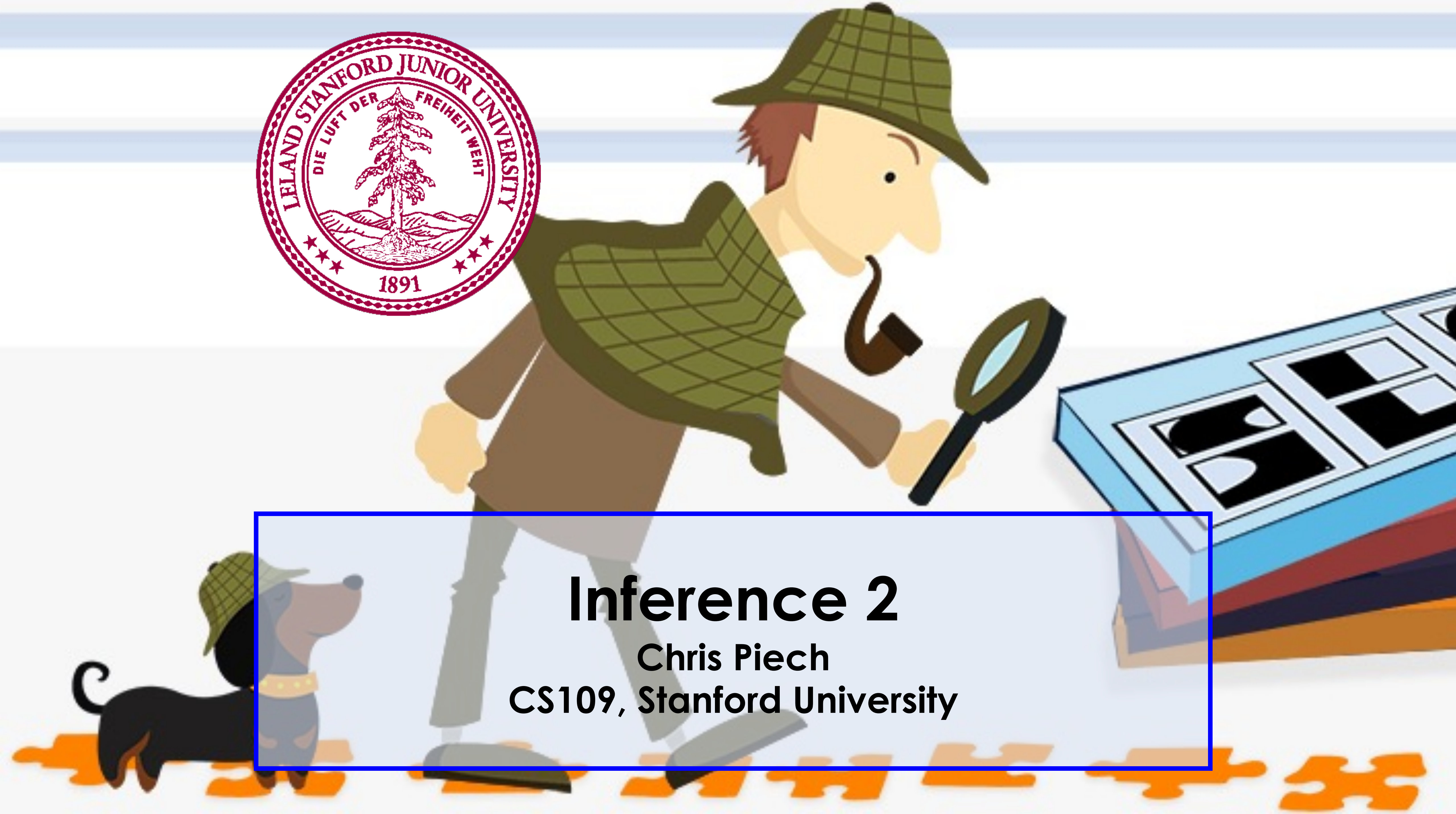
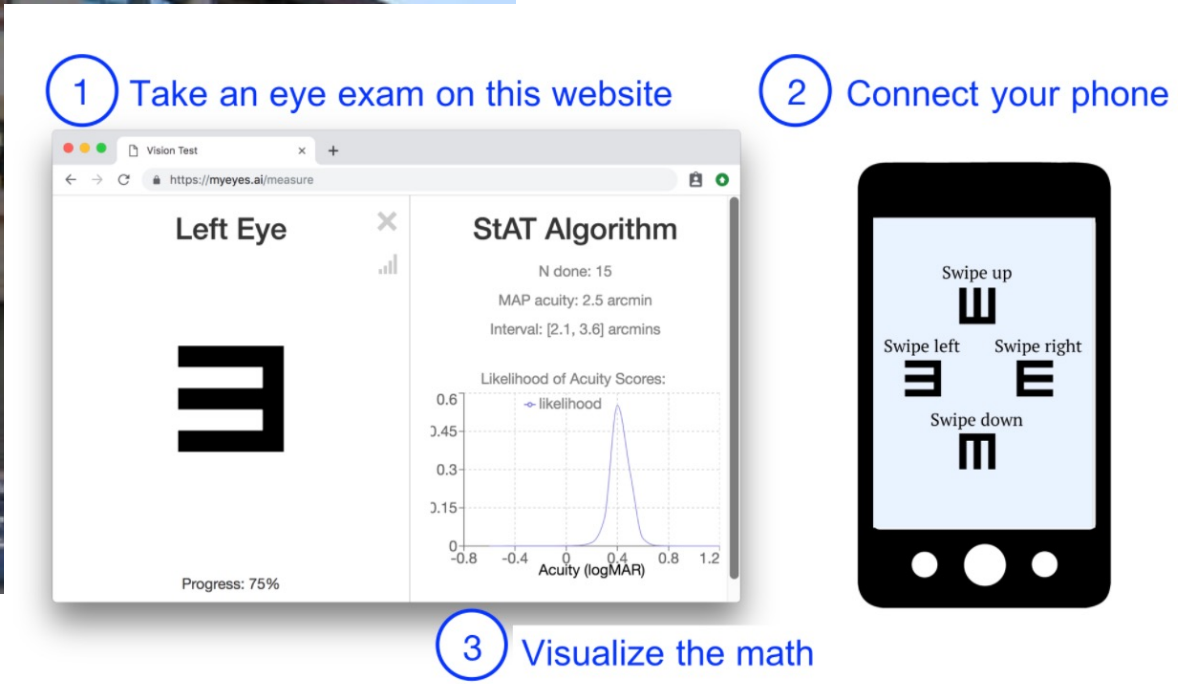


Inference 2

Chris Piech
CS109, Stanford University

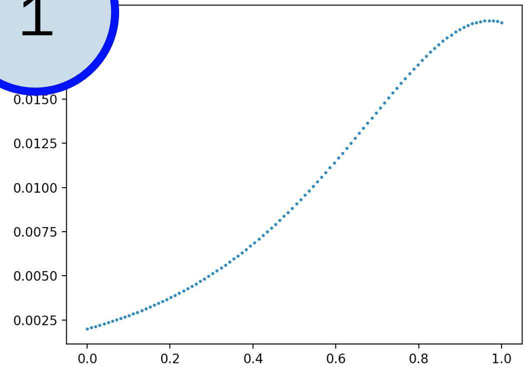
Today: Stanford Eye Test



I always wanted to make this a class demo

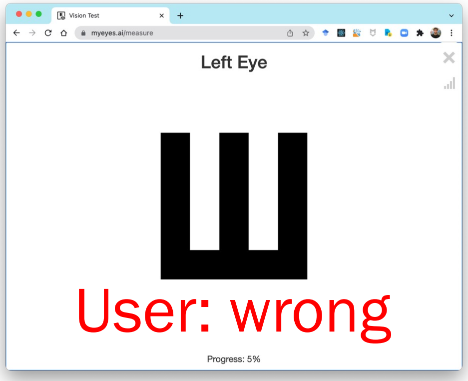
Stanford Eye Test

1



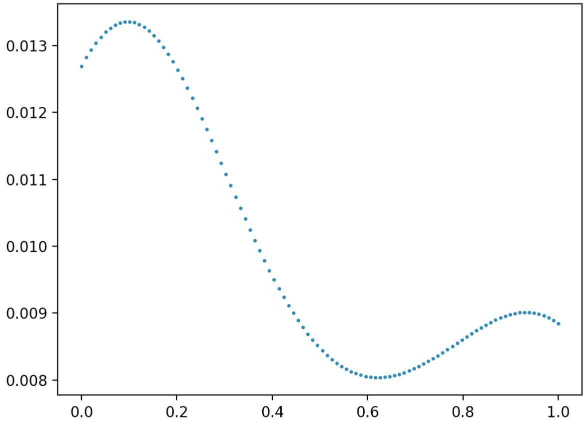
$$P(A = a)$$

2



Observation E

3



$$P(A = a | E)$$

Midterm **Tuesday** Nov 1st, 7pm

CEMEX:

A through L

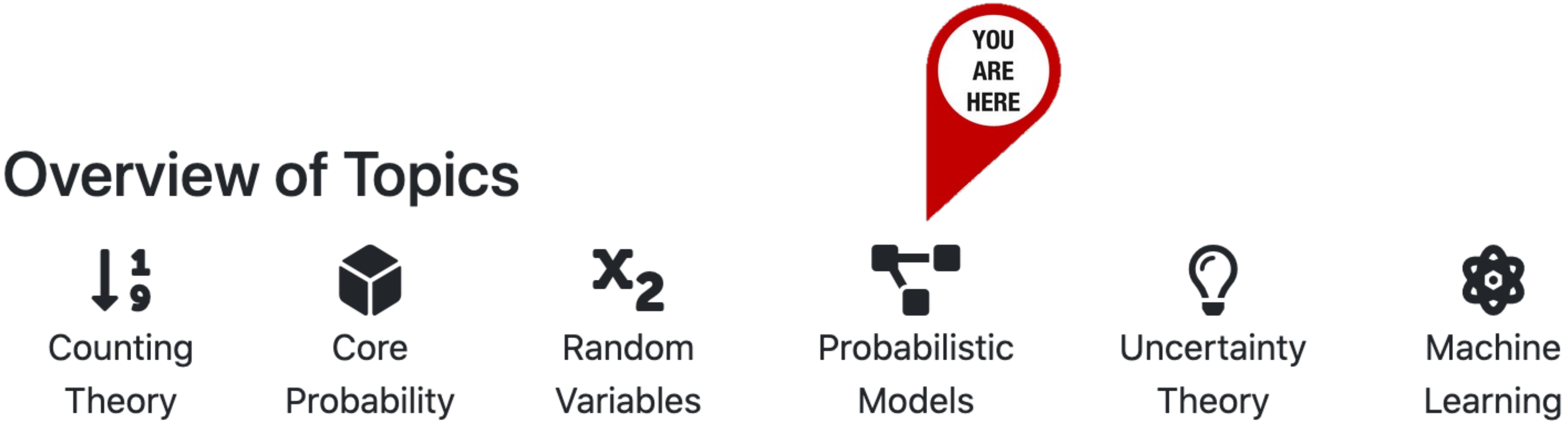
Hewlett 201:

M through Z

Midterm **Wednesday** Nov 2nd, 8pm

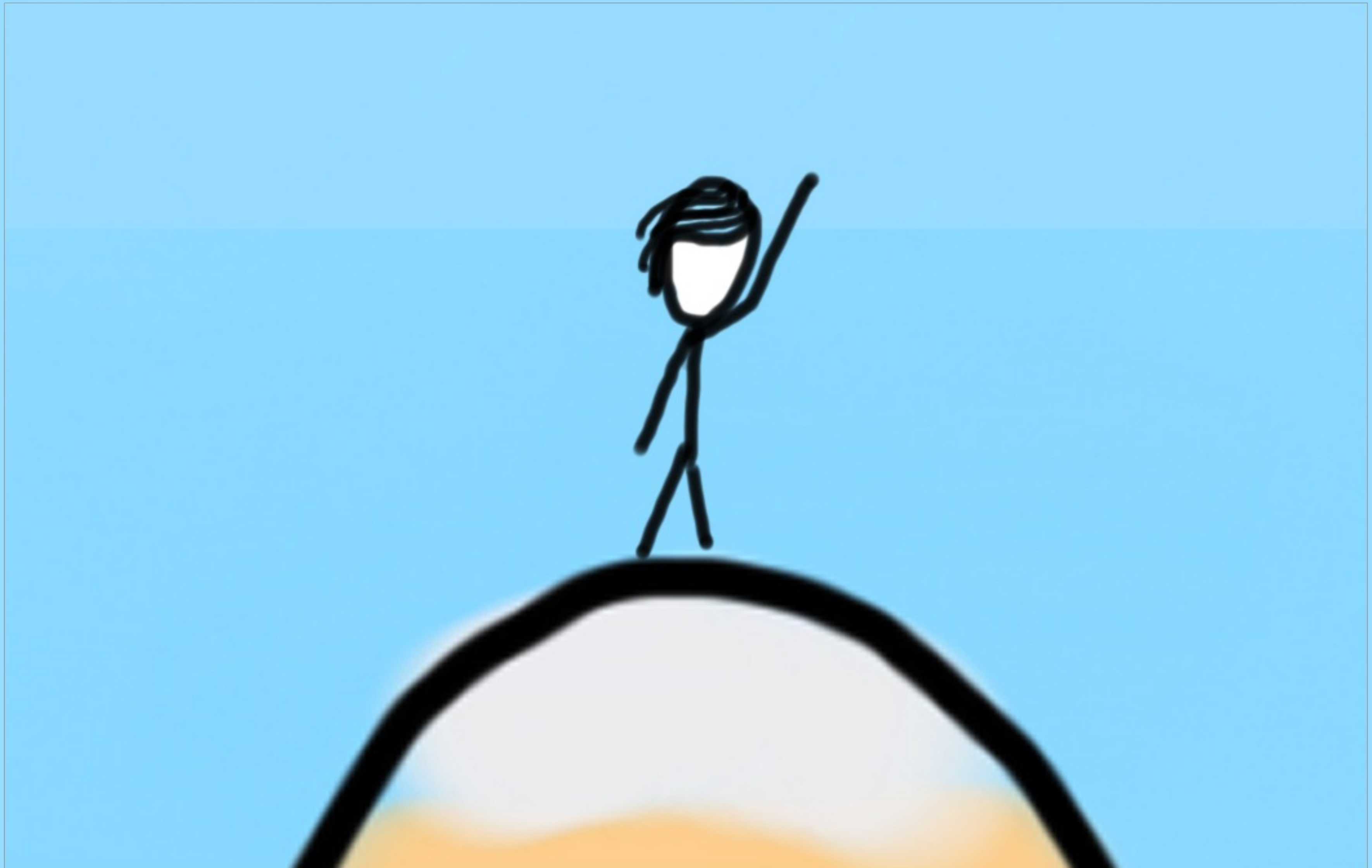
Hewlett 200:
Last Name A-Z

Where are we in CS109?



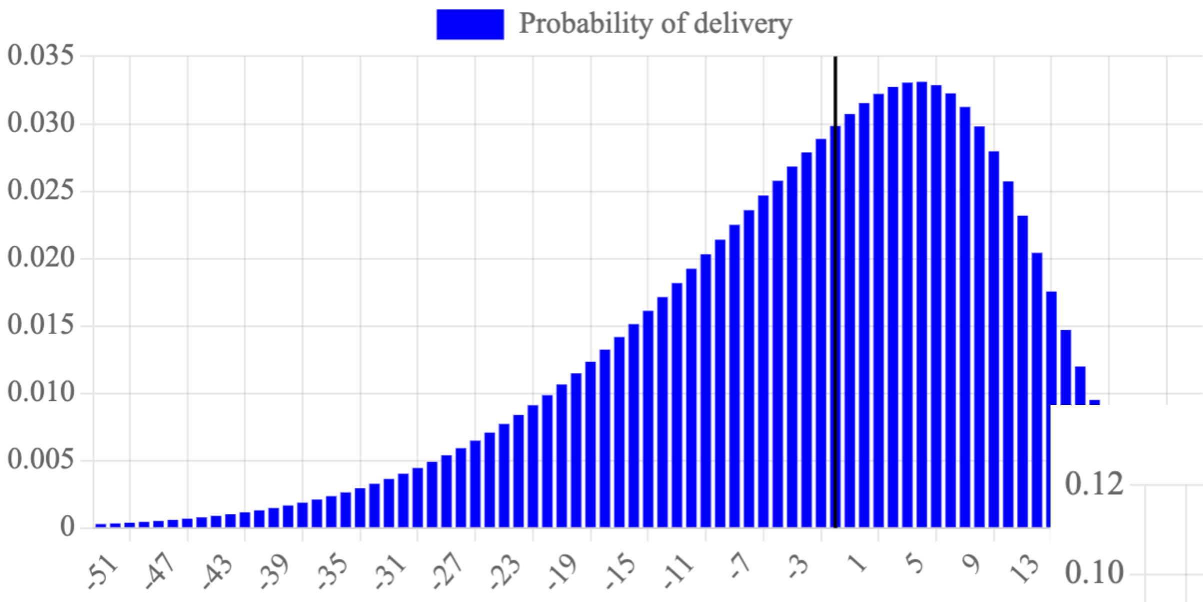
Learning Goals

1. Combine Bayes Theorem and Random Variables

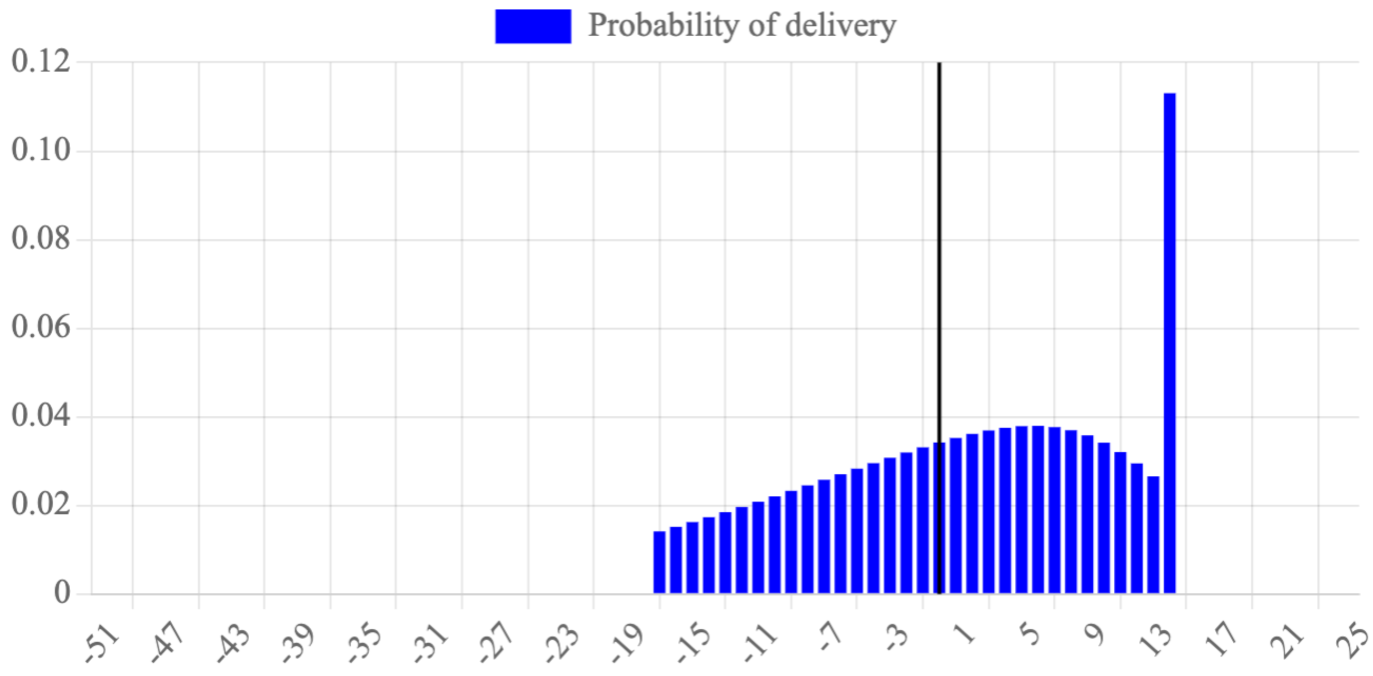


Review

Another example: Baby delivery



Its 19 days until the due date and no baby



For each value *d:*

$$P(D = d | \text{no child so far})$$

$$= \frac{P(\text{no child so far} | D = d) P(D = d)}{P(\text{no child so far})}$$

Bayes Theorm with Discrete

Let M be a **discrete** random variable

Let N be a **discrete** random variable

$$P(M=2|N=3) = \frac{P(N=3|M=2)P(M=2)}{P(N=3)}$$

More generally

$$P(M=m|N=n) = \frac{P(N=n|M=m)P(M=m)}{P(N=n)}$$

Shorthand notation

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

All the Bayes Belong to Us

M,N are discrete. X, Y are continuous

OG Bayes

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

Mix Bayes #1

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

Mix Bayes #2

$$P(n|x) = \frac{f(x|n)P(n)}{f(x)}$$

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

LOTP? Chain Rule? You can play too!

N is discrete. **X is continuous**

$$f(N = n, X = x) = f(X = x|N = n)P(N = n)$$

$$f(X = x) = \sum_n f(X = x|N = n)P(N = n)$$

Inference with Continuous

Q: At birth, girl elephant weights are distributed as a Gaussian with mean = 160kg, std = 7kg. At birth, boy elephant weights are distributed as a Gaussian with mean = 165kg, std = 3kg. All you know about a newborn elephant is that it is 163kg. What is the probability that it is a girl?

Inference with Continuous

Q: What is $P(G = 1 \mid X = 163)$

Let G be an indicator that the elephant is a girl. G is $\text{Bern}(p = 0.5)$

Let X be the distribution of weight of the elephant.

$X \mid G = 1$ is $N(\mu = 160, \sigma^2 = 7^2)$

$X \mid G = 0$ is $N(\mu = 165, \sigma^2 = 3^2)$

Inference with Continuous

Q: What is P(G = 1 | X = 163)

Let G be an indicator that the elephant is a girl. G is Bern(p = 0.5)

Let X be the distribution of weight of the elephant.

X | G = 1 is N(μ = 160, $\sigma^2 = 7^2$)

X | G = 0 is N(μ = 165, $\sigma^2 = 3^2$)

Joint Distribution is Implied:

$$f(G = 1, X = 72.3) = f(X = 72.3 | G = 1) P(G = 1)$$

More generally

$$f(G = g, X = x) = f(X = x | G = g) P(G = g)$$

Inference with Continuous

Q: What is $P(G = 1 \mid X = 163)$

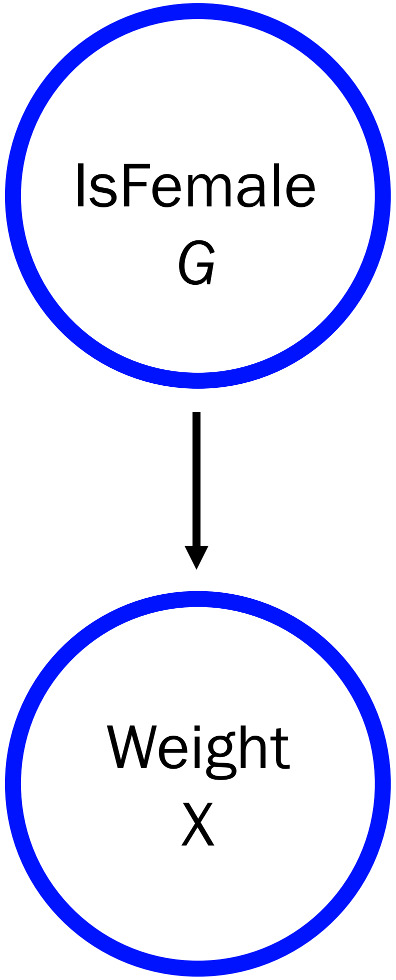
Let G be an indicator that the elephant is a girl. G is $\text{Bern}(p = 0.5)$

Let X be the distribution of weight of the elephant.

$X \mid G = 1$ is $N(\mu = 160, \sigma^2 = 7^2)$

$X \mid G = 0$ is $N(\mu = 165, \sigma^2 = 3^2)$

Model Shown Graphically



G = 1 is Bern(p = 0.5)

X | G = 1 is N(μ = 160, $\sigma^2 = 7^2$)
X | G = 0 is N(μ = 165, $\sigma^2 = 3^2$)

Does this define the joint?

$$f(G = g, X = x)$$

$$= f(X = x | G = g) P(G = g)$$

Q: What is P(G = 1 | X = 163)

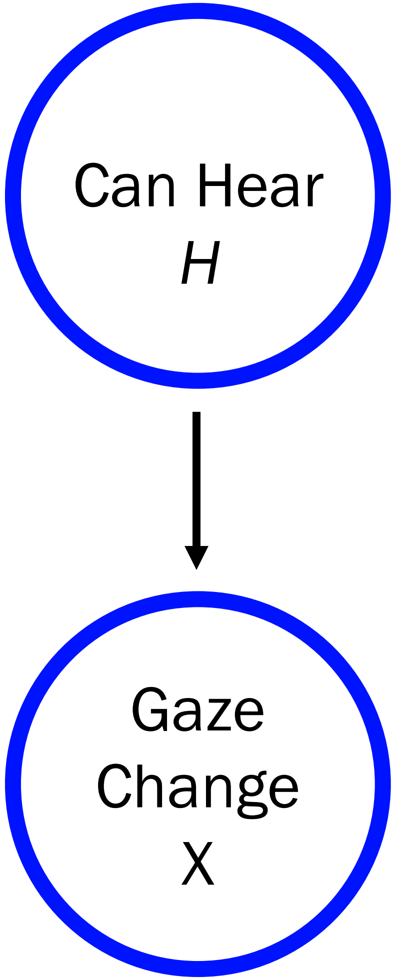
End Review

I Heard That Redux

Normal Assumption: We choose to approximate eye movements with normal distributions. For babies who can hear sounds, we approximate their gaze movement after the sound is played as: $N(\mu = 15, \sigma^2 = 50)$. For babies who can **not** hear sounds, we approximate gaze movement as $N(\mu = 8, \sigma^2 = 50)$.

For a new baby we observe a 14 degree movement after the sound is played. What is your belief that a baby can hear, under The **Normal Assumption**?

I Heard That Redux



$H = 1$ is Bern($p = 0.75$)

$X \mid H = 1$ is $N(\mu = 15, \sigma^2 = 50)$

$X \mid H = 0$ is $N(\mu = 8, \sigma^2 = 50)$

Q: What is $P(H = 1 \mid X = 14)$

Harder when neither random variable is a bernoulli

A Better Eye Test

https://www.thelancet.com/journals/lancet/article/PIIS0140-6736(21)02149-8/fulltext

https://www.science.org/content/article/eye-robot-artificial-intelligence-dramatically-improves-accuracy-classic-eye-exam

https://ojs.aaai.org/index.php/AAAI/article/view/5384/5240

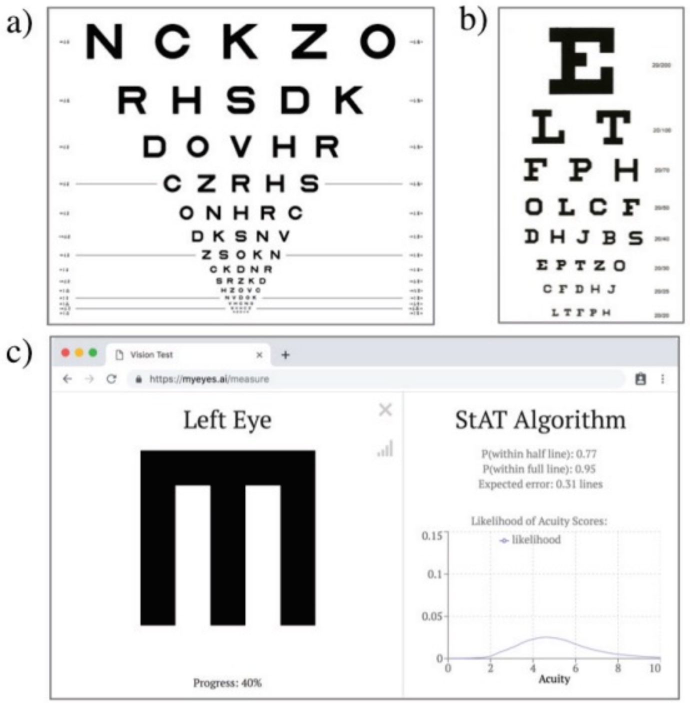
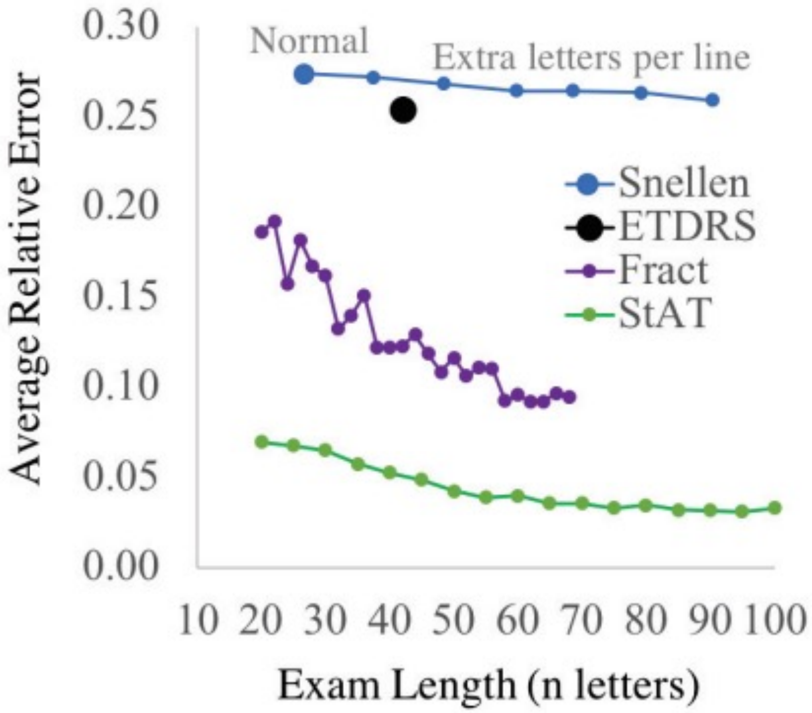
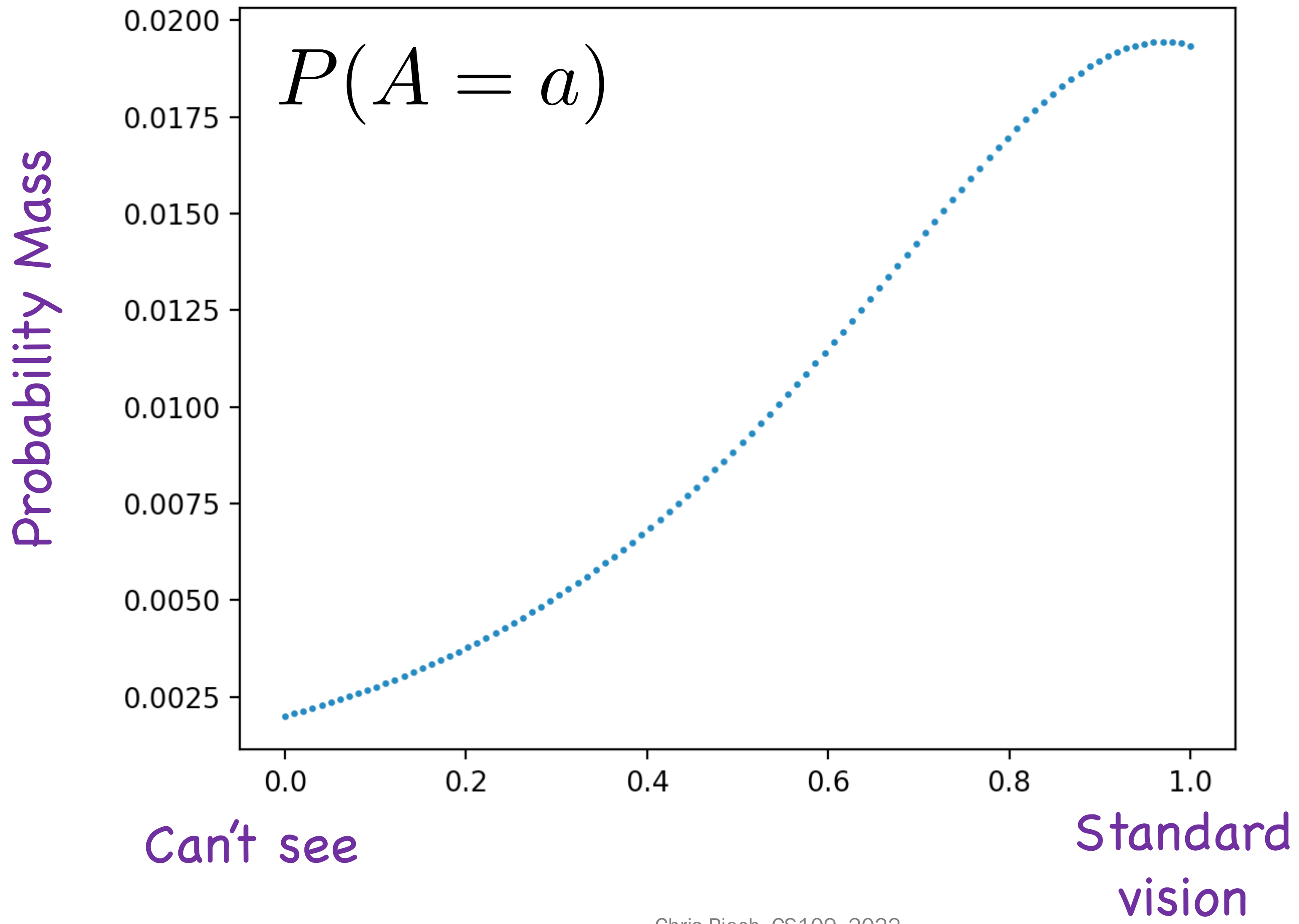


Figure 1: a) ETDRS, b) Snellen and c) StAT eye exams.



Prior Belief in Ability to See (Random Var A)



PMF is Actually Stored as a Dictionary

```python
def main():
    belief = get_prior_belief()
```

a	P(A=a)
0.00	0.00198
0.01	0.00205
0.02	0.00211
0.03	0.00218
0.04	0.00225
0.05	0.00233
0.06	0.0024
0.07	0.00248
0.08	0.00256
0.09	0.00264
0.10	0.00273
0.11	0.00281
0.12	0.0029
0.13	0.00299
0.14	0.00309
0.15	0.00319
0.16	0.00329
0.17	0.00339
0.18	0.0035
0.19	0.00361

a	P(A=a)
0.20	0.00372
0.21	0.00384
0.22	0.00396
0.23	0.00408
0.24	0.00421
0.25	0.00434
0.26	0.00447
0.27	0.00461
0.28	0.00475
0.29	0.00489
0.30	0.00504
0.31	0.00519
0.32	0.00535
0.33	0.00551
0.34	0.00567
0.35	0.00584
0.36	0.00601
0.37	0.00619
0.38	0.00637
0.39	0.00655

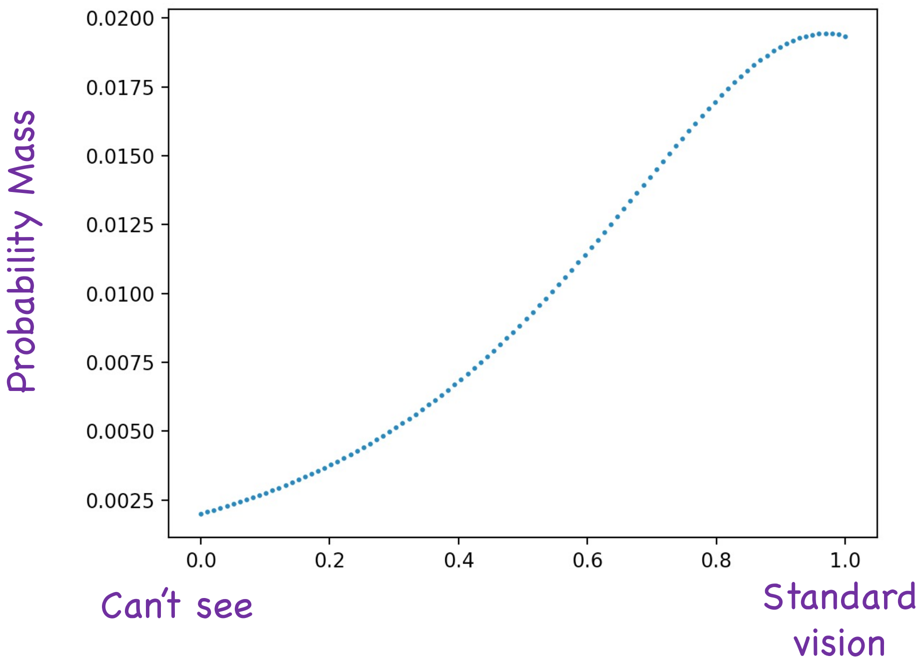
...

a	P(A=a)
0.80	0.01684
0.81	0.01708
0.82	0.01731
0.83	0.01753
0.84	0.01774
0.85	0.01795
0.86	0.01814
0.87	0.01832
0.88	0.01848
0.89	0.01864
0.90	0.01877
0.91	0.0189
0.92	0.019
0.93	0.01909
0.94	0.01916
0.95	0.01921
0.96	0.01924
0.97	0.01925
0.98	0.01924
0.99	0.01921

Prior Belief in Ability to See (Random Var A)

```
belief = get_prior_belief()
```

As a graph



As a dictionary

a	P(A=a)
0.00	0.00198
0.01	0.00205
0.02	0.00211
0.03	0.00218
0.04	0.00225
0.05	0.00233
0.06	0.0024
0.07	0.00248
0.08	0.00256
0.09	0.00264
0.10	0.00273
0.11	0.00281
0.12	0.0029
0.13	0.00299
0.14	0.00309
0.15	0.00319
0.16	0.00329
0.17	0.00339
0.18	0.0035
0.19	0.00361

a	P(A=a)
0.20	0.00372
0.21	0.00384
0.22	0.00396
0.23	0.00408
0.24	0.00421
0.25	0.00434
0.26	0.00447
0.27	0.00461
0.28	0.00475
0.29	0.00489
0.30	0.00504
0.31	0.00519
0.32	0.00535
0.33	0.00551
0.34	0.00567
0.35	0.00584
0.36	0.00601
0.37	0.00619
0.38	0.00637
0.39	0.00655

...

a	P(A=a)
0.80	0.01684
0.81	0.01708
0.82	0.01731
0.83	0.01753
0.84	0.01774
0.85	0.01795
0.86	0.01814
0.87	0.01832
0.88	0.01848
0.89	0.01864
0.90	0.01877
0.91	0.0189
0.92	0.019
0.93	0.01909
0.94	0.01916
0.95	0.01921
0.96	0.01924
0.97	0.01925
0.98	0.01924
0.99	0.01921

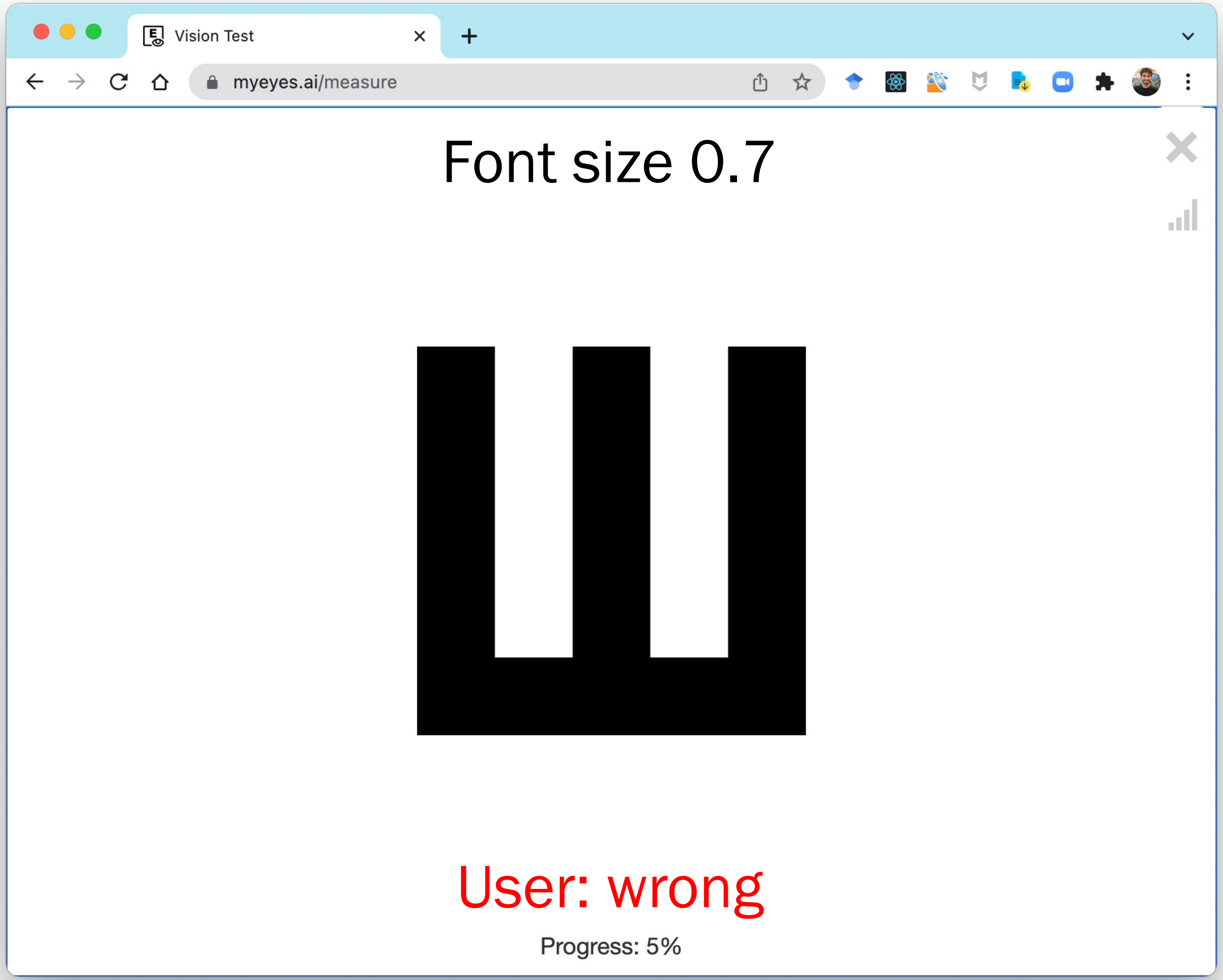
Today: I am going to simplify the units of vision

Normally doctors measure vision in logarithmic units. To make today's demo easier to understand:

I have translated ability to see

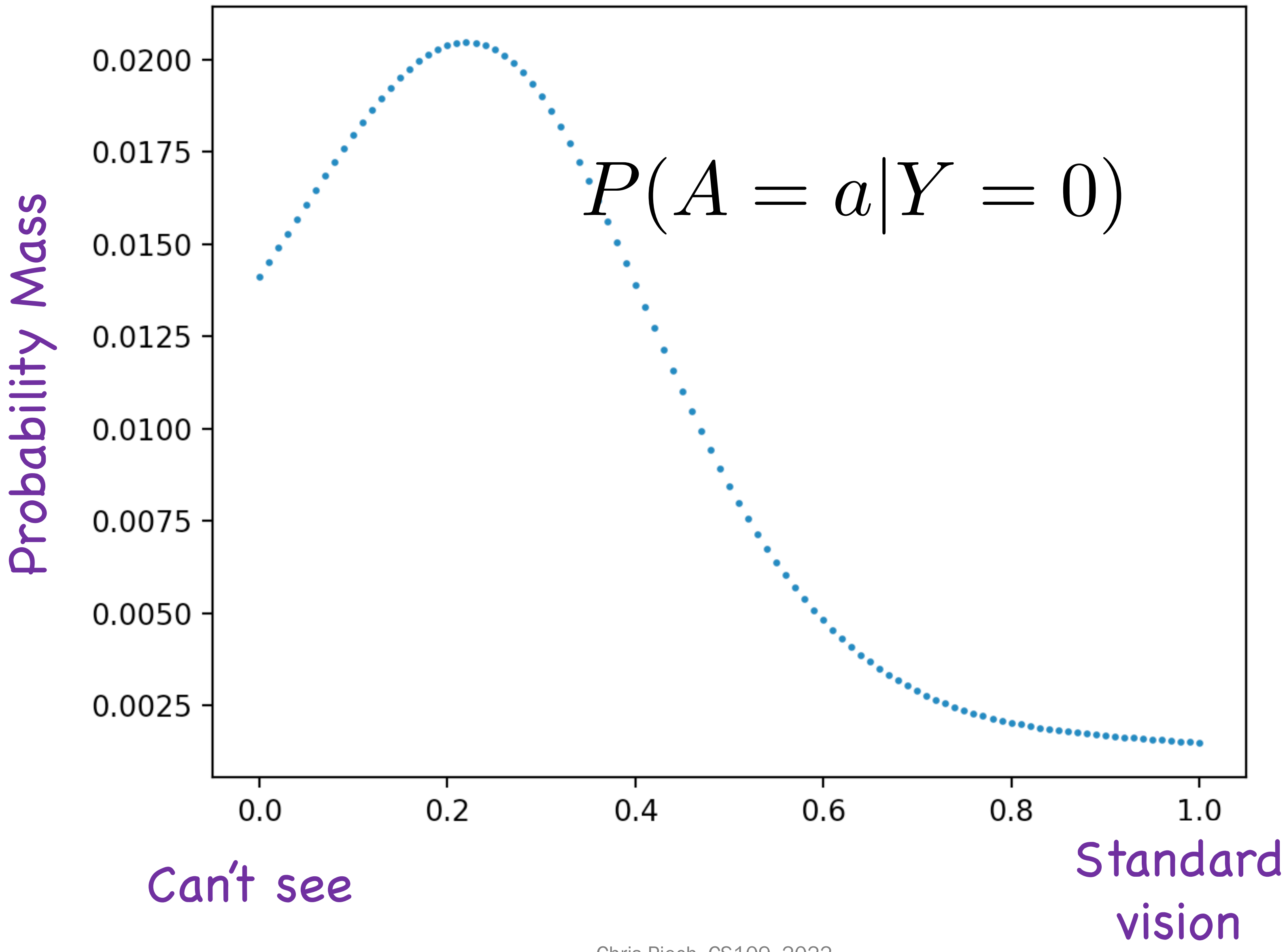
onto a [0, 1] scale.

The Patient is Shown One Letter and They Get it Wrong

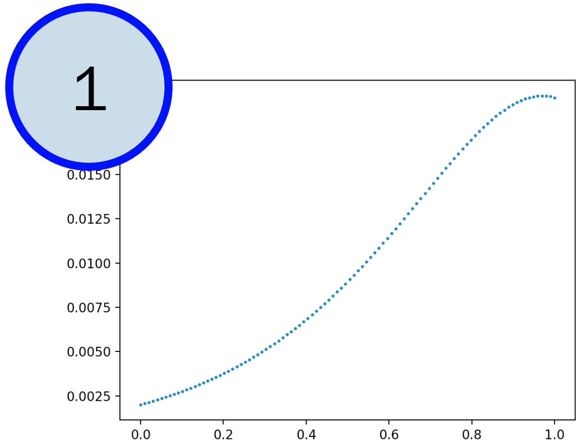


Observation $Y = 0$

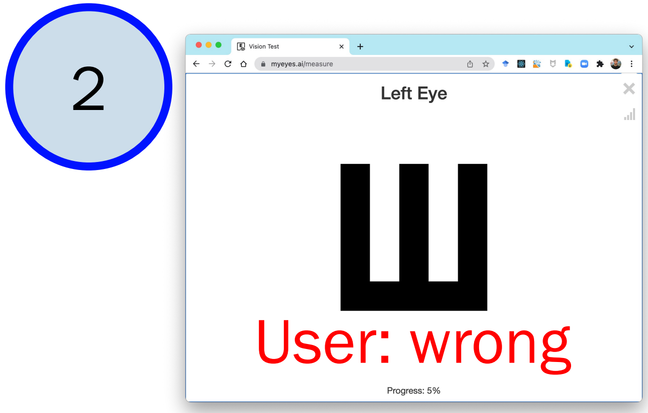
Posterior Belief in Ability to See (Random Var A)



Bayes with Random Variables

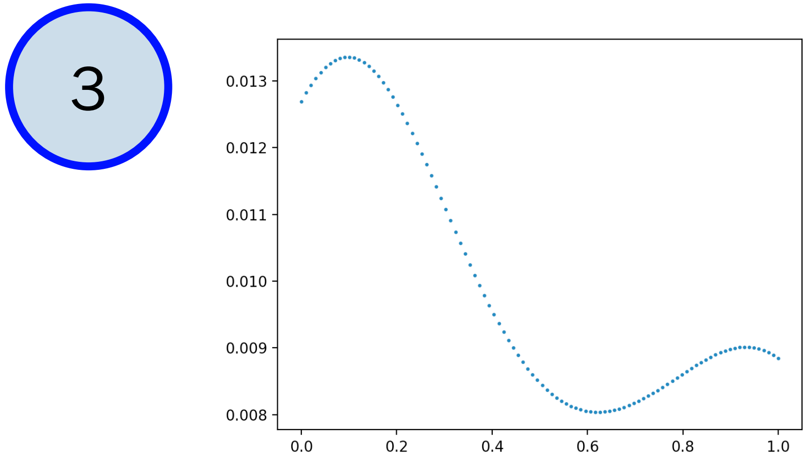


$$P(A = a)$$



Observation $Y = 0$

(At font size s_1)



$$P(A = a|Y = 0)$$

$$P(A = a|Y = 0) = \frac{P(Y = 0|A = a)P(A = a)}{P(Y = 0)}$$

Inference

In general Bayes theorem with a random variable is like the cellphone problem: multiple possible assignments to keep track of

Still true when some variables are continuous

Random Variables

Not all beliefs can be represented as a **function**. **Dictionary** / table is a great way to represent a random variable belief.

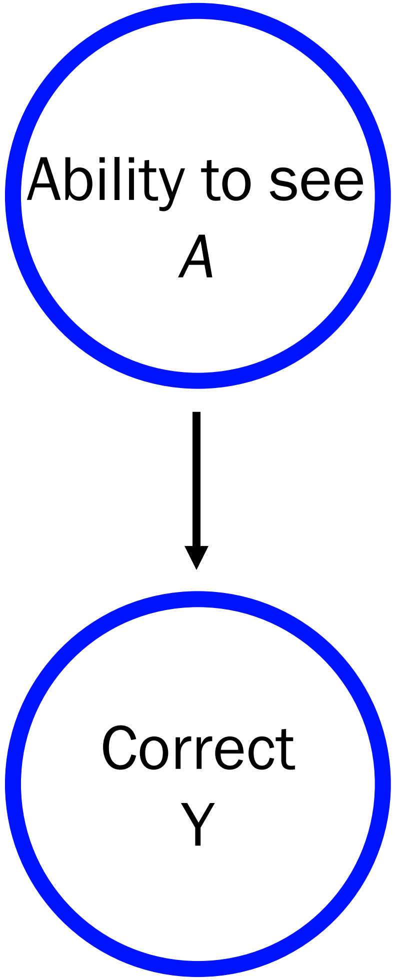
This is formally called non-parametric

Representing Continuous Variables

Dictionary can also be used to represent a discretization of a **continuous random var**

I do it all the time! Yay compute!

As a Causal Model



A is a non-parametric distribution (dictionary)

Font size is fixed

P(Y = 1 | A = a) is a Bernoulli where the parameter p is a known function of a and font size, s.

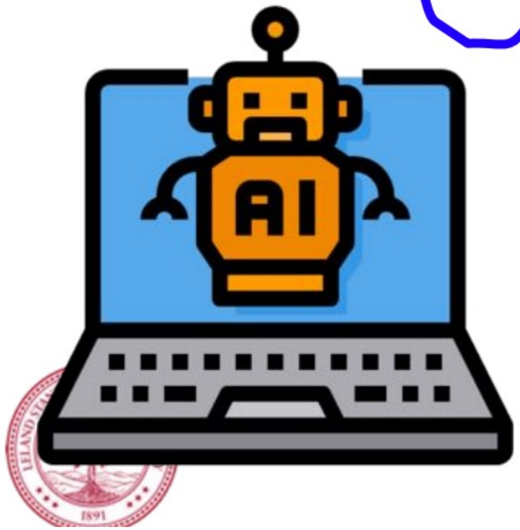
Q: What is P(A = a | Y = 0)

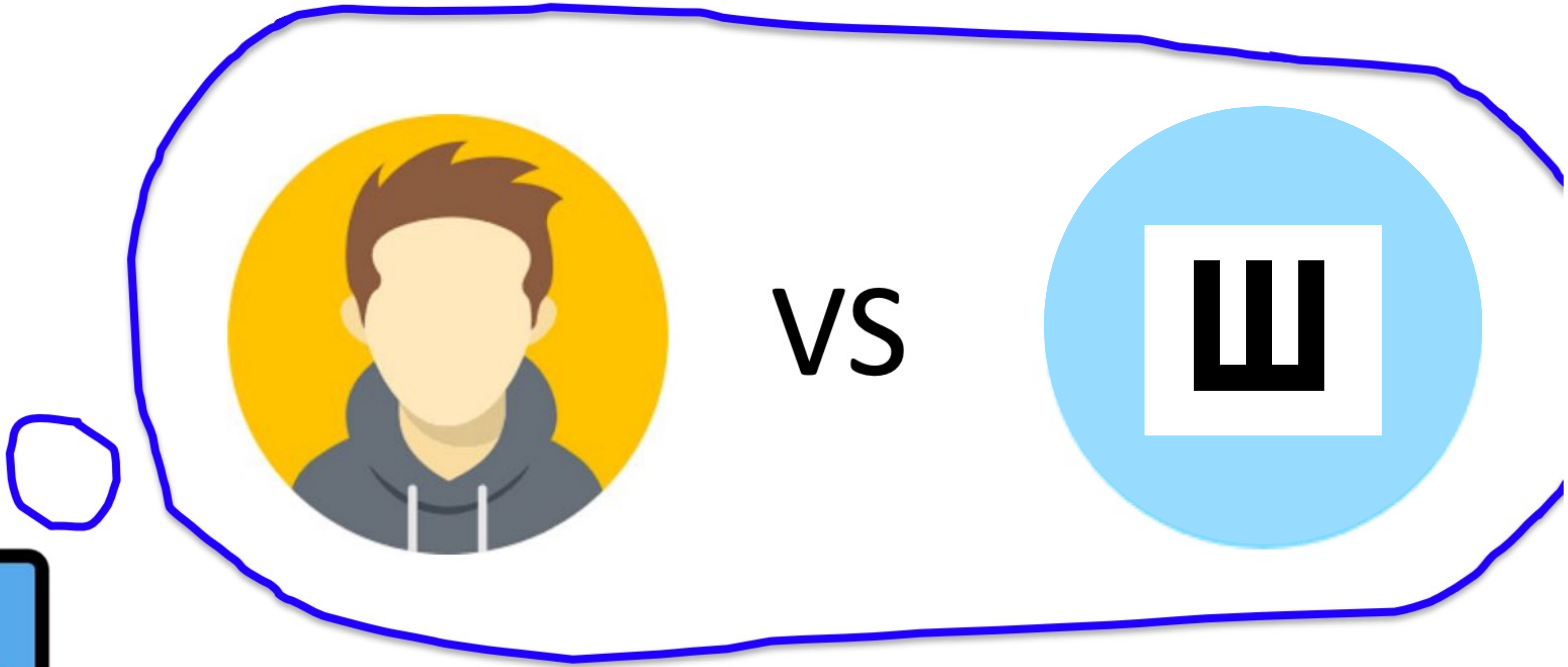
Aside: How to get

$$P(Y = 0|A = a)$$

Not required material…
(more education theory than CS109)

Likelihood: Item Response Theory





A **model** which gives a probability that a particular student will answer a particular question correctly

Likelihood: Item Response Theory

If student i attempts problem j, the likelihood they answer it correctly is…

Probability correct

$$p_{i,j} = \sigma(a_i - d_j)$$

Squashing function

Ability of student i

Difficulty of problem j (based on font size)

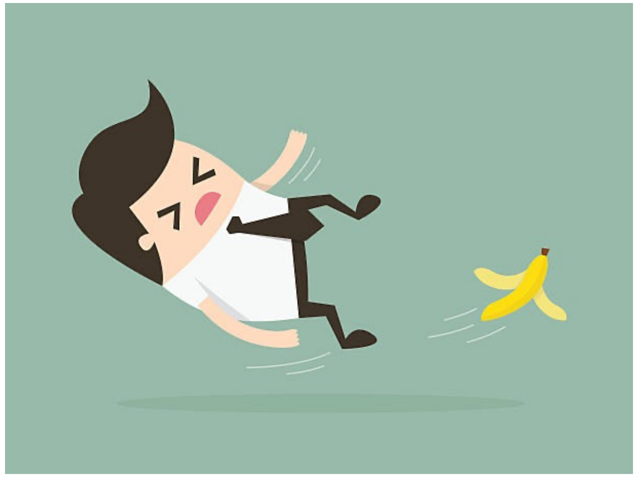
Slip and Fall

IRT says prob person *i* gets letter *j* correct is

$$p_{i,j} = \sigma(a_i - d_j)$$

What about guessing?

What about slips?



$$P(Y = 0|A = a) = 1 - P(Y = 1|A = a)$$

End Aside: How to get

$$P(Y = 0 | A = a)$$

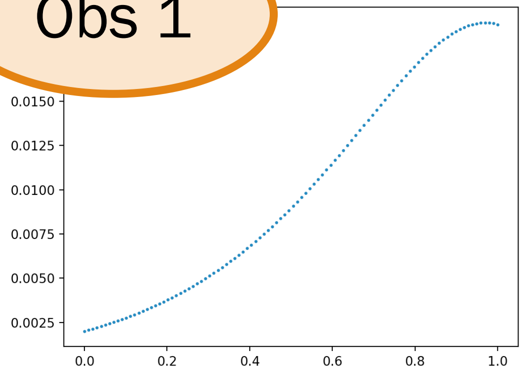
Not required material…

(more education theory than CS109)

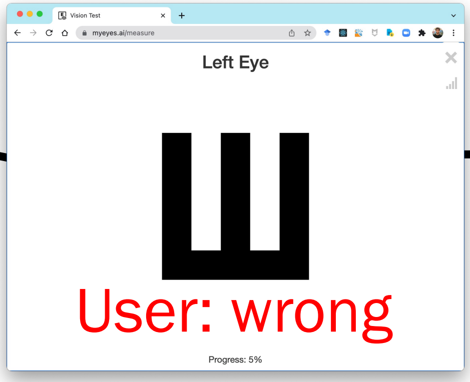
Multiple observations??

Multiple Observations

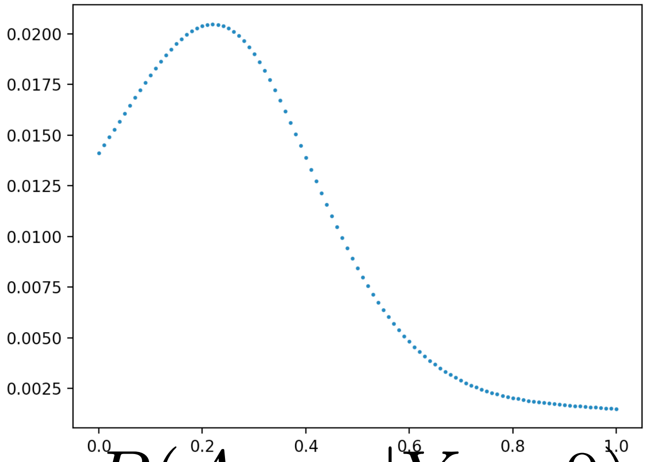
Obs 1



$P(A = a)$

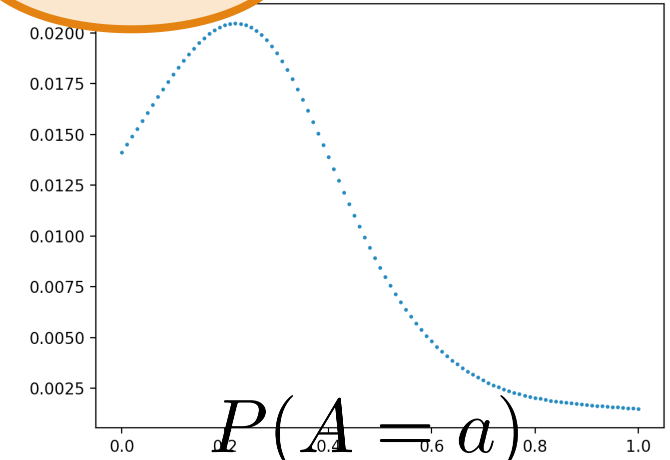


Observation $Y = 0$
(At font size 0.7)

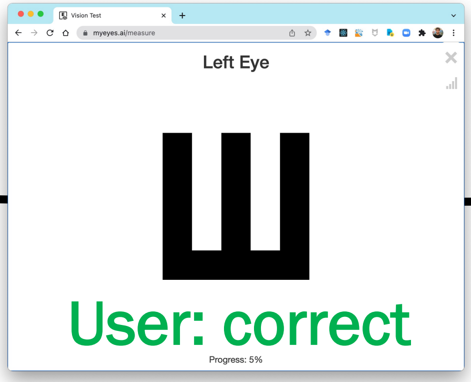


$P(A = a|Y = 0)$

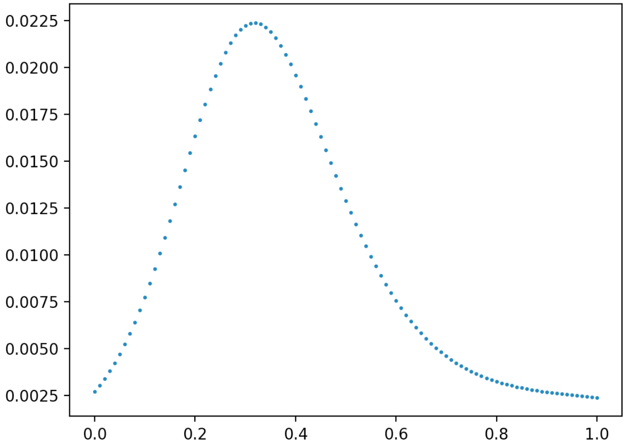
Obs 2



$P(A = a)$

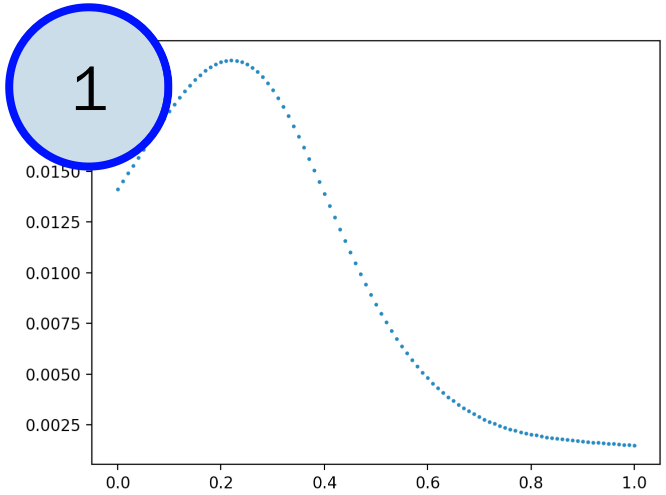


Observation $Y = 0$
(At font size 0.8)

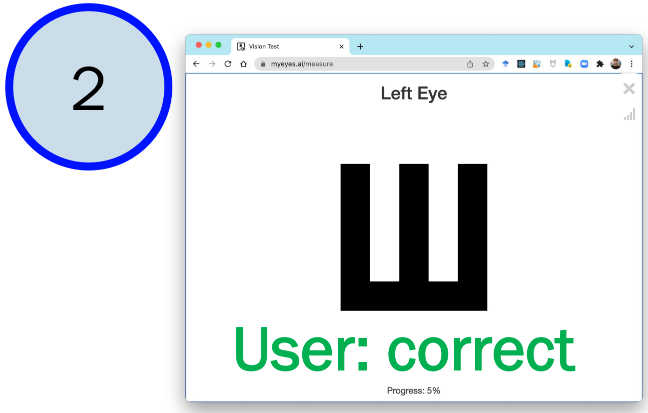


$P(A = a|Y = 1)$

Posterior becomes new prior

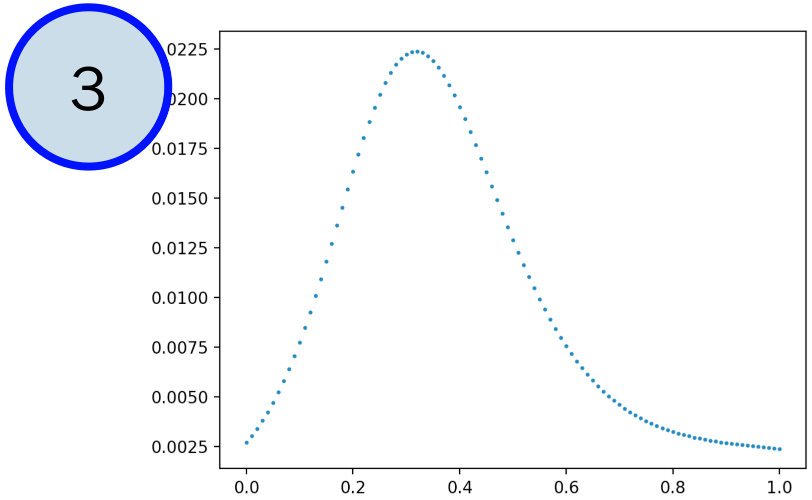


$$P(A = a)$$



Observation $Y = 1$

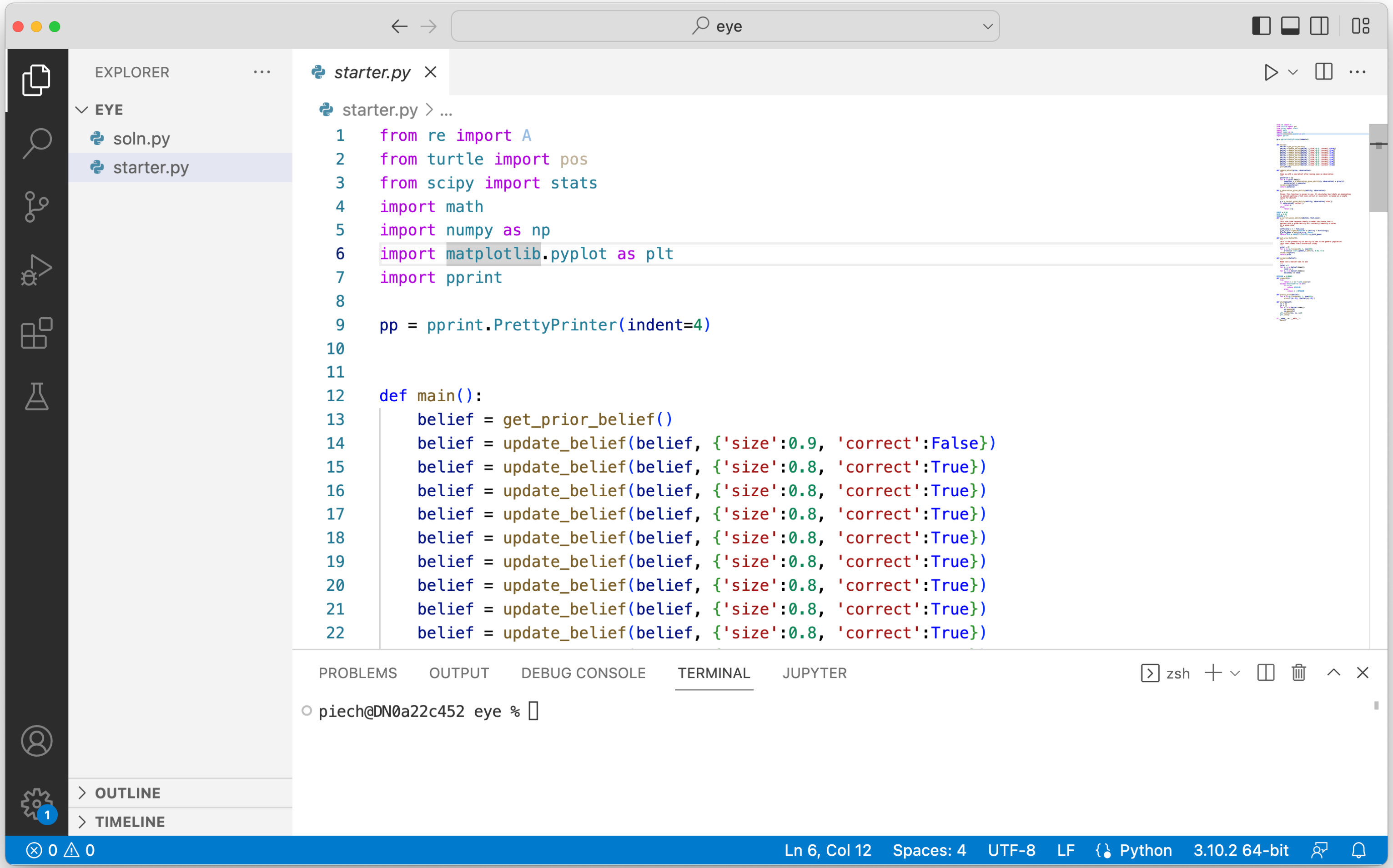
(At font size *0.8*)



$$P(A = a | Y = 1)$$

$$P(A = a | Y = 1) = \frac{P(Y = 1 | A = a) P(A = a)}{P(Y = 1)}$$

To the Code!


eye
EXPLORER
EYE
soln.py
starter.py
starter.py
starter.py > ...
from re import A
from turtle import pos
from scipy import stats
import math
import numpy as np
import matplotlib.pyplot as plt
import pprint
pp = pprint.PrettyPrinter(indent=4)
def main():
belief = get_prior_belief()
belief = update_belief(belief, {'size':0.9, 'correct':False})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
belief = update_belief(belief, {'size':0.8, 'correct':True})
PROBLEMS
OUTPUT
DEBUG CONSOLE
TERMINAL
JUPYTER
zsh
piech@DN0a22c452 eye %
OUTLINE
TIMELINE
0 0
Ln 6, Col 12
Spaces: 4
UTF-8
LF
Python
3.10.2 64-bit

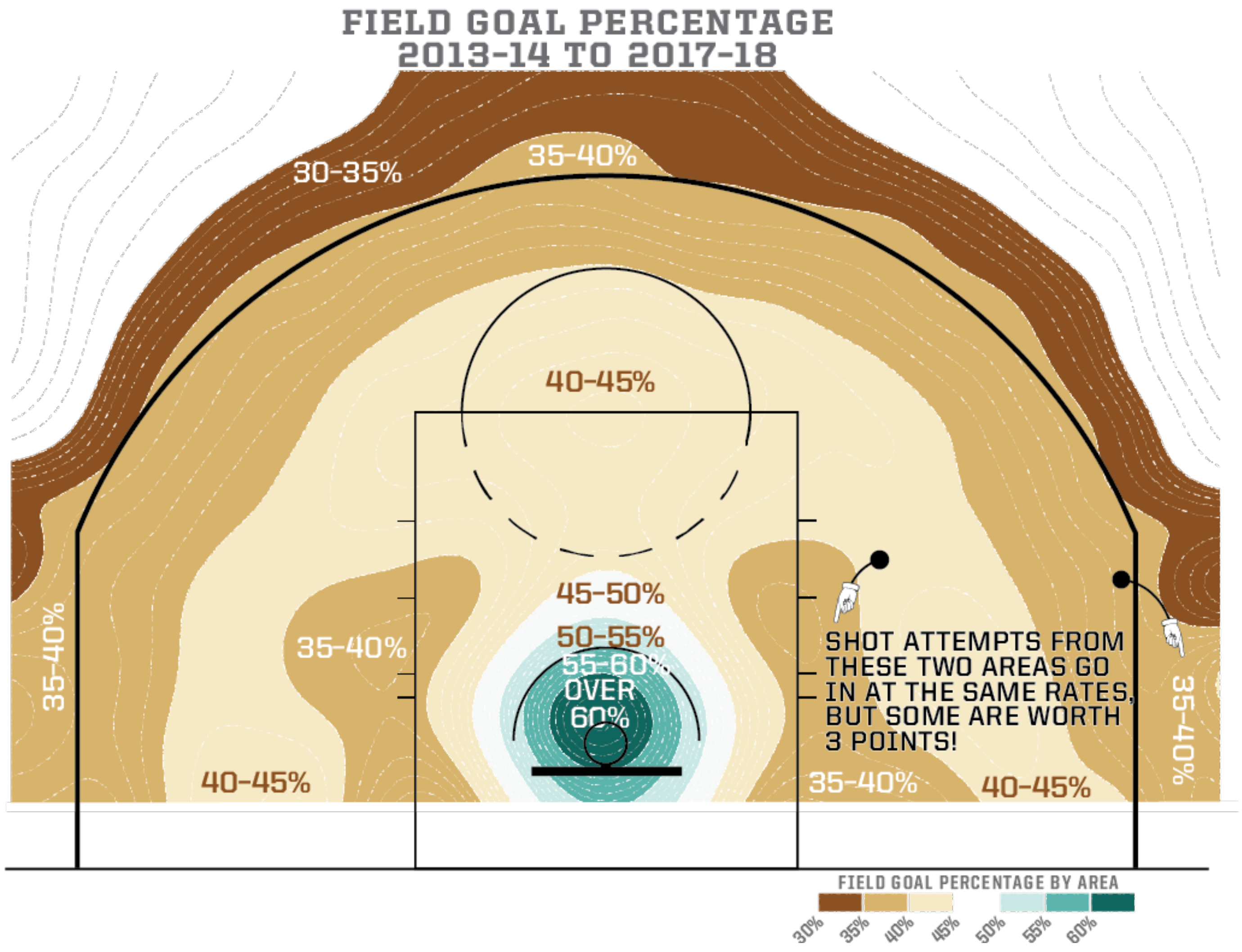
Aside, if time: Visualizing continuous joints

All the Bayes Belong to Us

X, Y are continuous

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

Probabilistic Models can have Continuous Random Vars



X location of the shot

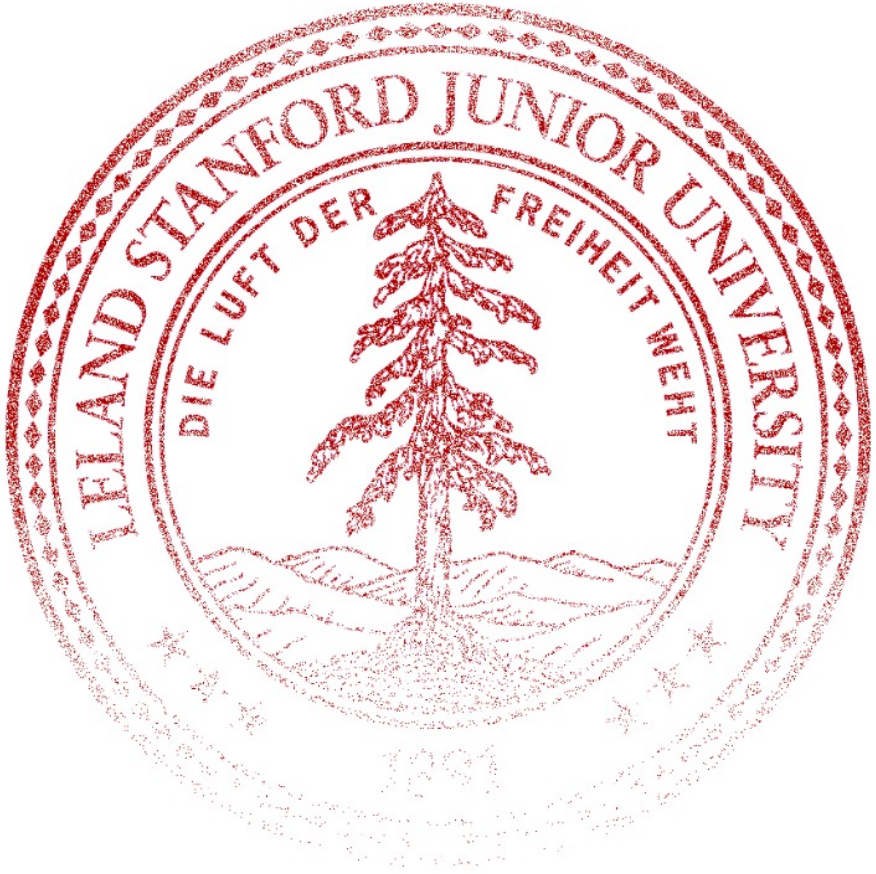
Y location of the shot

$$f(X = x, Y = y)$$

Joint: If any random variable is continuous, then we consider the joint a density

Probabilistic Models can have Continuous Random Vars

Dart Results



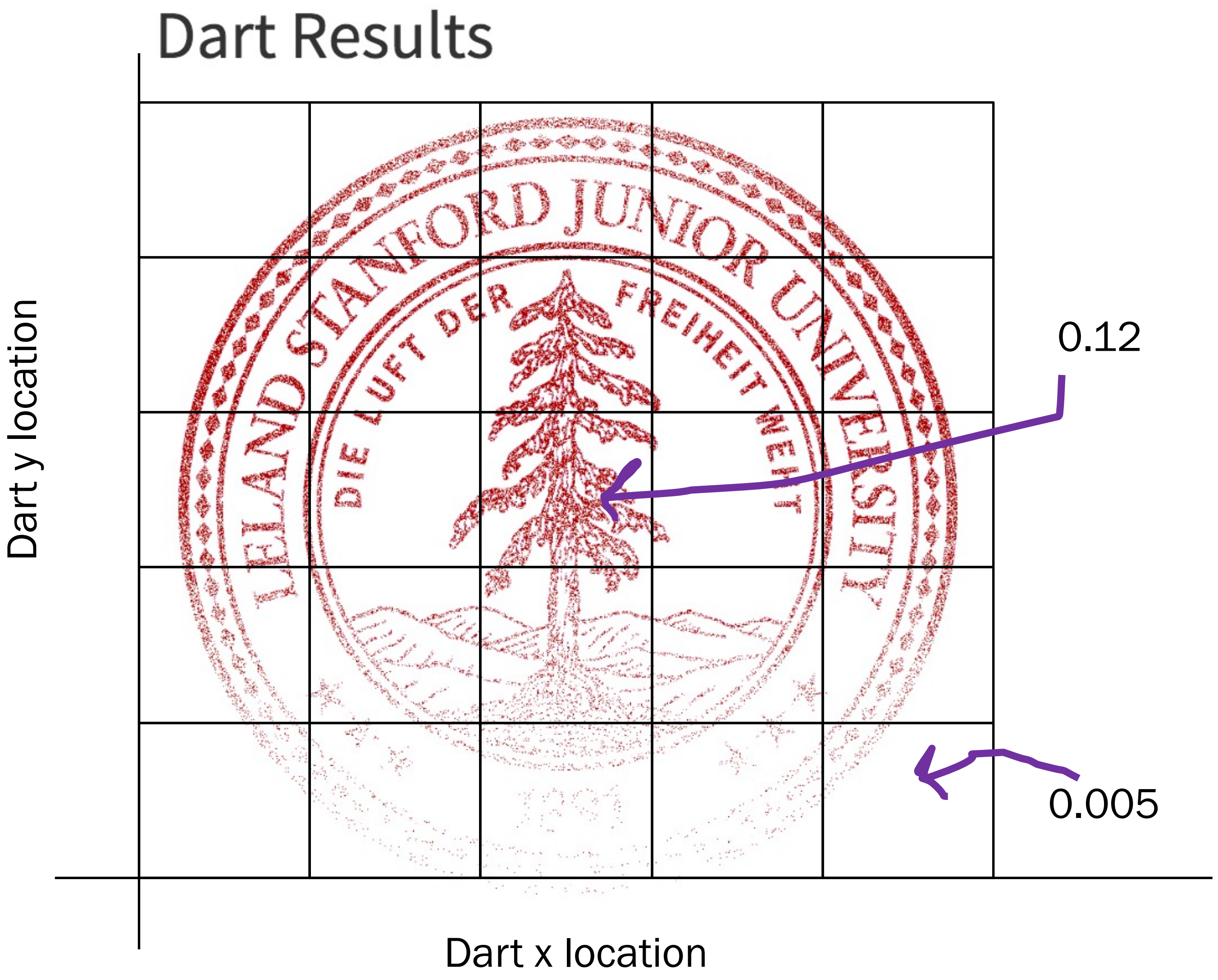
X location of the shot

Y location of the shot

$$f(X = x, Y = y)$$

Joint: If any random variable is continuous, then we consider the joint a density

Probabilistic Models can have Continuous Random Vars



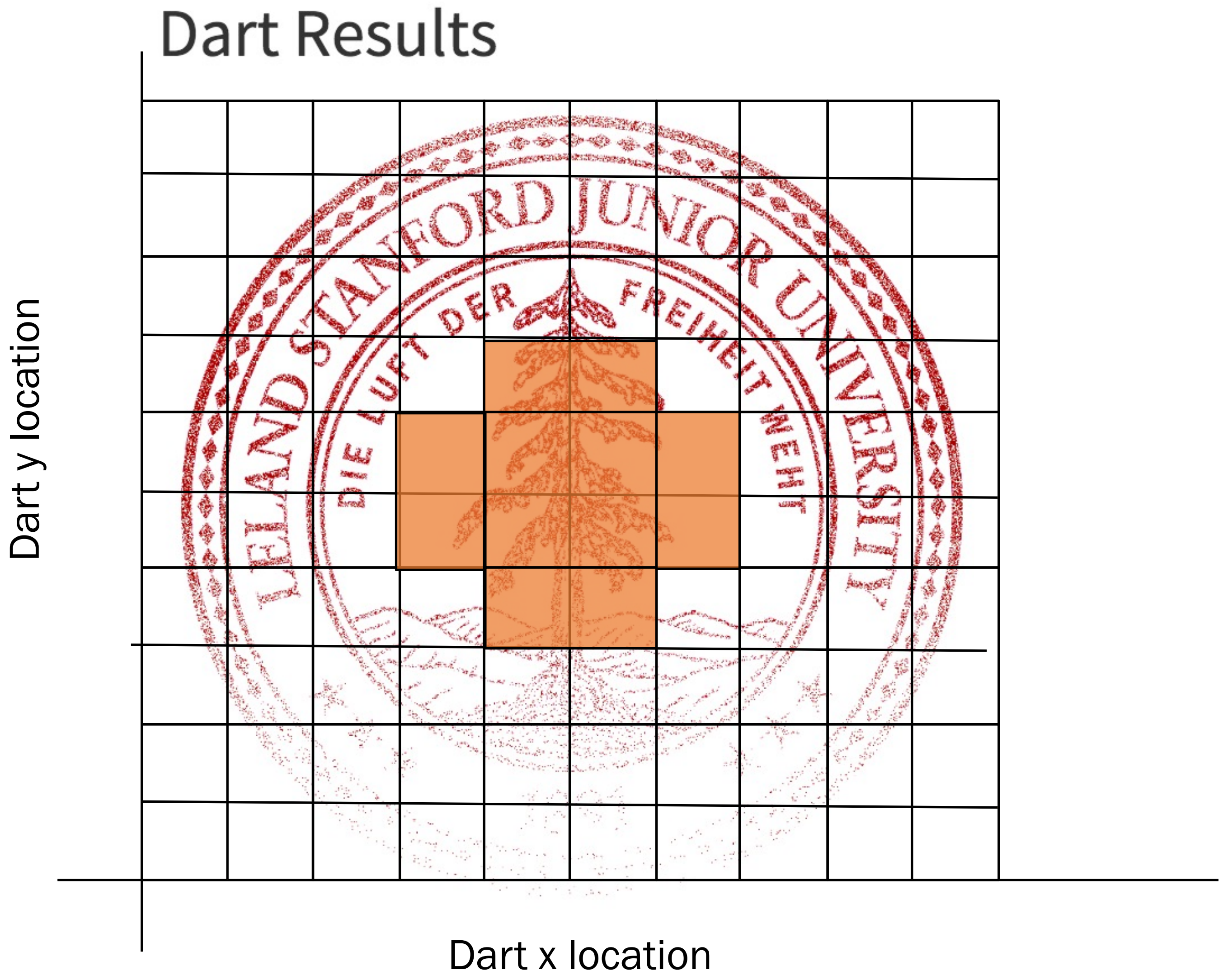
X location of the shot

Y location of the shot

$$f(X = x, Y = y)$$

Joint: If any random variable is continuous, then we consider the joint a density

Probabilistic Models can have Continuous Random Vars



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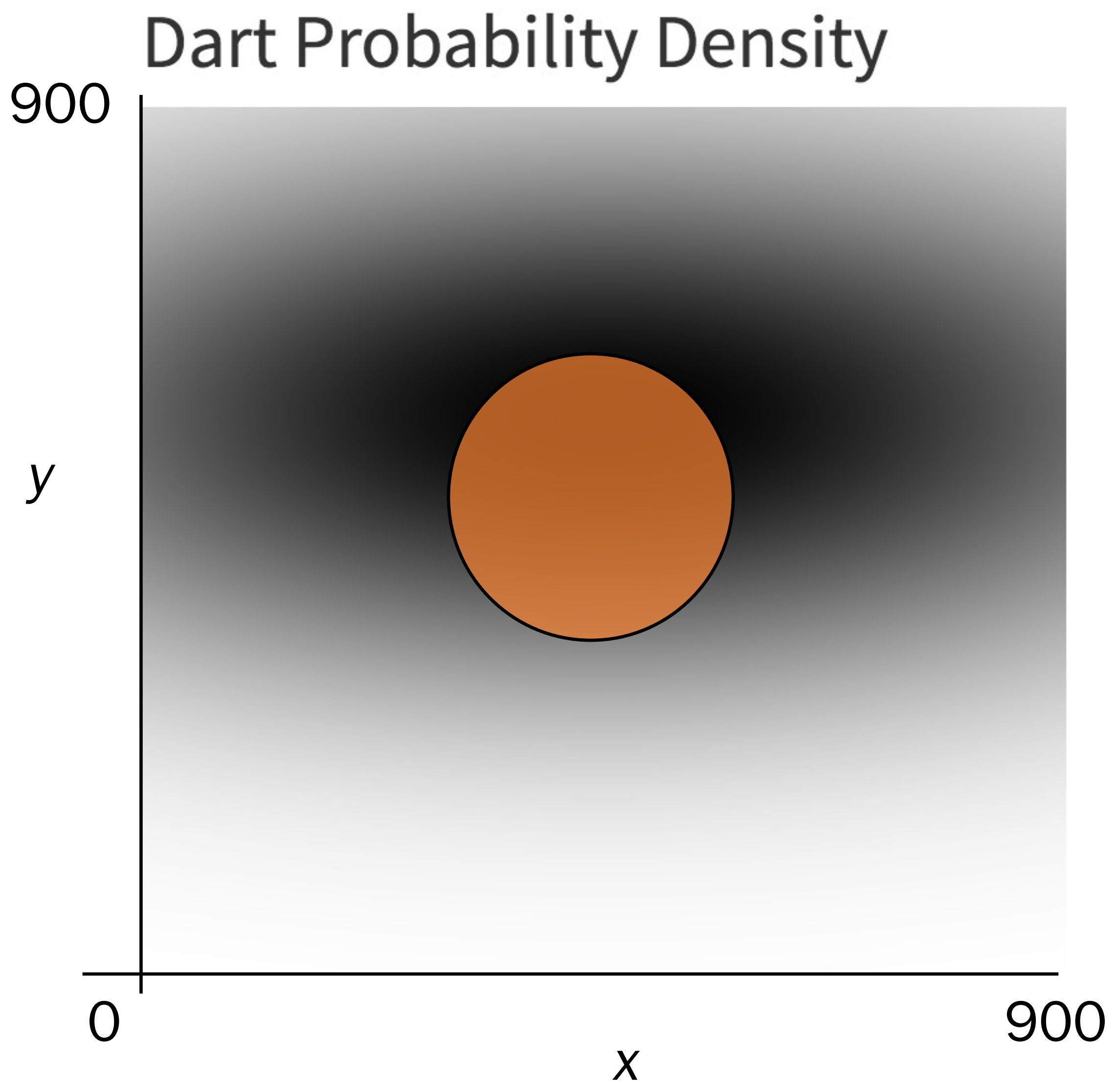


X location of the shot

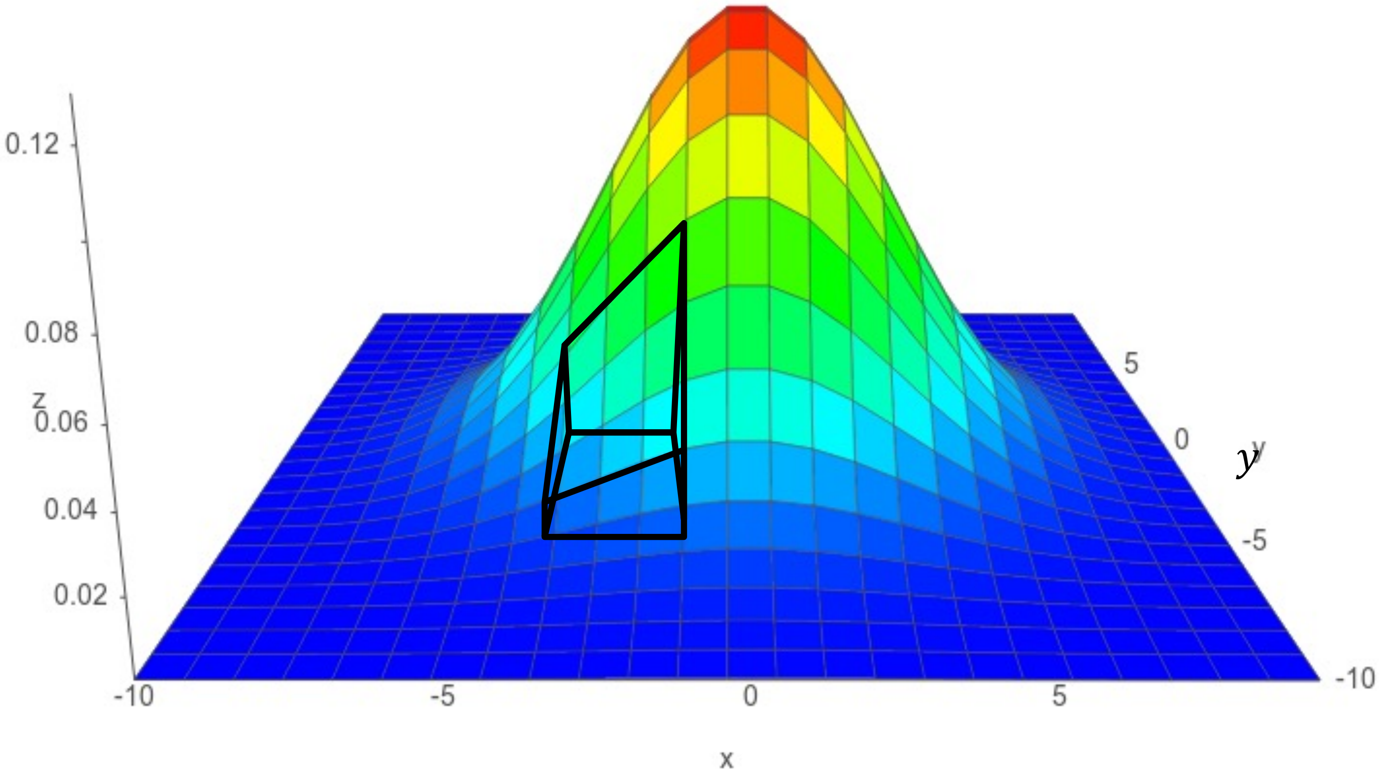
Y location of the shot

$$f(X = x, Y = y)$$

Joint: If any random variable is continuous, then we consider the joint a density

Joint Probability Density Function

$$P(a_1 < X < a_2, b_1 < Y < b_2) = \int_{x=a_1}^{a_2} \int_{y=b_1}^{b_2} f(X = x, Y = y) \, \partial y \partial x$$



plot by Academo

Joint is Complete Information!

A joint distribution is complete information. It can be used to answer any probability question.

Still true when some variables are continuous

Focus on Inference

When there are a mixture of discrete and continuous (or multiple continuous) I want you to focus on inference

All the Bayes Belong to Us

M,N are discrete. X, Y are continuous

OG Bayes

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

Mix Bayes #1

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

Mix Bayes #2

$$P(n|x) = \frac{f(x|n)P(n)}{f(x)}$$

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

End Aside

Joint Random Variables

- [x] Use a joint table, or joint function to solve probability question
- [x] Think about **conditional** probabilities with joint variables (which might be continuous)
- [ ] Use and find **independence** of random variables
- [ ] Use and find **expectation** of random variables