Independence

Chris Piech, CS109

Review

Review: Conditional Probability

$p(AB)$ vs $P(A|B)$

$$P(AB) = P(A|B)P(B)$$

Notation

And	Or	Given
$P(E \text{ and } F)$	$P(E \text{ or } F)$	$P(E\|F)$
$P(E, F)$	$P(E \cup F)$	$P(E\|F, G)$
$P(EF)$		Probability of E given F and G
$P(E \cap F)$		

Review: Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule (Product rule) ↑

↓ Definition of conditional probability

$$P(E|F)$$

Law of Total Probability ↙

↘ Bayes' Theorem

$$P(E)$$

$$P(F|E)$$

Bayes' Theorem

$P(E|F) \Rightarrow P(F|E)$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Proof

2 steps! See board

Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

Proof

1 more step! See board

Let’s Gamble

Telling in Cards

Your opponent gets excited if they are winning…

Probability of a tell given they have a winning hand: 0.5

Probability of a tell given they don't have a winning hand: 0.1

The Tell Game

```
You are playing a game. Your opponent has two unseen cards. If either is an ace you lose...
The following cards are seen by you (ie are not in opponent hand):
3H
2S
3D
5S
9D
KD
AC
Opponent does *not* have an excited tell...
Do you choose to play? yes
```

Probability of a tell given they **have a winning** hand: 0.5

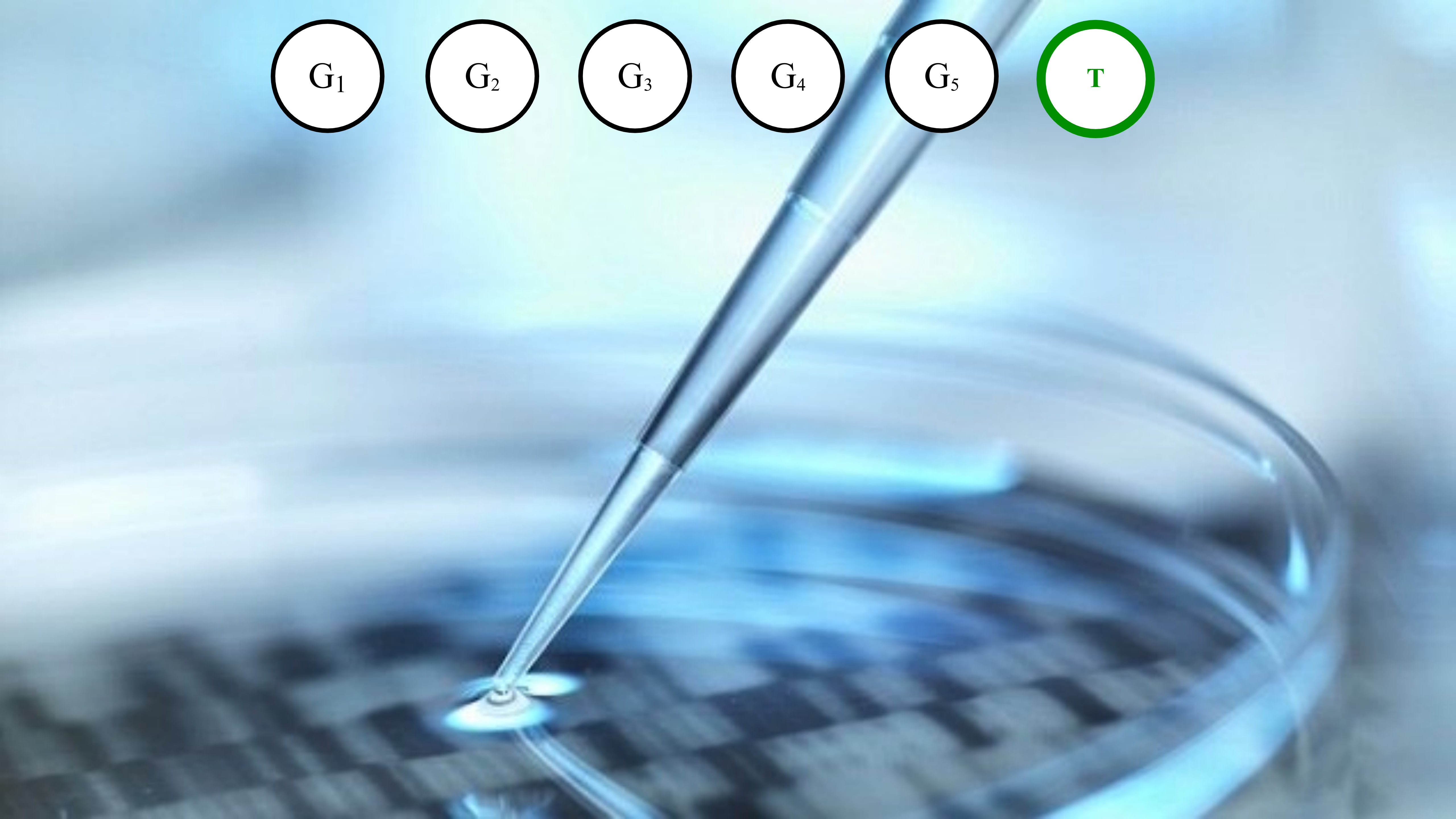
Probability of a tell given they **don't have** a winning hand: 0.1

End Review

Announcements

- **Sections** start today. Wahoo! Enjoy.
- **PSet #1** is due Friday 1p. Recall grace period.
- **PSet Party** tonight at 9p. Just log in to the app and hit **Learn with Others**

Today, start with a cool program


G_1
G_2
G_3
G_4
G_5
T

G_1 G_2 G_3 G_4 G_5 T

dna.txt — dna

dna.txt

```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
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True,True,False,True,True,True
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True,True,False,False,True,True
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False,False,False,True,False,False
```
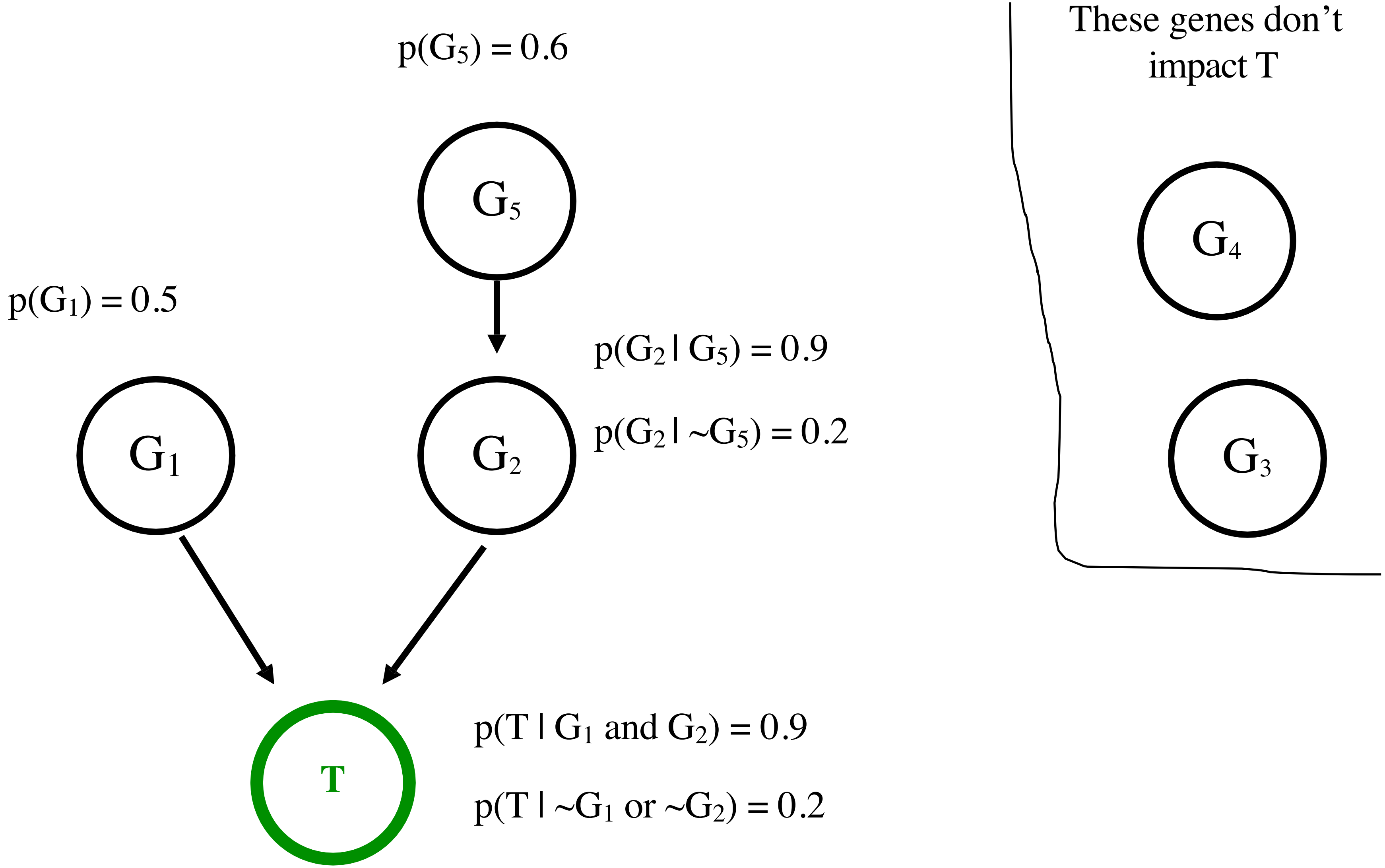

100,000 samples

6 observations per sample

Stanford University

Discovered Hypothesis



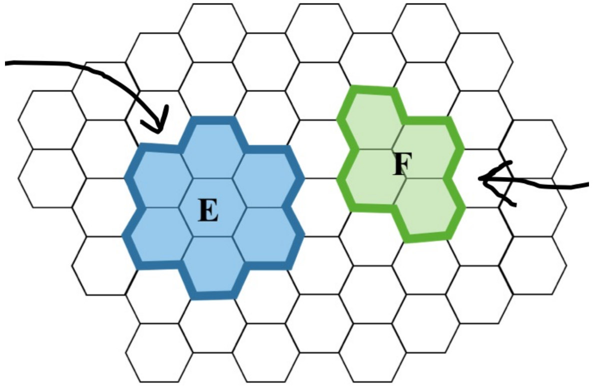
We’ve gotten ahead of ourselves

Source: The Hobbit

Start at the beginning

Source: The Hobbit

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes ***OR*** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$

Independent

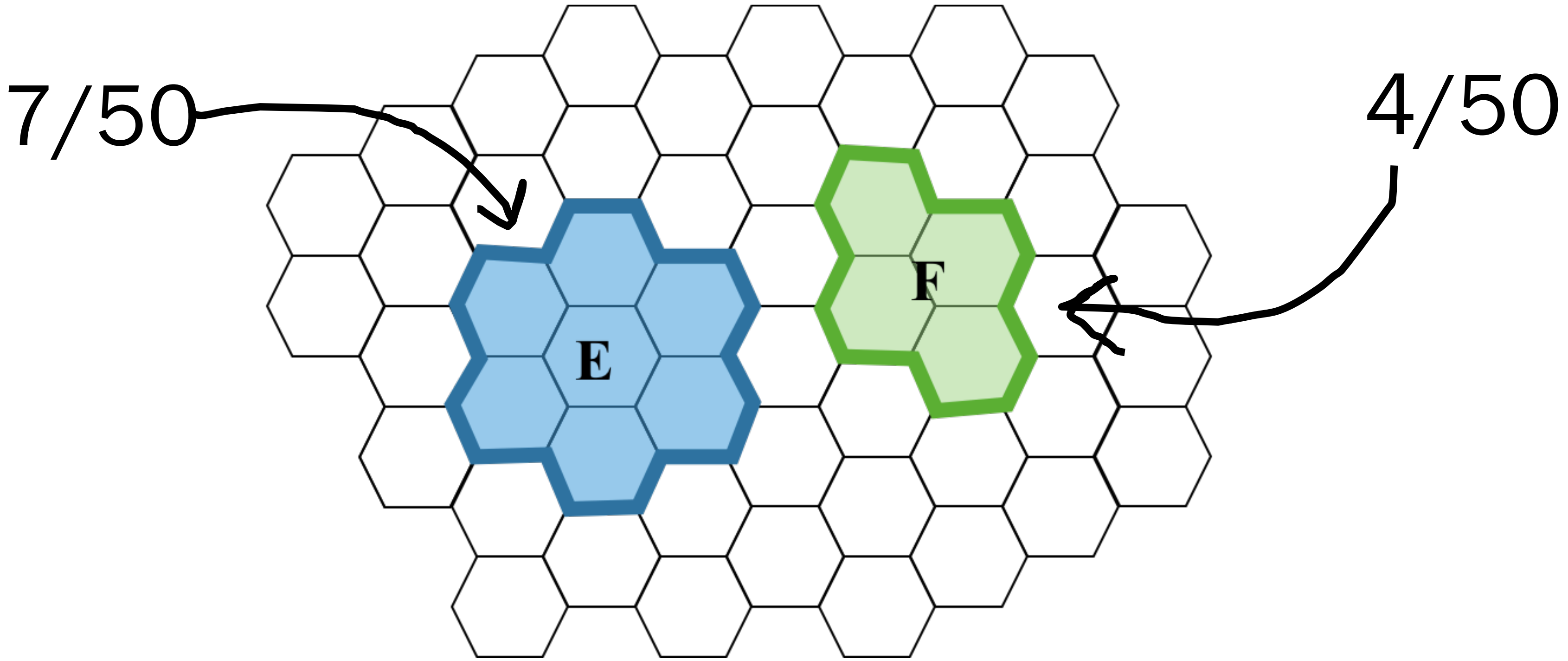
$$P(A) = P(A|B)$$

Makes ***AND*** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Probability of “OR”

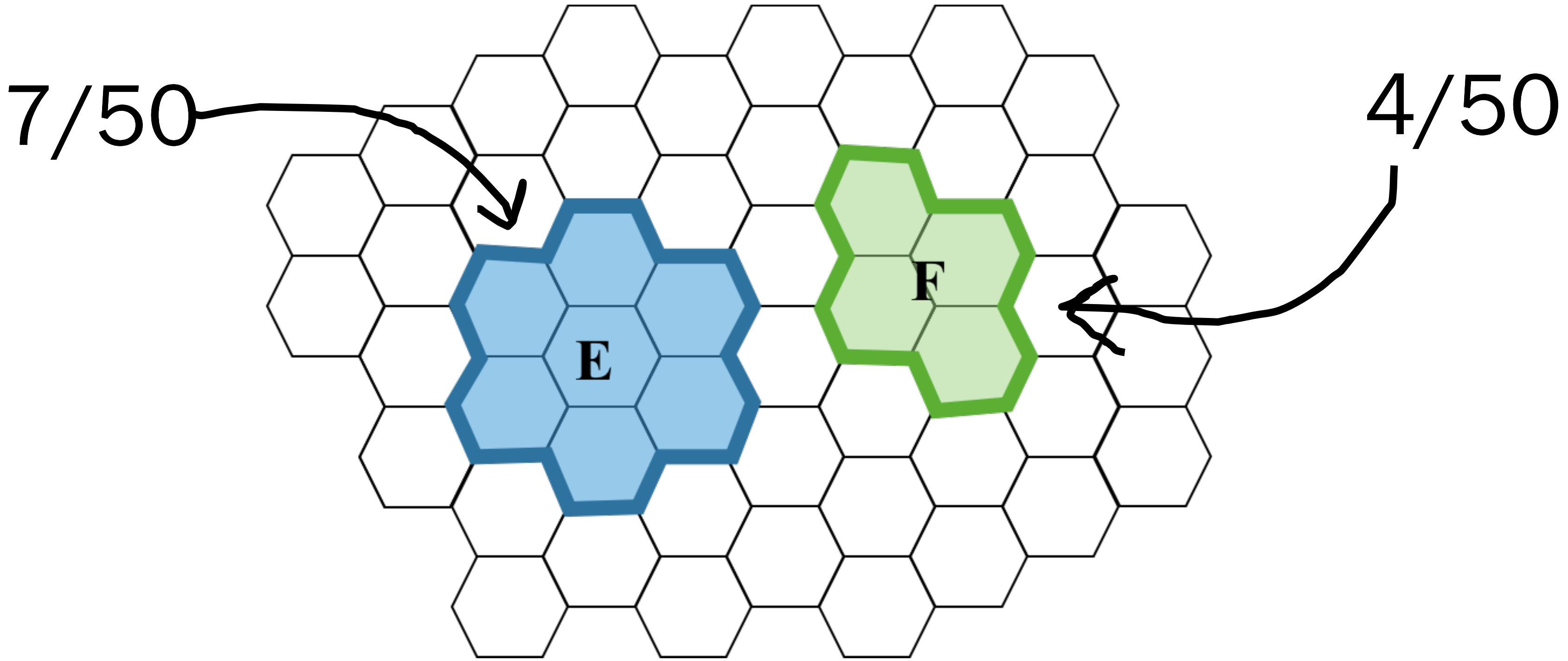
Review: OR with Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

Review: OR with Mutually Exclusive Events

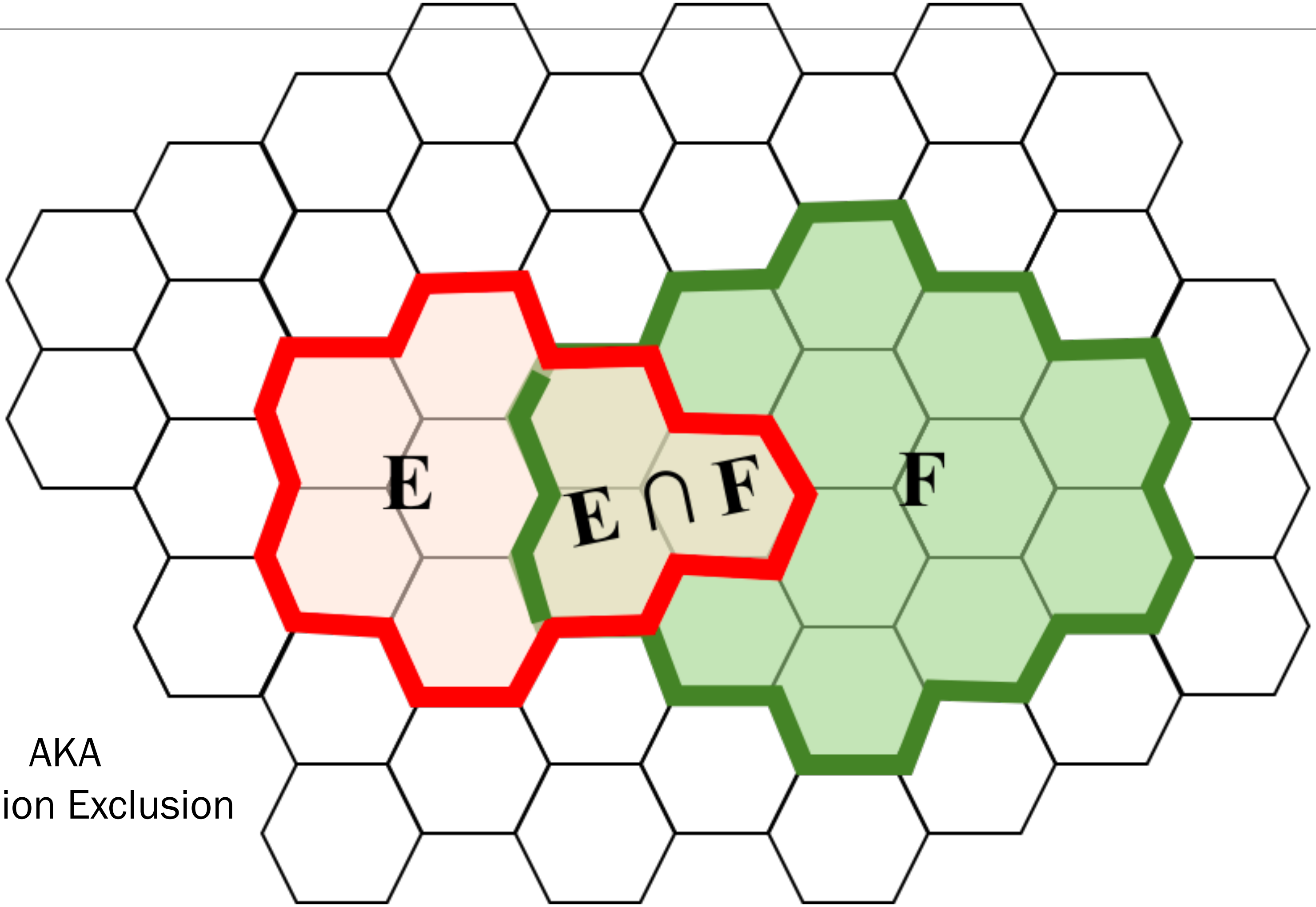


If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

What about when they are not *Mutually exclusive*?

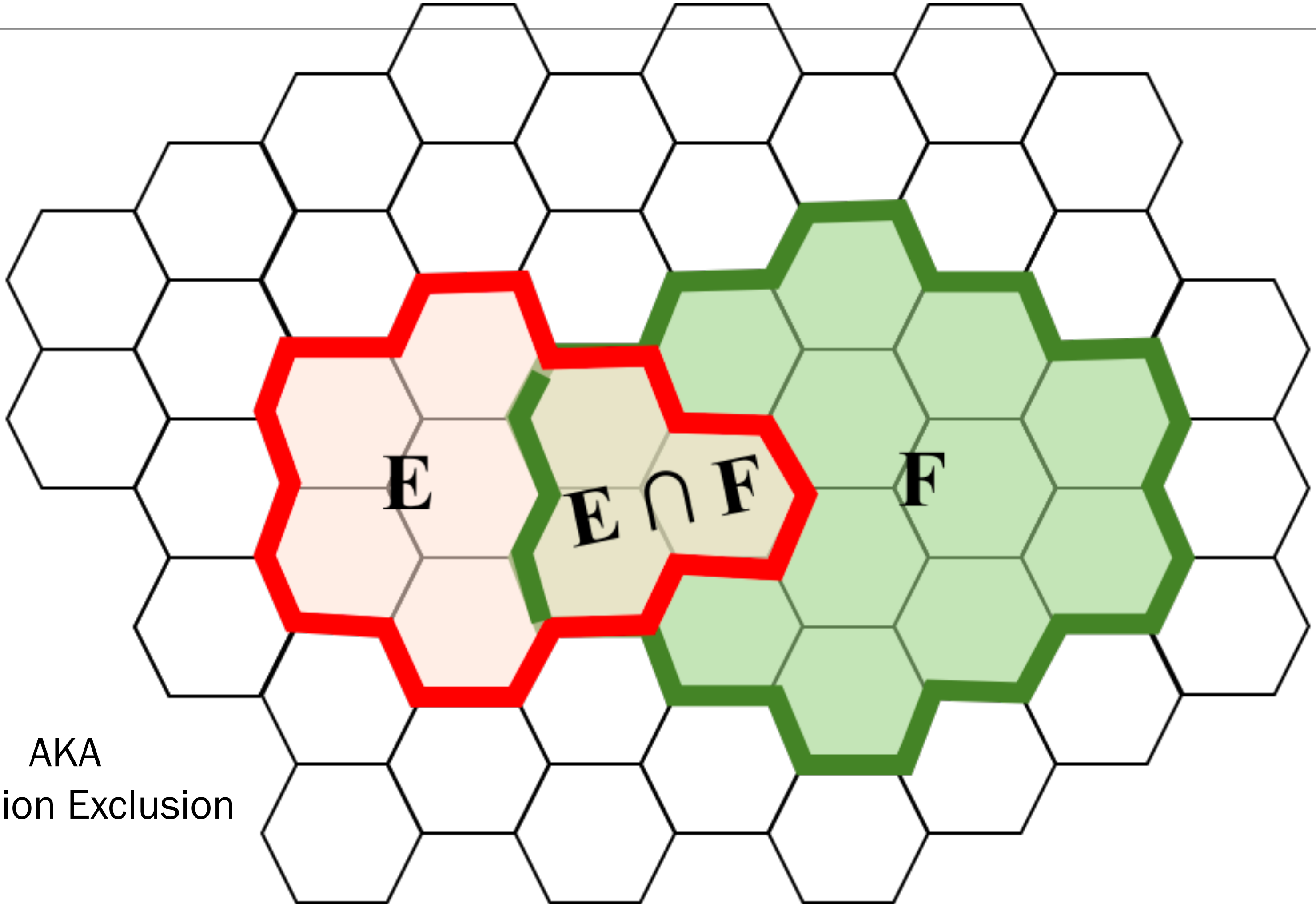
OR *without* Mutually Exclusive Events



AKA
Inclusion Exclusion

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

OR *without* Mutually Exclusive Events



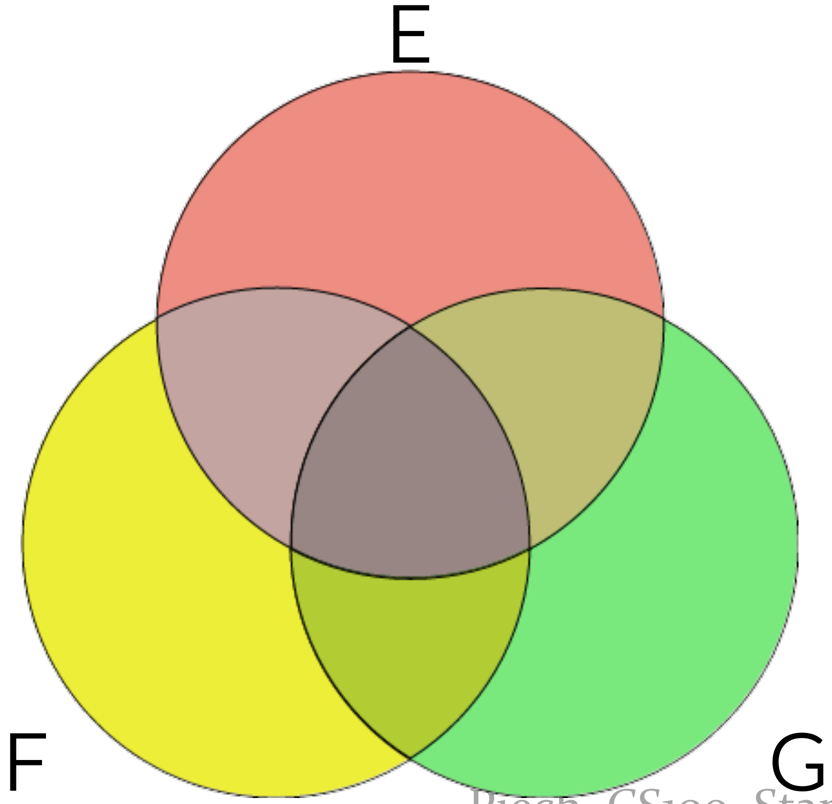
AKA
Inclusion Exclusion

$$P(E \cup F) = \frac{8}{50} + \frac{14}{50} - \frac{3}{50} = \frac{19}{50}$$

More than two sets?

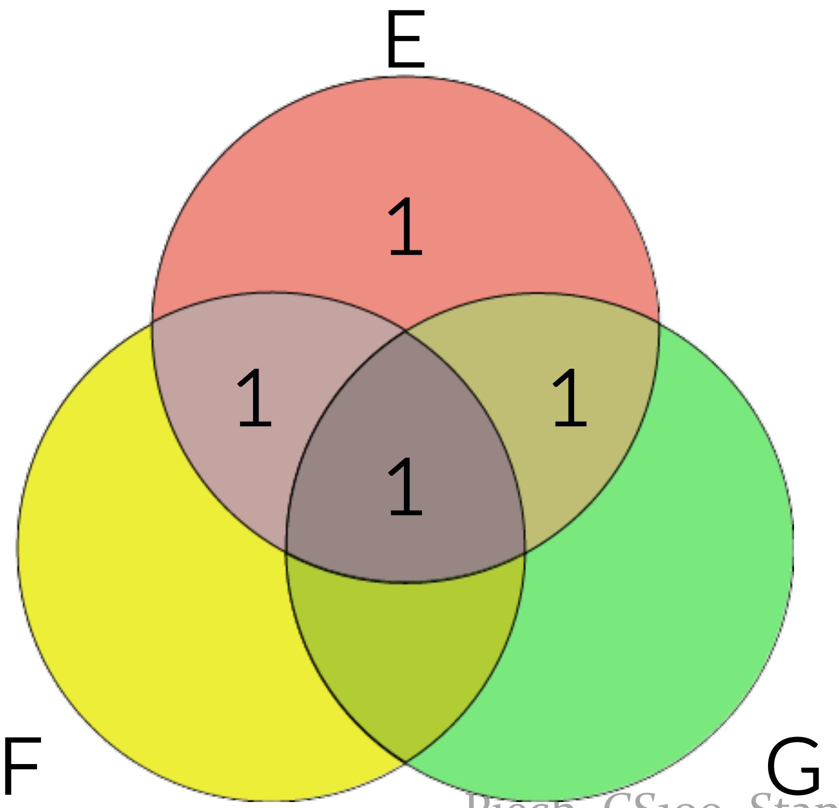
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) =$$



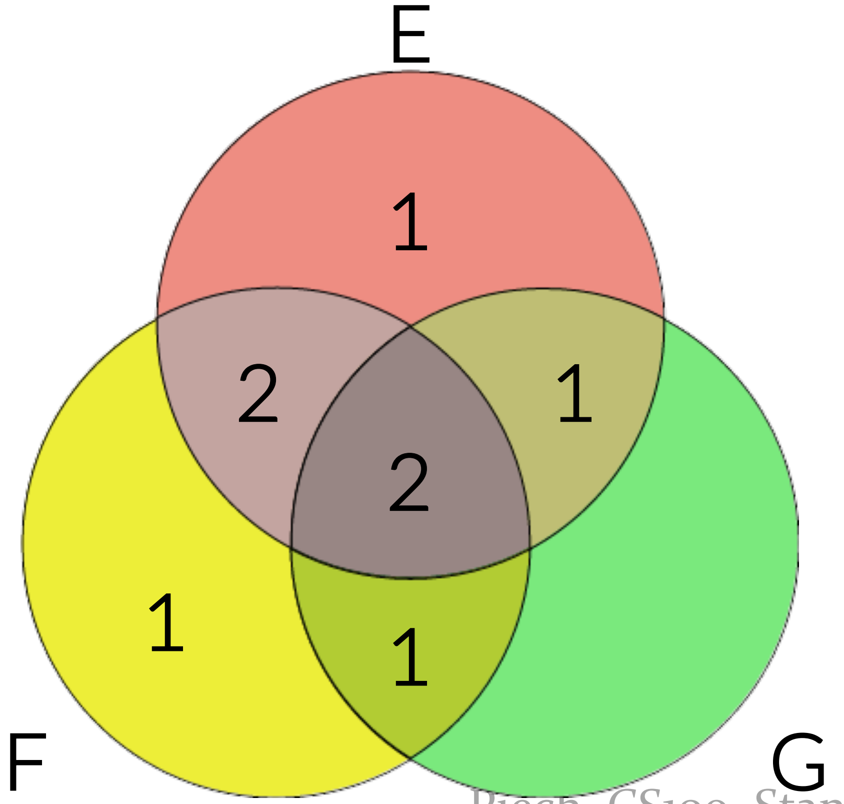
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E)$$



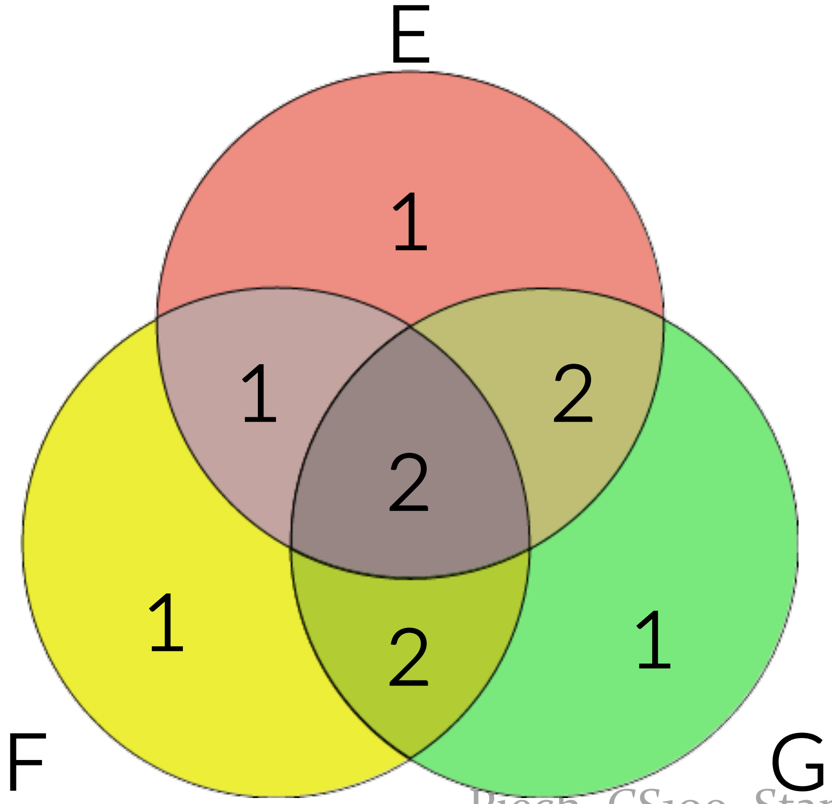
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F)$$



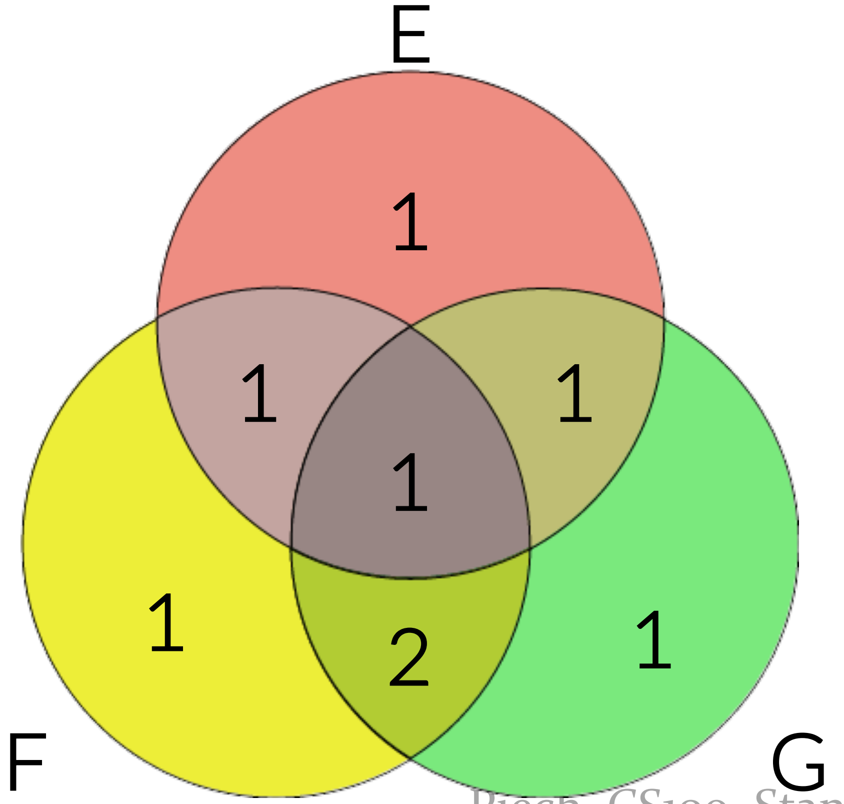
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF)$$



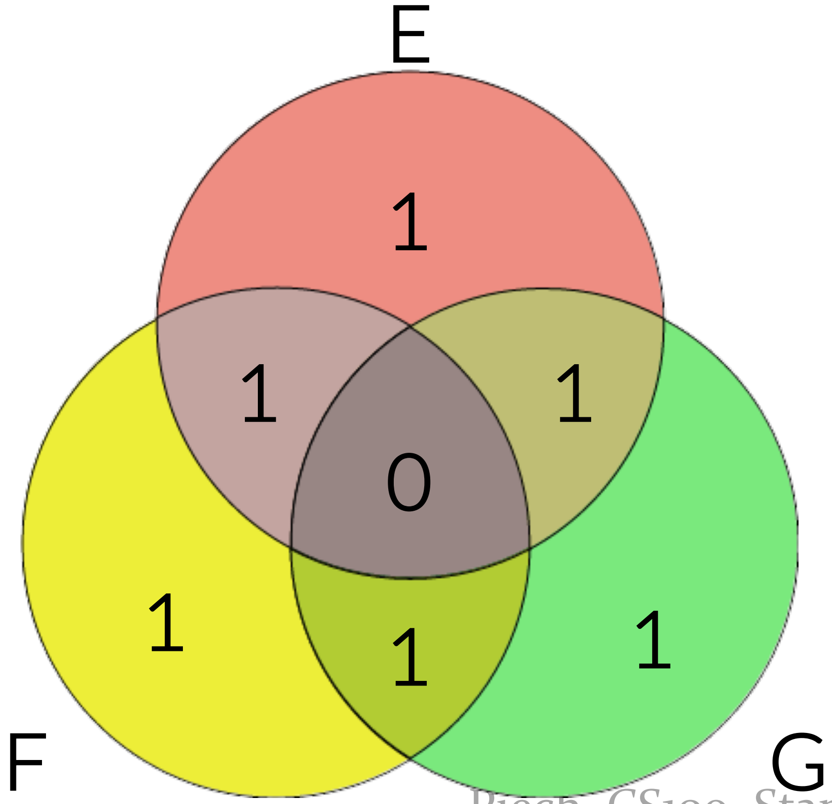
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG)$$



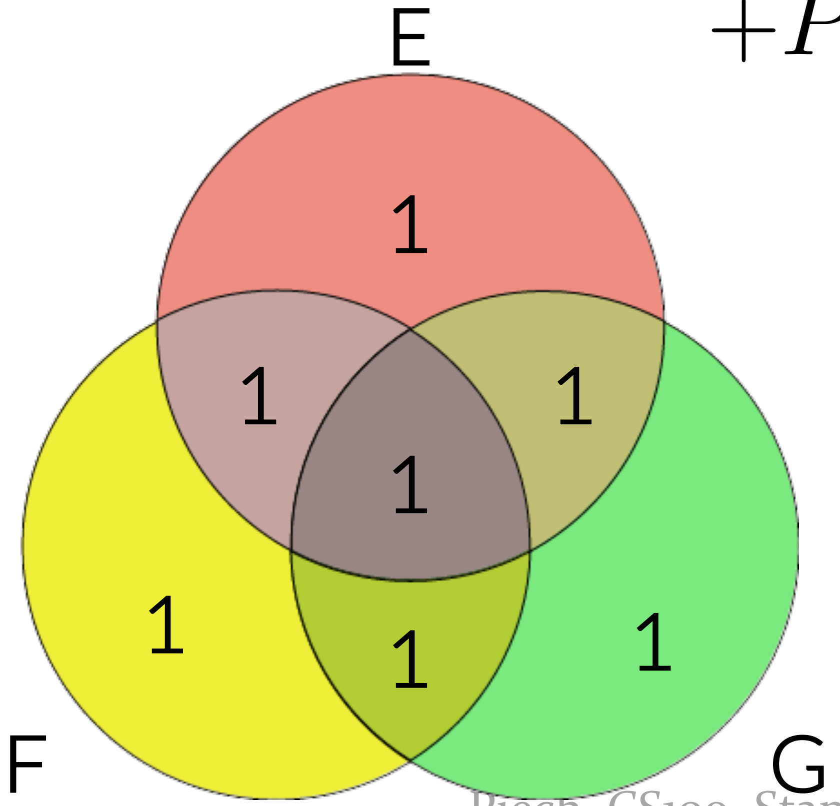
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG) - P(FG)$$



Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG) - P(FG)$$
$$+P(EFG)$$



General Inclusion / Exclusion

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} Y_r$$

Y_1 = Sum of all events on their own $\qquad \sum_i P(E_i)$

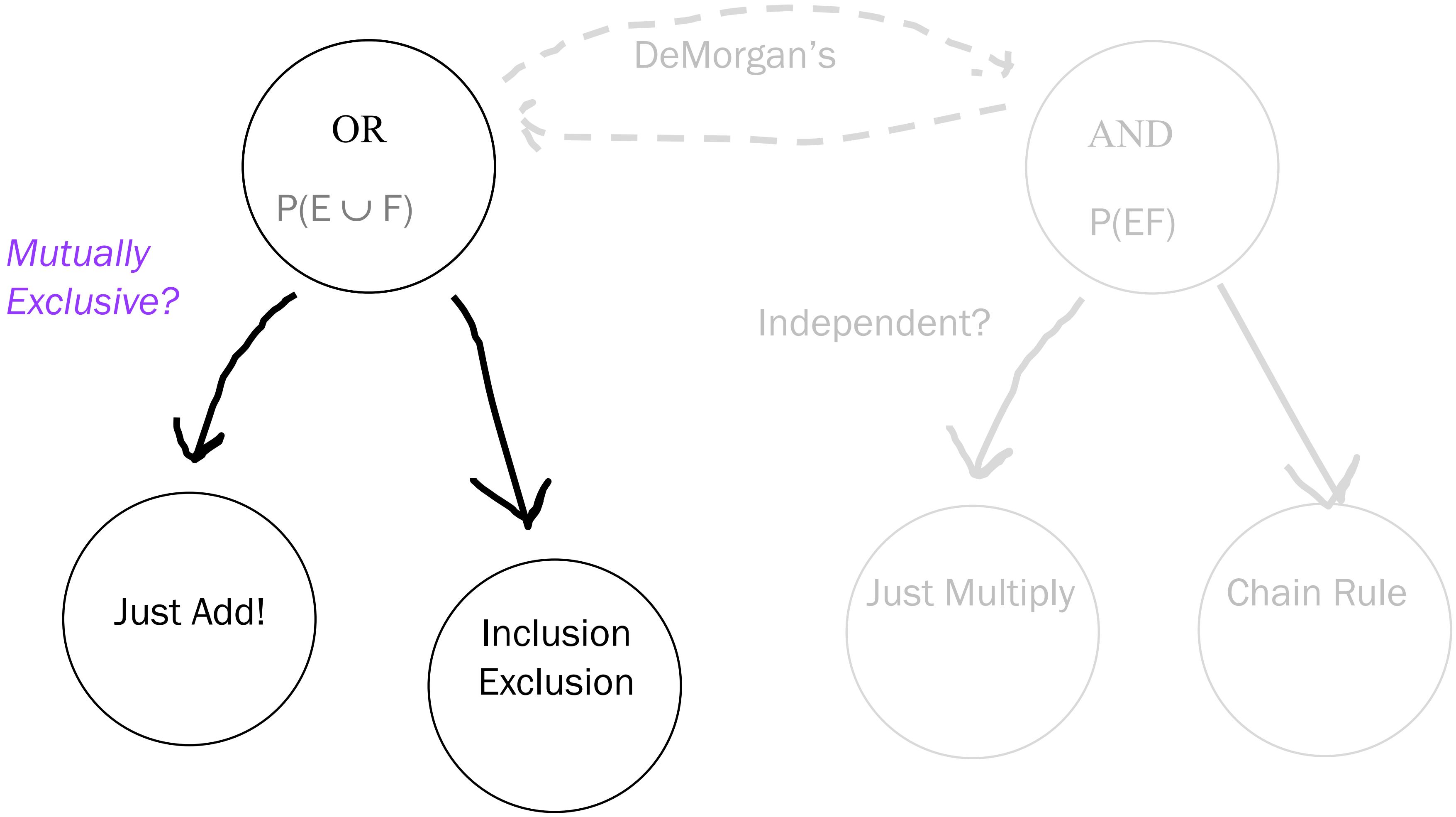
Y_2 = Sum of all pairs of events $\qquad \sum_{i,j} P(E_i \cap E_j) \quad \text{s.t.} i \neq j$

Y_3 = Sum of all triples of events $\qquad \sum_{i,j,k} P(E_i \cap E_j \cap E_k) \quad \text{s.t.} i \neq j, j \neq k, i \neq k$

* Where Y_r is the sum, for all combinations of *r* events, of the probability of the ~~union~~ those events.

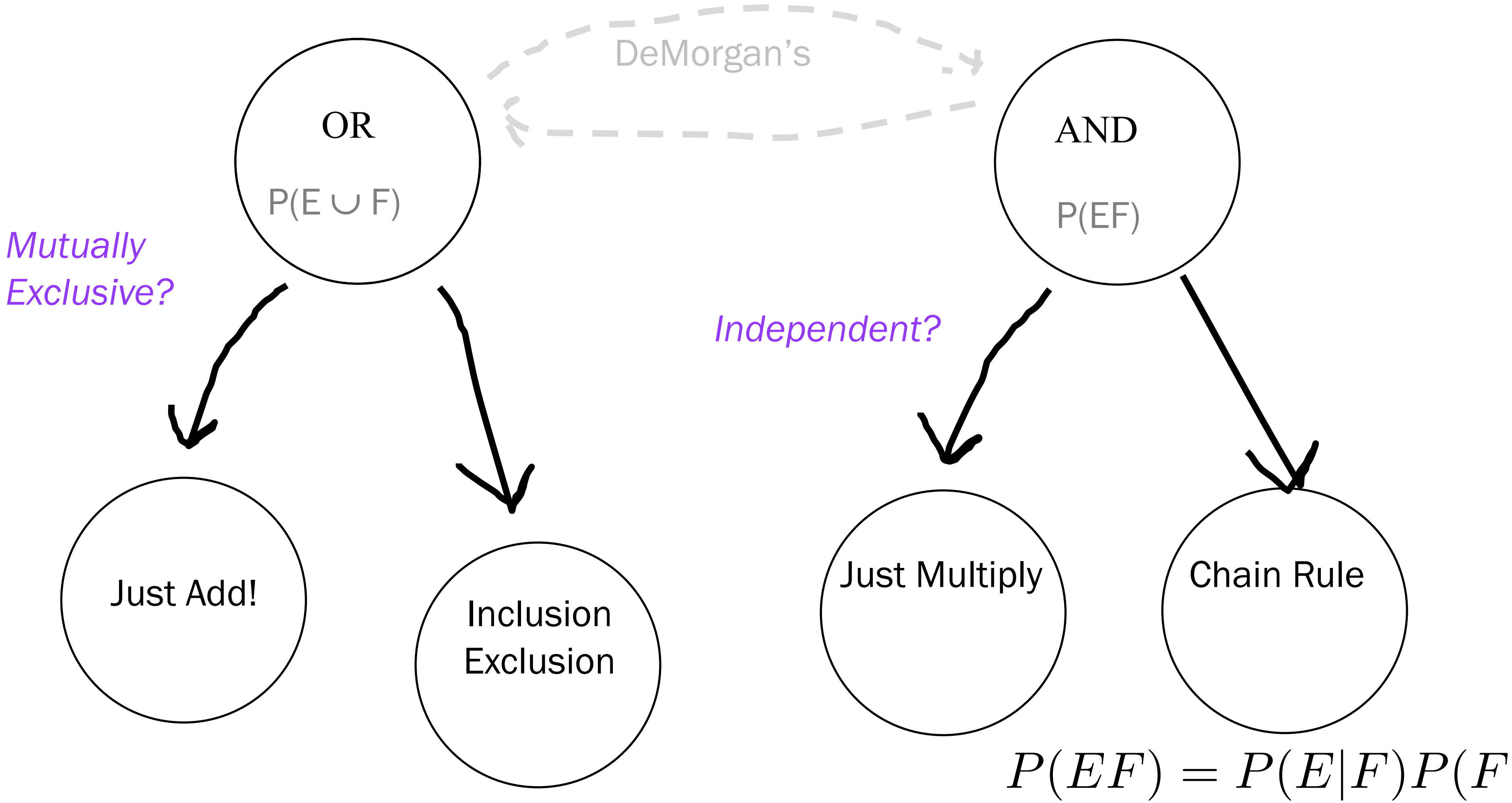
intersection

Today



OR
P(E ∪ F)
DeMorgan's
AND
P(EF)
Mutually Exclusive?
Independent?
Just Add!
Inclusion Exclusion
Just Multiply
Chain Rule

Today



$$P(EF) = P(E|F)P(F)$$

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Probability of “AND”


We the People
of the
insure domestic Tranquility, provide for the common defence
and our Posterity, do ordain and establish this
Article. I

Independence

Two events A and B are called **independent** if:

$$P(A) = P(A|B)$$

Knowing that event B happened, doesn't change our belief that A will happen.

Otherwise, they are called **dependent** events

Alternative Definition of Independence

Notation for *and*

$$P(A, B) = P(A) \cdot P(B|A)$$ Chain rule

$$= P(A) \cdot P(B)$$ Since B is independent of A

If you show this is true, you have proved the two events are independent!

If events are *independent* probability of AND is easy!

*You will need to use this “trick” with high probability

Independence is reciprocal

If A is independent of B, then B is independent of A

$$P(A) = P(A|B)$$

$$P(B|A) = P(B)$$

Proof:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' Thm.

$$= \frac{P(A)P(B)}{P(A)}$$

Because A is independent of B

$$= P(B)$$

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 1$

What is P(E), P(F), and P(EF)?

- $P(E) = 1/6$, $P(F) = 1/6$, $P(EF) = 1/36$
- $P(EF) = P(E)\,P(F)$ → E and F *independent*

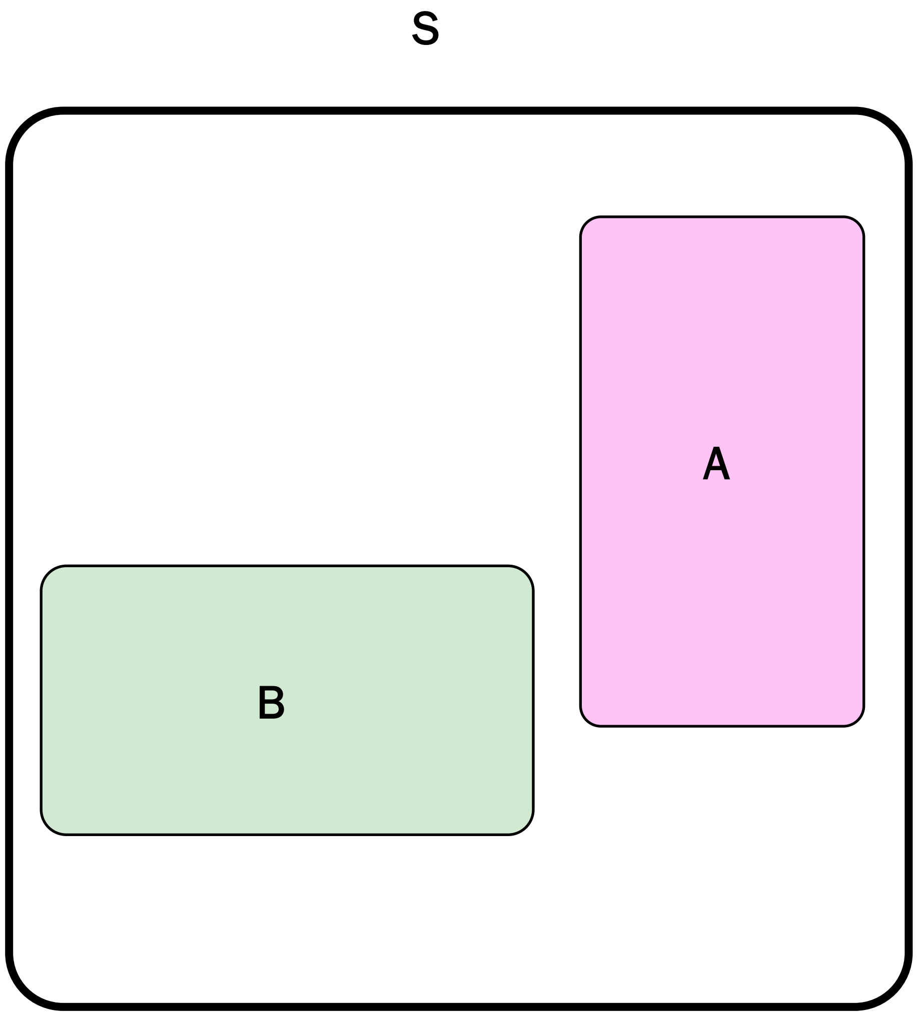
Let G be event: $D_1 + D_2 = 5$ {(1, 4), (2, 3), (3, 2), (4, 1)}

What is P(E), P(G), and P(EG)?

- $P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(EG) = 1/36$
- $P(EG) \neq P(E)\,P(G)$ → E and G *dependent*

What does independence look like?

Independence



Independence Definition 1:

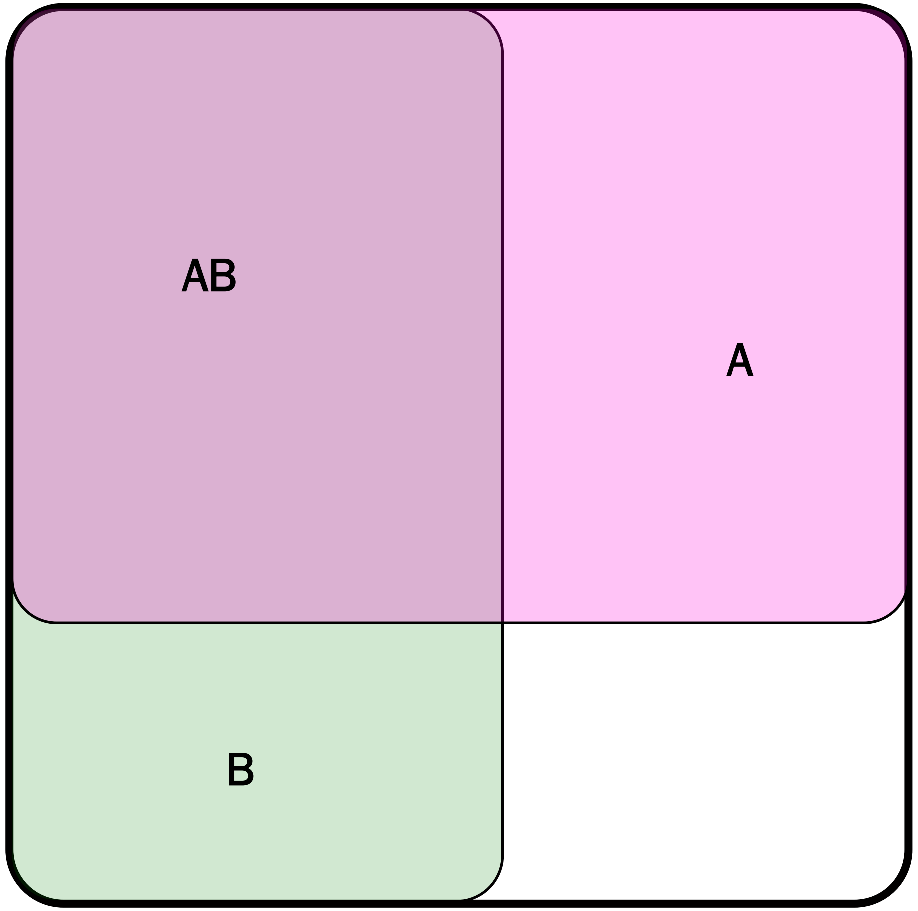
$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

0

Independence

S



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

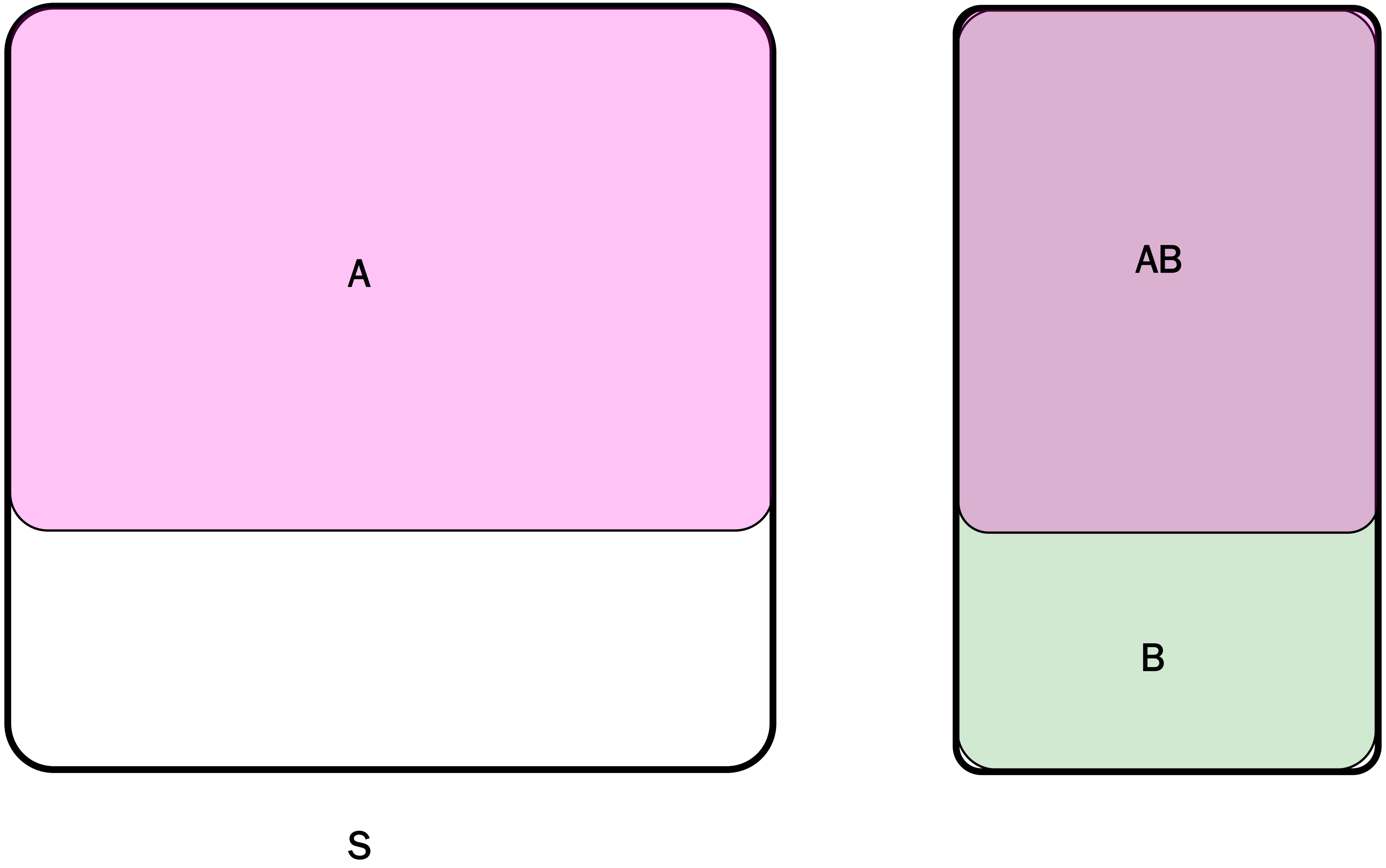
Independence Definition 2:

$$P(A|B) = P(A)$$

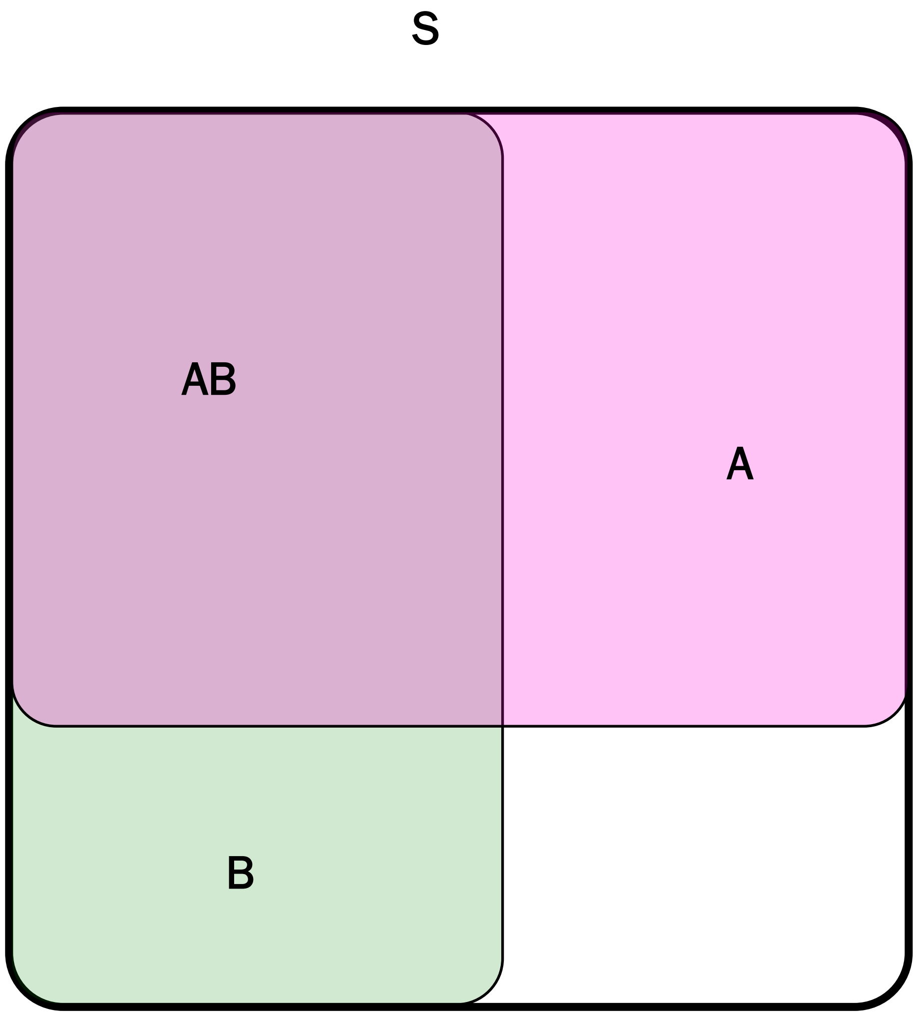
$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Independence

This ratio, P(A)... ... is the same as this one, P(A|B)



Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

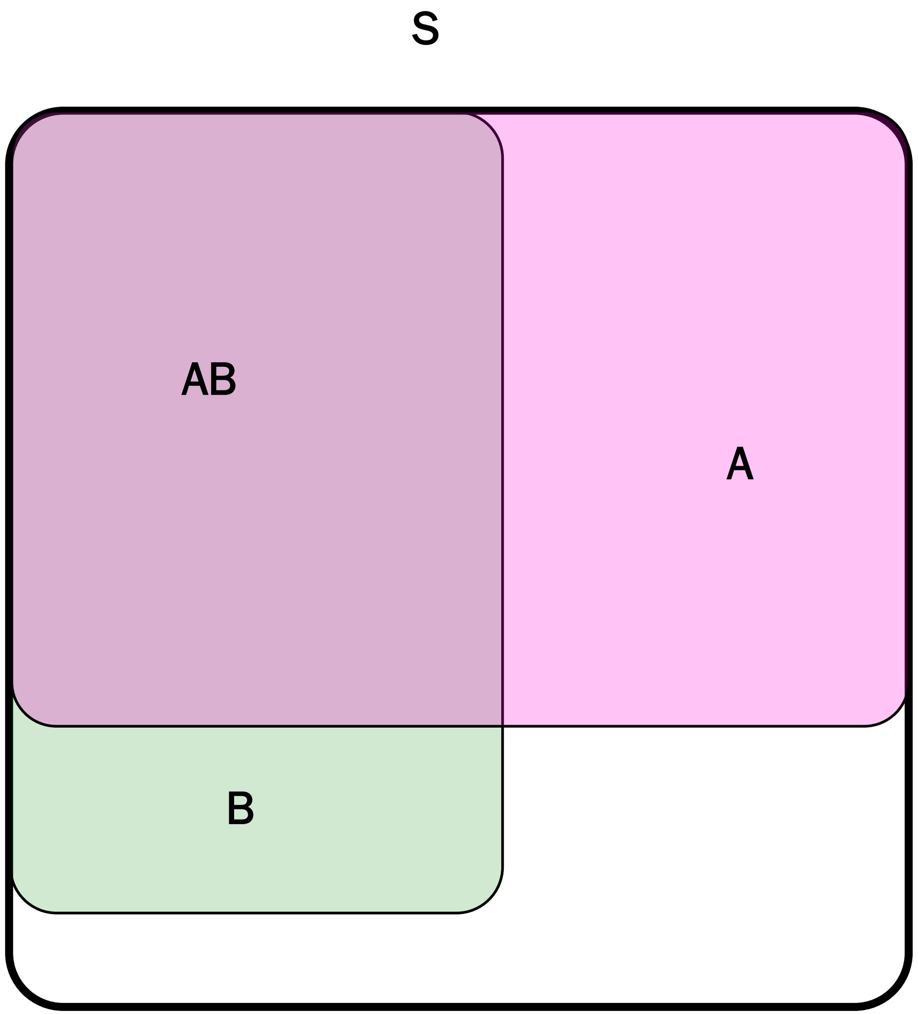
$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Dependence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

More Intuition through proofs:

Independence

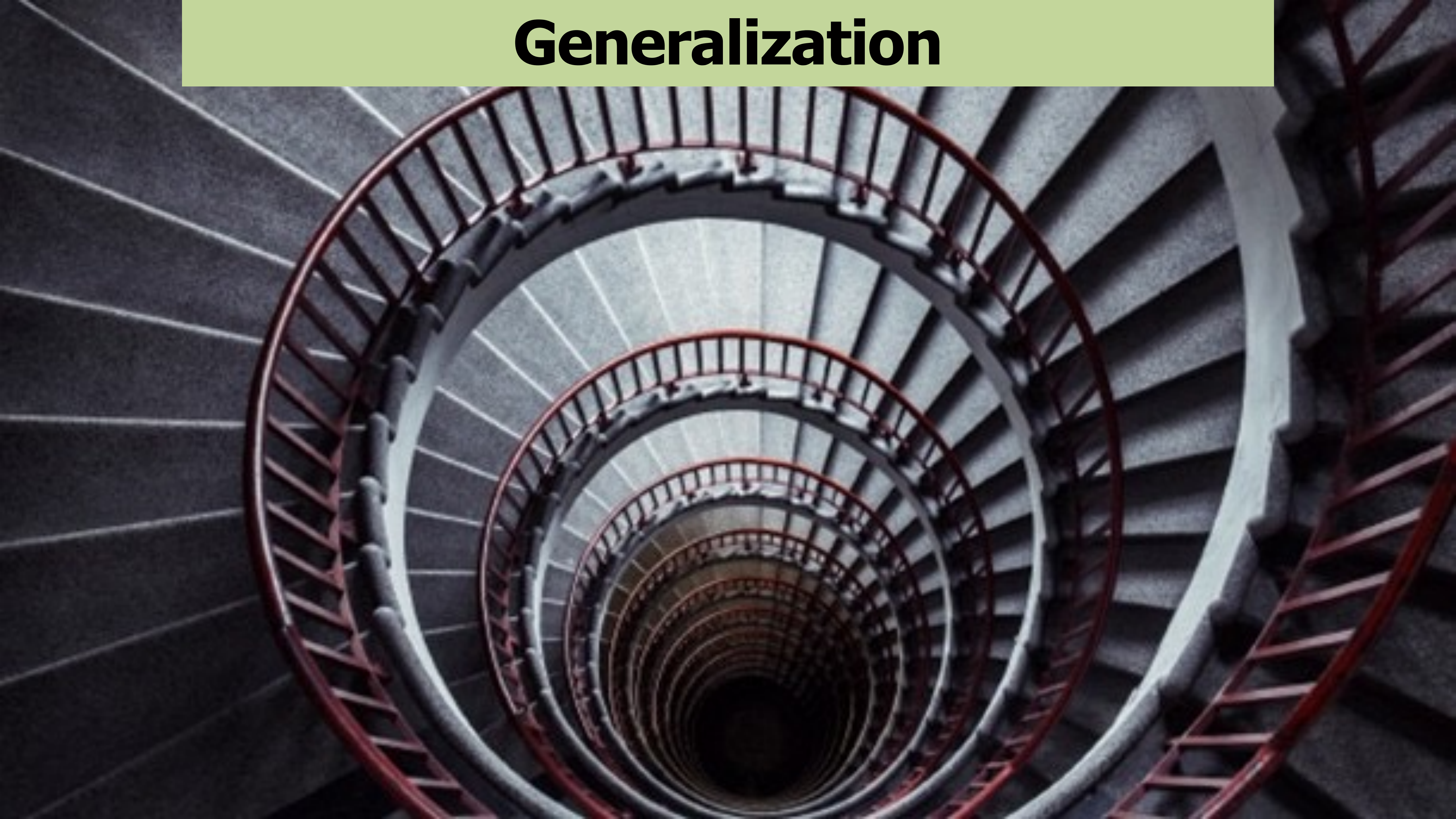
Given independent events A and B, prove that A and B^C are independent

We want to show that $P(AB^C) = P(A)P(B^C)$

$$\begin{aligned} P(AB^C) &= P(A) - P(AB) && \text{By Total Law of Prob.} \\ &= P(A) - P(A)P(B) && \text{By independence} \\ &= P(A)[1 - P(B)] && \text{Factoring} \\ &= P(A)P(B^C) && \text{Since } P(B) + P(B^C) = 1 \end{aligned}$$

So if A and B are independent A and B^C are also independent

Generalization



Generalized Independence

General definition of Independence:

Events E_1 , E_2 , ..., E_n are independent if **for every subset** with r elements (where $r \leq n$) it holds that:

$$P(E_{1'}E_{2'}E_{3'}\ldots E_{r'}) = P(E_{1'})P(E_{2'})P(E_{3'})\ldots P(E_{r'})$$

Example: outcomes of n separate flips of a coin are all independent of one another

- Each flip in this case is called a “trial” of the experiment

Math > Intuition

Two Dice

Roll two 6-sided dice, yielding values D_1 and D_2

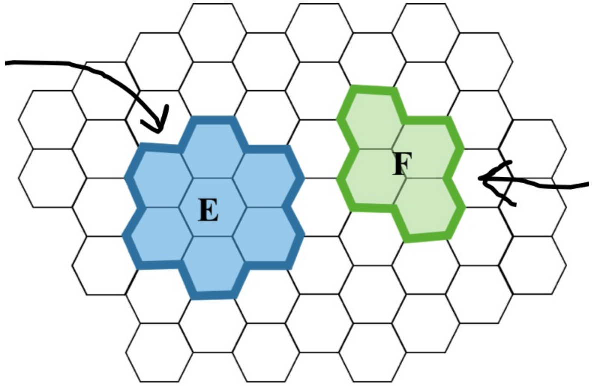
- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 6$
- Are E and F independent? Yes!

Let G be event: $D_1 + D_2 = 7$

- Are E and G independent? Yes!
- $P(E) = 1/6$, $P(G) = 1/6$, $P(E\ G) = 1/36$ [roll (1, 6)]
- Are F and G independent? Yes!
- $P(F) = 1/6$, $P(G) = 1/6$, $P(F\ G) = 1/36$ [roll (1, 6)]
- Are E, F and G independent? No!
- $P(EFG) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

New Ability

Properties of Pairs of Events



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

also:

$$P(A \text{ or } B) = P(A) + P(B)$$

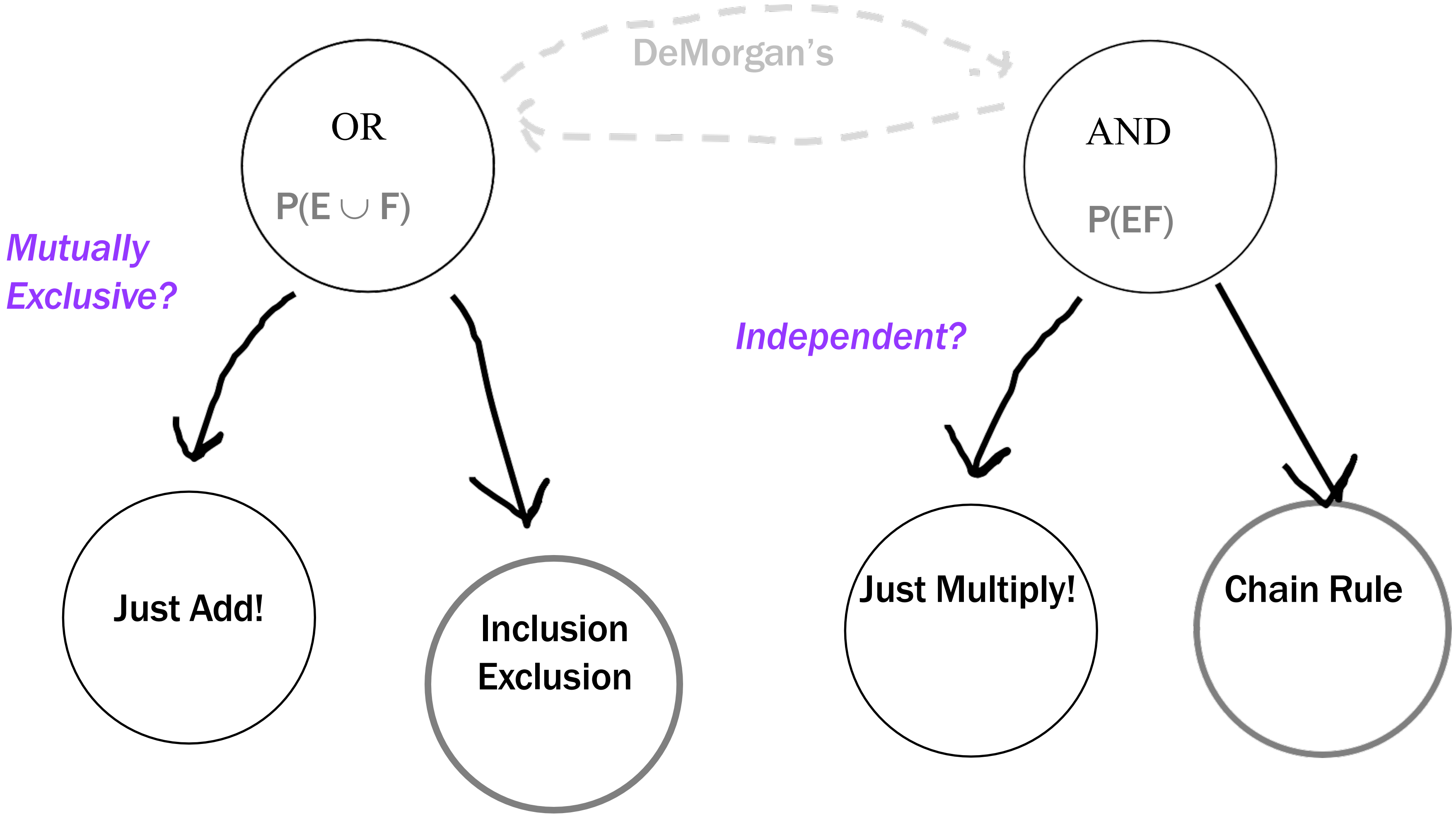
Independent

$$P(A) = P(A|B)$$

also:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Today


DeMorgan's
OR
P(E ∪ F)
AND
P(EF)
Mutually Exclusive?
Independent?
Just Add!
Inclusion Exclusion
Just Multiply!
Chain Rule

Think of the children as independent trials

Independence:
$P(A \text{ and } B) = P(A) \cdot P(B)$

Two parents both have an (A, a) gene pair.

- Each parent will pass on one of their genes (each gene equally likely) to a child.
- The probability of **any single child** having curly hair (the recessive trait) is 0.25, independent of other siblings.
- There are three children.

What is the probability that all three children have curly hair?

Let E_1, E_2, E_3 be the events that child 1, 2, and 3 have curly hair, respectively.

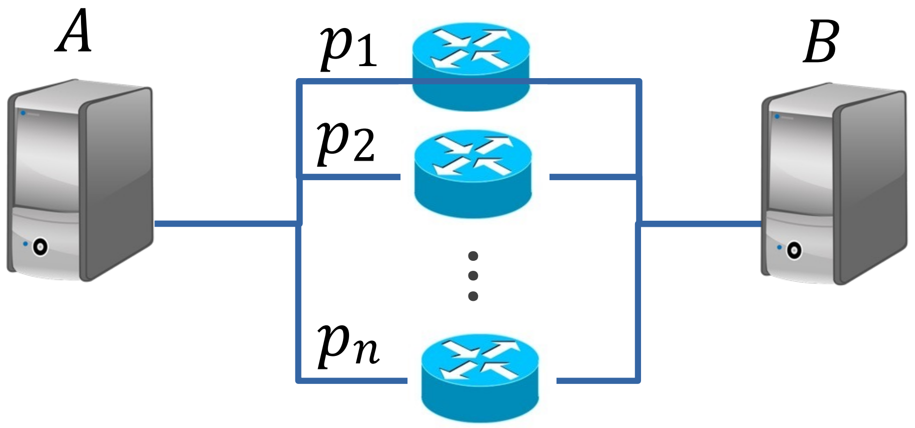
$$P(E_1E_2E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)$$
$$= P(E_1)P(E_2)P(E_3)$$

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

What is $P(E)$?



Network reliability

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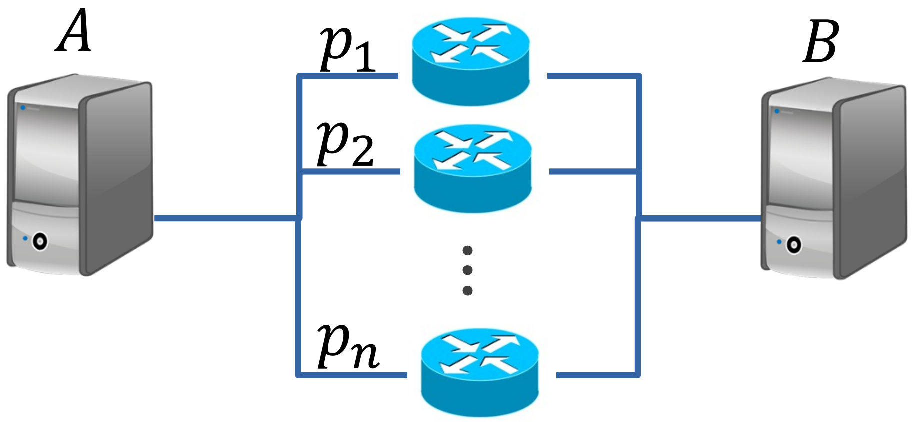


$$\begin{aligned} P(E) &= P(\geq 1 \text{ one router works}) \\ &= 1 - P(\text{all routers fail}) \\ &= 1 - (1-p_1)(1-p_2)\cdots(1-p_n) \\ &= 1 - \prod_{i=1}^{n}(1-p_i) \end{aligned}$$

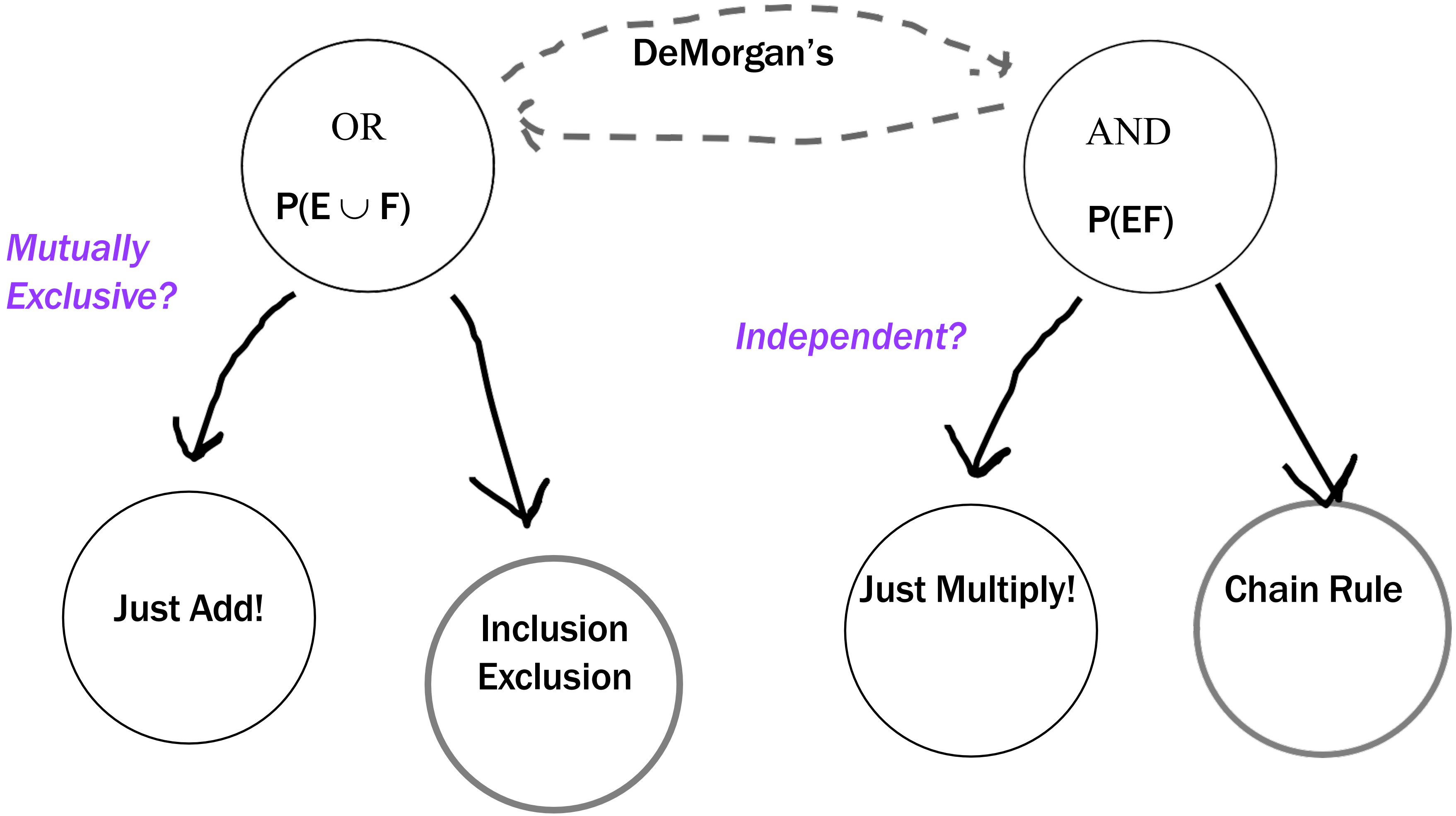
≥ 1 with independent trials: take complement

The Most Important Core Probability Question

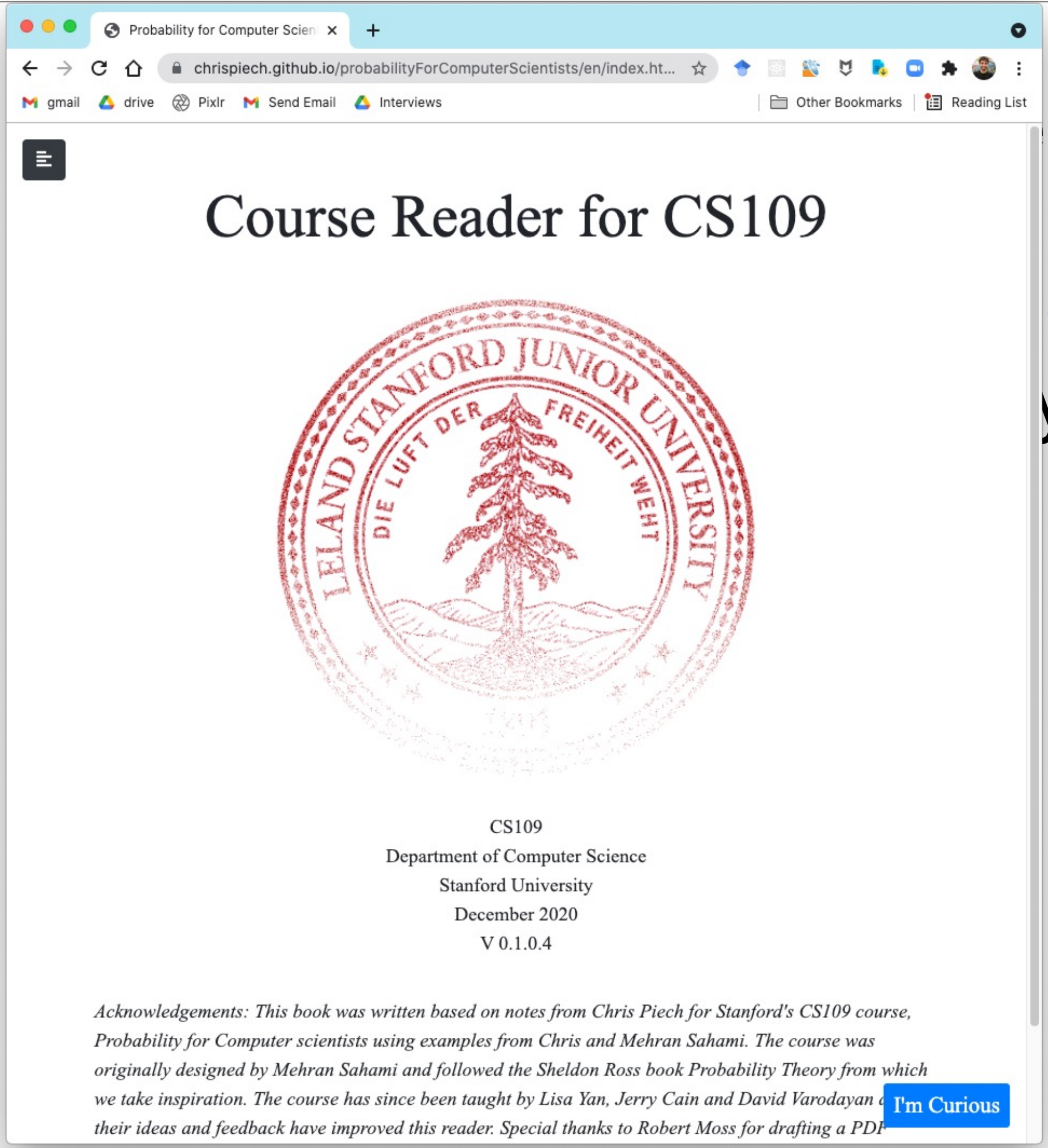
Say a coin comes up heads with probability p

- Flip the coin n times
- Each coin flip is an **independent** trial
- What is the probability of exactly k heads?

Pedagogical Pause



The Most Important Core Probability Question



Course Reader for CS109

CS109
Department of Computer Science
Stanford University
December 2020
V 0.1.0.4

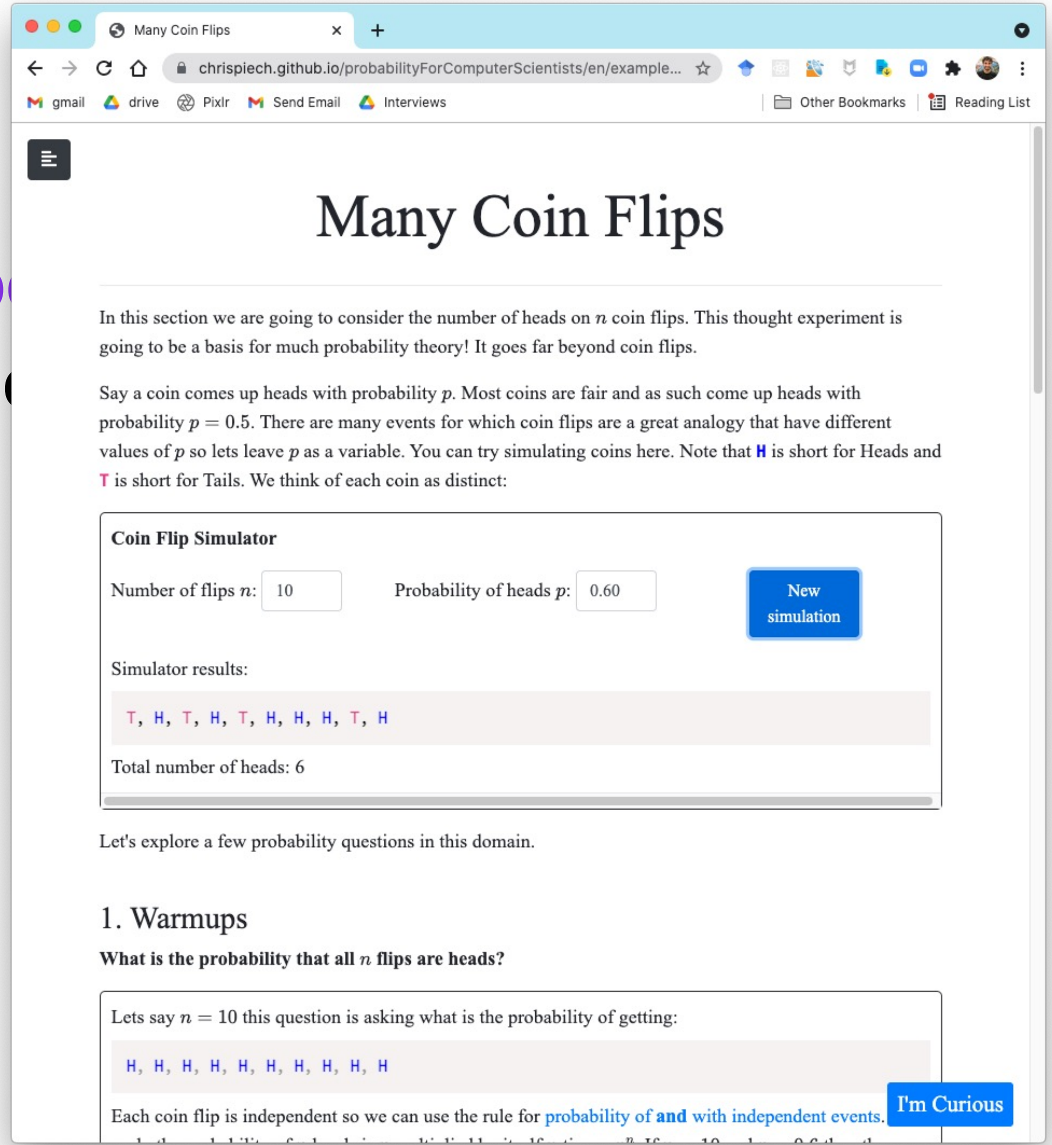
Acknowledgements: This book was written based on notes from Chris Piech for Stanford's CS109 course, Probability for Computer scientists using examples from Chris and Mehran Sahami. The course was originally designed by Mehran Sahami and followed the Sheldon Ross book Probability Theory from which we take inspiration. The course has since been taught by Lisa Yan, Jerry Cain and David Varodayan ... their ideas and feedback have improved this reader. Special thanks to Robert Moss for drafting a PDF

I'm Curious

ads

dep

y of



Many Coin Flips

In this section we are going to consider the number of heads on n coin flips. This thought experiment is going to be a basis for much probability theory! It goes far beyond coin flips.

Say a coin comes up heads with probability p. Most coins are fair and as such come up heads with probability $p = 0.5$. There are many events for which coin flips are a great analogy that have different values of p so lets leave p as a variable. You can try simulating coins here. Note that **H** is short for Heads and **T** is short for Tails. We think of each coin as distinct:

Coin Flip Simulator

Number of flips n: 10 Probability of heads p: 0.60 New simulation

Simulator results:

T, H, T, H, T, H, H, H, T, H

Total number of heads: 6

Let's explore a few probability questions in this domain.

1. Warmups

What is the probability that all n flips are heads?

Lets say $n = 10$ this question is asking what is the probability of getting:

H, H, H, H, H, H, H, H, H, H

Each coin flip is independent so we can use the rule for probability of **and** with independent events.

I'm Curious

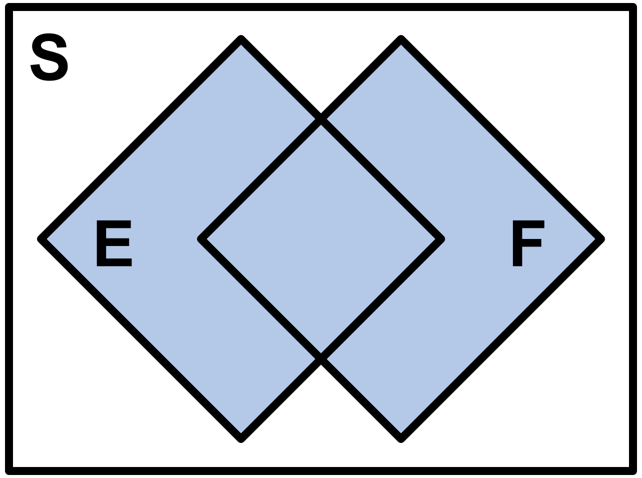
Sets Review

Sets Review

Say E and F are events in S

Event that is in E <u>or</u> F

E ∪ F



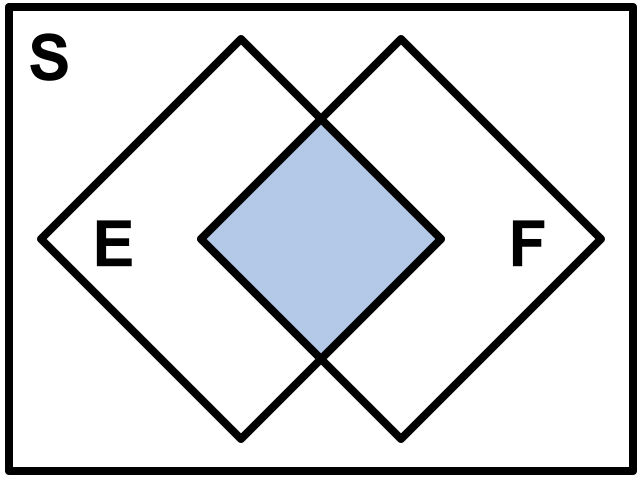
- S = {1, 2, 3, 4, 5, 6} die roll outcome
- E = {1, 2} F = {2, 3} E ∪ F = {1, 2, 3}

Sets Review

Say E and F are events in S

Event that is in E and F

$E \cap F$

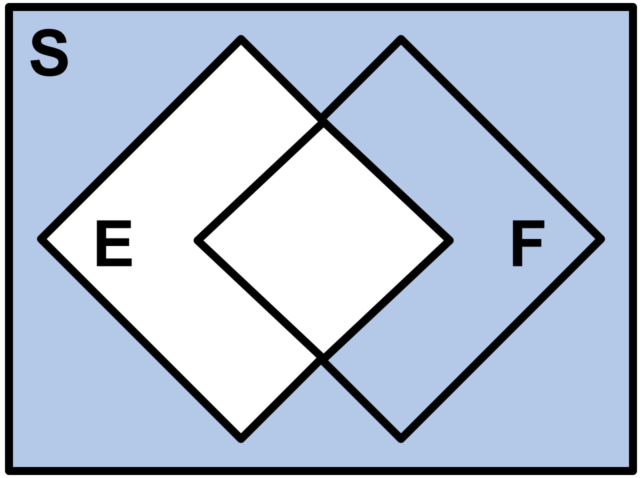


Sets Review

Say E and F are events in S

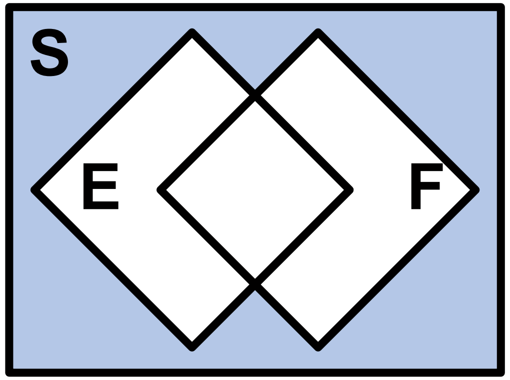
Event that is not in E (called complement of E)

E^c *or* ~E



Sets Review

Say E and F are subsets of S



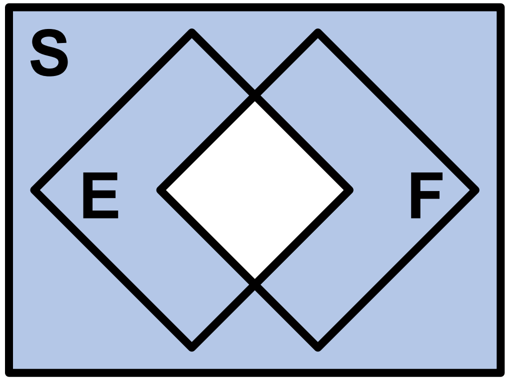
Which of these two is it?

a) $(E \text{ or } F)^C$

b) $(E^C \text{ and } F^C)$

Sets Review

Say E and F are subsets of S



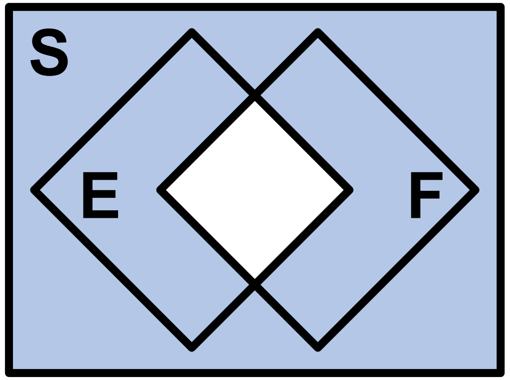
Which of these two is it?

a) $(E \text{ and } F)^C$

b) $(E^C \text{ or } F^C)$

De Morgan's Laws

De Morgan's Law lets you alternate between AND and OR.

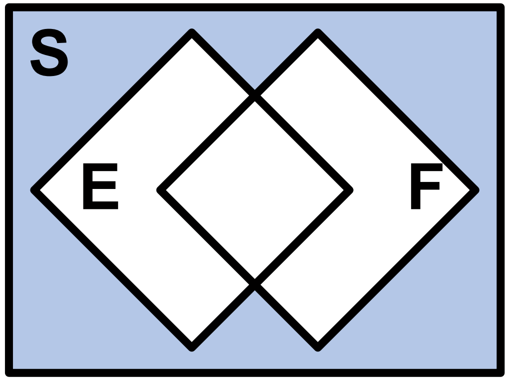


$$(E \cap F)^C = E^C \cup F^C$$

In probability:

$$P(E_1 E_2 \cdots E_n)$$
$$= 1 - P\left((E_1 E_2 \cdots E_n)^C\right)$$
$$= 1 - P\left(E_1^C \cup E_2^c \cup \cdots \cup E_n^c\right)$$

Great if E_i^C mutually exclusive!



$$(E \cup F)^C = E^C \cap F^C$$

In probability:

$$P(E_1 \cup E_2 \cup \cdots \cup E_n)$$
$$= 1 - P\left((E_1 \cup E_2 \cup \cdots \cup E_n)^C\right)$$
$$= 1 - P(E_1^c E_2^c \cdots E_n^c)$$

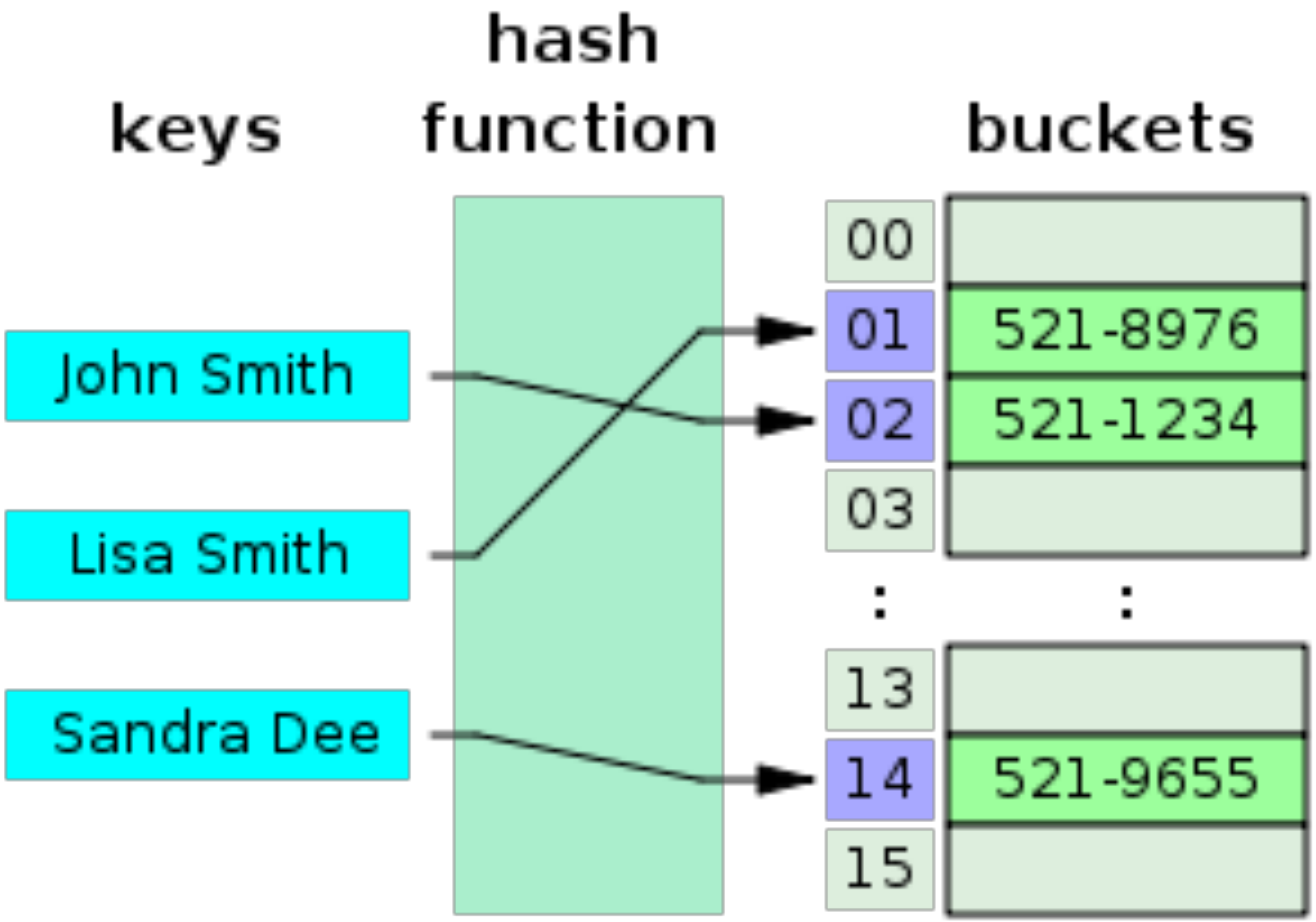
Great if E_i independent!

Augustin Demorgan

Jason Alexander

- British Mathematician who wrote the book “Formal Logic”in 1847
- Celebrity lookalike is Jason Alexander from Seinfeld.

Hash Tables



keys
hash function
buckets
John Smith
Lisa Smith
Sandra Dee
00
01
02
03
13
14
15
521-8976
521-1234
521-9655

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is **independent** with probability p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define: S_i = **string** i **hashes to bucket** 1
S_i^C = **string** i **doesn't hash to bucket** 1

$$P(S_i) = p_1$$
$$P(S_i^C) = 1 - p_1$$

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it?

Define: S_i = string i hashes to bucket 1
S_i^C = string i doesn't hash to bucket 1

$P(S_i) = p_1$
$P(S_i^C) = 1 - p_1$

WTF (not-real acronym for Want To Find):

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$

$$= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right)$$ **Complement**

$$= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right)$$ **De Morgan's Law**

$$= 1 - P(S_1^C)P(S_2^C) \cdots P(S_m^C) = 1 - \left(P(S_1^C)\right)^m$$ **S_i independent trials**

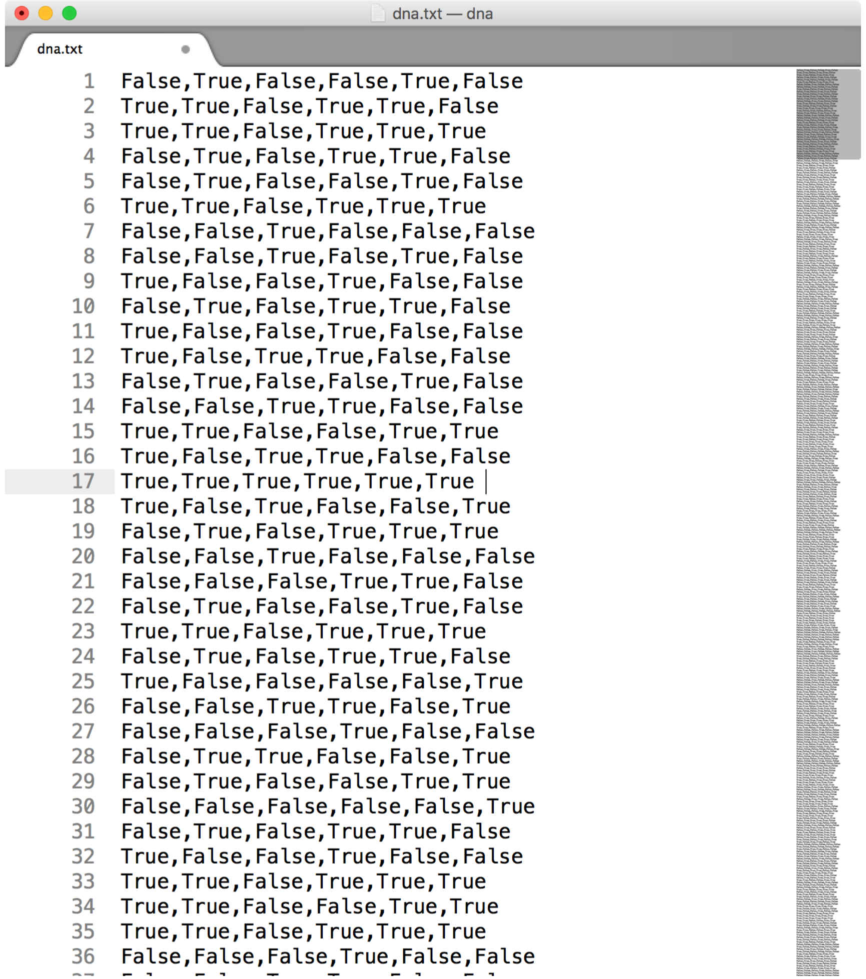
$$= 1 - (1 - p_1)^m$$

Here we are

Source: The Hobbit

G_1 G_2 G_3 G_4 G_5 **T**

G_1 G_2 G_3 G_4 G_5 T


dna.txt — dna

dna.txt

```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
True,False,False,True,False,False
False,True,False,True,True,False
True,False,False,True,False,False
True,False,True,True,False,False
False,True,False,False,True,False
False,False,True,True,False,False
True,True,False,False,True,True
True,False,True,True,False,False
True,True,True,True,True,True
True,False,True,False,False,True
False,True,False,True,True,True
False,False,True,False,False,False
False,False,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,True,False,True,True,False
True,False,False,False,False,True
False,False,True,True,False,True
False,False,False,True,False,False
False,True,True,False,False,True
False,True,False,False,True,True
False,False,False,False,False,True
False,True,False,True,True,False
True,False,False,True,False,False
True,True,False,True,True,True
True,True,False,False,True,True
True,True,False,True,True,True
False,False,False,True,False,False
```

100,000 samples

6 observations per sample

Discovered Pattern

```
Piech-2:dna piech$ python findStructure.py
size data =  100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

...

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

Discovered Pattern

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...

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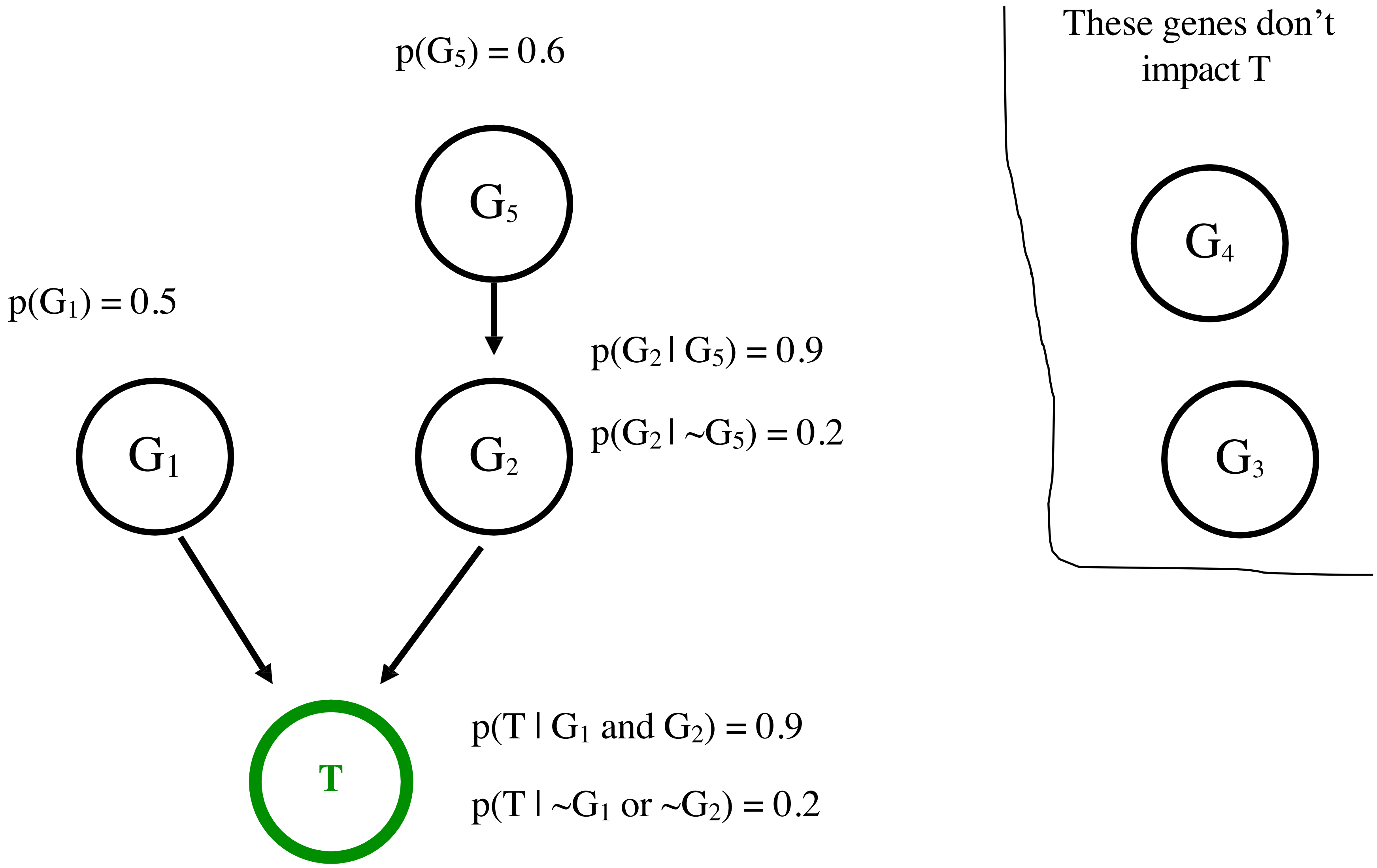
Only Causal Structure That Fits



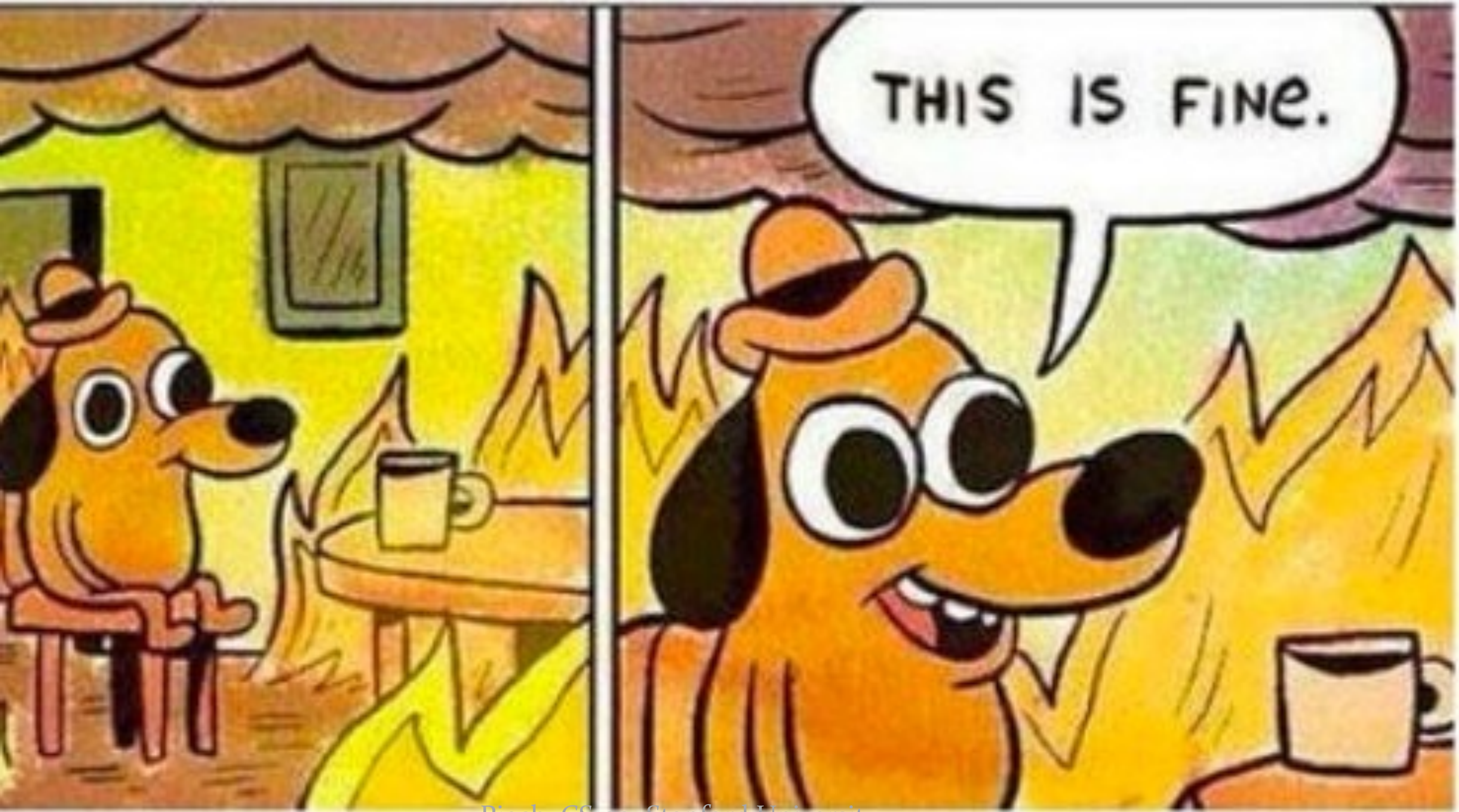
Want extra challenges???

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it?
3. E = each of buckets 1 to k has ≥ 1 string hashed into it?


THIS IS FINE.

More hash table fun: Possible approach?

- m strings are hashed (not uniformly) into a hash table with n buckets.
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What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
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$$P(E) \quad =$$

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$P(E) \quad =$

Define F_i = bucket i has at least one string in it

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$$P(E) \quad = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$

Define F_i = bucket i has at least one string in it

⚠ F_i bucket events are *dependent*! So we cannot just add.

More hash table fun: Possible approach?

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Define F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$
$$= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right)$$
$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$
$$=$$

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$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$
$$=$$

$= P(\textbf{buckets } 1 \textbf{ to } k \textbf{ all denied strings})$

$= \left(P(\text{each string hashes to } k+1 \text{ or higher})\right)$

$= (1 - p_1 - p_2 \ldots - p_k)^m$

⚠ F_i **bucket events are *dependent*! So we cannot just add.**

More hash table fun: Possible approach?

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

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The fun never stops with hash tables

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it? ✅
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it? ✅

Looking for another challenge? ☺

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it?
3. E = each of buckets 1 to k has ≥ 1 string hashed into it?

Hint: Use Part 2's event definition:

Define F_i = **bucket i has at least one string in it**

Hint: Try $k = 2$, then $k = 3$, then generalize.

The fun never stops with hash tables

Solution

- F_i = at least one string hashed into *i*-th bucket
- P(E) $= P(F_1 F_2 \ldots F_k) = 1 - P((F_1 F_2 \ldots F_k)^c)$

  $= 1 - P(F_1^c \cup F_2^c \cup \ldots \cup F_k^c)$ (DeMorgan's Law)

  $= 1 -$

where $$P\left(\bigcup_{i=1}^{k} F_i^c\right) = 1 - \sum_{r=1}^{k} (-1)^{(r+1)} \sum_{i_1 < \ldots < i_r} P(F_{i_1}{}^c F_{i_2}{}^c \ldots F_{i_r}{}^c)$$

$$P(F_{i_1}{}^c F_{i_2}{}^c \ldots F_{i_r}{}^c) = (1 - p_{i_1} - p_{i_2} - \ldots - p_{i_r})^m$$