

03: Intro to Probability

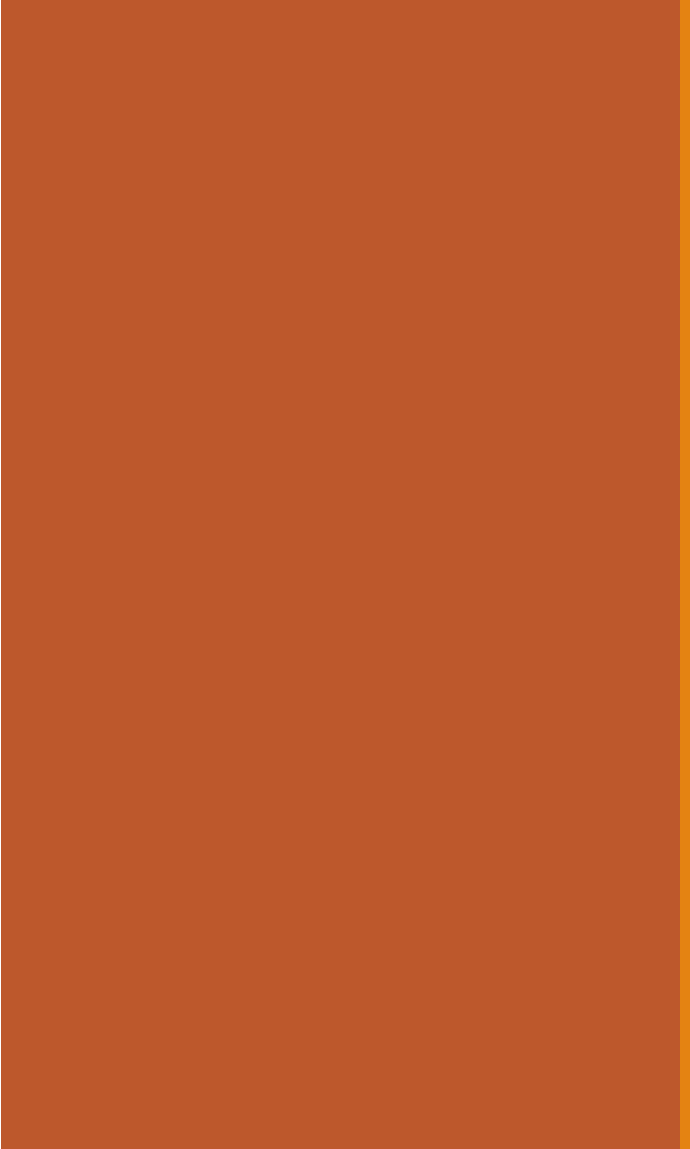
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January 13, 2023

Ed Discussion: <https://edstem.org/us/courses/32220/discussion/2380963>

Quick slide reference

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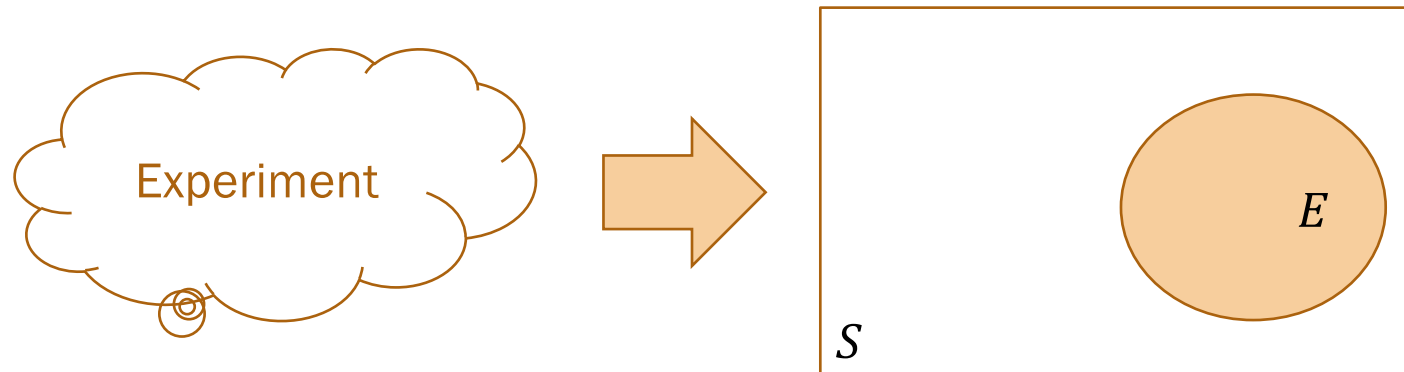




Defining Probability

Key definitions

An experiment in probability:



Sample Space, S : The set of all possible **outcomes** of an **experiment**

Event, E : Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
 $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
 $S = \{(H,H), (H,T), (T,H), (T,T)\}$
- Roll of 6-sided die
 $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
 $S = \{x \mid x \in \mathbb{Z}, x \geq 0\}$
- TikTok hours in a day
 $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
 $E = \{\text{Heads}\}$
- ≥ 1 head in two coin flips
 $E = \{(H,H), (H,T), (T,H)\}$
- Roll is 3 or less:
 $E = \{1, 2, 3\}$
- Low email day (≤ 20 emails)
 $E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 20\}$
- Lost day (≥ 5 TT hours):
 $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

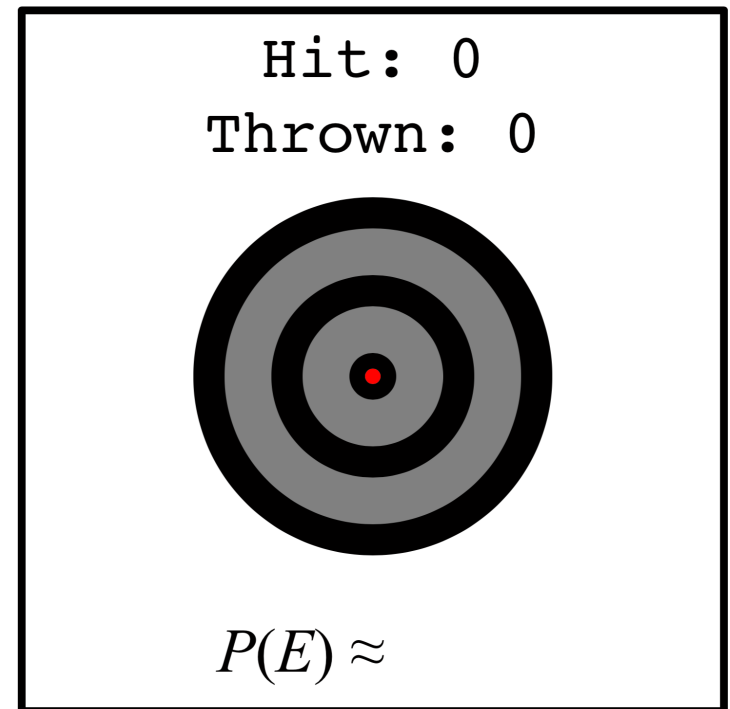
What is a probability?

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes
where you hit the target.



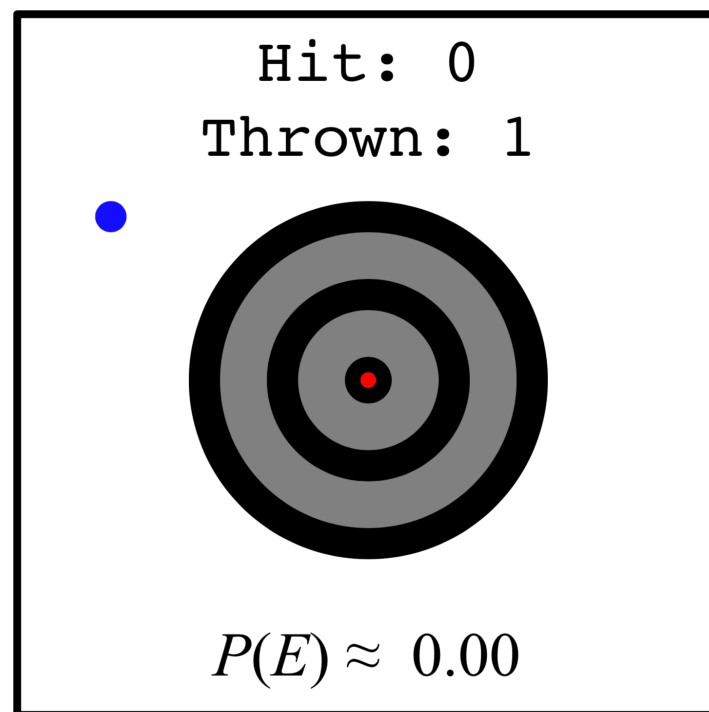
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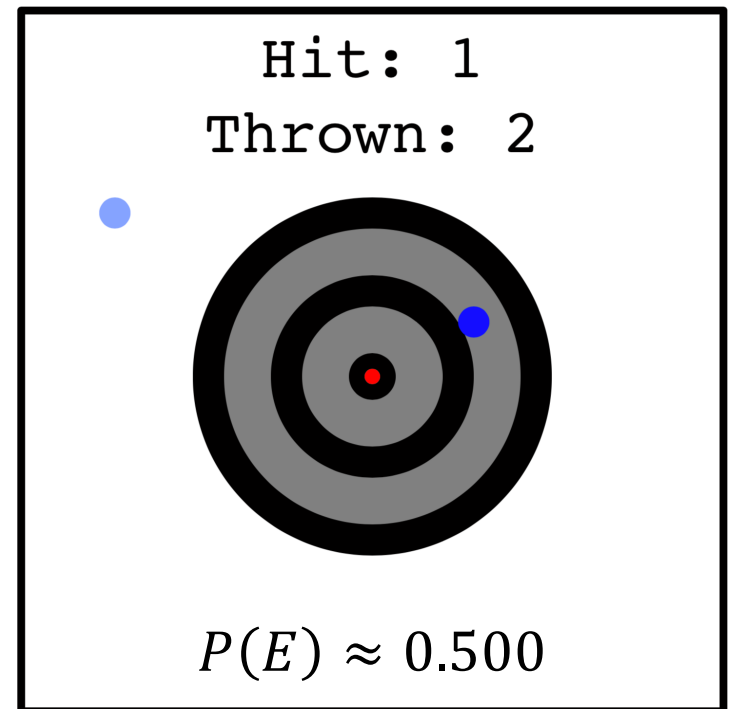
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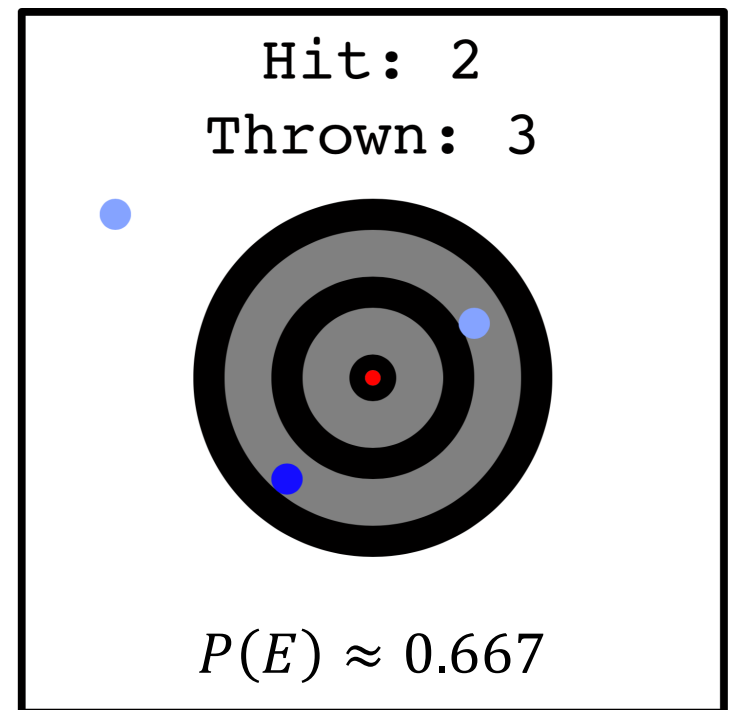
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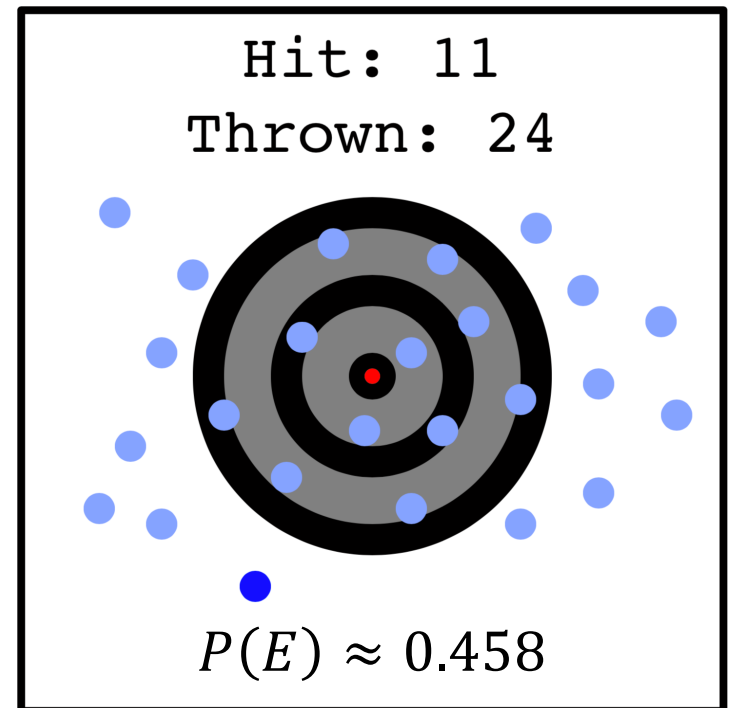
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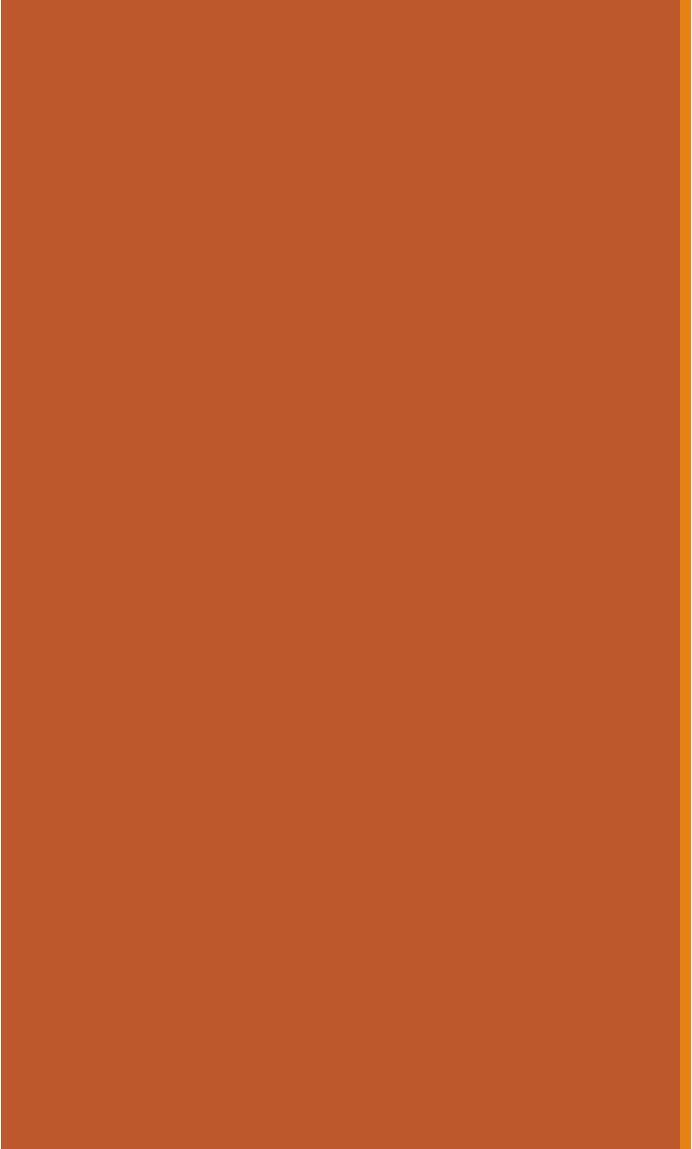
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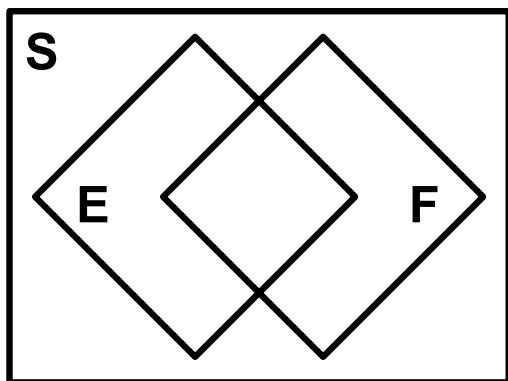




Axioms of Probability

Quick review of sets

Review of Sets



E and F are events in S .

Experiment:

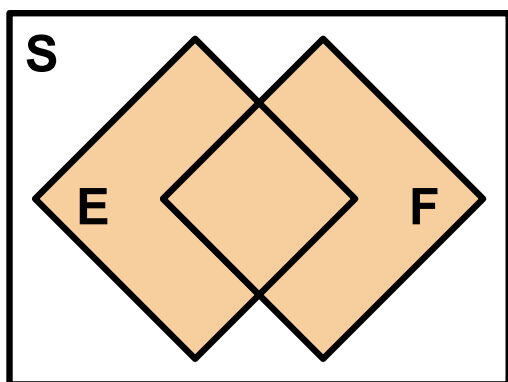
Die roll

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$\text{Let } E = \{1, 2\}, \text{ and } F = \{2, 3\}$$

Quick review of sets

Review of Sets



E and F are events in S .

Experiment:

Die roll

$$S = \{1, 2, 3, 4, 5, 6\}$$

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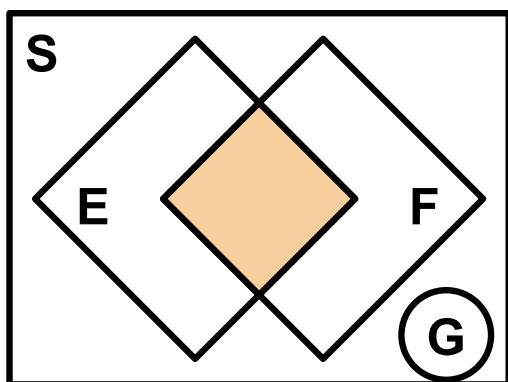
def **Union** of events, $E \cup F$

The event containing all outcomes
in E **or** F .

$$E \cup F = \{1, 2, 3\}$$

Quick review of sets

Review of Sets



E and F are events in S .

Experiment:

Die roll

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$\text{Let } E = \{1, 2\}, \text{ and } F = \{2, 3\}$$

def **Intersection** of events, $E \cap F$

The event containing all outcomes in E **and** F .

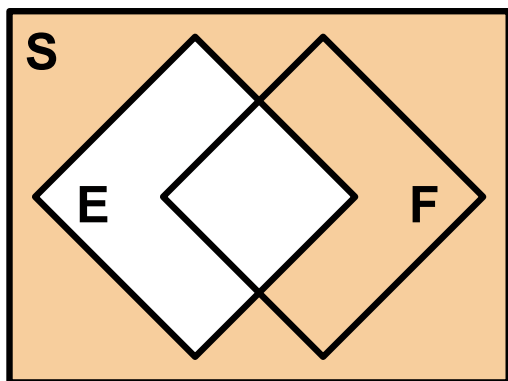
$$E \cap F = EF = \{2\}$$

def **Mutually exclusive** events F

and G means that $F \cap G = \emptyset$

Quick review of sets

Review of Sets



E and F are events in S .

Experiment:

Die roll

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$\text{Let } E = \{1, 2\}, \text{ and } F = \{2, 3\}$$

def **Complement** of event E , E^C

The event containing all outcomes in that are not in E .

$$E^C = \{3, 4, 5, 6\}$$

3 Axioms of Probability

Definition of probability: $P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

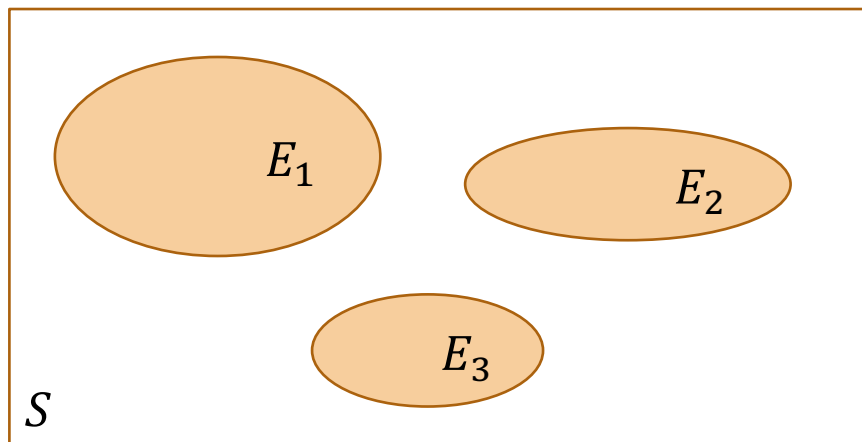
Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

Axiom 3 is the (analytically) useful axiom

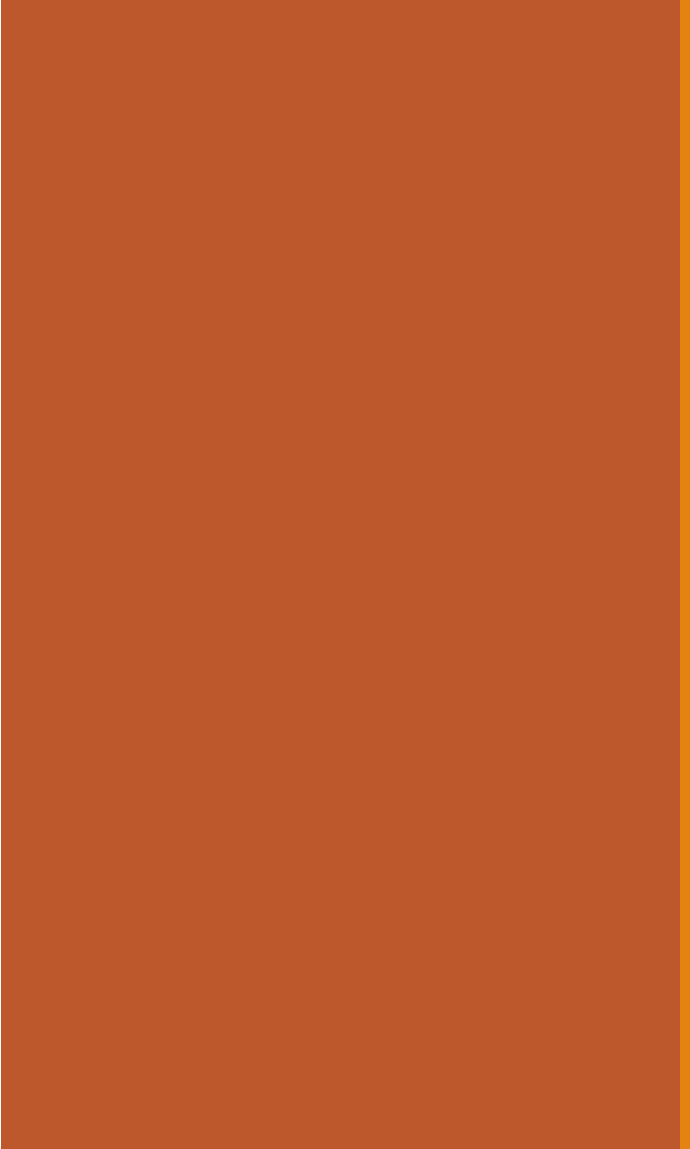
Axiom 3: If E and F are mutually exclusive—that is, if $E \cap F = \emptyset$ —then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \dots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$



(like the Sum Rule of Counting, but for probabilities)



Equally Likely Outcomes

Equally Likely Outcomes

Some sample spaces have **equally likely outcomes**.

- Flipping one coin: $S = \{\text{Head}, \text{Tails}\}$
- Flipping two coins: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
- Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

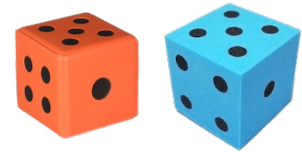
If we have equally likely outcomes, then $P(\text{Each outcome}) = \frac{1}{|S|}$

Therefore $P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{|E|}{|S|}$ (by Axiom 3)

Roll two dice

$$P(E) = \frac{|E|}{|S|} \text{ Equally likely outcomes}$$

Roll two 6-sided fair dice. What is $P(\text{sum} = 7)$?

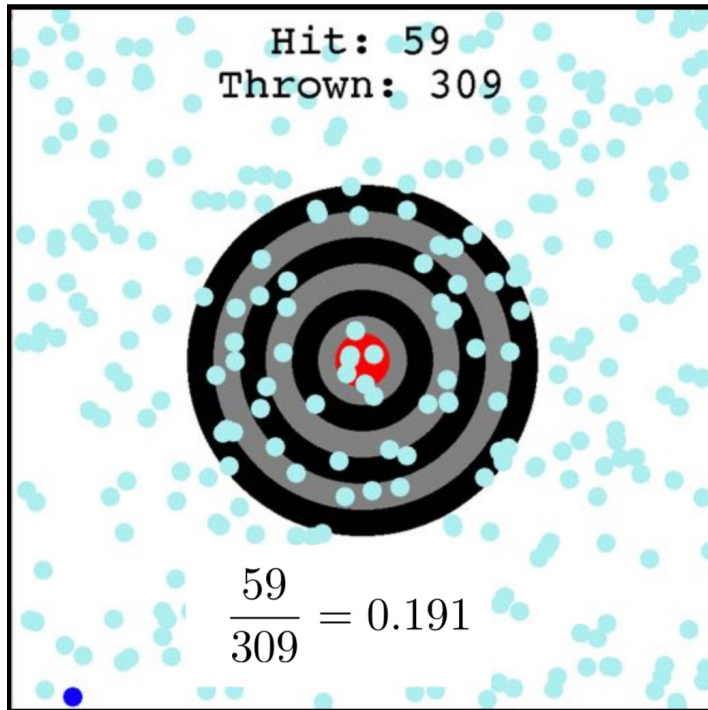

$$S = \{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\ (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\ (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\ (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\ (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \}$$

$E =$

Target revisited

$$P(E) = \frac{|E|}{|S|} \quad \begin{array}{l} \text{Equally likely} \\ \text{outcomes} \end{array}$$

Let E = the set of outcomes
where you hit the target.



Screen size = 800×800

Radius of target: 200

The dart is equally likely to land
anywhere on the screen. What is $P(E)$,
the probability of hitting the target?

$$|S| = 800^2 \qquad |E| \approx \pi \cdot 200^2$$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Cats and sharks (note: stuffed animals)

$$P(E) = \frac{|E|}{|S|} \text{ Equally likely outcomes}$$

4 cats and 3 sharks in a bag. 3 drawn.
What is $P(1 \text{ cat and } 2 \text{ sharks drawn})$?

Question: Do indistinct objects give you an equally likely sample space?

(No)

Make indistinct items distinct to get equally likely outcomes.

- A. $\frac{3}{7}$
- B. $\frac{1}{4} \cdot \frac{2}{3}$
- C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$
- D. $\frac{12}{35}$
- E. 0



Cats and sharks (ordered solution)

$$P(E) = \frac{|E|}{|S|} \text{ Equally likely outcomes}$$

4 cats and 3 sharks in a bag. 3 drawn.
What is $P(1 \text{ cat and } 2 \text{ sharks drawn})$?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items
- E = 1 distinct cat, 2 distinct sharks

Cats and sharks (unordered solution)

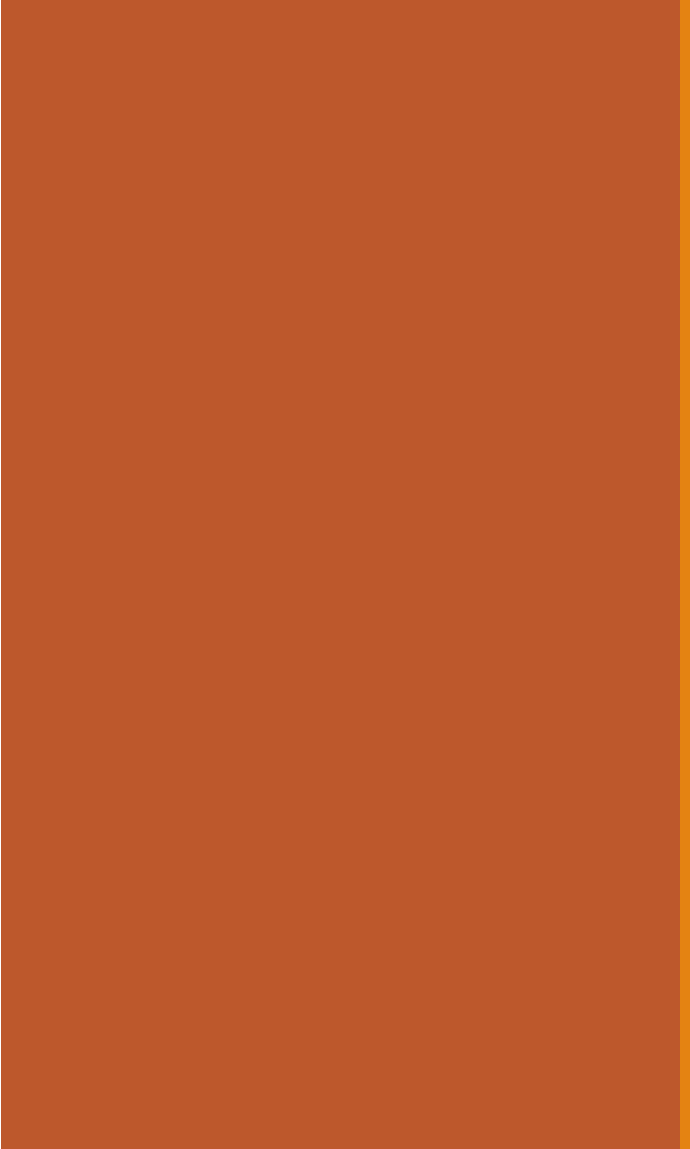
$$P(E) = \frac{|E|}{|S|} \text{ Equally likely outcomes}$$

4 cats and 3 sharks in a bag. 3 drawn.
What is $P(1 \text{ cat and } 2 \text{ sharks drawn})$?

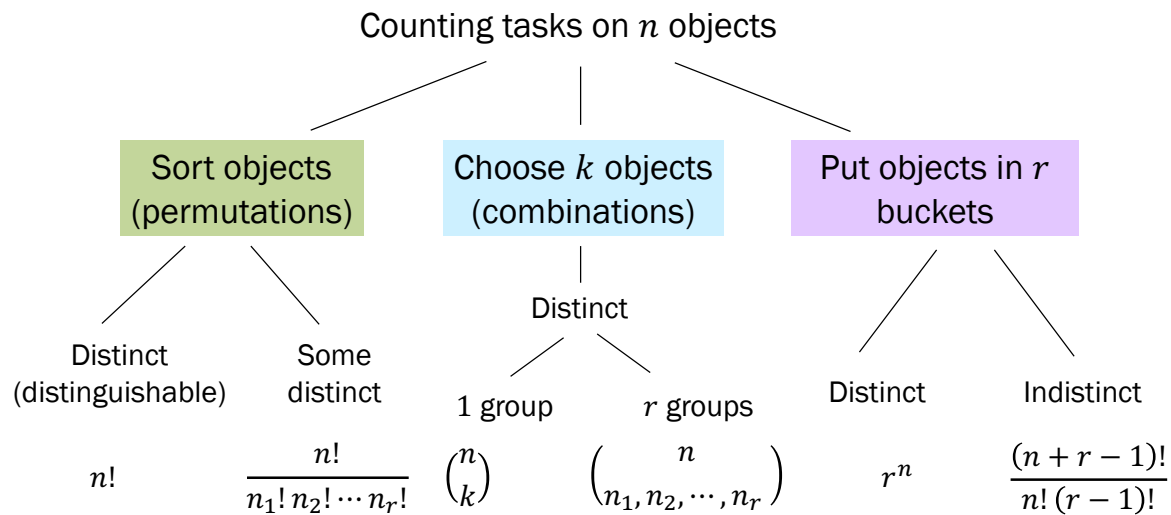
Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items
- E = 1 distinct cat, 2 distinct sharks



Exercises



Combinatorics

Equally likely
outcomes:

$$P(E) = \frac{|E|}{|S|}$$

Probability

Counting? Probability? Distinctness?

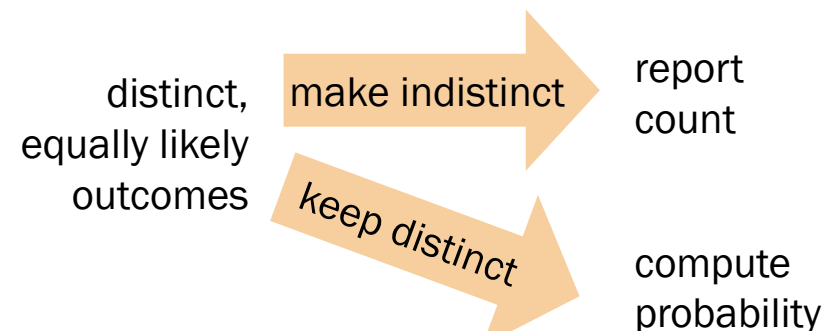
Review

We choose **3 books** from a set of **4 distinct** (distinguishable) and **2 indistinct** (indistinguishable) books. Each set of 3 books is equally likely.

Let event E = our choice excludes one or both indistinct books.

1. How many distinct outcomes are in E ?

2. What is $P(E)$?



Poker Straights and Computer Chips

1. Consider equally likely 5-card poker hands.
 - Define "poker straight" as 5 consecutive rank cards of any suit

What is $P(\text{poker straight})$?

- What is an example of an equally likely outcome?
- Should objects be ordered or unordered?

2. Computer chips: n chips are manufactured, 1 of which is defective. k chips are randomly selected from n for testing.

What is $P(\text{defective chip is in } k \text{ selected chips})$?



1. Any Poker Straight

Consider equally likely 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is $P(\text{Poker straight})$?

Define

- S (unordered)
- E (unordered,
consistent with S)

Compute $P(\text{Poker straight}) =$

2. Chip defect detection

n chips are manufactured, 1 of which is defective.
 k chips are randomly selected from n for testing.

What is $P(\text{defective chip is in } k \text{ selected chips?})$

Define

- S (unordered)
- E (unordered,
consistent with S)

Compute $P(E) =$

2. Chip defect detection, solution #2

n chips are manufactured, 1 of which is defective.
 k chips are randomly selected from n for testing.

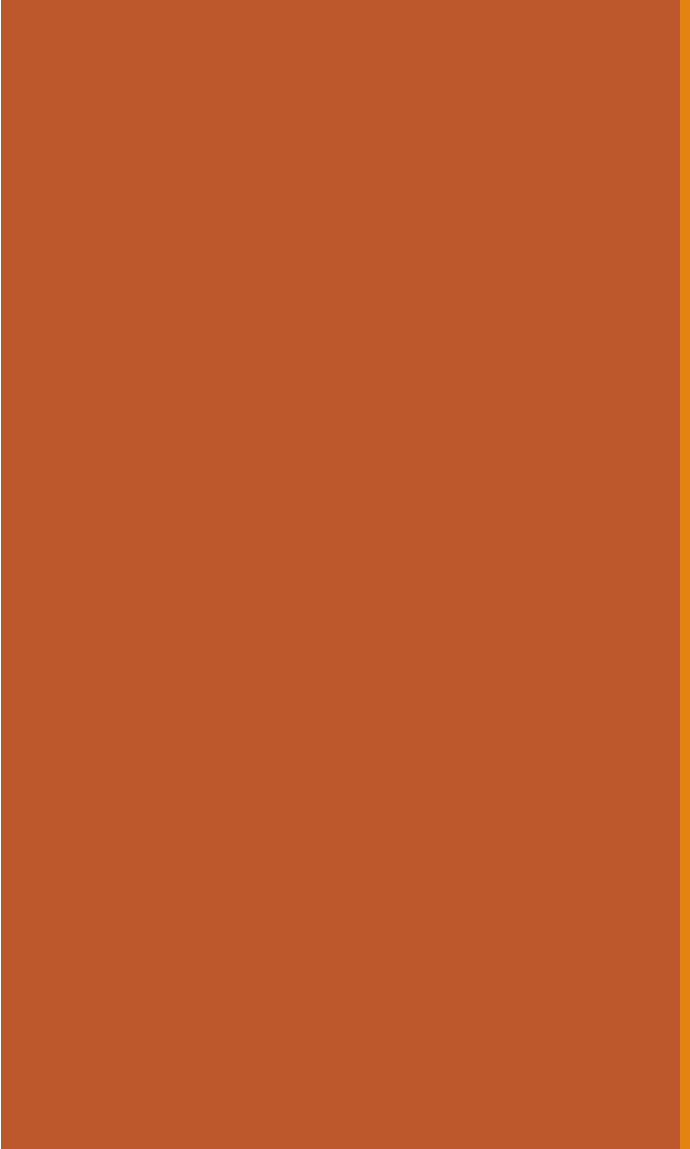
What is $P(\text{defective chip is in } k \text{ selected chips?})$

Redefine experiment

1. Choose k indistinct chips (1 way)
2. Throw a dart and make one defective

Define

- S (unordered)
- E (unordered,
consistent with S)



Corollaries of Probability

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^c) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$
(Inclusion-Exclusion Principle for Probability)

Selecting Programmers

- $P(\text{student programs in Python}) = 0.28$
- $P(\text{student programs in C++}) = 0.07$
- $P(\text{student programs in Python and C++}) = 0.05.$

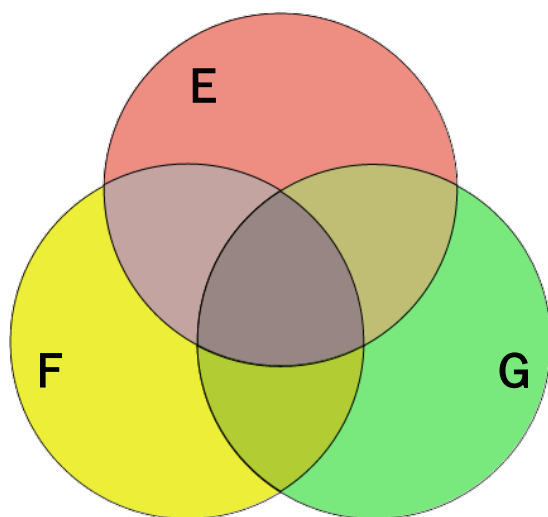
What is $P(\text{student does not program in (Python or C++)})$?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

General form:
$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{r=1}^n (-1)^{r+1} \sum_{i_1 < \dots < i_r} P\left(\bigcap_{j=1}^r E_{i_j}\right)$$



$$P(E \cup F \cup G) =$$

$$r = 1: \quad P(E) + P(F) + P(G)$$

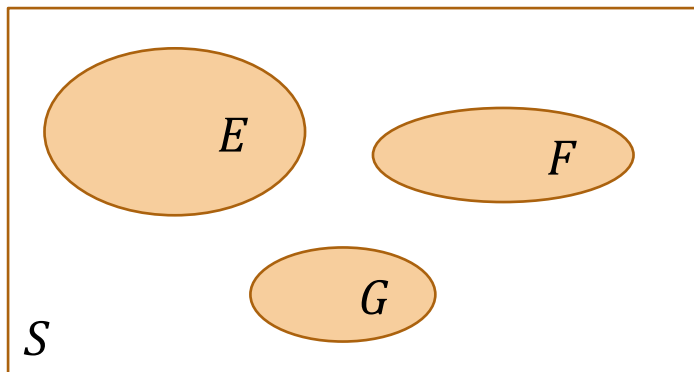
$$r = 2: \quad - P(E \cap F) - P(E \cap G) - P(F \cap G)$$

$$r = 3: \quad + P(E \cap F \cap G)$$

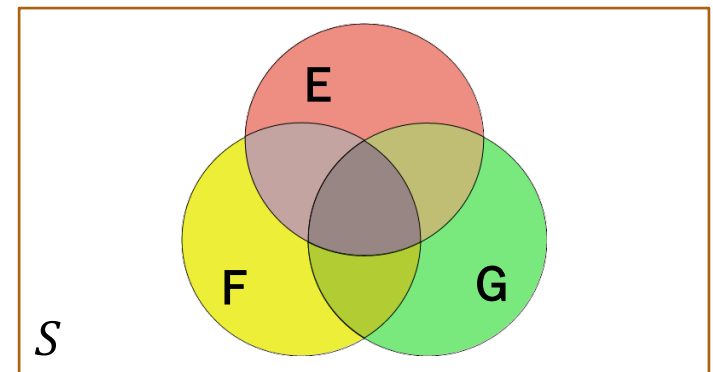
Takeaway: Union of events

Review

Axiom 3,
Mutually exclusive events



Corollary 3,
Inclusion-Exclusion Principle



The challenge of probability is in defining events.
Some event probabilities are easier to compute than others.

Serendipity

Let it find you.

SERENDIPITY

the effect by which one accidentally stumbles upon something truly wonderful, especially while looking for something entirely unrelated.



WHEN YOU MEET YOUR BEST FRIEND

Somewhere you didn't expect to.

Serendipity

- The population of Stanford is $n = 17,000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in the room.

What is the probability that you see someone you know in the room?

<http://web.stanford.edu/class/cs109/demos/serendipity.html>

CS109 Mixer

Slide 43 is a question to think over by yourself (~30 seconds).



Serendipity

- The population of Stanford is $n = 17,000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- $E: \geq 1$ friend in the room

What strategy would you use?

A. $P(\text{exactly } 1) + P(\text{exactly } 2) + P(\text{exactly } 3) + \dots$

B. $1 - P(\text{see no friends})$



Serendipity

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What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- $E: \geq 1$ friend in the room

It is often much easier to compute $P(E^c)$.

The Birthday Paradox Problem

What is the probability that in a set of n people, at least one pair of them share the same birthday?

For you to think about (and discuss in your first section)



Card Flipping

In a 52-card deck, cards are flipped one at a time.

After the first ace (of any suit) appears, consider the next card.

Is $P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = 2 \text{ Clubs})$?



Card Flipping

In a 52-card deck, cards are flipped one at a time.

After the first ace (of any suit) appears, consider the next card.

Is $P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = 2 \text{ Clubs})$?

Sample space $S = 52$ in-order cards (shuffle deck)

Event E_{AS} , next card
is Ace Spades

1. Take out Ace of Spades.
2. Shuffle leftover 51 cards.
3. Add Ace Spades after first ace.

$$|E_{AS}| = 51! \cdot 1$$

E_{2C} , next card
is 2 Clubs

1. Take out 2 Clubs.
2. Shuffle leftover 51 cards.
3. Add 2 Clubs after first ace.

$$|E_{2C}| = 51! \cdot 1$$

$$P(E_{AS}) = P(E_{2C})$$

Have a good
weekend!

