02: Combinatorics

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Lecture Discussion on Ed
Summary of Combinatorics

Counting tasks on \( n \) objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - \( n! \)

- Choose \( k \) objects (combinations)
  - Some distinct

- Put objects in \( r \) buckets
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations of distinct objects = permutations considering some objects are indistinct × Permutations of just the indistinct objects
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{Permutations of distinct objects} = \frac{\text{Permutations of just the indistinct objects}}{\text{permutations considering some objects are indistinct}}
\]
General approach to counting permutations

When there are $n$ objects such that
- $n_1$ are the same (indistinguishable or indistinct), and
- $n_2$ are the same, and
- ...
- $n_r$ are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1! n_2! \cdots n_r!}.$$

For each group of indistinct objects, Divide by the overcounted permutations.

Simple example:

How many strings can be formed from the letters TENSE?

Answer is

$$\frac{5!}{2! 1! 1! 1!} = 60.$$
Sort semi-distinct objects

How many permutations?

Order $n$ semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

- Coke
- Coke0
- Coke
- Coke0
- Coke0
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $n! / n_1! n_2! \cdots n_r!$

- Choose $k$ objects (combinations)

- Put objects in $r$ buckets

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Strings

How many letter orderings are possible for the following strings?

1. **SUBCOMMITTEE**

   - 12 letters
   - 2 M's, 2 T's, 2 E
   - Order of semi-distinct objects: \[ \frac{n!}{n_1! n_2! \cdots n_r!} \]

   \[ \Rightarrow \frac{12!}{2! 2! 2!} \]

2. **MISSISSIPPI**

   - 11 letters
   - 4 I's, 4 S's, 2 P's

   \[ \Rightarrow \frac{11!}{4! 4! 2!} \]
Strings

How many letter orderings are possible for the following strings?

1. **SUBCOMMITTEE**
   
   \[ \frac{12!}{2!2!2!} = 59,875,200 \]

2. **MISSISSIPPI**
   
   \[ \frac{11!}{4!4!2!} = 34,650 \]
Unique 6-digit passcodes with **six** smudges

How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

Total = 6!

= 720 passcodes
Unique 6-digit passcodes with five smudges

How many unique 6-digit passcodes are possible if a phone password uses each of five distinct numbers?

Steps:
1. Choose digit to repeat
   - 5 outcomes
2. Create passcode
   - (sort 6 digits: 4 distinct, 2 indistinct)

Total = \(5 \times \frac{6!}{2!}\) = 1,800 passcodes
Combinations I
Summary of Combinatorics

Counting tasks on $n$ objects

Sort objects (permutations)

- Distinct (distinguishable)
  - $n!$

Choose $k$ objects (combinations)

- Some distinct
  - $\frac{n!}{n_1!n_2!\cdots n_r!}$
- Distinct

Put objects in $r$ buckets

- This more general formula devolves into $n!$ when $n_1, n_2, n_3, \text{etc.}$ are all 1 - i.e., all elements/objects are truly distinct.
Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...
Combinations with cake

There are \( n = 20 \) people. How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line

\( n! \) ways
Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

$n!$ ways

1. $n$ people get in line
2. Put first $k$ in cake room

1 way
Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

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$n!$ ways 1 way
Combinations with cake

There are \( n = 20 \) people.

How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line
   \( n! \) ways

2. Put first \( k \) in cake room
   1 way

3. Allow cake group to mingle
   \( k! \) different permutations lead to the same mingle
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   $n!$ ways

2. Put first $k$ in cake room
   1 way

3. Allow cake group to mingle
   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

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Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   $n!$ ways

2. Put first $k$ in cake room
   1 way

3. Allow cake group to mingle
   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle
   $(n - k)!$ different permutations lead to the same mingle
Combinations

A combination is an unordered selection of \(k\) objects from a set of \(n\) distinct objects.

The number of ways of making this selection is

\[
\frac{n!}{k! (n - k)!} = n! \times \frac{1}{k!} \times \frac{1}{(n - k)!}
\]

1. Order \(n\) distinct objects
2. Take first \(k\) as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice
Combinations

A combination is an unordered selection of \( k \) objects from a set of \( n \) distinct objects.

The number of ways of making this selection is

\[
\frac{n!}{k!(n-k)!} = n! \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}
\]

Binomial coefficient

Note: \( \binom{n}{n-k} = \binom{n}{k} \)

number of ways to choose 5 children from a classroom of 20 is simply \( \binom{20}{5} = \frac{20!}{5!15!} \)
How many ways are there to choose a set of 3 from a set of 6 distinct books? Assume order doesn’t matter.

\[
\binom{6}{3} = \frac{6!}{3! \cdot 3!} = 20 \text{ ways}
\]
Combinations II
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$ if distinct
    - $\frac{n!}{n_1! n_2! \cdots n_r!}$ if some distinct

- Choose $k$ objects (combinations)
  - Distinct
    - $\binom{n}{k}$

- Put objects in $r$ buckets
  - 1 group
  - $r$ groups
General approach to combinations

The number of ways to choose $r$ groups of $n$ distinct objects such that
For all $i = 1, ..., r$, group $i$ has size $n_i$, and
$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is
\[
\frac{n!}{n_1! n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}
\]
Multinomial coefficient
13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

A. \( \binom{13}{6,4,3} = 60,060 \)

B. \( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)

C. \( 6 \cdot 1001 \cdot 10 = 60,060 \)

D. A and B

E. All of the above
Datacenters

13 different computers are to be allocated to 3 datacenters as shown in the table:

<table>
<thead>
<tr>
<th>Datacenter</th>
<th># machines</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
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Strategy: Combinations into 3 groups

Group 1 (datacenter A): \( n_1 = 6 \)
Group 2 (datacenter B): \( n_2 = 4 \)
Group 3 (datacenter C): \( n_3 = 3 \)
Datacenters

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B. \( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)

Strategy: Product rule with 3 steps

1. Choose 6 computers for A \( \binom{13}{6} \)
2. Choose 4 computers for B \( \binom{7}{4} \)
3. Choose 3 computers for C \( \binom{3}{3} \)

\[
\frac{13!}{6!7!} \cdot \frac{7!}{4!3!} \cdot \frac{3!}{3!0!} \Rightarrow \frac{13!}{6!4!3!1!3!0!} = 60,060
\]
Datacenters

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How many different divisions are possible?

A. \[\binom{13}{6,4,3} = 60,060\]

Strategy: Combinations into 3 groups

Group 1 (datacenter A): \(n_1 = 6\)
Group 2 (datacenter B): \(n_2 = 4\)
Group 3 (datacenter C): \(n_3 = 3\)

Both approaches equally sound, equally valid.

B. \[\binom{13}{6} \cdot \binom{7}{4} \cdot \binom{3}{3} = 60,060\]

Strategy: Product rule with 3 steps

1. Choose 6 computers for A \(\binom{13}{6}\)
2. Choose 4 computers for B \(\binom{7}{4}\)
3. Choose 3 computers for C \(\binom{3}{3}\)

Your approach will determine if you use binomial/multinomial coefficients or factorials.
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[ \binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways} \]

2. Two are by the same author. What if we don’t want to choose both?

A. \( \binom{6}{3} - \binom{6}{2} = 5 \text{ ways} \)

B. \( \frac{6!}{3!3!2!} = 10 \)

C. \( 2 \cdot \binom{4}{2} + \binom{4}{3} = 16 \)

D. \( \binom{6}{3} - \binom{4}{1} = 16 \)

E. Both C and D

F. Something else
1. How many ways are there to choose 3 books from a set of 6 distinct books? 
\[ \binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways} \]

2. Two are by the same author. What if we don’t want to choose both?

Strategy 1: Sum Rule

- Two books are by author with last name Steinbeck.
- Others by authors A, B, C, and D.

Cases partitioned into three:

- \( \binom{4}{2} \) choose any two, and Steinbeck
- \( \binom{4}{2} \) choose any two, and not Steinbeck
- \( \binom{3}{3} \) choose any three
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[
\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways}
\]

2. Two are by the same author. What if we don’t want to choose both?

Strategy 2: "Forbidden method" (unofficial name)

- Count number of illegal/forbidden subsets
- "Select one -"

Forbidden method: It is sometimes easier to exclude invalid cases than to include cases.
Buckets and The Divider Method
Summary of Combinatorics

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- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1!n_2!\cdots n_r!}$

- Choose $k$ objects (combinations)
  - Distinct
    - 1 group
      - $\binom{n}{k}$
    - $r$ groups
      - $\binom{n}{n_1, n_2, \ldots, n_r}$

- Put objects in $r$ buckets
  - Distinct
  - Indistinct
Balls and urns  Hash tables and distinct strings

How many ways are there to hash $n$ distinct strings to $r$ buckets?

Steps:
1. Bucket 1$^{st}$ string $\rightarrow r$ choices
2. Bucket 2$^{nd}$ string $\rightarrow r$ choices
   ...
3. Bucket $n^{th}$ string $\rightarrow r$ choices

$r^n$ outcomes
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Choose $k$ objects (combinations)
- Some distinct
  - $\frac{n!}{n_1!n_2!\cdots n_r!}$
- Distinct
  - $\binom{n}{k}$

Put objects in $r$ buckets
- Distinct
  - $\binom{n}{n_1, n_2, \cdots, n_r}$
- Indistinct
  - $r^n$
Servers and indistinct requests

How many ways are there to distribute $n$ indistinct web requests to $r$ servers?

Goal
Server 1 has $x_1$ requests,
Server 2 has $x_2$ requests,
...
Server $r$ has $x_r$ requests (the rest)
Bicycle helmet sales

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?
Bicycle helmet sales

1 possible assignment outcome:

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

Consider the following generative process...
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct
The divider method: A generative proof

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**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\[ (n + r - 1)! \]
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers
   \[ (n + r - 1)! \]

2. Make \( n \) objects indistinct
   \[ \frac{1}{n!} \]

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The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

**Goal** Order $n$ indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

1. Order $n$ distinct objects and $r - 1$ distinct dividers

\[(n + r - 1)!\]

2. Make $n$ objects indistinct

\[\frac{1}{n!}\]

3. Make $r - 1$ dividers indistinct

\[\frac{1}{(r - 1)!}\]
The divider method

The number of ways to distribute $n$ indistinct objects into $r$ buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that $n$ are indistinct objects, and $r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$
You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don’t have to invest all your money?

Venture capitalists

Divide method
\( \binom{n + r - 1}{r - 1} \)
(n indistinct objects, \( r \) buckets)
You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?

Set up

\[ x_1 + x_2 + x_3 + x_4 = 10 \]

\( x_i \): amount invested in company \( i \)

\( x_i \geq 0 \)

\( x_i \) are integers

Solve

\[ \binom{n + r - 1}{r - 1} = \binom{10 + 4 - 1}{4 - 1} = 286 \]
You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?

2. What if you want to invest at least $3 million in company 1?

Set up

\[ x_1 + x_2 + x_3 + x_4 = 10 \]

\( x_i \): amount invested in company \( i \)

constraint that $13 million goes to company 1, but remaining $7 million can be freely allocated. We're really solving

\[ x_1 + x_2 + x_3 + x_4 = 7 \]

Solve

\[
\binom{n + r - 1}{r - 1}
\]

\[
\binom{7 + 4 - 1}{4 - 1}
\]

\[ = 120 \]

Divider method

\((n \text{ indistinct objects}, r \text{ buckets})\)
Venture capitalists. #3

You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don’t have to invest all your money?

Set up

\[ x_1 + x_2 + x_3 + x_4 \leq 10 \]

\[ x_i: \text{amount invested in company } i \]

\[ x_i \geq 0 \]

You are really solving thus

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 10, \text{ where} \]

\[ x_5 \text{ counts the money you don’t invest} \]

Solve

\[ \binom{n + r - 1}{r - 1} \]

\[ \binom{10 + 5 - 1}{5 - 1} = 1001 \]
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Counting tasks on \( n \) objects

- Sort objects (permutations)
  - Distinct (distinguishable)

- Choose \( k \) objects (combinations)
  - Some distinct
  - Distinct
  
- Put objects in \( r \) buckets
  - Distinct
  - Indistinct

- 1 group
- \( r \) groups

- Determine if objects are distinct
- Use Product Rule if several steps
- Use Inclusion-Exclusion if different cases
Combinatorial Proofs
Combinatorial Proofs

A **combinatorial proof**—sometimes called a **story proof**—is a proof that counts the same thing in two different ways, forgoing any tedious algebra.

Combinatorial proofs aren’t as formal as CS103 proofs, but they still need to convince the reader something is true in an absolute sense.

An algebraic proof of, say, \( \binom{n}{k} = \binom{n}{n-k} \) is straightforward if you just write combinations in terms of factorials.

A combinatorial proof makes an identity like \( \binom{n}{k} = \binom{n}{n-k} \) easier to believe and understand intuitively.

**Combinatorial Proof:**

Consider choosing a set of \( k \) CS109 CAs from a total of \( n \) applicants. We know that there are \( \binom{n}{k} \) such possibilities. Another way to choose the \( k \) CS109 CAs is to **disqualify** \( n-k \) applicants. There are \( \binom{n}{n-k} \) ways to choose which \( n-k \) don’t get the job. Specifying who **is** on CS109 course staff is the same as specifying who **isn’t**. That means that \( \binom{n}{k} \) and \( \binom{n}{n-k} \) must be counting the same thing.
Let's provide another combinatorial proof, this time proving that

\[ n \binom{n-1}{k-1} = k \binom{n}{k} \]

This is easy to prove algebraically (provided \(k\) and \(n\) are positive integers, with \(k \leq n\)). A combinatorial/story proof, however, is more compelling!

Combinatorial Proof:

Consider \(n\) candidates for college admission, where \(k\) candidates can be accepted, and precisely one of the \(k\) is selected for a full scholarship. We can first choose the lucky recipient of the full scholarship and then select an additional \(k - 1\) applicants from the remaining \(n - 1\) applicants to round out the set of admits. Or we can select which \(k\) applicants are accepted and then choose which of those \(k\) gets the full ride.