03: Intro to Probability

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Lecture Discussion on Ed
Defining Probability
Key definitions

An experiment in probability:

Sample Space, $S$: The set of all possible outcomes of an experiment
Event, $E$: Some subset of $S$ ($E \subseteq S$).
Key definitions

Sample Space, $S$

- Coin flip
  $S = \{\text{Heads, Tails}\}$

- Flipping two coins
  $S = \{(H,H), (H,T), (T,H), (T,T)\}$

- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$

- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, x \geq 0\}$

- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, $E$

- Flip lands heads
  $E = \{\text{Heads}\}$

- $\geq 1$ head in two coin flips
  $E = \{(H,H), (H,T), (T,H)\}$

- Roll is 3 or less:
  $E = \{1, 2, 3\}$

- Low email day ($\leq 100$ emails)
  $E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 100\}$

- Lost day ($\geq 5$ TikTok hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$
What is a probability?

A number between 0 and 1 to which we ascribe meaning.*

*our belief that an event $E$ occurs.
What is a probability?

Let $E$ = the set of outcomes where you hit the target.

$n = \#$ of total trials

$n(E) = \#$ trials where $E$ occurs

$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$
What is a probability?

\[ P(E) = \lim_{{n \to \infty}} \frac{n(E)}{n} \]

- \(n\) = \# of total trials
- \(n(E)\) = \# trials where \(E\) occurs

Let \(E\) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.00 \]
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- \( n \) = # of total trials
- \( n(E) \) = # trials where \( E \) occurs

Let \( E \) = the set of outcomes where you hit the target.

\[ P(E) \approx 0.500 \]
What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$n = \# \text{ of total trials}$

$n(E) = \# \text{ trials where } E \text{ occurs}$

Let $E = \text{ the set of outcomes where you hit the target.}$

$P(E) \approx 0.667$
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

\( n = \# \text{ of total trials} \)
\( n(E) = \# \text{ trials where } E \text{ occurs} \)

Let \( E \) = the set of outcomes where you hit the target.

\( P(E) \approx 0.458 \)
Axioms of Probability
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
   Die roll
   $S = \{1, 2, 3, 4, 5, 6\}$
   Let $E = \{1, 2\}$, and $F = \{2,3\}$

2 is in both $E$ and $F$
4, 5, and 6 are in neither
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:

Die roll

$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def Union** of events, $E \cup F$

The event containing all outcomes in $E$ or $F$.

$E \cup F = \{1, 2, 3\}$

*Set theory doesn’t ask or even allow us to count 2 twice. 2 is simply present/included.*
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
- Die roll
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def Intersection** of events, $E \cap F$

The event containing all outcomes in $E$ and $F$.

**def Mutually exclusive** events $F$ and $G$ means that $F \cap G = \emptyset$

\[
E \cap F = EF = \{2\}
\]

Note: easier to write this way so it's written like this more often.

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2024
Quick review of sets

$E$ and $F$ are events in $S$.

Experiment:
Die roll

$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2, 3\}$

**def Complement** of event $E$, $E^C$

The event containing all outcomes in that are *not* in $E$.

$E^C = \{3, 4, 5, 6\}$

*The complement is everything in the world that isn't in $E$.*
Three Axioms of Probability

Definition of probability: \[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

Axiom 1: \( 0 \leq P(E) \leq 1 \)

Axiom 2: \( P(S) = 1 \)

Axiom 3: If \( E \) and \( F \) are mutually exclusive \( (E \cap F = \emptyset) \), then \( P(E \cup F) = P(E) + P(F) \)
Axiom 3 is the (analytically) useful axiom

**Axiom 3:** If $E$ and $F$ are mutually exclusive—that is, if $E \cap F = \emptyset$—then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i)$$

just like the Sum Rule of Counting, but for probabilities.
Equally Likely Outcomes
Equally Likely Outcomes

Some sample spaces have **equally likely outcomes**.

- Flipping one coin: \( S = \{\text{Head, Tails}\} \)
- Flipping two coins: \( S = \{(H, H), (H, T), (T, H), (T, T)\} \)
- Roll of 6-sided die: \( S = \{1, 2, 3, 4, 5, 6\} \)

If we have equally likely outcomes, then \( P(\text{Each outcome}) = \frac{1}{|S|} \)

Therefore

\[
P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{|E|}{|S|} \quad \text{(by Axiom 3)}
\]
Roll two dice

Roll two 6-sided fair dice. What is \( P(\text{sum} = 7) \)?

\[
S = \{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \}
\]

\[
E = \{ (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) \}
\]

\[
P(E) = \frac{|E|}{|S|} = \frac{1}{6}
\]
Target revisited

Let $E$ = the set of outcomes where you hit the target.

Screen size = 800 × 800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

Let $S$ = the set of outcomes where you hit the target, and $E$ = the set of outcomes where you hit the target.

Think of each pixel as an equally likely target, and count pixels.

$|S| = 800^2$

$|E| \approx \pi \cdot 200^2$

$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$

59/309 = 0.191
Cats and sharks (note: stuffed animals)

4 cats and 3 sharks in a bag. 3 drawn.
What is \( P(1 \text{ cat and 2 sharks drawn}) \)?

**Question:** Do indistinct objects give you an equally likely sample space?

(No)

\[
P(E) = \frac{|E|}{|S|} \quad \text{Equally likely outcomes}
\]

Make indistinct items distinct to get equally likely outcomes.

\[
\begin{align*}
A. & \quad \frac{3}{7} \\
B. & \quad \frac{1}{4} \cdot \frac{2}{3} \\
C. & \quad \frac{4}{7} + 2 \cdot \frac{3}{6} \\
D. & \quad \frac{12}{35} \\
E. & \quad 0
\end{align*}
\]
Cats and sharks (ordered solution)

4 cats and 3 sharks in a bag. 3 drawn.
What is \( P(1 \text{ cat and } 2 \text{ sharks drawn})? \)

pretend all stuffed animal are unique

Define
- \( S = \) Pick 3 distinct items
  and retain order
- \( E = 1 \) distinct cat, 2 distinct sharks

\[
|S| = 7 \cdot 6 \cdot 5
\]

\[
|E| = \text{sum of three distinct cases}
\]

\[
P(E) = \frac{|E|}{|S|} = \frac{72}{210} = \frac{12}{35}
\]

\( P(E) = \frac{|E|}{|S|} \) Equally likely outcomes

Make indistinct items distinct to get equally likely outcomes.
Cats and sharks (unordered solution)

4 cats and 3 sharks in a bag. 3 drawn.
What is $P(1$ cat and $2$ sharks drawn)?

Define

- $S =$ Pick 3 distinct items, ignore order
- $E =$ 1 distinct cat, 2 distinct sharks

because we're ignoring order with this approach, we rely on combinations and choose terms instead of multiplication

Thus, make indistinct items distinct to get equally likely outcomes.

$$|S| = \binom{7}{3} = 35$$
$$|E| = \binom{4}{1} \cdot \binom{3}{2} = 4 \cdot 3 = 12$$

$$P(E) = \frac{|E|}{|S|} = \frac{12}{35}$$
Exercises
CS109 so far

Review

Equally likely outcomes:

\[ P(E) = \frac{|E|}{|S|} \]

Combinatorics

Probability

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2024
We choose 3 books from a set of 4 distinct (distinguishable) and 2 indistinct (indistinguishable) books. Each set of 3 books is equally likely.

Let event $E =$ our choice excludes one or both indistinct books.

1. How many distinct outcomes are in $E$?

   \[
   |E| = 2 \left( \binom{4}{1} + \binom{4}{3} \right) = 16
   \]

2. What is $P(E)$?

   \[
   P(E) = \frac{16}{20} = \frac{4}{5}
   \]
Poker Straights and Computer Chips

1. Consider equally likely 5-card poker hands.
   - Define "poker straight" as 5 consecutive rank cards of any suit. What is \( P(\text{poker straight}) \)?

2. Computer chips: \( n \) chips are manufactured, 1 of which is defective. \( k \) chips are randomly selected from \( n \) for testing. What is \( P(\text{defective chip is in } k \text{ selected chips?}) \)?
1. Any Poker Straight

Consider equally likely 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is $P(\text{Poker straight})$?

Define

- $S$ (unordered)

- $E$ (unordered, consistent with $S$)

Compute

\[
P(\text{Poker straight}) = \frac{|E|}{|S|} = \frac{10 \cdot 4^5}{\binom{52}{5}} = 0.00394
\]
2. Chip defect detection

$n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing.

What is $P$(defective chip is in $k$ selected chips?)

Define

- $S$ (unordered)
- $E$ (unordered, consistent with $S$)

Compute

$$P(E) = \frac{|E|}{|S|} = \frac{(n-1)\ldots(k-1)}{n\ldots(n-k+1)} = \frac{(n-1)\ldots k}{n\ldots(k-1)} = \frac{k}{n}$$

$\binom{n}{k}$ is the number of ways we can choose the one defective chip!

$\binom{n-1}{k-1}$ is the number of ways to choose an additional $k-1$ chips from the $n-1$ good ones.
2. Chip defect detection, solution #2

$n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing. What is $P($defective chip is in $k$ selected chips$)$?

Redefine experiment

1. Choose $k$ indistinct chips (1 way)
2. Throw a dart and make one defective

Define

• $S$ (unordered)
• $E$ (unordered, consistent with $S$)
Corollaries of Probability
### 3 Corollaries of Axioms of Probability

**Corollary 1:**  \( P(E^C) = 1 - P(E) \)

**Corollary 2:**  If \( E \subseteq F \), then \( P(E) \leq P(F) \)

**Corollary 3:**  \( P(E \cup F) = P(E) + P(F) - P(EF) \)  
(Inclusion-Exclusion Principle for Probability)
Selecting Programmers

- \( P(\text{student programs in Python}) = 0.28 \)
- \( P(\text{student programs in C++}) = 0.07 \)
- \( P(\text{student programs in Python and C++}) = 0.05 \).

What is \( P(\text{student does not program in (Python or C++))} \)?

1. Define events & state goal
   - \( Y = \text{student codes in Python} \)
   - \( D = \text{student codes in C++} \)

   We want \( P((Y \cup D)^C) = 1 - P(Y \cup D) \)
   \[
   = 1 - (0.28 + 0.07 - 0.05) = 0.17
   \]
Inclusion-Exclusion Principle (Corollary 3)

**Corollary 3:**
\[ P(E \cup F) = P(E) + P(F) - P(EF) \]

**General form:**
\[ P\left( \bigcup_{i=1}^{n} E_i \right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \ldots < i_r} P\left( \bigcap_{j=1}^{r} E_{i_j} \right) \]

\[ P(E \cup F \cup G) = \]
\[ r = 1: \quad P(E) + P(F) + P(G) \]
\[ r = 2: \quad - P(E \cap F) - P(E \cap G) - P(F \cap G) \]
\[ r = 3: \quad + P(E \cap F \cap G) \]
Takeaway: Union of events

Axiom 3, Mutually exclusive events

Corollary 3, Inclusion-Exclusion Principle

The challenge of probability is in defining events. Some event probabilities are easier to compute than others.
Serendipity

Let it find you.

SERENDIPITY
the effect by which one accidentally stumbles upon something truly wonderful, especially while looking for something entirely unrelated.

WHEN YOU MEET YOUR BEST FRIEND
Somewhere you didn't expect to.
Serendipity

• The population of Stanford is \( n = 17,000 \) people.
• You are friends with \( r = 100 \) people.
• Walk into a room, see \( k = 223 \) random people.
• Assume each group of \( k \) Stanford people is equally likely to be in the room.

What is the probability that you see someone you know in the room?

http://web.stanford.edu/class/cs109/demos/serendipity.html
Serendipity

- The population of Stanford is $n = 17,000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of $k$ Stanford people is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

Define

- $S$ (unordered)
- $E: \geq 1$ friend in the room

What strategy would you use?

A. $P($exactly 1$) + P($exactly 2$) + P($exactly 3$) + ...$

B. $1 - P($see no friends$)$
The population of Stanford is $n = 17,000$ people.
You are friends with $r = 100$ people.
Walk into a room, see $k = 223$ random people.
Assume each group of $k$ Stanford people is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

Define

- $S$ (unordered)
- $E$: $\geq 1$ friend in the room

$$P(E) = 1 - P(E^c) = 1 - \frac{\binom{16900}{223}}{\binom{17000}{223}} = 0.7340$$

It is often much easier to compute $P(E^c)$. 
The Birthday Paradox Problem

What is the probability that in a set of \( n \) people, at least one pair of them share the same birthday?

For you to think about (and discuss in your first section)
Card Flipping

In a 52-card deck, cards are flipped one at a time. After the first ace (of any suit) appears, consider the next card.

Is $P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = 2 \text{ Clubs})$?
Card Flipping

In a 52-card deck, cards are flipped one at a time. After the first ace (of any suit) appears, consider the next card.

Is \( P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = 2 \text{ Clubs})? \)

**Sample space** \( S = 52 \) in-order cards (shuffle deck)

<table>
<thead>
<tr>
<th>Event</th>
<th>( E_{AS} ), next card is Ace Spades</th>
<th>( E_{2C} ), next card is 2 Clubs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Take out Ace of Spades.</td>
<td>1. Take out 2 Clubs.</td>
</tr>
<tr>
<td>2.</td>
<td>Shuffle leftover 51 cards.</td>
<td>2. Shuffle leftover 51 cards.</td>
</tr>
<tr>
<td>3.</td>
<td>Add Ace Spades after first ace.</td>
<td>3. Add 2 Clubs after first ace.</td>
</tr>
</tbody>
</table>

\[ |E_{AS}| = 51! \cdot 1 \]

\[ |E_{2C}| = 51! \cdot 1 \]

\[ P(E_{AS}) = P(E_{2C}) \]