06: Random Variables

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Lecture Discussion on Ed
Conditional Independence
Conditional Paradigm

For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

Axiom 1

\[ 0 \leq P(A|E) \leq 1 \]

Corollary 1 (complement)

\[ P(A|E) = 1 - P(A^c|E) \]

Transitivity

\[ P(AB|E) = P(BA|E) \]

Chain Rule

\[ P(AB|E) = P(B|E)P(A|BE) \]

Bayes’ Theorem

\[ P(A|BE) = \frac{P(B|AE)P(A|E)}{P(B|E)} \]

BAE’s theorem?
Conditional Independence

Two events $A$ and $B$ are defined as **conditionally independent given $E$** if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$
Conditional Independence

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An equivalent definition:

A. $P(A|B) = P(A)$

B. $P(A|BE) = P(A)$

C. $P(A|BE) = P(A|E)$

E is the "new sample space", so left and right side must both be conditioned on E.
Netflix and Condition

Let $E =$ a user watches Life is Beautiful.
Let $F =$ a user watches Amelie.
What is $P(E)$?

$$P(E) \approx \frac{\# \text{ people who have watched movie}}{\# \text{ people on Netflix}} = \frac{10,234,231}{50,923,123} \approx 0.20$$

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\# \text{ people who have watched both}}{\# \text{ people who have watched Amelie}} \approx 0.42$$
Netflix and Condition

Let $E$ be the event that a user watches the given movie. Let $F$ be the event that the same user watches Amelie.

$P(E) = 0.19$  
$P(E) = 0.32$  
$P(E) = 0.20$  
$P(E) = 0.09$  
$P(E) = 0.20$

$P(E|F) = 0.14$  
$P(E|F) = 0.35$  
$P(E|F) = 0.20$  
$P(E|F) = 0.72$  
$P(E|F) = 0.42$

Independent!
Netflix and Condition (on many movies)

Watched: $E_1$, $E_2$, $E_3$, $E_4$

What if $E_1E_2E_3E_4$ are not independent? (e.g., all international emotional comedies)

$$P(E_4|E_1E_2E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)} = \frac{\# \text{ people who have watched all 4}}{\# \text{ people who have watched those 3}}$$

We need to keep track of an exponential number of movie-watching statistics

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2024
Netflix and Condition (on many movies)

Assume: $E_1 E_2 E_3 E_4$ are conditionally independent given $K$

$P(E_4 | E_1 E_2 E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)}$

An easier probability to store and compute!

$P(E_4 | E_1 E_2 E_3 K) = P(E_4 | K)$
Dependent events can be conditionally independent.

(And vice versa: Independent events can be conditionally dependent.)

Challenge: How do we determine $K$? Stay tuned in 6 weeks’ time!
Random Variables
Random variables are like typed variables

**int**  \( a = 5; \)

- **type**: int
- **name**: \( a \)
- **value**: 5

**double**  \( b = 4.2; \)

- **type**: double
- **name**: \( b \)
- **value**: 4.2

**bit**  \( c = 1; \)

- **type**: bit
- **name**: \( c \)
- **value**: 1

\( A \) is the number of Pokemon we bring to our future battle.
\[ A \in \{1, 2, ..., 6\} \]

\( B \) is the amount of money we get after we win a battle.
\[ B \in \mathbb{R}^+ \]

\( C \) is 1 if we successfully beat the Elite Four. 0 otherwise.
\[ C \in \{0,1\} \]
Random Variable

A random variable is a real-valued function defined on a sample space.

Example:

3 coins are flipped. Let $X = \#$ of heads. $X$ is a random variable.

1. What is the value of $X$ for the outcomes:
   - $(T,T,T)$?
   - $(H,H,T)$?

2. What is the event (set of outcomes) where $X = 2$?

3. What is $P(X = 2)$?
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Random variables are **NOT** events!

It is confusing that random variables and events use the same notation.

- **Random variables ≠ events.**
- We can define an event to be a particular assignment of a random variable, or more generally, in terms of a random variable.

Example:

3 coins are flipped. Let $X = \#$ of heads. $X$ is a **random variable**.

$X = 2$ event $P(X = 2)$ probability (number b/t 0 and 1)
Random variables are **NOT** events!

It is confusing that random variables and events use the same notation.

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Example:

<table>
<thead>
<tr>
<th>$X = x$</th>
<th>Set of outcomes</th>
<th>$P(X = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>${(T, T, T)}$</td>
<td>$1/8$</td>
</tr>
<tr>
<td>$X = 1$</td>
<td>${(H, T, T), (T, H, T), (T, T, H)}$</td>
<td>$3/8$</td>
</tr>
<tr>
<td>$X = 2$</td>
<td>${(H, H, T), (H, T, H), (T, H, H)}$</td>
<td>$3/8$</td>
</tr>
<tr>
<td>$X = 3$</td>
<td>${(H, H, H)}$</td>
<td>$1/8$</td>
</tr>
<tr>
<td>$X \geq 4$</td>
<td>${}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

3 coins are flipped. Let $X =$ # of heads. $X$ is a random variable.
PMF/CDF
So far

3 coins are flipped. Let $X = \# \text{ of heads}$. $X$ is a random variable.

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Can we get a "shorthand" for this last step? Seems like it might be useful!
Probability Mass Function

3 coins are flipped. Let $X = \#$ of heads. $X$ is a random variable.

A function on $k$
with range $[0,1]$

$P(X = k)$

What would be a *useful* function to define?
The probability of the event that a random variable $X$ takes on the value $k$! For *discrete random variables*, this is a *probability mass function*. 

Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain, CS109, Spring 2024
Probability Mass Function

3 coins are flipped. Let \( X = \# \) of heads. \( X \) is a random variable.

- **parameter/input** \( k \): a value of \( X 
- **return value/output**: probability of the event \( X = 2 \)

A function on \( k \) with range \([0,1]\)

**probability mass function**

```python
def prob_x(n, k, p):
    n_ways = math.comb(n, k)
    p_way = p ** k * (1 - p) ** (n - k)
    return n_ways * p_way
```
Discrete RVs and Probability Mass Functions

A random variable $X$ is **discrete** if it can take on countably many values.

- $X = x$, where $x \in \{x_1, x_2, x_3, \ldots\}$

The **probability mass function** (PMF) of a discrete random variable is

$$ P(X = x) = p(x) = p_X(x) $$

shorthand notation

- Probabilities must sum to 1:

$$ \sum_{i=1}^{\infty} p(x_i) = 1 $$

This last point is a good way to verify any PMF you create is valid.
Let $X$ be a random variable that represents the result of a single dice roll.

- **Support** of $X$ : $\{1, 2, 3, 4, 5, 6\}$
- Therefore, $X$ is a discrete random variable.
- **PMF of X:**
  
  $p(x) = \begin{cases} 
  1/6 & x \in \{1, \ldots, 6\} \\
  0 & \text{otherwise}
  \end{cases}$
Cumulative Distribution Functions

For a random variable $X$, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{ where } -\infty < a < \infty$$

For a discrete RV $X$, the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$
CDFs as graphs

Let $X$ be a random variable that represents the result of a single dice roll.

CDF (cumulative distribution function) $F(a) = P(X \leq a)$

**PMF of $X$**

<table>
<thead>
<tr>
<th>$X = x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1/6</td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
</tr>
<tr>
<td>3</td>
<td>1/6</td>
</tr>
<tr>
<td>4</td>
<td>1/6</td>
</tr>
<tr>
<td>5</td>
<td>1/6</td>
</tr>
<tr>
<td>6</td>
<td>1/6</td>
</tr>
</tbody>
</table>

**CDF of $X$**

$P(X \leq 0) = 0$

$P(X \leq 6) = 1$
Expectation
Discrete random variables

Definition

Properties

Experiment outcomes

PMF
\[ P(X = x) = p(x) \]

CDF \( F(x) \)

Discrete Random Variable, \( X \)

Without performing the experiment:

- The support tells us which values our random variable might produce
- Next up: How do we report the "average" value?
Expectation

The expectation of a discrete random variable $X$ is defined as:

$$E[X] = \sum_{x: p(x) > 0} p(x) \cdot x$$

• Note: sum over all values of $X = x$ that have non-zero probability.

• Other names: mean, expected value, weighted average, center of mass, first moment
Expectation of a die roll

What is the expected value of a 6-sided die roll?

1. Define random variables
   \[ X = \text{RV for value of roll} \]
   \[ P(X = x) = \begin{cases} 
   1/6 & x \in \{1, \ldots, 6\} \\
   0 & \text{otherwise} 
\end{cases} \]

2. Solve
   \[ E[X] = 1 \left( \frac{1}{6} \right) + 2 \left( \frac{1}{6} \right) + 3 \left( \frac{1}{6} \right) + 4 \left( \frac{1}{6} \right) + 5 \left( \frac{1}{6} \right) + 6 \left( \frac{1}{6} \right) = \frac{7}{2} \]
Important properties of expectation

1. Linearity:
   \[ E[aX + b] = aE[X] + b \]

   - Let \( X \) = 6-sided dice roll,
     \( Y = 2X - 1 \).
   - \( E[X] = 3.5 \)
   - \( E[Y] = 6 \)

2. Expectation of a sum = sum of expectation:
   \[ E[X + Y] = E[X] + E[Y] \]

   Sum of two dice rolls:
   - Let \( X \) = roll of die 1
     \( Y = \) roll of die 2
   - \( E[X + Y] = 3.5 + 3.5 = 7 \)

3. Unconscious statistician:
   \[ E[g(X)] = \sum_x g(x)p(x) \]

These properties let you avoid defining difficult PMFs.
Linearity of Expectation proof

\[ E[aX + b] = aE[X] + b \]

Proof:

\[
E[aX + b] = \sum_x (ax + b)p(x) = \sum_x axp(x) + bp(x) \\
= a \sum_x xp(x) + b \sum_x p(x) \\
= a E[X] + b \cdot 1
\]
Expectation of Sum intuition

\[ E[X + Y] = E[X] + E[Y] \]

we’ll prove this in a few lectures

<table>
<thead>
<tr>
<th>Intuition for now:</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
<td>X + Y</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>6</td>
<td>9</td>
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<td>2</td>
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<tr>
<td>-1</td>
<td></td>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>16</td>
<td>24</td>
</tr>
</tbody>
</table>

Average:

\[ \frac{1}{n} \sum_{i=1}^{n} x_i + \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} (x_i + y_i) \]

\[ \frac{1}{7} (28) + \frac{1}{7} (56) = \frac{1}{7} (84) \]
LOTUS proof

Let $Y = g(X)$, where $g$ is a real-valued function.

$$E[g(X)] = E[Y] = \sum_j y_j p(y_j)$$

$$= \sum_j y_j \sum_{i: g(x_i) = y_j} p(x_i)$$

$$= \sum_j \sum_{i: g(x_i) = y_j} y_j p(x_i)$$

$$= \sum_j \sum_{i: g(x_i) = y_j} g(x_i) p(x_i)$$

$$= \sum g(x_i) p(x_i)$$

For you to review so that you can sleep tonight!
Exercises
A Whole New World with Random Variables

Event-driven probability

- Relate only binary events
  - Either something happens ($E$)
  - or it doesn’t happen ($E^C$)

- Can only report probability

- Lots of combinatorics

Random Variables

- Link multiple similar events together ($X = 1, X = 2, \ldots, X = 6$)

- Can compute statistics: report the "average" outcome

- Once we have the PMF (for discrete RVs), we can do regular math
Example random variable

Consider 5 flips of a coin which comes up heads with probability $p$. Each coin flip is an independent trial. Let $Y$ = # of heads on 5 flips.

1. What is the support of $Y$? In other words, what are the values that $Y$ can take on with non-zero probability?

2. Define the event $Y = 2$. What is $P(Y = 2)$?

3. What is the PMF of $Y$? In other words, what is $P(Y = k)$, for $k$ in the support of $Y$?
Consider 5 flips of a coin which comes up heads with probability $p$. Each coin flip is an independent trial. **Let $Y = \# \text{ of heads on 5 flips}$.**

1. What is the **support** of $Y$? In other words, what are the values that $Y$ can take on with non-zero probability? $\{0, 1, 2, 3, 4, 5\}$

2. Define the event $Y = 2$. What is $P(Y = 2)$? 
   
   $P(Y = 2) = \binom{5}{2} p^2 (1 - p)^3$

3. What is the **PMF** of $Y$? In other words, what is $P(Y = k)$, for $k$ in the support of $Y$? 

   $P(Y = k) = \binom{5}{k} p^k (1 - p)^{5-k}$
Lying with statistics

A school has 3 classes with 5, 10, and 150 students. What is the average class size?

1. Interpretation #1
   • Randomly choose a class with equal probability.
   • \( X = \text{size of chosen class} \)
   \[
   E[X] = 5 \left( \frac{1}{3} \right) + 10 \left( \frac{1}{3} \right) + 150 \left( \frac{1}{3} \right)
   \]
   \[
   = \frac{165}{3} = 55
   \]

2. Interpretation #2
   • Randomly choose a student with equal probability.
   • \( Y = \text{size of chosen class} \)
   \[
   E[Y] = 5 \left( \frac{5}{165} \right) + 10 \left( \frac{10}{165} \right) + 150 \left( \frac{150}{165} \right)
   \]
   \[
   = \frac{22635}{165} \approx 137
   \]

What alumni relations usually reports

Average student perception of class size
Being a statistician unconsciously

Let $X$ be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 = 0$

B. $E[Y] = E[0] = 0$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3} \cdot |1| = \frac{2}{3}$

E. C and D
Being a statistician unconsciously

Let $X$ be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 = 0$ \hspace{1cm} \times \hspace{1cm} E[X]

B. $E[Y] = E[0] = 0$ \hspace{1cm} \times \hspace{1cm} E[E[X]]

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3} |1| = \frac{2}{3}$

E. C and D

1. Find PMF of $Y$: $p_Y(0) = \frac{1}{3}, p_Y(1) = \frac{2}{3}$
2. Compute $E[Y]$

Use LOTUS by using PMF of $X$:

1. $P(X = x) \cdot |x|$
2. Sum up