12: Independent RVs

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April 26th, 2024

Lecture Discussion on Ed
Sums of independent Binomial RVs
Independent discrete RVs

Recall the definition of independent events $E$ and $F$:

Two discrete random variables $X$ and $Y$ are independent if:

\[
P(X = x, Y = y) = P(X = x)P(Y = y)
\]

\[
p_{X,Y}(x, y) = p_X(x)p_Y(y)
\]

- Intuitively: knowing value of $X$ tells us nothing about the distribution of $Y$ (and vice versa)
- If two variables are not independent, they are termed dependent.
Sum of independent Binomials

\[ X \sim \text{Bin}(n_1, p) \]
\[ Y \sim \text{Bin}(n_2, p) \]

\[ X, Y \text{ independent} \]

Intuition:
- Each trial in \( X \) and \( Y \) is independent and has same success probability \( p \)
- Define \( Z = \# \) successes in \( n_1 + n_2 \) independent trials, each with success probability \( p \). \( Z \sim \text{Bin}(n_1 + n_2, p) \) and \( Z = X + Y \) as well.

Holds in general case:

\[ X_i \sim \text{Bin}(n_i, p) \]
\[ X_i \text{ independent for } i = 1, \ldots, n \]

\[ \sum_{i=1}^{n} X_i \sim \text{Bin} \left( \sum_{i=1}^{n} n_i, p \right) \]
**Coin flips**

Flip a coin with probability $p$ of heads a total of $n + m$ times.

Let

- $X = \text{number of heads in first } n \text{ flips. } X \sim \text{Bin}(n, p)$
- $Y = \text{number of heads in next } m \text{ flips. } Y \sim \text{Bin}(m, p)$
- $Z = \text{total number of heads in } n + m \text{ flips.}$

1. Are $X$ and $Z$ independent?
2. Are $X$ and $Y$ independent?
Coin flips

Flip a coin with probability $p$ of heads a total of $n + m$ times.

Let

- $X = \text{number of heads in first } n \text{ flips. } X \sim \text{Bin}(n, p)$
- $Y = \text{number of heads in next } m \text{ flips. } Y \sim \text{Bin}(m, p)$
- $Z = \text{total number of heads in } n + m \text{ flips.}$

1. Are $X$ and $Z$ independent? ✗
   Countereample: What if $Z = 0$?

2. Are $X$ and $Y$ independent? ✓

\[
P(X = x, Y = y) = P\left(\text{first } n \text{ flips have } x \text{ heads and next } m \text{ flips have } y \text{ heads}\right) \\
= \binom{n}{x} p^x (1 - p)^{n-x} \binom{m}{y} p^y (1 - p)^{m-y} \\
= P(X = x)P(Y = y)
\]

This probability (found through counting) is the product of the marginal PMFs.

# of mutually exclusive outcomes in event: $\binom{n}{x}\binom{m}{y}$

$P(\text{each outcome}) = p^x (1 - p)^{n-x} p^y (1 - p)^{m-y}$
Convolution: Sum of independent Poisson RVs
Convolution: Sum of independent random variables

For any discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_k P(X = k, Y = n - k)$$

In particular, for independent discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

the convolution of $p_X$ and $p_Y$
Insight into convolution

For independent discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

the convolution of $p_X$ and $p_Y$

Suppose $X$ and $Y$ are independent, both with support $\{0, 1, \ldots, n, \ldots\}$:

- $\checkmark$: event where $X + Y = n$
- Each event has probability:
  $$P(X = k, Y = n - k) = P(X = k)P(Y = n - k)$$
  (because $X, Y$ are independent)
- $P(X + Y = n) = \text{sum of mutually exclusive events}$
The distribution of a sum of 2 dice rolls is a convolution of 2 PMFs.

Example:
\[
P(X + Y = 4) = P(X = 1)P(Y = 3) + P(X = 2)P(Y = 2) + P(X = 3)P(Y = 1)
\]
Sum of 10 dice rolls (fun preview)

The distribution of a sum of 10 dice rolls is a convolution of 10 PMFs.

Looks kinda Normal...???
(more on this in a few weeks)
Sum of independent Poissons

**Theorem:**

If \( X \sim \text{Poi}(\lambda_1) \) and \( Y \sim \text{Poi}(\lambda_2) \) are independent, then \( X + Y \sim \text{Poi}(\lambda_1 + \lambda_2) \).

**Proof:**

\[
P(X + Y = n) = \sum_{k=0}^{n} P(X = k)P(Y = n - k)
\]

\[
= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!}
= e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k! (n-k)!}
\]

\[
= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n}
\]

\[
= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n
\]

This shows that the sum of independent Poisson variables is also a Poisson variable with the sum of the rates.

**Binomial Theorem:**

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}
\]

\[
P(m) = e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}
\]

\[
= \text{Poi}(\lambda_1 + \lambda_2)
\]

**Convolution:**

\[
P(X + Y = n) = \sum_{k=0}^{n} P(X = k)P(Y = n - k)
\]
Sum of independent Poissons

\[ X \sim \text{Poi}(\lambda_1), \ Y \sim \text{Poi}(\lambda_2) \]
\[ X, Y \text{ independent} \quad \Rightarrow \quad X + Y \sim \text{Poi}(\lambda_1 + \lambda_2) \]

- \( n \) servers with independent number of requests/minute
- Server \( i \)'s requests each minute can be modeled as \( X_i \sim \text{Poi}(\lambda_i) \)

What is the probability that the total number of web requests received at all servers in the next minute exceeds 10?
Exercises
Independent questions

1. Let $X \sim \text{Bin}(30, 0.01)$ and $Y \sim \text{Bin}(50, 0.02)$ be independent RVs.
   - How do we compute $P(X + Y = 2)$ using a Poisson approximation?
   - How do we compute $P(X + Y = 2)$ exactly?

2. Let $N = \#$ of requests to a web server per day. Suppose $N \sim \text{Poi}(\lambda)$.
   - Each request independently comes from a human (prob. $p$), or bot ($1 - p$).
   - Let $X$ be $\#$ of human requests/day, and $Y$ be $\#$ of bot requests/day.
   Are $X$ and $Y$ independent? What are their marginal PMFs?
1. Approximating the sum of independent Binomial RVs

Let $X \sim \text{Bin}(30, 0.01)$ and $Y \sim \text{Bin}(50, 0.02)$ be independent RVs.

• How do we compute $P(X + Y = 2)$ using a Poisson approximation?

• How do we compute $P(X + Y = 2)$ exactly?

\begin{align*}
P(X + Y = 2) &= \sum_{k=0}^{2} P(X = k)P(Y = 2 - k) \\
&= \sum_{k=0}^{2} \binom{30}{k} 0.01^k (0.99)^{30-k} \binom{50}{2-k} 0.02^{2-k} 0.98^{50-(2-k)} \\
&\approx 0.2327
\end{align*}
2. Web server requests

Let \( N = \# \) of requests to a web server per day. Suppose \( N \sim \text{Poi}(\lambda) \).

- Each request independently comes from a human (prob. \( p \)), or bot (1 - \( p \)).
- Let \( X \) be \# of human requests/day, and \( Y \) be \# of bot requests/day.

Are \( X \) and \( Y \) independent? What are their marginal PMFs?

\[
P(X = x, Y = y) = P(X = x, Y = y \mid N = x + y)P(N = x + y) + P(X = x, Y = y \mid N \neq x + y)P(N \neq x + y)
\]

\[
= P(X = x \mid N = x + y)P(Y = y \mid X = x, N = x + y)P(N = x + y) + P(X = x \mid N \neq x + y)P(Y = y \mid N \neq x + y)P(N \neq x + y)
\]

\[
= \binom{x + y}{x} p^x (1 - p)^y \cdot 1 \cdot e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!}
\]

\[
= \frac{(x+y)!}{x!y!} e^{-\lambda p} \frac{(\lambda p)^x}{x!} \cdot e^{-\lambda(1-p)} \frac{\lambda^{1-p}}{y!}
\]

\[
= P(X = x)P(Y = y) \quad \text{where } X \sim \text{Poi}(\lambda p), Y \sim \text{Poi}(\lambda(1 - p))
\]

Yes, \( X \) and \( Y \) are independent!
Expectation of Common RVs
Linearity of Expectation: Important

Expectation is a linear mathematical operation. If \( X = \sum_{i=1}^{n} X_i \):

\[
E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]
\]

- Even if you don’t know the distribution of \( X \) (e.g., because the joint distribution of \( (X_1, ..., X_n) \) is unknown), you can still compute expectation of \( X \).

- Problem-solving key: Define \( X_i \) such that

\[
X = \sum_{i=1}^{n} X_i
\]

Most common use cases:
- \( E[X_i] \) easy to calculate
- Sum of dependent RVs
Expectations of common RVs: Binomial

X ~ Bin(n, p)  \quad E[X] = np

Recall: Bin(1, p) = Ber(p)

\[ X = \sum_{i=1}^{n} X_i \]

Let \( X_i = \text{ith trial is heads} \)
\( X_i \sim \text{Ber}(p), E[X_i] = p \)

\[ E[X] = E \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np \]
Expectations of common RVs: Negative Binomial

\( Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p} \)

# of independent trials with probability of success \( p \) until \( r \) successes

Recall: \( \text{NegBin}(1, p) = \text{Geo}(p) \)

\[
Y = \sum_{i=1}^{?} Y_i
\]

1. How should we define \( Y_i \)?

2. How many terms are in our summation?
Expectations of common RVs: Negative Binomial

\[ Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p} \]

# of independent trials with probability of success \( p \) until \( r \) successes

Recall: \( \text{NegBin}(1, p) = \text{Geo}(p) \)

\[ Y = \sum_{i=1}^{r} Y_i \]

Let \( Y_i = \) # trials to get \( i \)th success (after \( (i-1) \)th success)

\( Y_i \sim \text{Geo}(p), E[Y_i] = \frac{1}{p} \)

\[ E[Y] = E\left[ \sum_{i=1}^{r} Y_i \right] = \sum_{i=1}^{r} E[Y_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p} \]