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CS109 Winter 2024

CS109 Week 4 Exam Solution

This is a closed calculator/computer exam. You are, however, permitted to consult the two double-sided sheets of notes you've prepared ahead of time. You're otherwise not permitted to refer to any other notes.

In the event of an incorrect answer, any explanation you provide of how you obtained your answer can potentially allow us to give you partial credit for a problem. It is fine for your answers to include summations, products, factorials, exponents, and combinations unless stated otherwise.

I acknowledge and accept the letter and spirit of the honor code.

Signature: ______________________________

Full Name [print]: ______________________________

SunetID [i.e., your @stanford.edu email]: ______________________________

SUID [i.e., your seven or eight-digit student ID number]: ______________________________

1 Doris and Valentine's Day [15 points]

Valentine's Day is just two weeks away, and since Doris isn't dating anyone at the moment, she's planning on baking 100 biscuits and giving them to her 6 best friends.

a. [3 points] Assuming all 100 biscuits look precisely the same, how many ways can all 100 be distributed and shared with her 6 best friends. (The 100 biscuits are indistinguishable, but the friends are all distinguishable. And never give Doris any biscuits! Just share them with her six friends.)

> **Solution.** This is classic divider method, where 100 indistinct objects are distributed across 6 distinct buckets.
>
> $$\binom{100+6-1}{6-1} = \binom{105}{5} = \binom{105}{100}$$

b. [4 points] Assuming all 100 biscuits look precisely the same, how many ways can all 100 be distributed and shared with her 6 best friends, subject to the constraint that each friend be given at least one biscuit. (The 100 biscuits are still indistinguishable, and the six friends are all distinguishable.)

> **Solution.** This is similar, except that 6 cookies are constrained and 94, not 100, are free to be distributed across the 6 friends.
>
> $$\binom{94+6-1}{6-1} = \binom{99}{5} = \binom{99}{94}$$

c. [3 points] Now assume that all of the biscuits are distinguishable from one another. How many ways can the biscuits be distributed between her six friends? (Note that it's possible once again that one or more friends are denied biscuits.)

> **Solution.** Because biscuits are distinct, each one can be directed to any one of six friends, independently of all other biscuits. This is classic product rule for 100 different "experiments", each of which has six different outcomes.
>
> $$6^{100}$$

d. [5 points] The biscuits are still distinguishable from one another, but each friend must be given at least one biscuit. How many ways can the biscuits be distributed between her six friends now? Your answer should leverage Inclusion-Exclusion, and it should be left in the form of a summation.

Solution. Our answer from part c overcounts, because it includes distributions where one or more of her friends go without biscuits. We need to subtract all the ways exactly one friend might be denied biscuits, recognizing that doing so will overcompensate by subtracting some distributions more than once (e.g., those where the first two of her six friends go without). That means we need to add all of those distributions back, continuing to subtract, add, subtract, add until all legitimate distributions have been counted exactly once.

This is precisely what the Inclusion-Exclusion principle is designed for!

$$6^{100} - \binom{6}{1} \cdot 5^{100} + \binom{6}{2} \cdot 4^{100} - \binom{6}{3} \cdot 3^{100} + \binom{6}{4} \cdot 2^{100} - \binom{6}{5} \cdot 1^{100}$$

Note that $\binom{6}{6} \cdot 0^{100} = 0$ could have been included as well, particularly if you expressed your answer using Σ notation instead of an expanded sum as I have.

2 Combinatorial Proofs [10 points]

Consider the following combinatorial identity for all integers $n \geq 0$:

$$\sum_{s=0}^{n} \sum_{t=0}^{n-s} \binom{n}{s} \binom{n-s}{t} = 3^n$$

Present a **combinatorial** proof of the above identity, without relying on any tedious algebra. As a hint, consider all of the ways to distribute n distinct items across three different subsets S, T, and V.

Solution. The right-hand side counts the number of ways each of the n elements can be distributed to each of the three subsets. There are three options for the first element, three for the second, three for the third, and so forth.

The left-hand side partitions all 3^n possible distributions by the sizes of S and T. $\binom{n}{s}$ counts the number of ways s of the n elements can be directed to S, and $\binom{n-s}{t}$ counts the number of ways the $n - s$ elements not assigned to S can be split between T (of size t) and V (of size $v = n - s - t$). This describes a full partition of all possible distributions, and we double sum over all of them.

3 Metallic Tastes [20 points]

Whenever a patient complains of a metallic taste in their mouth, doctors are concerned the patient may be suffering from acid reflux, Bell's Palsy, or perhaps both.

Acid reflux is the backflow of stomach acid into the esophagus, and Bell's Palsy is a temporary weakening of the facial muscles that prompts one side of the patient's face to droop. Both can impact what taste buds perceive.

Let D_1 be the event that a patient suffers from acid reflex, let D_2 be the event that a patient suffers from Bell's Palsy, and let M be the event that someone is detecting a metallic taste in their mouth. We'll assume that D_1 and D_2 are independent events where $P(D_1) = p_1$, $P(D_2) = p_2$, and that someone with neither of these two conditions might still be experiencing a metallic taste with a probability of m.

We'll make the reasonable assumptions that $0 < p_1, p_2 < 1$ and that there's at least one person in the population who is healthy and symptom-free. We'll also assume that someone suffering from either or both of these conditions always, always, always complains of a metallic taste in their mouth, so that $P(M|D_1) = P(M|D_2) = P(M|D_1D_2) = 1$.

a. [5 points] Present an expression for $P(M)$.

> **Solution.** I let $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. We use the Law of Total Probability to add up the many different pieces contributing to our $P(M)$ calculation.
>
> $$\begin{aligned} P(M) &= P(M|D_1)p_1 + P(M|D_2)p_2 - P(M|D_1D_2)p_1p_2 + mq_1q_2 \\ &= p_1 + p_2 - p_1p_2 + mq_1q_2 \end{aligned}$$

b. [8 points, 4 and 4] Present expressions for $P(D_1|M)$ and $P(D_1D_2|M)$. We'll assume your expression for $P(D_2|M)$ is analogous to that presented $P(D_1|M)$. If your answer relies on your answer from part a, simply refer to that probability value as p_a.

> **Solution.** Using Bayes' Rule, we have the following:
>
> $$\begin{aligned} P(D_1|M) &= \frac{P(M|D_1)P(D_1)}{P(M)} \\ &= \frac{P(M|D_1)p_1}{p_a} \\ &= \frac{p_1}{p_a} = \frac{p_1}{p_1 + p_2 - p_1p_2 + mq_1q_2} \end{aligned}$$
>
> $$\begin{aligned} P(D_1D_2|M) &= \frac{P(M|D_1D_2)P(D_1)P(D_1)}{P(M)} \\ &= \frac{P(M|D_1D_2)p_1p_2}{p_a} \\ &= \frac{p_1p_2}{p_a} = \frac{p_1p_2}{p_1 + p_2 - p_1p_2 + mq_1q_2} \end{aligned}$$

c. [4 points] Are D_1 and D_2 conditionally independent of M when $0 < m < 1$?

> **Solution.**
>
> Assuming $P(M|D_1) = P(M|D_2) = P(M|D_1D_2) = 1$, it's impossible for $P(D_1|M)P(D_2|M)$ to equal $P(D_1D_2|M)$—i.e. that D_1 and D_2 are conditionally independent given M)—since equality would require the denominator common to all of your answers to part b. be equal to its own square root. That's only possible if the denominator is either 0 or 1, and it's neither.

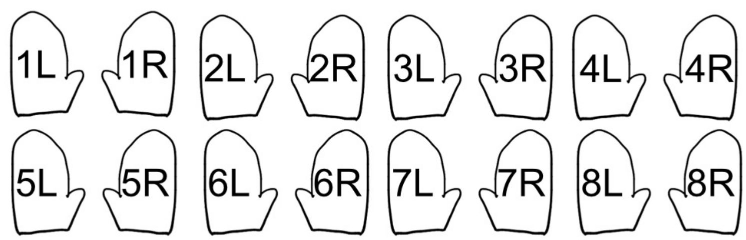
d. [3 points] When $m = 0$, it can be shown that D_1 and D_2 are not conditionally independent given M. Present an intuitive explanation as to why that's the case without relying on algebra.

> **Solution.**
>
> If we know that $m = 0$, we can confine ourselves to the world where everyone has a metallic taste in their mouth, and everyone in this world therefore suffers from one or both of the diseases. In this world, knowing that someone doesn't have acid reflux means they must have Bell's Palsy, or vice versa.
>
> That means:
>
> $$1 = P(D_2|D_1^c M) \neq P(D_2|M) = p_2$$
>
> so knowing whether D_1 occurs informs our belief of whether D_2 occurs.

4 Mittens and Kittens [15 points]

The fourth problem from the Winter 2024 midterm was cannibalized for your current problem set, so I grabbed this problem from a very, very old take-home exam to test similar things. Because the exam was a take-home, students were expected to manage a little more algebra than I would during an in-person, no-calculators-allowed exam. Still, it's a cool problem.

You have 8 pairs of mittens, each a different pattern. Left and right mittens are also distinct. Suppose that you are fostering kittens, and you leave them alone for a few hours with your mittens. When you return, you discover that they have hidden 4 mittens! Suppose that your kittens are equally likely to hide any 4 of your 16 distinct mittens. Let X be the number of complete, distinct ***pairs*** of mittens that remain.



a. [7 points] Compute the probability mass function of X, $p_X(x)$. (Hint: Note the support of X is $\{4, 5, 6\}$, since only 4, 5, or 6 complete pairs are possible after the kitten fiasco.)

> **Solution.** There are three possible scenarios involving 4 mittens. Assuming unordered outcomes:
>
> - $\binom{8}{4}\binom{2}{1}^4 = 1120$: 4 mittens from 4 different pairs are gone, leaving 4 complete pairs. Choose 4 pairs out of the 8 pairs, then for each pair choose either the left or right mitten to give to your kittens.
> - $\binom{8}{3}\binom{3}{1}\binom{2}{1}^2 = 672$: 1 complete pair is gone, and 2 mittens from 2 different pairs are gone, leaving 5 complete pairs. Choose 3 out of the 8 pairs, then choose the one complete pair to give to your kittens. Then, for the remaining 2 patterns, choose the left or right mitten to give to your kittens.
> - $\binom{8}{2} = 28$: 2 complete pairs are gone, leaving 6 complete pairs. Choose 2 out of the 8 pairs, and give all four of the chosen mittens.
>
> The sample space has a total of $\binom{16}{4} = 1820$ outcomes. If each outcome is equally likely, we have the following, brute-force PMF
>
> $$p_X(4) = \frac{1120}{1820}, p_X(5) = \frac{672}{1820}, p_X(6) = \frac{28}{1820}$$

b. [2 points] Compute $E[X]$ using the definition of expectation and your answer to part a.

> **Solution.**
> $$E[X] = 4 \cdot \frac{1120}{1820} + 5 \cdot \frac{672}{1820} + 6 \cdot \frac{28}{1820} = \frac{22}{5} = 4.4 \text{ mittens.}$$

c. [6 points] Define the random variable X_i to be 1 if your i^{th} pair of mittens is complete after the kitten fiasco, and 0 otherwise, Using this definition of X_i for $i = 1, \ldots, 8$ and the linearity of expectation, compute $E[X]$ again.

Solution. We would let $X_i = 1$ with probability $p = \binom{14}{4}/\binom{16}{4} = \frac{11}{20}$. Our probability here is the fraction of all ways to choose four mittens from the 14 that are not of the i^{th} pair. That means the expected value of X_i is $1 \cdot \frac{11}{20} + 0 \cdot \frac{9}{20} = \frac{11}{20}$ for all values of $i = 1, 2, 3, 4, 5, 6, 7, 8$.

By defining $X = X_1 + X_2 + \cdots + X_8$ and using linearity of expectation, we compute $E[X]$ to be $8 \cdot \frac{11}{20} = \frac{22}{5}$, which matches the value generated in part b.