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CS109

Winter 2024

CS109 Week 4 Exam

This is a closed calculator/computer exam. You are, however, permitted to consult the two double-sided sheets of notes you've prepared ahead of time. You're otherwise not permitted to refer to any other notes.

In the event of an incorrect answer, any explanation you provide of how you obtained your answer can potentially allow us to give you partial credit for a problem. It is fine for your answers to include summations, products, factorials, exponents, and combinations unless stated otherwise.

I acknowledge and accept the letter and spirit of the honor code.

Signature: ______________________________

Full Name [print]: ______________________________

SunetID [i.e., your @stanford.edu email]: ______________________________

SUID [i.e., your seven or eight-digit student ID number]: ______________________________

1 Doris and Valentine's Day [15 points]

Valentine's Day is just two weeks away, and since Doris isn't dating anyone at the moment, she's planning on baking 100 biscuits and giving them to her 6 best friends.

a. [3 points] Assuming all 100 biscuits look precisely the same, how many ways can all 100 be distributed and shared with her 6 best friends. (The 100 biscuits are indistinguishable, but the friends are all distinguishable. And never give Doris any biscuits! Just share them with her six friends.)

b. [4 points] Assuming all 100 biscuits look precisely the same, how many ways can all 100 be distributed and shared with her 6 best friends, subject to the constraint that each friend be given at least one biscuit. (The 100 biscuits are still indistinguishable, and the six friends are all distinguishable.)

c. [3 points] Now assume that all of the biscuits are distinguishable from one another. How many ways can the biscuits be distributed between her six friends? (Note that it's possible once again that one or more friends are denied biscuits.)

d. [5 points] The biscuits are still distinguishable from one another, but each friend must be given at least one biscuit. How many ways can the biscuits be distributed between her six friends now? Your answer should leverage Inclusion-Exclusion, and it should be left in the form of a summation.

2 Combinatorial Proofs [10 points]

Consider the following combinatorial identity for all integers $n \geq 0$:

$$\sum_{s=0}^{n} \sum_{t=0}^{n-s} \binom{n}{s} \binom{n-s}{t} = 3^n$$

Present a **combinatorial** proof of the above identity, without relying on any tedious algebra. As a hint, consider all of the ways to distribute n distinct items across three different subsets S, T, and V.

3 Metallic Tastes [20 points]

Whenever a patient complains of a metallic taste in their mouth, doctors are concerned the patient may be suffering from acid reflux, Bell's Palsy, or perhaps both.

Acid reflux is the backflow of stomach acid into the esophagus, and Bell's Palsy is a temporary weakening of the facial muscles that prompts one side of the patient's face to droop. Both can impact what taste buds perceive.

Let D_1 be the event that a patient suffers from acid reflex, let D_2 be the event that a patient suffers from Bell's Palsy, and let M be the event that someone is detecting a metallic taste in their mouth. We'll assume that D_1 and D_2 are independent events where $P(D_1) = p_1$, $P(D_2) = p_2$, and that someone with neither of these two conditions might still be experiencing a metallic taste with a probability of m.

We'll make the reasonable assumptions that $0 < p_1, p_2 < 1$ and that there's at least one person in the population who is healthy and symptom-free. We'll also assume that someone suffering from either or both of these conditions complains of a metallic taste in their mouth, so that $P(M|D_1) = P(M|D_2) = P(M|D_1D_2) = 1$.

a. [5 points] Present an expression for $P(M)$.

b. [8 points, 4 and 4] Present expressions for $P(D_1|M)$ and $P(D_1D_2|M)$. We'll assume your expression for $P(D_2|M)$ is analogous to that presented $P(D_1|M)$. If your answer relies on your answer from part a, simply refer to that probability value as p_a.

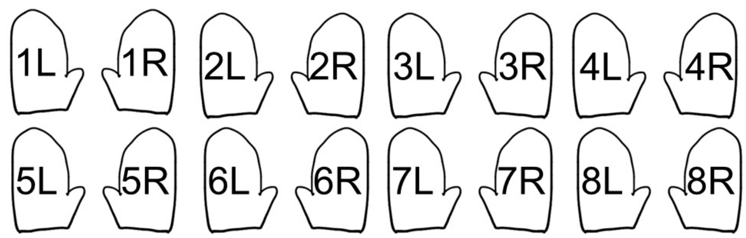
c. [4 points] Are D_1 and D_2 conditionally independent of M when $0 < m < 1$?

d. [3 points] When $m = 0$, it can be shown that D_1 and D_2 are not conditionally independent given M. Present an intuitive explanation as to why that's the case without relying on algebra.

4 Mittens and Kittens [15 points]

The fourth problem from the Winter 2024 midterm was cannibalized for your current problem set, so I grabbed this problem from a very, very old take-home exam to test similar things. Because the exam was a take-home, students were expected to manage a little more algebra than I would during an in-person, no-calculators-allowed exam. Still, it's a cool problem.

You have 8 pairs of mittens, each a different pattern. Left and right mittens are also distinct. Suppose that you are fostering kittens, and you leave them alone for a few hours with your mittens. When you return, you discover that they have hidden 4 mittens! Suppose that your kittens are equally likely to hide any 4 of your 16 distinct mittens. Let X be the number of complete, distinct ***pairs*** of mittens that remain.



a. [7 points] Compute the probability mass function of X, $p_X(x)$. (Hint: Note the support of X is $\{4, 5, 6\}$, since only 4, 5, or 6 complete pairs are possible after the kitten fiasco.)

b. [2 points] Compute $E[X]$ using the definition of expectation and your answer to part a.

c. [6 points] Define the random variable X_i to be 1 if your i^{th} pair of mittens is complete after the kitten fiasco, and 0 otherwise, Using this definition of X_i for $i = 1, \ldots, 8$ and the linearity of expectation, compute $E[X]$ again. **Do not use your answer to parts a or b.**.