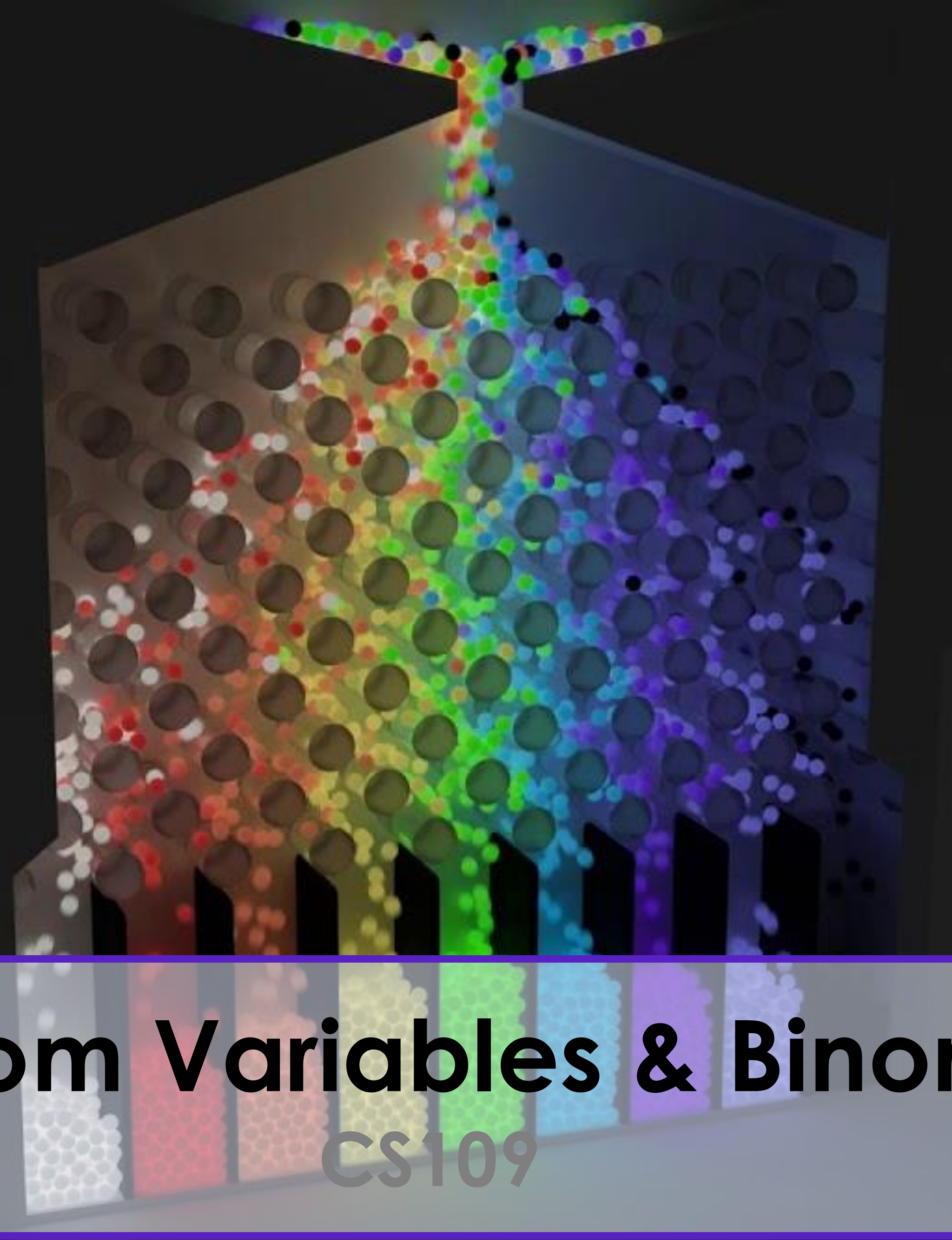


Random Variables & Binomial

CS109

Review

The Core Probability Toolkit

The Law of Total Probability

$$P(E) = P(E \text{ and } F) + P(E \text{ and } F^C)$$

$$P(E) = P(E|F)\,P(F) + P(E|F^C)\,P(F^C)$$

$$P(E) = \sum_{i=1}^{n} P(E \text{ and } B_i)$$

$$= \sum_{i=1}^{n} P(E|B_i)\,P(B_i)$$

Bayes' Theorem

$$P(B|E) = \frac{P(E|B) \cdot P(B)}{P(E)}$$

$$P(B|E) = \frac{P(E|B) \cdot P(B)}{P(E|B) \cdot P(B) + P(E|B^C) \cdot P(B^C)}$$

Definition of Conditional Probability

$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}$$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

$$P(E^C) = 1 - P(E)$$

De Morgan's Laws

$(A \text{ or } B)^C = A^C \text{ and } B^C$

$(A \text{ and } B)^C = A^C \text{ or } B^C$

Axiom 3: If E and F are mutually exclusive, then $P(E \text{ or } F) = P(E) + P(F)$

Otherwise, use Inclusion-Exclusion:

$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$

Chain Rule

$$P(E \text{ and } F) = P(E|F) \cdot P(F)$$

$$= P(F|E) \cdot P(E)$$

Independence

$$P(E|F) = P(E)$$

$$P(E \text{ and } F) = P(E)\,P(F)$$

Review: Conditional Probability Formulas

$P(E \text{ and } F)$

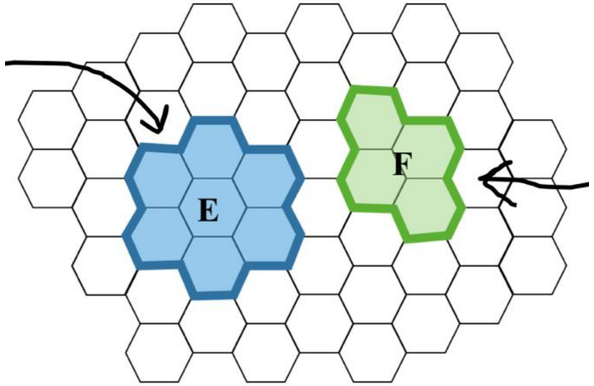
Definition of Conditional Probability ↓ ↑ Chain Rule

$P(E|F)$

Law of Total Probability ↙ ↘ Bayes' Theorem

$P(E)$ $P(F|E)$

Last Class



Mutually Exclusive Events

make ***OR*** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$

Independent Events

make ***AND*** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Independence

Two events A and B are **independent** if:

$$P(A) = P(A|B)$$

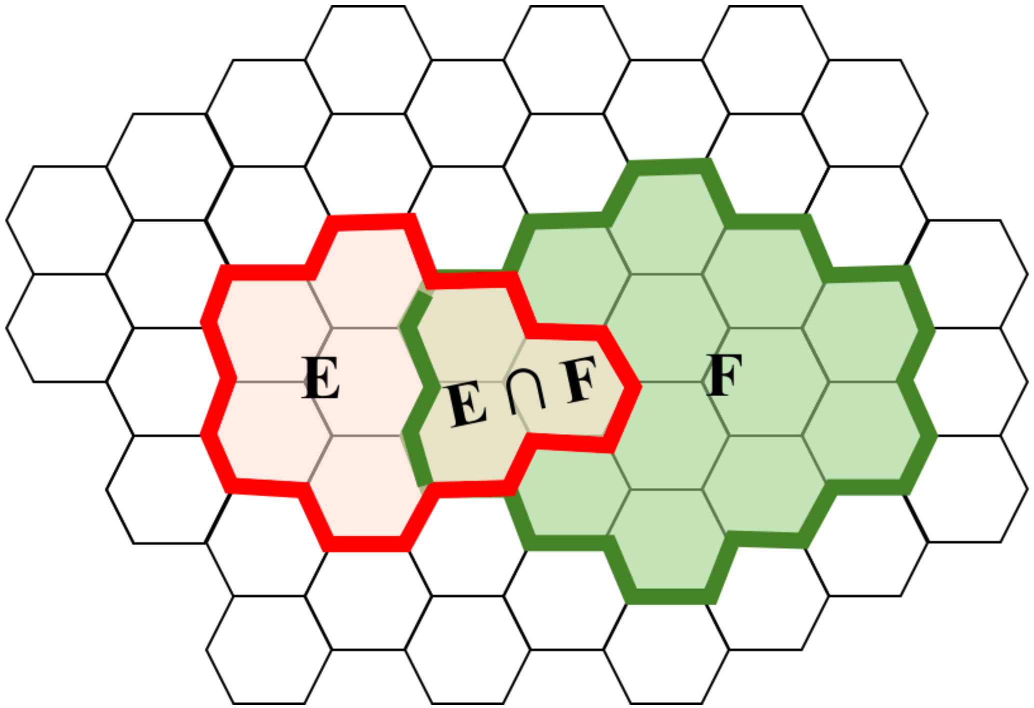
Intuitive Definition:

Knowing that event B happened doesn't change our belief that A happens.

With independence, we can simplify the chain rule:

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(A) \cdot P(B) \end{aligned}$$

Probability of Or *Without* Mutually Exclusive Events



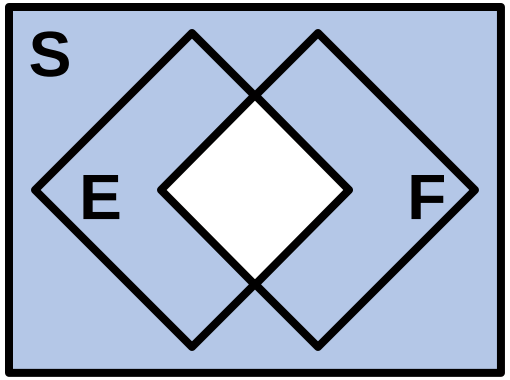
If events have outcomes in common, we correct for double-counting them:

$$P(E \text{ or } F) = P(E) + P(F) - P(EF)$$

The "Inclusion Exclusion" Rule

DeMorgan's Laws

These rules let you alternate between AND and OR.

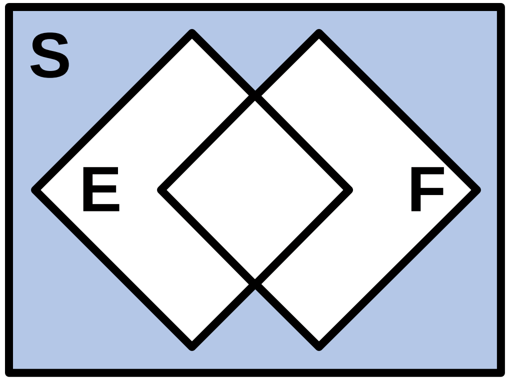


$$(E \text{ and } F)^C = E^C \text{ or } F^C$$

If events are *dependent*:

$$P(E_1 E_2 \cdots E_n) = 1 - P((E_1 E_2 \cdots E_n)^C)$$
$$= 1 - P(E_1^C \text{ or } E_2^C \text{ or } \cdots E_n^C)$$

Great if E^C mutually exclusive!



$$(E \text{ or } F)^C = E^C \text{ and } F^C$$

If events are *not* mutually exclusive:

$$P(E_1 \text{ or } E_2 \text{ or } \cdots E_n) = 1 - P((E_1 \text{ or } E_2 \text{ or } \cdots E_n)^C)$$
$$= 1 - P(E_1^C E_2^C \cdots E_n^C)$$

Great if E_is are independent!

ULTIMATE PROBABILITY

https://www.maikaisogawa.com/ultimate-frisbee-probability/ https://www.youtube.com/watch?v=H2IfTwGisOg

Warmup: The Frisbee Problem

You flip two frisbees.

The probability that each frisbee lands "**heads**" is **0.6**.

The two frisbees are considered "**even**" if **both** frisbees are heads or tails.

What is the probability that the frisbees are even?

Warmup: The Frisbee Problem

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The probability that each frisbee lands "**heads**" is **0.6**.

The two frisbees are considered "**even**" if **both** frisbees are heads or tails.

What is the probability that the frisbees are even?

Let H_1 = event that 1st frisbee is heads, H_2 = event that the 2nd frisbee is heads.

$$\begin{aligned} P((H_1 \cap H_2) \text{ or } (H_1^C \cap H2^C)) &= P(H_1 \cap H_2) + P(H_1^C \cap H2^C) \\ &= P(H_1) \cdot P(H_2) + P(H_1^C) \cdot P(H2^C) \\ &= 0.6^2 + (1 - 0.6)^2 \\ &= 0.36 + 0.16 = 0.52 \end{aligned}$$

You have now started PSet 2!

We define a function fair_random(), which uses unknown_random() to produce a 1 or 0.

```
def fair_random():
    """
    There are four outcomes for assignments to r1 and r2:
      [0, 0], [0, 1], [1, 0], [1, 1]. Return 1 if the
      outcomes are [0, 0] or [1, 1]
    """
    r1 = unknown_random()
    r2 = unknown_random()
    return r1 == r2
```

Is this function truly fair? Determine P(fair_random returns 1) in terms of p.

End Review

What’s Missing?

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Events are nice, but…

- They’re binary: events collapse outcomes into “did E happen, yes or no”

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Example: let's write out all the outcomes + probabilities of rolling a dice.

Let E_1 be the event we get a 1. $P(E_1) = 1/6$

Let E_2 be the event we get a 2. $P(E_2) = 1/6$

…

…this feels silly

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This doesn't express that E_1, E_2, etc. are mutually exclusive outcomes *from the same experiment*

This clunkiness gets in the way of us recognizing patterns in probability problems

With just events, it would be hard to scale all the way up to machine learning.

It’s Time…

It’s Time…

X

It's Time…

X

…For Random Variables

Random Variables Are Variables…That Are Random

Random Variables Are Variables...That Are Random

Check out the variable **`result`** in the code below.

```
import random

def flip_coin():
    # returns 0 or 1 with prob. 0.5
    return random.choice([0,1])

result = flip_coin()
```

Random Variables Are Variables…That Are Random

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- Do we know the value of **`result`** before we run the code? Nope!
- Is the value of **`result`** the same every time we run the code? Nope!

Like **`result`**, a random variable is a variable whose value is uncertain.

Random Variables Are Variables…That Are Random

A **random variable** is a variable whose value is uncertain.

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"Let ***X*** be the result of flipping a coin."

$$P(X = 0) = 0.5$$
$$P(X = 1) = 0.5$$

Random Variables Are Variables…That Are Random

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"Let X be the result of flipping a coin."

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- Random variables store the outcome of an experiment
- Random variables can be described by their possible outcomes + probabilities
  - Note: random variables can only be numbers (not "heads" or "tails")

Random variables are an abstraction on top of events

Random variables are *not* events

Random Variables vs. Events

$$X$$

Let X be a random variable

Random Variables vs. Events

It is an event when
X takes on a value

$$X \qquad\qquad X = 2$$

Let X be a
random variable

Random Variables vs. Events

It is an event when
X takes on a value

$$X \qquad X = 2 \qquad P(X = 2)$$

Let X be a
random variable

So we can still work with
probabilities of events

Examples of Random Variables

"Let X be the result of rolling a dice."

- $P(X = 1) = 1/6$
- $P(X = 2) = 1/6$
- $P(X = 3) = 1/6$
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- $P(X = 5) = 1/6$
- $P(X = 6) = 1/6$

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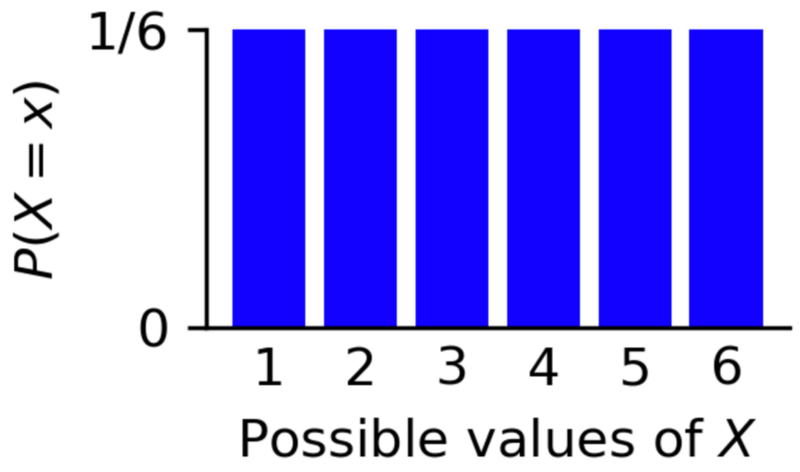

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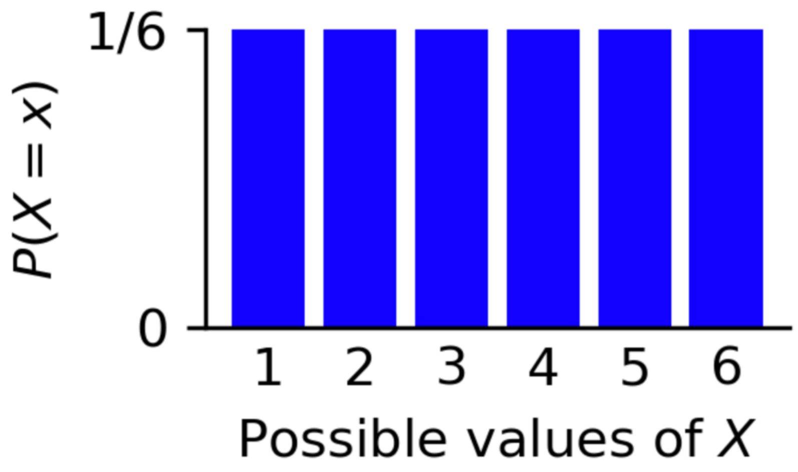

"Let Y be the number of heads seen in 2 coin flips."

- $P(Y = 0) = 1/4$ (T, T)
- $P(Y = 1) = 1/2$ (H, T), (T, H)
- $P(Y = 2) = 1/4$ (H, H)

Examples of Random Variables

"Let X be the result of rolling a dice."

- $P(X = 1) = 1/6$
- $P(X = 2) = 1/6$
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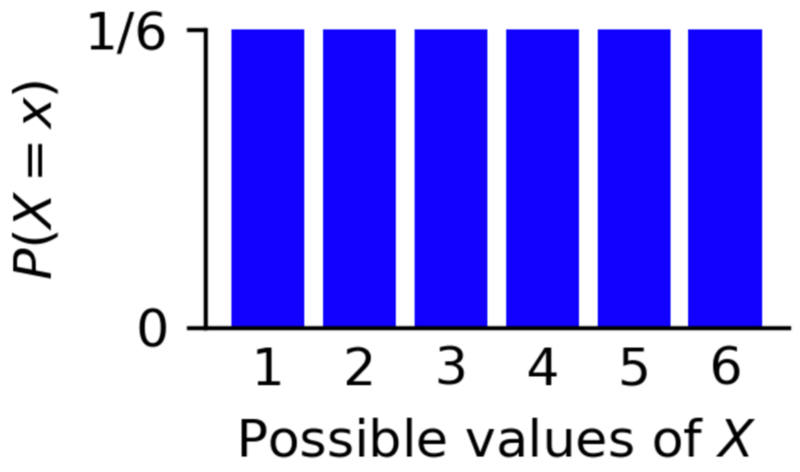

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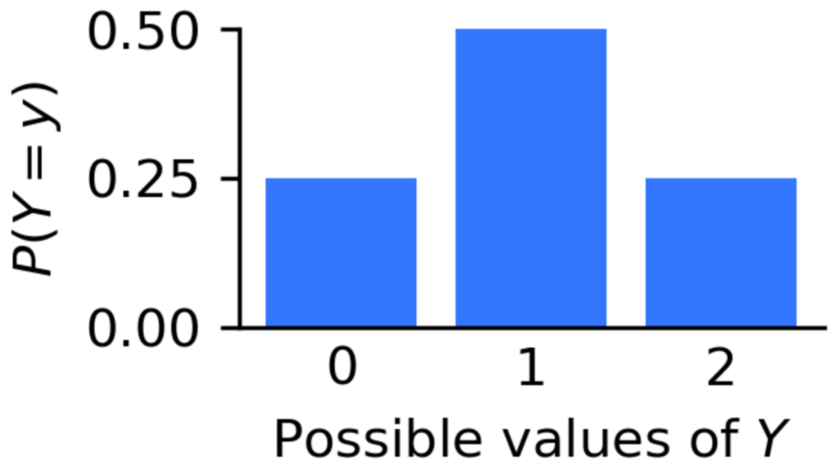
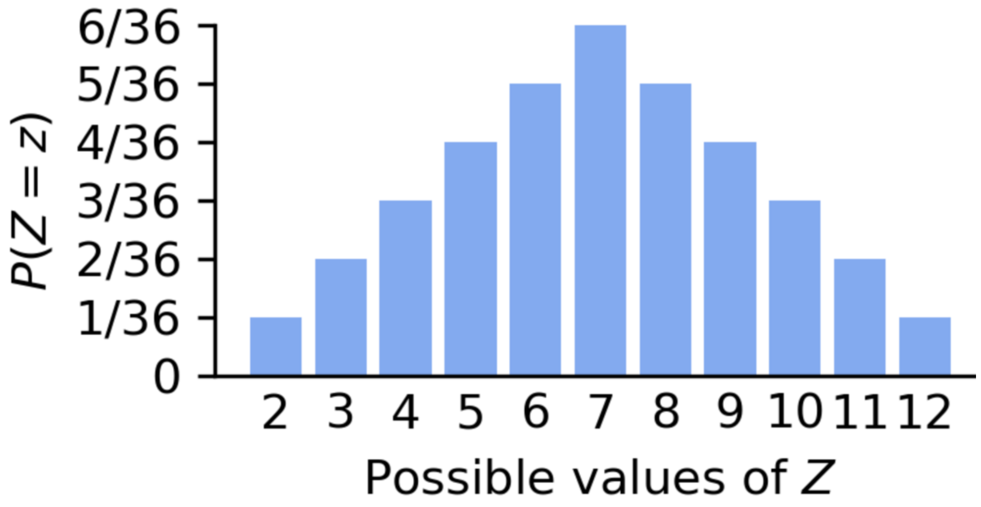


Examples of Random Variables

"Let Z be the sum of rolling two dice."

- $P(Z = 2) = 1/36$
- $P(Z = 3) = 2/36$
- $P(Z = 4) = 3/36$
- $P(Z = 5) = 4/36$
- $P(Z = 6) = 5/36$
- $P(Z = 7) = 6/36$
- $P(Z = 8) = 5/36$
- $P(Z = 9) = 4/36$
- $P(Z = 10) = 3/36$
- $P(Z = 11) = 2/36$
- $P(Z = 12) = 1/36$



$$P(Z = z) = \begin{cases} \frac{z-1}{36} & z \in \mathbb{Z}, 1 \leq z \leq 6 \\ \frac{13-z}{36} & z \in \mathbb{Z}, 7 \leq z \leq 12 \\ 0 & \text{else} \end{cases}$$

Examples of Random Variables

"Let Z be the sum of rolling two dice."

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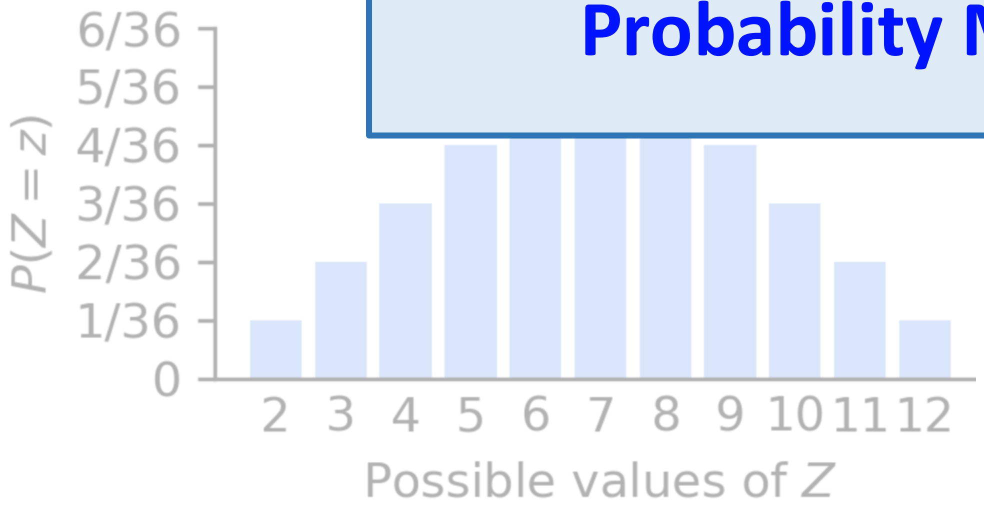
- $P(Z =$ …
- $P(Z =$ …
- $P(Z = 6) = 5/36$
- $P(Z = 7) = 6/36$
- $P(Z = 10) = 3/36$
- $P(Z = 11) = 2/36$
- … /36

There's a name for what we're describing, when we list out all possible outcomes + their probabilities:

Probability Mass Function (PMF)



$$P(Z = z) = \begin{cases} \ldots & \ldots \mathbb{Z}, 1 \le z \le 6 \\ \frac{13-z}{36} & z \in \mathbb{Z}, 7 \le z \le 12 \\ 0 & \text{else} \end{cases}$$

Probability Mass Functions

Random Variables & Functions

"Let Y be the number of heads seen in 2 coin flips."

If this is a number

$$P(Y = 2)$$

Then this is a number (between 0 and 1)

Random Variables & Functions

"Let Y be the number of heads seen in 2 coin flips."

If this is a variable

$$P(Y = k)$$

Then this is a function

Random Variables & Functions

"Let Y be the number of heads seen in 2 coin flips."

...and get out their probabilities!

$$0.5$$

$$P(Y = k)$$

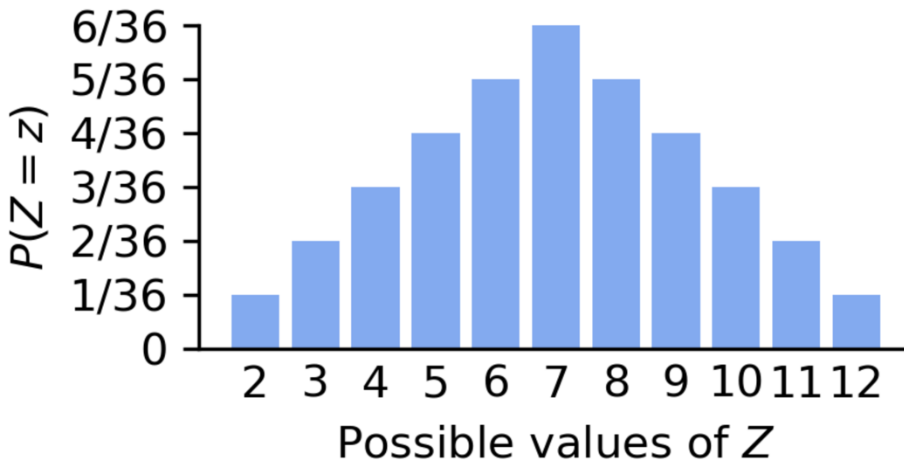
$$k = 1$$

We can put in different inputs...

The relationship between values a random variable can take on, and the corresponding probability, is a ***function***!

Probability Mass Function: Representations

$$P(Z=z) = \begin{cases} \frac{z-1}{36} & z \in \mathbb{Z}, 1 \leq z \leq 6 \\ \frac{13-z}{36} & z \in \mathbb{Z}, 7 \leq z \leq 12 \\ 0 & \text{else} \end{cases}$$




```
def event_probability(z):
    # probability mass function of Z
    if not z.is_integer() or z > 12 or z < 1:
        return 0

    if z < 7:
        return (z – 1) / 36
    else:
        return (13 - z) / 36
```

All of these are different ways we can represent probability mass functions!

Quick Understanding Check

$$\sum_{\text{all } k} P(Y = k) \stackrel{?}{=}$$

What is the sum of the probabilities of all possible outcomes for Y?

Quick Understanding Check

$$\sum_{\text{all } k} P(Y = k) = 1$$

What is the sum of the probabilities of all possible outcomes for Y?

1!

Can You Calculate A PMF From Data? Yes

Say this is your dataset, a list of observations of random variable Y:

[2, 4, 5, 12, 9, 7, 9, 4, 6, 4, 3, 7, 5, 8, 9, 6, 5, 10, 10, 10, 7, 11, 4,
6, 7, 9, 10, 6, 11, 6, 5, 12, 7, 3, 11, 6, 4, 7, 8, 2, 7, 8, 6, 6, 8, 3, 2,
8, 6, 9, 5, 11, 8, 6, 9, 7, 10, 10, 10, 6, 5, 9, 4, 5, 8, 8, 6, 6, 6, 10,
4, 7, 7, 5, 7, 9, 12, 6, 7, 5, 5, 10, 7, 5, 4, 7, 6, 6, 5, 5, 8, 9, 7, 7, 7,
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3, 8, 10, 6, 10, 8, 10, 8, 4, 11, 4, 12, 6, 7, 3, 9, 5, 11, 5, 7, 4, 7, 8,
12, 9, 8, 10, 4, 4, 5, 6, 4, 5, 6, 7, 3, 3, 11, 8, 9, 2, 8, 4, 8, 7, 8, 9,
10, 5, 10, 7, 9, 8, 8, 6, 7, 5, 6, 11, 2, 5, 3, 8, 4, 7, 7, 4, 7, 2, 7, 10,
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7, 3, 12, 7, 10, 4, 6, 7, 4, 11, 3, 3, 6, 6, 12, 11, 12, 10, 11, 7, 9, 7,
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9, 12, 9, 8, 8, 8, 9, 6, 8, 5, 11, 6, 8, 6, 5, 8, 5, 8, 6, 11, 5, 8, 3, 7,
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7, 10, 9, 9, 4, 7, 8, 9, 8, 7, 7, 6, 12, 2, 7, 11, 10, 8, 7, 9, 11, 7, 9,
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Can You Calculate A PMF From Data? Yes

Say this is your dataset, a list of observations of random variable Y:

[2, 4, 5, 12, 9, 7, 9, 4, 6, 4, 3, 7, 5, 8, 9, 6, 5, 10, 10, 10, 7, 11, 4,
6, 7, 9, 10, 6, 11, 6, 5, 12, 7, 3, 11, 6, 4, 7, 8, 2, 7, 8, 6, 6, 8, 3, 2,
8, 6, 9, 5, 11, 8, 6, 9, 7, 10, 10, 10, 6, 5, 9, 4, 5, 8, 8, 6, 6, 6, 10,
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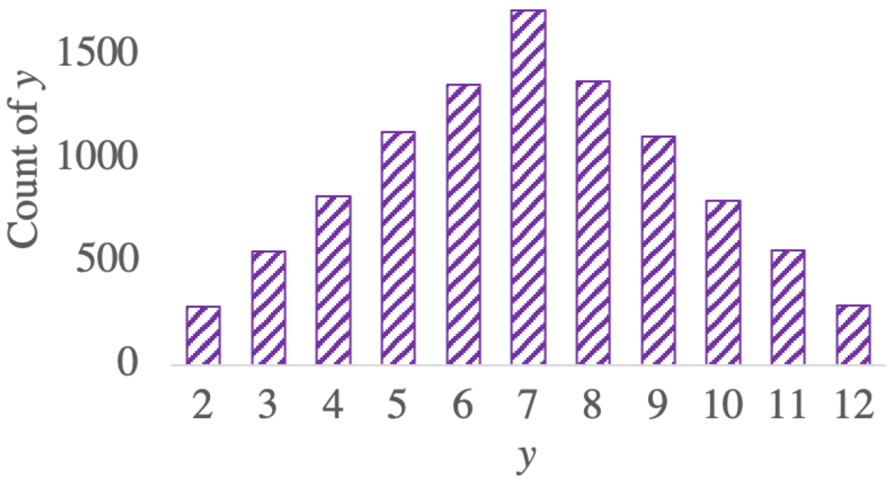
8, 9, 7, 5, 10, 5, 8, 5, 5, 5, 8, 9, 3, 9, 10, 10, 6, 4, 6, 2, 6, 2, 8, 7,
4, 6, 6, 7, 9, 4, 6, 8, 5, 7, 7, 7, 9, 2, 6, 7, 3, 10, 10, 7, 3, 5, 3, 6, 6,
7, 12, 9, 9, 11, 9, 4, 4, 10, 8, 8, 9, 8, 4, 4, 6, 2, 7, 5, 7, 7, 10, 4,
11, 5, 7, 8, 8, 2, 8, 6, 9, 8, 7, 8, 8, 10, 4, 7, 10, 10, 10, 4, 6, 12,
11, 4, 9, 12, 2, 3, 5, 3, 3, 11, 7, 8, 8, 5, 10, 8, 9, 4, 7, 7, 2, 5, 10,
7, 10, 9, 9, 4, 7, 8, 9, 8, 7, 7, 6, 12, 2, 7, 11, 10, 8, 7, 9, 11, 7, 9,
6, 8, 9, 10, 7, 3, 8, 10, 6, 6, 4, 2, 7, 11, 5, 6, 5, 4, 3, 2, 8]

Just convert your data into counts:

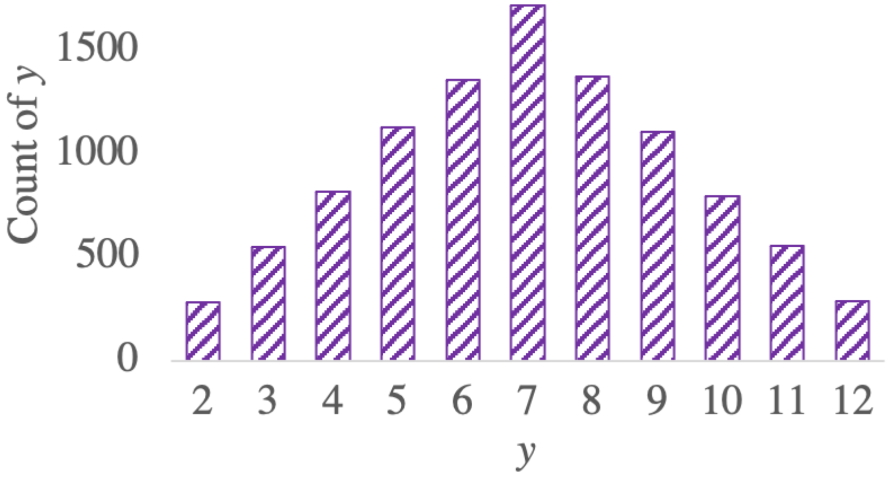


Can You Calculate A PMF From Data? Yes

Say this is your dataset, a list of observations of random variable Y:

[2, 4, 5, 12, 9, 7, 9, 4, 6, 4, 3, 7, 5, 8, 9, 6, 5, 10, 10, 10, 7, 11, 4,
6, 7, 9, 10, 6, 11, 6, 5, 12, 7, 3, 11, 6, 4, 7, 8, 2, 7, 8, 6, 6, 8, 3, 2,
8, 6, 9, 5, 11, 8, 6, 9, 7, 10, 10, 10, 6, 5, 9, 4, 5, 8, 8, 6, 6, 6, 10,
4, 7, 7, 5, 7, 9, 12, 6, 7, 5, 5, 10, 7, 5, 4, 7, 6, 6, 5, 5, 8, 9, 7, 7, 7,
9, 9, 8, 9, 11, 11, 10, 5, 3, 8, 10, 9, 7, 11, 6, 12, 6, 3, 8, 6, 3, 11,
11, 9, 6, 5, 7, 9, 7, 9, 6, 8, 9, 3, 7, 9, 10, 8, 9, 9, 7, 6, 9, 7, 5, 5, 5,
3, 8, 10, 6, 10, 8, 10, 8, 4, 11, 4, 12, 6, 7, 3, 9, 5, 11, 5, 7, 4, 7, 8,
12, 9, 8, 10, 4, 4, 5, 6, 4, 5, 6, 7, 3, 3, 11, 8, 9, 2, 8, 4, 8, 7, 8, 9,
10, 5, 10, 7, 9, 8, 8, 6, 7, 5, 6, 11, 2, 5, 3, 8, 4, 7, 7, 4, 7, 2, 7, 10,
10, 7, 9, 3, 5, 8, 6, 4, 8, 7, 7, 6, 8, 6, 11, 7, 3, 6, 6, 6, 9, 11, 6, 5,
7, 3, 12, 7, 10, 4, 6, 7, 4, 11, 3, 3, 6, 6, 12, 11, 12, 10, 11, 7, 9, 7,
5, 12, 6, 3, 6, 4, 5, 10, 6, 11, 11, 7, 6, 8, 11, 5, 12, 4, 7, 9, 9, 9,
10, 7, 9, 7, 4, 4, 6, 8, 6, 3, 4, 9, 7, 11, 8, 6, 11, 5, 7, 11, 7, 7, 6, 4,
9, 12, 9, 8, 8, 8, 9, 6, 8, 5, 11, 6, 8, 6, 5, 8, 5, 8, 6, 11, 5, 8, 3, 7,
8, 8, 10, 9, 8, 9, 8, 4, 7, 9, 5, 8, 8, 9, 7, 3, 9, 3, 4, 6, 9, 9, 5, 6, 4,
8, 9, 7, 5, 10, 5, 8, 5, 5, 5, 8, 9, 3, 9, 10, 10, 6, 4, 6, 2, 6, 2, 8, 7,
4, 6, 6, 7, 9, 4, 6, 8, 5, 7, 7, 7, 9, 2, 6, 7, 3, 10, 10, 7, 3, 5, 3, 6, 6,
7, 12, 9, 9, 11, 9, 4, 4, 10, 8, 8, 9, 8, 4, 4, 6, 2, 7, 5, 7, 7, 10, 4,
11, 5, 7, 8, 8, 2, 8, 6, 9, 8, 7, 8, 8, 10, 4, 7, 10, 10, 10, 4, 6, 12,
11, 4, 9, 12, 2, 3, 5, 3, 3, 11, 7, 8, 8, 5, 10, 8, 9, 4, 7, 7, 2, 5, 10,
7, 10, 9, 9, 4, 7, 8, 9, 8, 7, 7, 6, 12, 2, 7, 11, 10, 8, 7, 9, 11, 7, 9,
6, 8, 9, 10, 7, 3, 8, 10, 6, 6, 4, 2, 7, 11, 5, 6, 5, 4, 3, 2, 8]

Just convert your data into counts:



And then use those counts to calculate probabilities for each outcome:

$$\frac{\text{count}(Y=3)}{n} = \frac{552}{10000} = 0.0552$$

You Can Use PMFs Other People Give You

Let X be the number of earthquakes that happen in California next year.

Here's the PMF for X:

$$P(X = x) = \frac{69^x e^{-69}}{x!}$$

What is the probability that there are 60 earthquakes in California next year?

You Can Use PMFs Other People Give You

Let X be the number of earthquakes that happen in California next year.

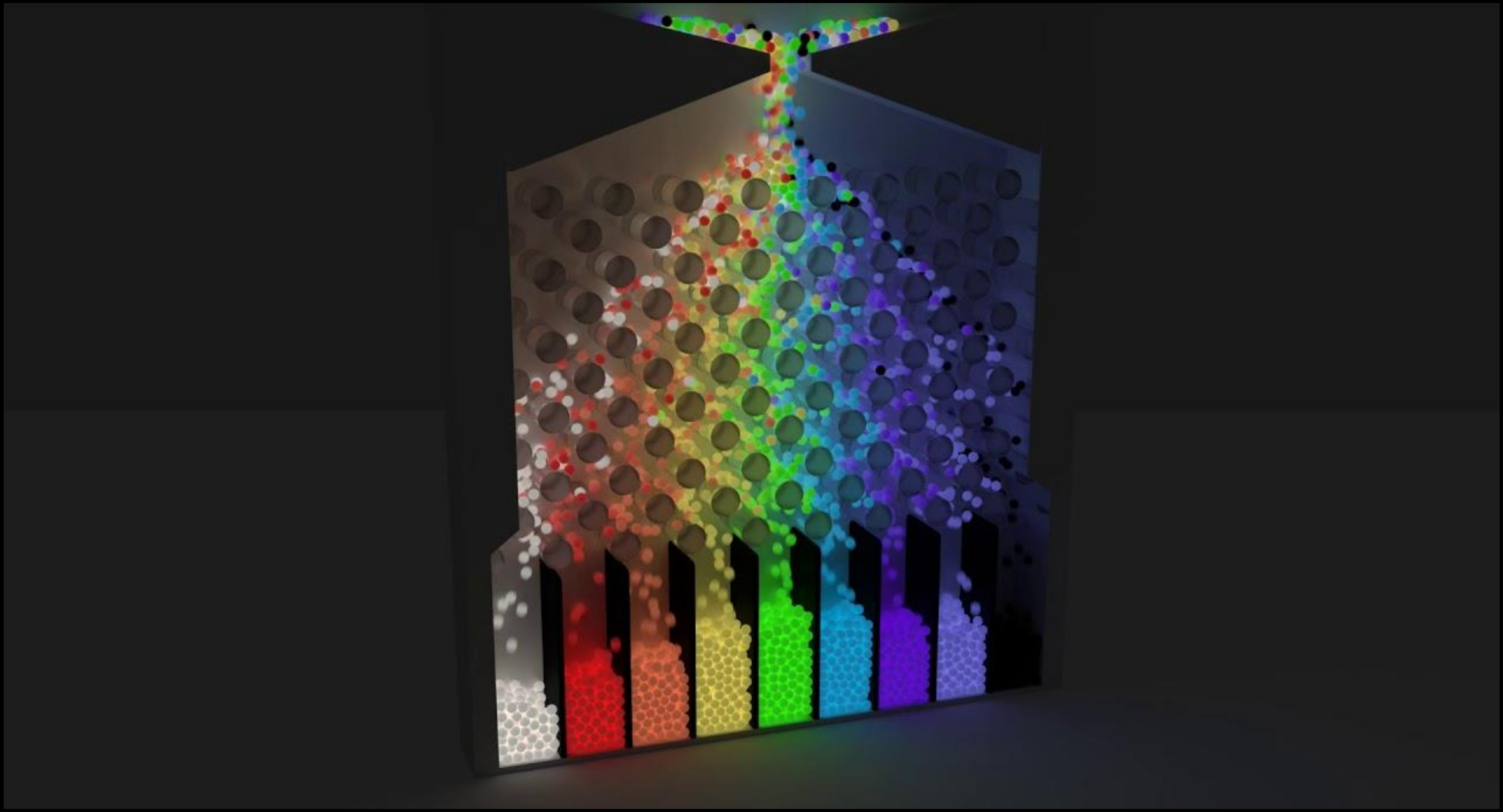
Here's the PMF for X: $$P(X = x) = \frac{69^x e^{-69}}{x!}$$

What is the probability that there are 60 earthquakes in California next year?

$$P(X = 60) = \frac{69^{60} e^{-69}}{60!} \approx 0.028$$

Just plug numbers in!

Random Variables Are Awesome



Some Random Variables Are “Classics”


THE BINOMIAL
DISTRIBUTION

Let's Derive The Most Famous Random Variable

Imagine flipping a coin n times and counting the number of heads.

1. We will flip a coin ***n*** times: ***n* independent trials** of the same experiment
2. Each coin flip has a **probability *p*** of being heads
3. What we want to model: what is the probability of **exactly *k* heads**?

Let's Derive The Most Famous Random Variable

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3. What we want to model: what is the probability of **exactly *k* heads**?

(This isn't really about flipping coins, though.)

Lots of scenarios fit the same description:

- # of 1's in randomly generated in length *n* bit string
- # of servers working in a large computer cluster
- # of people who vote for one of two candidates in an election
- # of jury members selected from a particular demographic

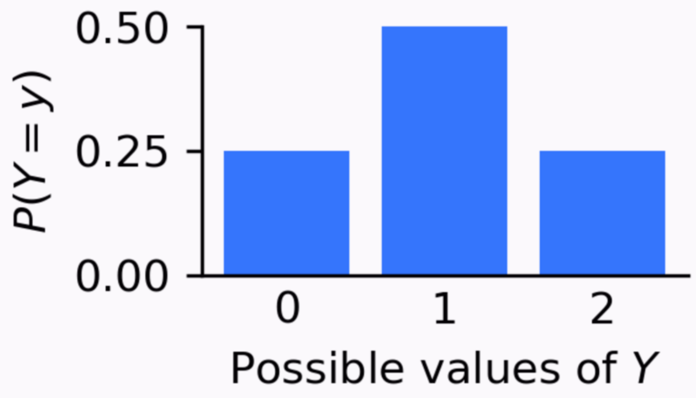
Let's Derive The Most Famous Random Variable

Imagine flipping a coin *n* times and counting the number of heads.

1. We will flip a coin ***n*** times: ***n*** **independent trials** of the same experiment
2. Each coin flip has a **probability *p*** of being heads
3. What we want to model: what is the probability of **exactly *k* heads**?

"Let Y be the # of heads in 2 coin flips."

- $P(Y = 0) = 1/4$ (T, T)
- $P(Y = 1) = 1/2$ (H, T), (T, H)
- $P(Y = 2) = 1/4$ (H, H)



This is the binomial for n = 2. Can we generalize from this?

Probability of Exactly *k* Heads in *n* Coin Flips

To start:

- Let's say we flip the coin 10 times. Probability of heads is *p*.
- For now, focus on the probability of 4 heads.

What is the probability of the outcome below?

```
(H, H, H, H, T, T, T, T, T, T)
```

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What is the probability of the outcome below?

(H, H, H, H, T, T, T, T, T, T)

$$p^4(1-p)^6$$

(H, H, H, T, H, T, T, T, T, T)

Probability of Exactly *k* Heads in *n* Coin Flips

To start:

- Let's say we flip the coin 10 times. Probability of heads is *p*.
- For now, focus on the probability of 4 heads.

What is the probability of the outcome below?

(H, H, H, H, T, T, T, T, T, T) $p^4(1-p)^6$

(H, H, H, T, H, T, T, T, T, T) $p^4(1-p)^6$

All of the outcomes with exactly 4 heads have the same probability

Probability of Exactly *k* Heads in *n* Coin Flips

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

Then, the probability of getting k heads in any ordering is the "**or**" of all of these **mutually exclusive** cases

How many cases are there?

Each outcome has probability $p^k(1-p)^{10-k}$

Probability of Exactly *k* Heads in *n* Coin Flips

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

Then, the probability of getting k heads in any ordering is the "**or**" of all of these **mutually exclusive** cases

How many cases are there? $\binom{10}{k}$

Each outcome has probability $p^k(1-p)^{10-k}$

Probability of Exactly k Heads in n Coin Flips

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

Then, the probability of getting k heads in any ordering is the “**or**” of all of these **mutually exclusive** cases

How many cases are there? $\binom{10}{k}$

Each outcome has probability $p^k(1-p)^{10-k}$

$$P(k \text{ heads}) = \binom{10}{k} p^k (1-p)^{10-k}$$

We Have Invented The Binomial

Here yee. This type of random variable is so common it needs a name so that I can talk about it generally.

I shall call it: the ***Binomial*** *Random Variable. Huzzah.*

Jacob "James" Bernoulli (1654-1705): Swiss mathematician

One of many mathematicians in the Bernoulli family

We Have Invented The Binomial

Here yee.
variable is s
name so th

I shall call it:
Vari

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One of many mathematicians in the Bernoulli family

Celebrity look alike: Ice Cube

Declaring a Random Variable to be Binomial

Our random variable

Num trials

Probability of success on each trial

$$X \sim \text{Bin}(n, p)$$

Is distributed as a

Binomial

With these parameters

Then We Automatically Know the PMF!

Probability Mass Function for a Binomial

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

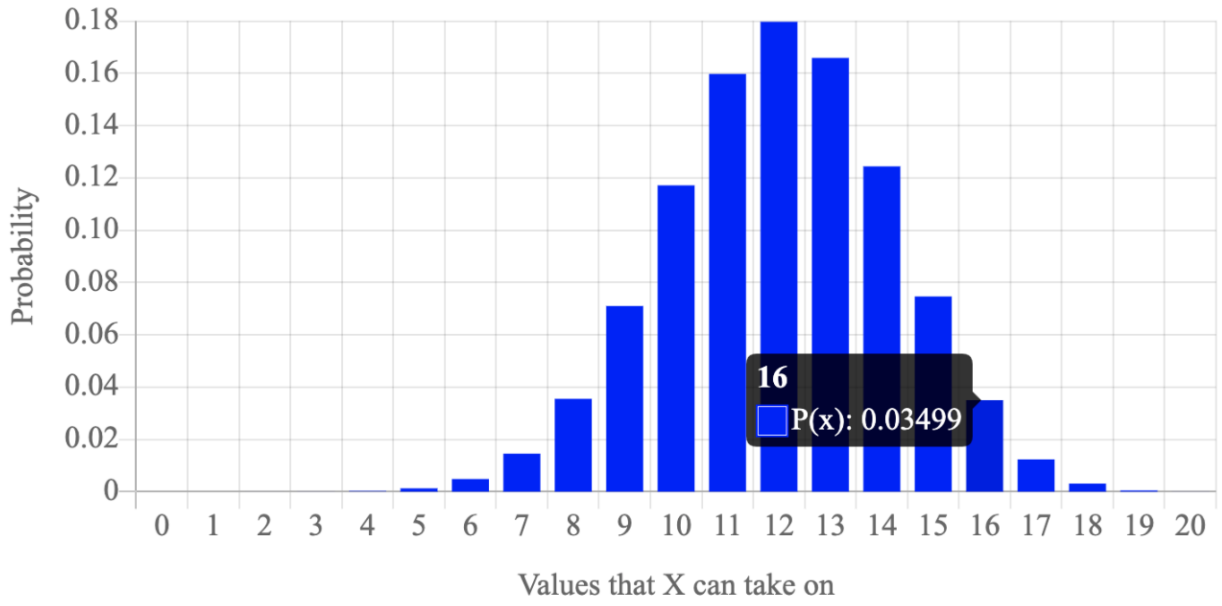
Probability that our variable takes on the value k

You Get So Much For Free!

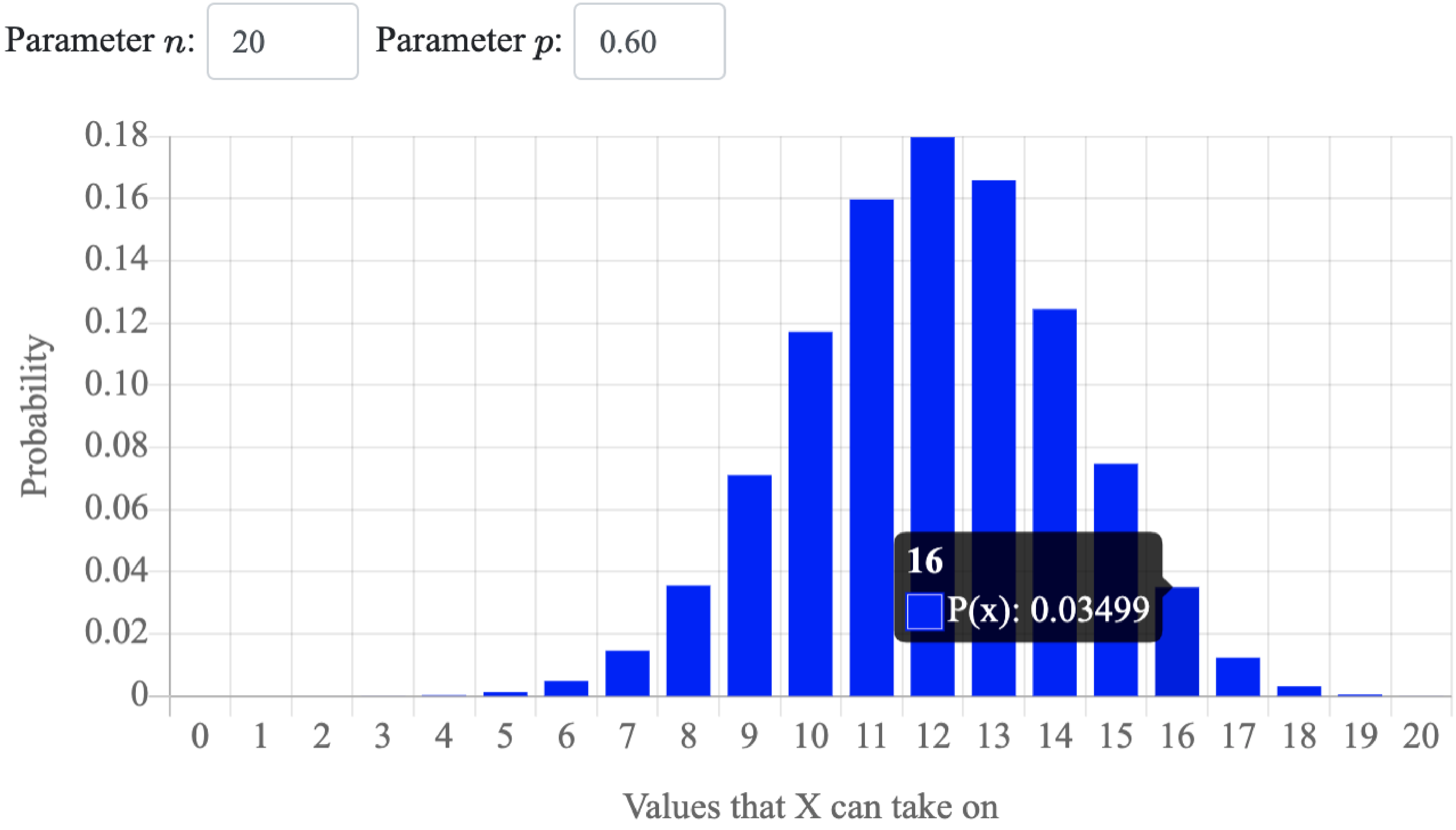
Binomial Random Variable

Notation:	$X \sim \text{Bin}(n, p)$
Description:	Number of "successes" in n identical, independent experiments each with probability of success p.
Parameters:	$n \in \{0, 1, \ldots\}$, the number of experiments. $p \in [0, 1]$, the probability that a single experiment gives a "success".
Support:	$x \in \{0, 1, \ldots, n\}$
PMF equation:	$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
Expectation:	$\text{E}[X] = n \cdot p$
Variance:	$\text{Var}(X) = n \cdot p \cdot (1-p)$
PMF graph:	

Parameter n: 20 Parameter p: 0.60

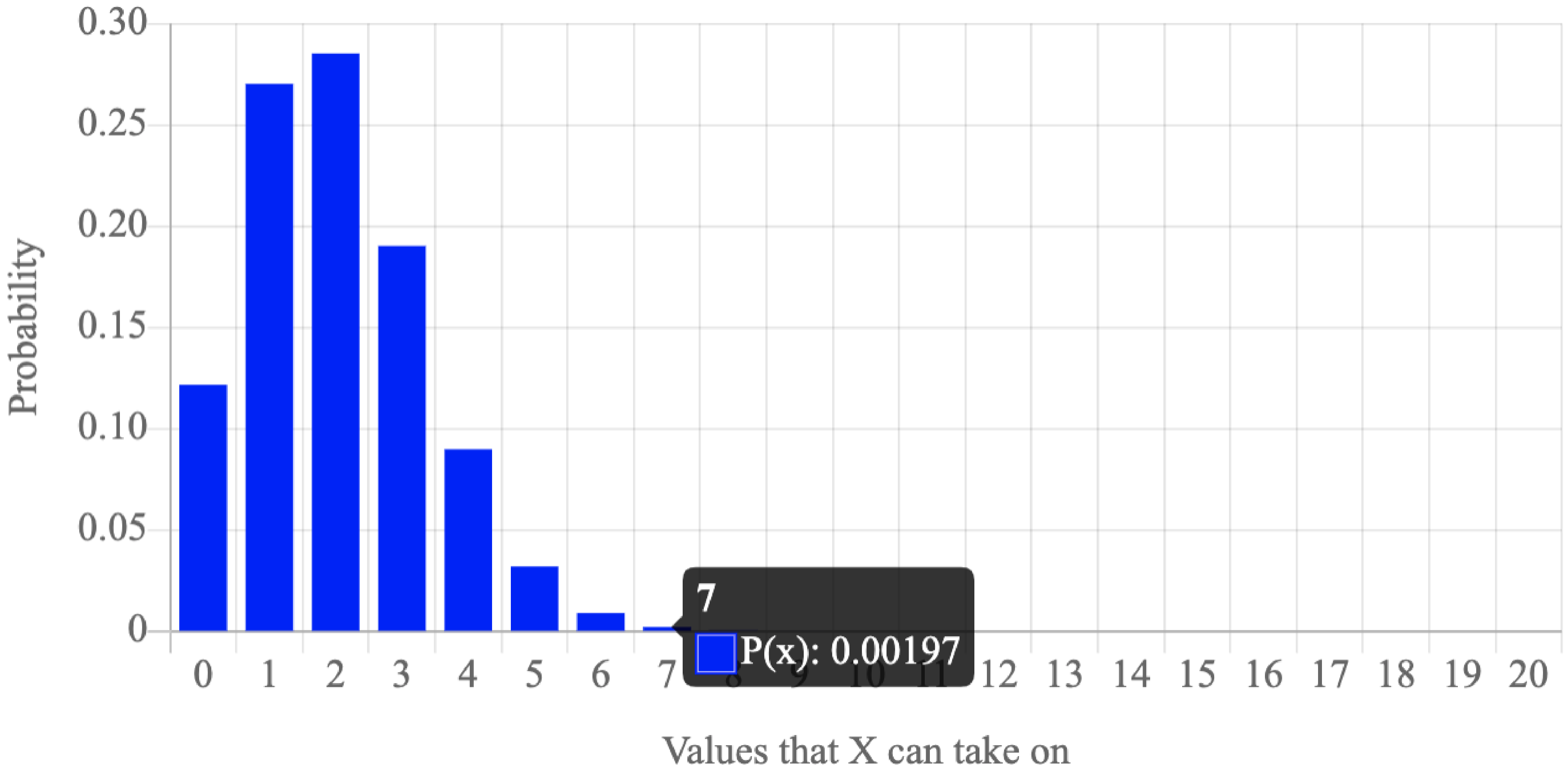


The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.6)$



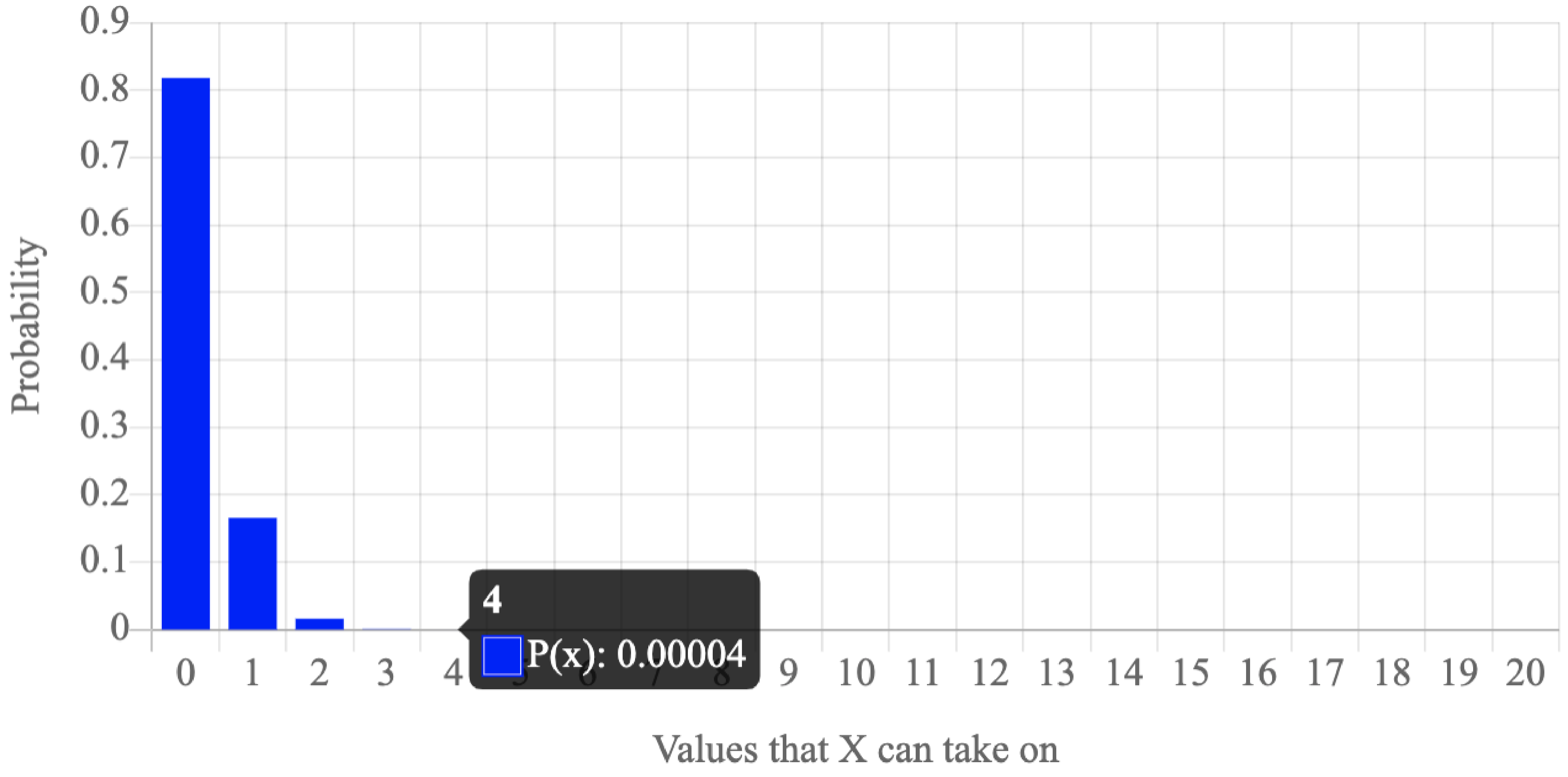
The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.1)$

Parameter n: 20 Parameter p: 0.1



The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.01)$

Parameter n: 20 Parameter p: 0.01



Probability of k Heads In n Flips: Now With Binomial

Three fair (p = 0.5 of heads) coins are flipped.

Let X be the number of heads.

$$X \sim \text{Bin}(n = 3, p = 0.5)$$

Probability of k Heads In n Flips: Now With Binomial

Three fair (p = 0.5 of heads) coins are flipped.

Let X be the number of heads.

$$X \sim \text{Bin}(n = 3, p = 0.5)$$

What is the probability of…

… 0 heads?

… 1 heads?

… 2 heads?

… 3 heads?

Probability of k Heads In n Flips: Now With Binomial

Three fair (p = 0.5 of heads) coins are flipped.

Let X be the number of heads.

$$X \sim \text{Bin}(n = 3, p = 0.5)$$

What is the probability of…

… 0 heads?

$$P(X = 0) = \binom{3}{0} p^0 (1 - p)^3 = \frac{1}{8}$$

… 1 heads?

$$P(X = 1) = \binom{3}{1} p^1 (1 - p)^2 = \frac{3}{8}$$

… 2 heads?

$$P(X = 2) = \binom{3}{2} p^2 (1 - p)^1 = \frac{3}{8}$$

… 3 heads?

$$P(X = 3) = \binom{3}{3} p^3 (1 - p)^0 = \frac{1}{8}$$

New Recipe For Solving Problems!

1. Recognize a classic random variable type

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2. Define a random variable to be that type, with parameters

$$X \sim \mathrm{Bin}(n, p)$$

New Recipe For Solving Problems!

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2. Define a random variable to be that type, with parameters

$$X \sim \text{Bin}(n, p)$$

3. Profit off the PMF

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with $p = 0.01$ (otherwise it's ignored).

What is the probability of this ad getting 10 clicks?

Concept Check

Practice: Ad Clicks

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Each ad served is clicked with $p = 0.01$ (otherwise it's ignored).

What is the probability of this ad getting 10 clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

Concept Check

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with $p = 0.01$ (otherwise it's ignored).

What is the probability of this ad getting 10 clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

$$P(X = k) = \binom{1000}{k}(0.01)^k(0.99)^{1000-k}$$

$$P(X = 10) = \binom{1000}{10}(0.01)^{10}(0.99)^{990} \approx 0.125$$

Concept Check

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with $p = 0.01$ (otherwise it's ignored).

What is the probability of this ad getting **20** clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

$$P(X = k) = \binom{1000}{k}(0.01)^k(0.99)^{1000-k}$$

$$P(X = 20) = \binom{1000}{20}(0.01)^{20}(0.99)^{980} \approx 0.0018$$

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

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What is the probability of this ad getting **20** clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

```
>>> from scipy import stats
>>> stats.binom.pmf(10, 1000, 0.01)
0.1257402111262075
>>> stats.binom.pmf(20, 1000, 0.01)
0.0017918782400182195
```

k n p

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

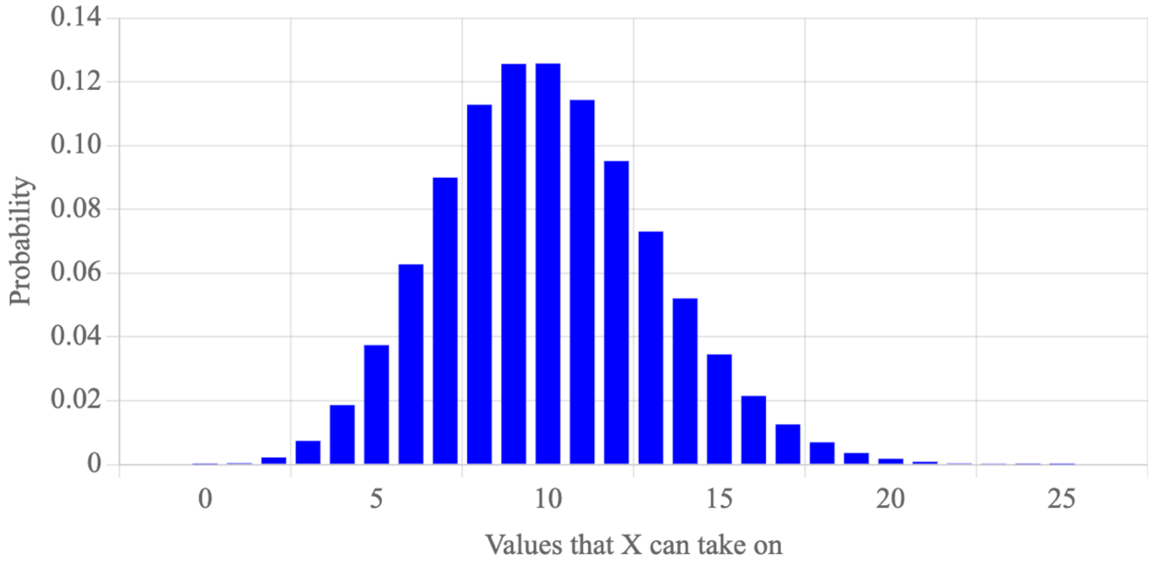
Each ad served is clicked with $p = 0.01$ (otherwise it's ignored).

What is the probability of this ad getting **20** clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

PMF graph:

Parameter n: 1000 Parameter p: 0.01



Server Redundancy

A network can remain functional as long as at least 2 out of 7 servers are alive.

The probability of any server working is 0.8.

What is the probability that less than 2 servers are alive?

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Let ***X*** be the number of servers alive. $X \sim \text{Bin}(n = 7, p = 0.8)$.

$$P(X = k) = \binom{7}{k}(0.8)^k(0.2)^{7-k}$$

Server Redundancy

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The probability of any server working is 0.8.

What is the probability that less than 2 servers are alive?

Let ***X*** be the number of servers alive. ***X*** ~ Bin($n = 7$, $p = 0.8$).

$$P(X = k) = \binom{7}{k}(0.8)^k(0.2)^{7-k}$$

$$P(X < 2) = P(X = 0) + P(X = 1)$$

Server Redundancy

A network can remain functional as long as at least 2 out of 7 servers are alive.

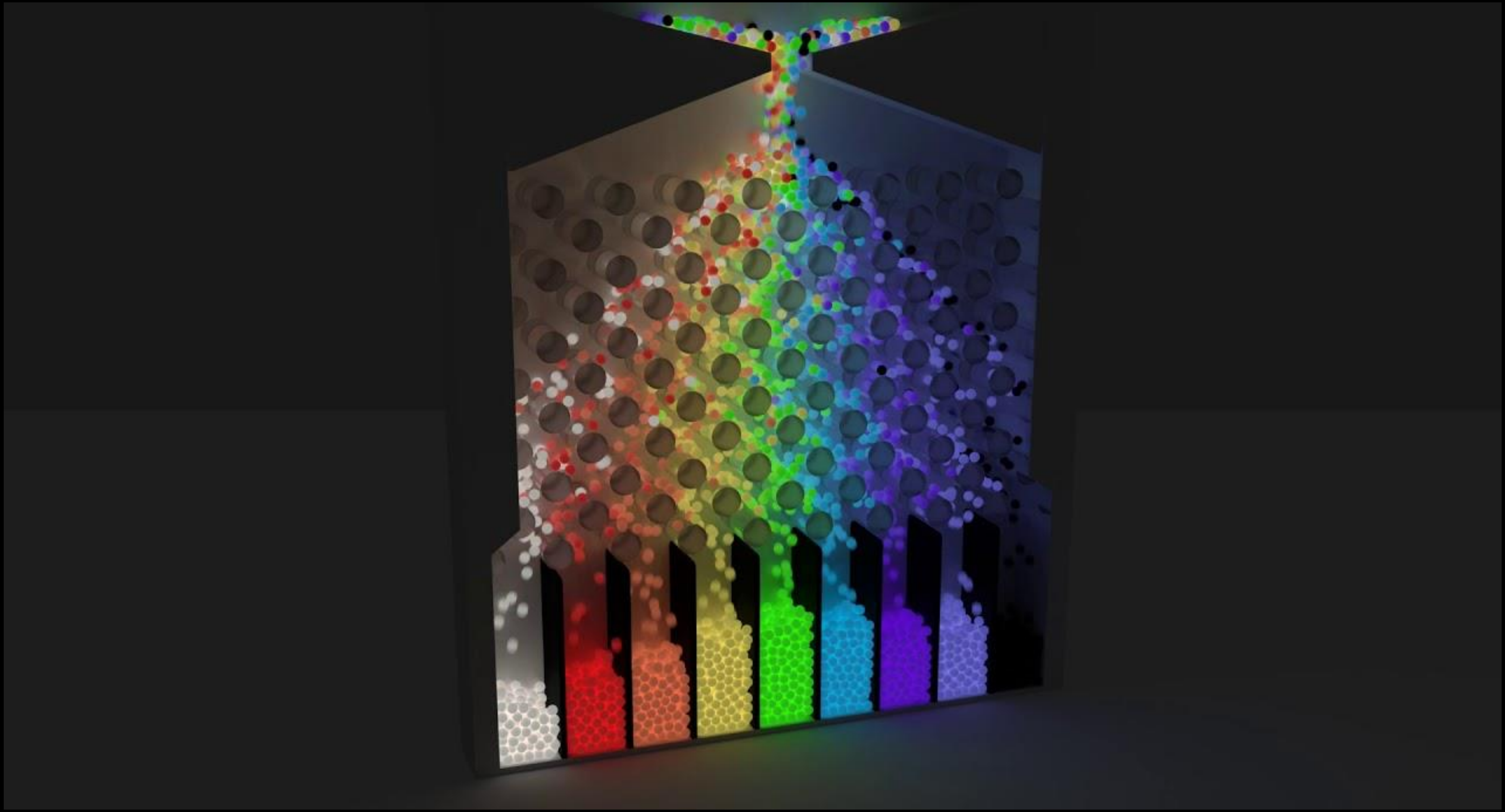
The probability of any server working is 0.8.

What is the probability that less than 2 servers are alive?

Let X be the number of servers alive. $X \sim \text{Bin}(n = 7, p = 0.8)$.

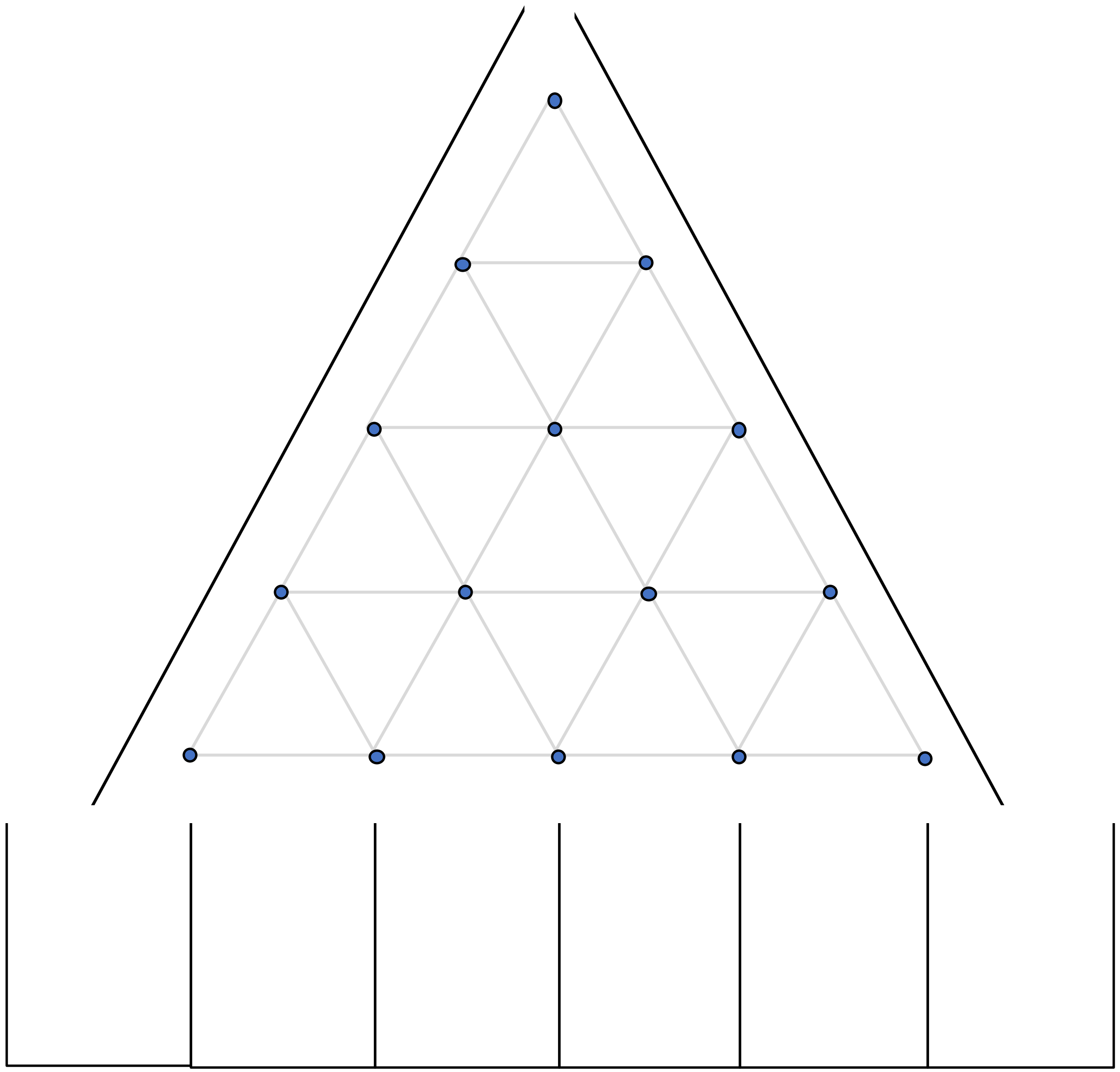
$$P(X = k) = \binom{7}{k}(0.8)^k(0.2)^{7-k}$$

$$P(X < 2) = P(X = 0) + P(X = 1) = \binom{7}{0}(0.8)^0(0.2)^{7-0} + \binom{7}{1}(0.8)^1(0.2)^{7-1} \approx 0.0004$$

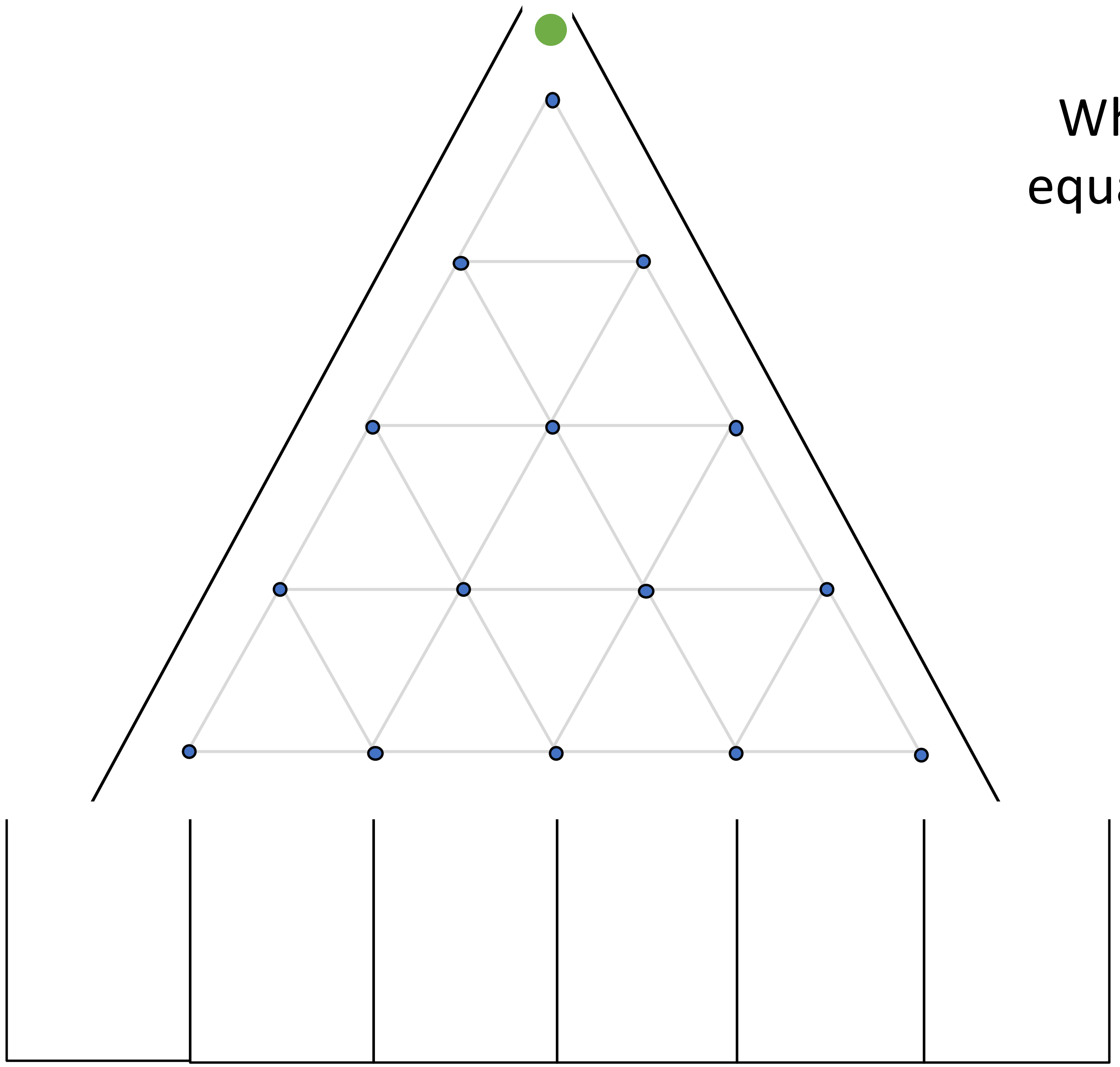


Galton Board Time!

Galton Board Fun

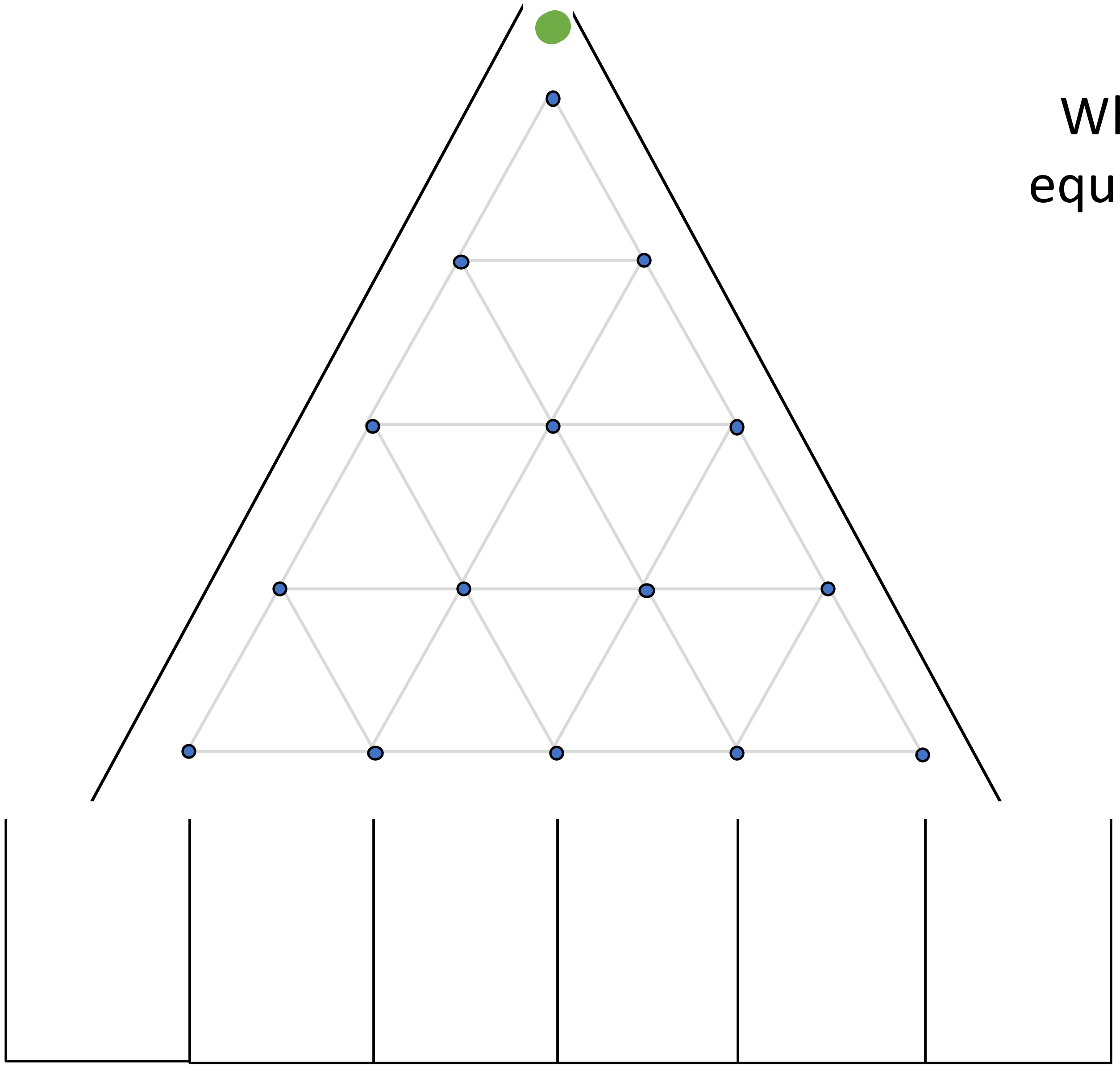


Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

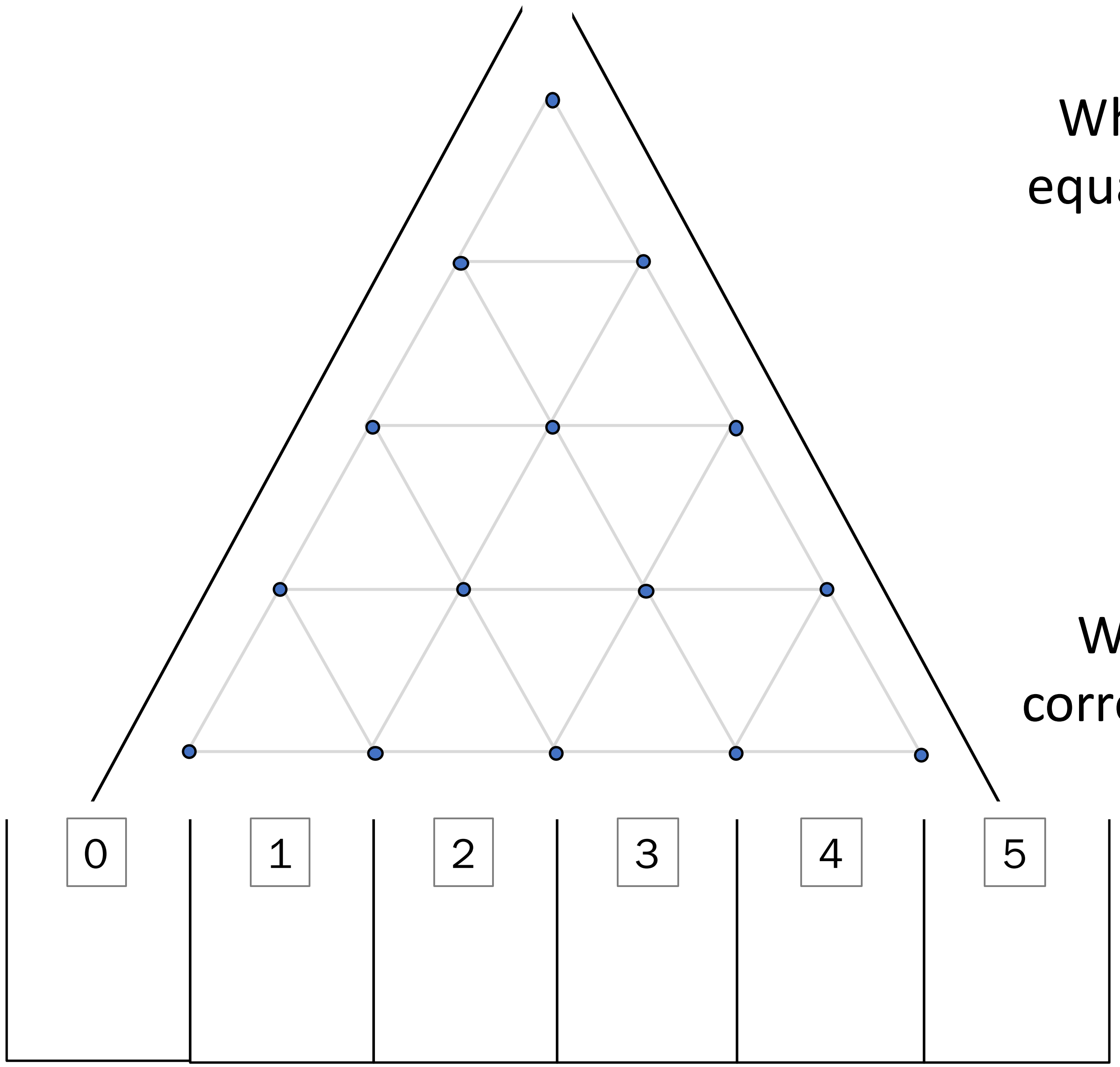
Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Which bucket a marble lands in corresponds to the number of times the marble went right.

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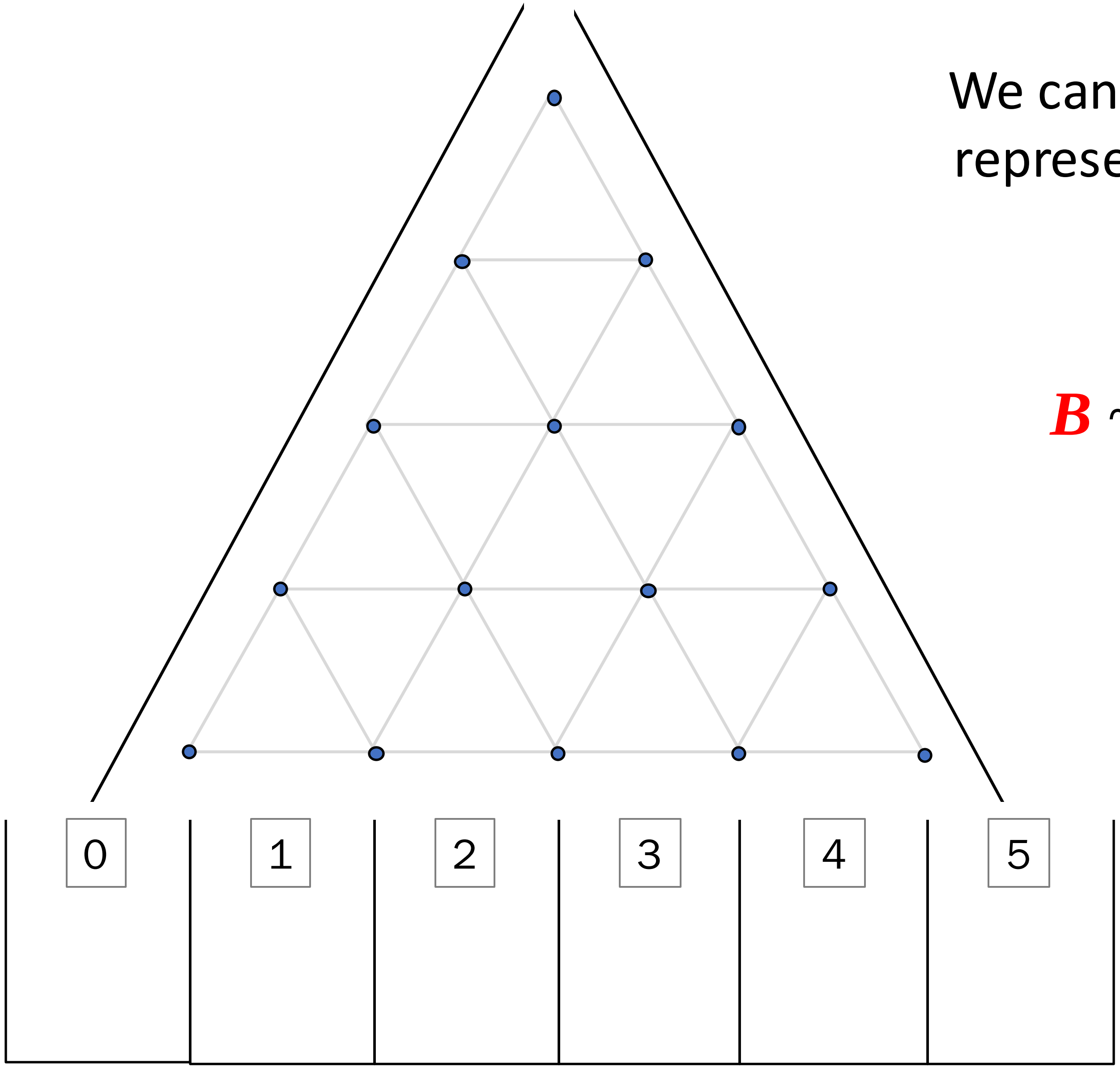

We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(\text{n} = \text{levels}, p = 0.5)$$

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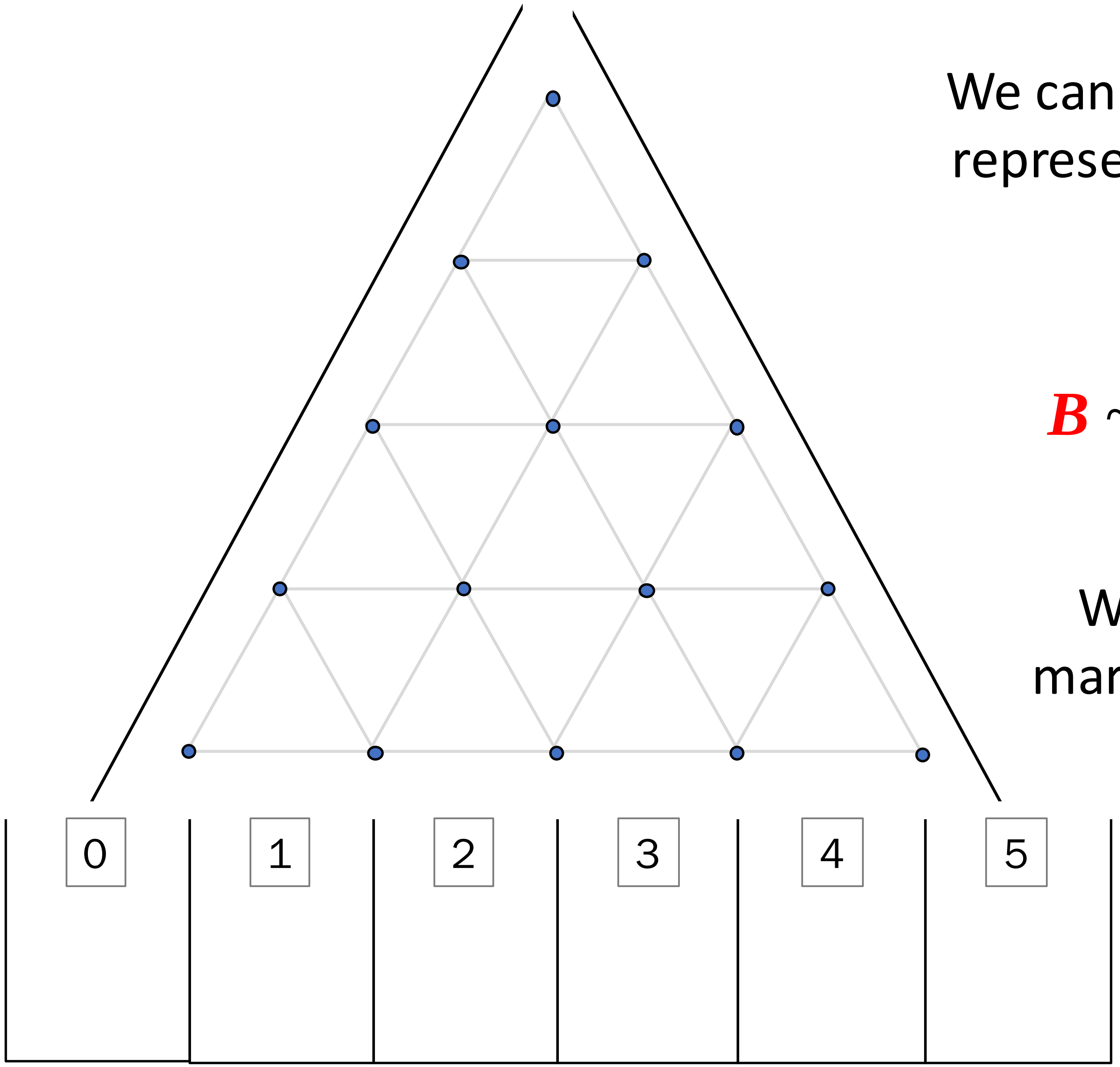

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What is the probability of a marble landing in each bucket?

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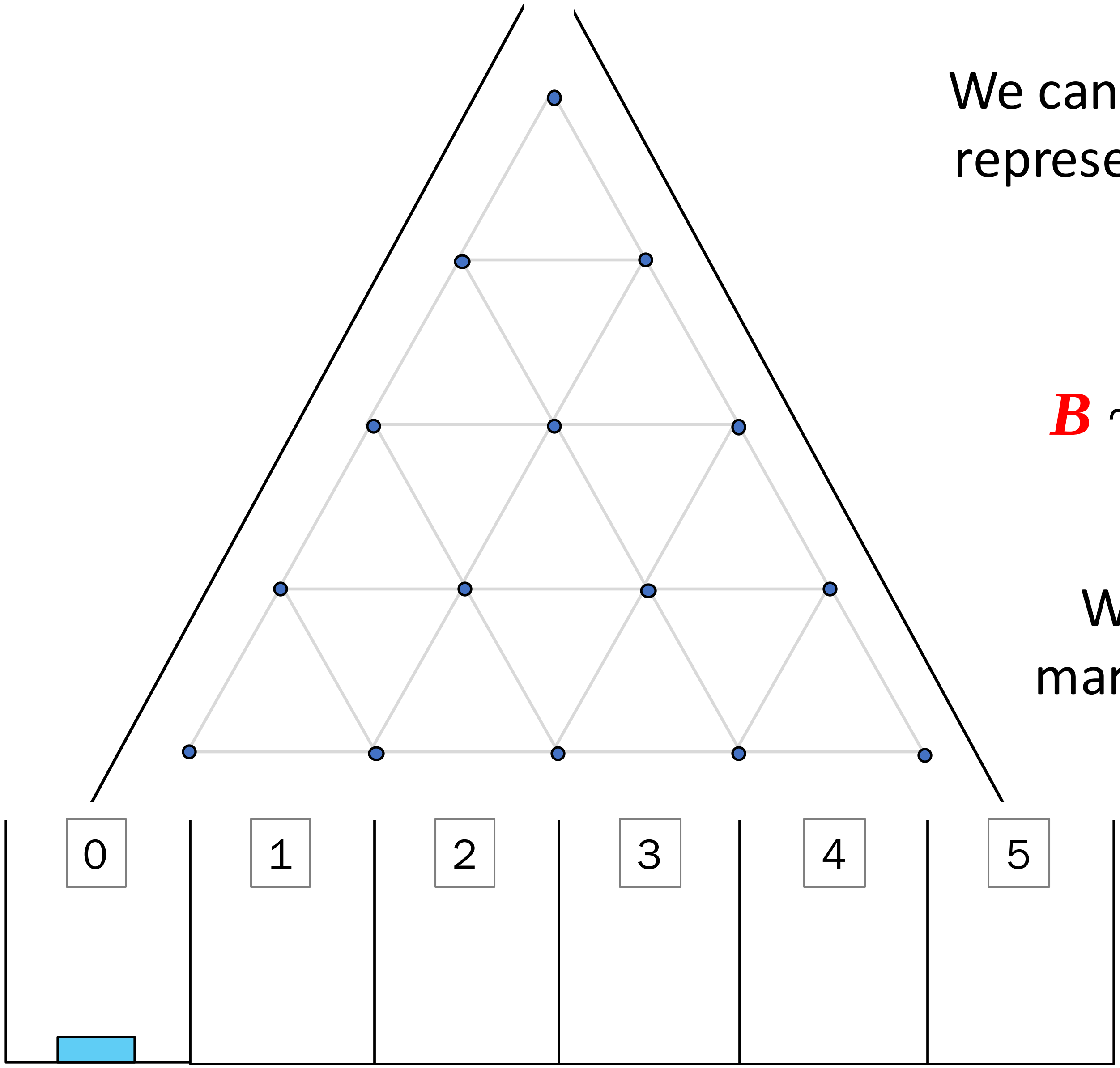

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$$P(B = 0) = \binom{5}{0}\frac{1}{2}^5 \approx 0.03$$

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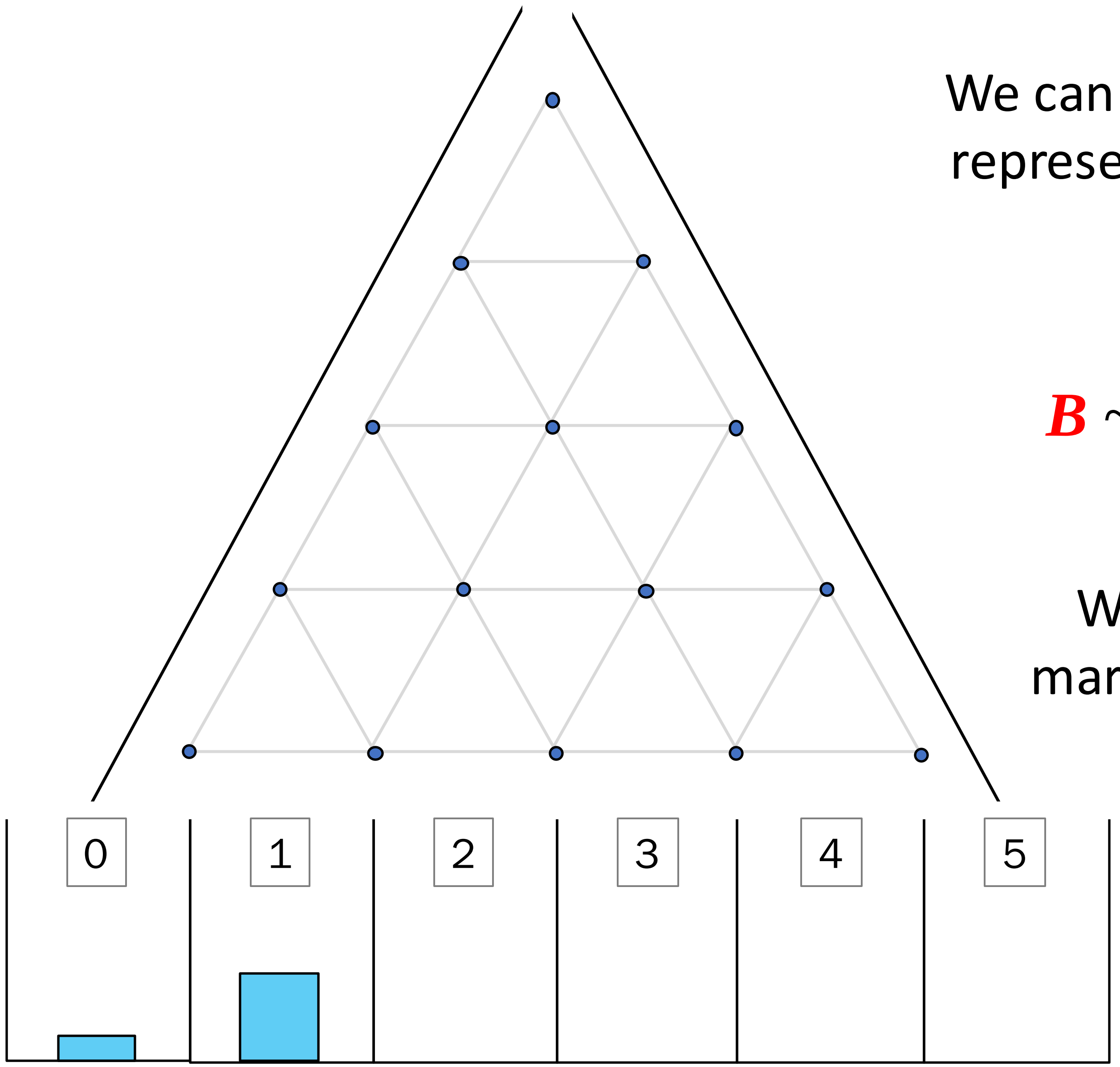

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What is the probability of a marble landing in each bucket?

$$P(B=0) = \binom{5}{0}\frac{1}{2}^5 \approx 0.03$$

$$P(B=1) = \binom{5}{1}\frac{1}{2}^5 \approx 0.16$$

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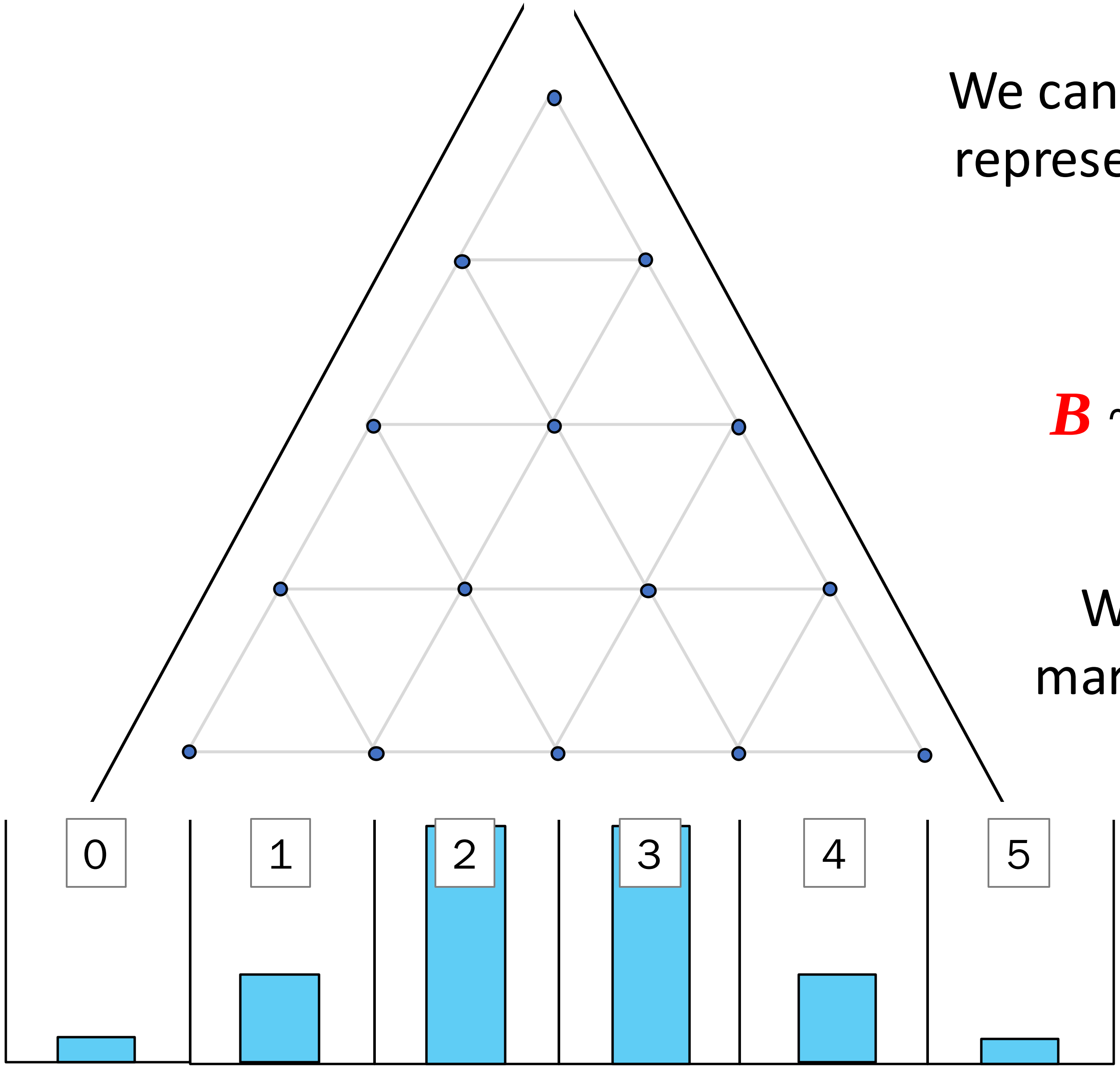

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What is the probability of a marble landing in each bucket?

This is the PMF of the binomial

From Chaos to Order

Probability is *Everywhere*

Have a wonderful weekend!