

Section 2: Random Variables

1 Warmups

1. Definitions: Cite Bayes' Theorem. Can you explain to your partner why $P(A|B)$ is different than $P(B|A)$?
2. True or False: $P(AB|C) = P(B|C)P(A|BC)$.

1. Bayes' Theorem: $P(E|F) = \frac{P(F|E)P(E)}{P(F)}$

Examples of events where $P(B|A) \neq P(A|B)$:

- Let A be the event that I'm hungry and B be the event that I just ate lunch
 - Let A be the event there is a rainbow and B be the event it rained.
 - Let A be the event of being late for work and B be the event there is heavy traffic.
 - (Feel free to add more here)
2. True. If we imagine a world where we already know that C happened, this statement is just the chain rule; we formalize being in that world by including C in what we condition on consistently for every term. You can also prove that this is true:

$P(AB C)$	Left side
$\frac{P(ABC)}{P(C)}$	Def'n Cond'n Prob
$\frac{P(A BC)P(BC)}{P(C)}$	Chain Rule
$\frac{P(A BC)P(B C)P(C)}{P(C)}$	Chain Rule
$P(A BC)P(B C)$	Cancellation
$P(B C)P(A BC)$	■

2 Conditional Probabilities: Missing Not at Random

Preamble: We have three big tools for manipulating conditional probabilities:

- Definition of conditional probability: $P(EF) = P(E|F)P(F)$
- Law of Total Probability: $P(E) = P(EF) + P(EF^C) = P(E|F)P(F) + P(E|F^C)P(F^C)$
- Bayes Rule: $P(E|F) = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{P(F|E)P(E)+P(F|E^C)P(E^C)}$

This is a good time to commit these three to memory and start thinking about when each of them is useful.

Problem: You recently tried out a new collaborative note-taking tool in class and want to find out if students like it. You send an email to all 100 people in your class, asking them to reply with whether or not they liked it.

User Response	Count
Responded that they liked your tool	40
Responded that they didn't like your tool	45
Did not respond	15

Let L be the event that a person did like your tool. Let M be the event that a person did not respond to your email. We are interested in estimating $P(L)$, however that is hard given the 15 people who did not respond.

- What is the probability that a user liked your tool and that they responded to the poll $P(L \text{ and } M^C)$?
- Which formula from class would you use to calculate $P(L)$? Your formula should rely on the context that people who like your tool are in one of two (mutually exclusive) groups: those that did reply, and those that did not.
- Calculate $P(L)$. You estimate that the probability that someone did not reply, given that they liked the tool is $P(M|L) = \frac{1}{5}$.

a. $P(L \text{ and } M^C) = \frac{40}{100}$

- b. The law of total probability. It breaks down $P(L)$ into two parts, the part which intersects with M and the part that intersections with M^C .

$$P(L) = P(L \text{ and } M) + P(L \text{ and } M^C)$$

c.

$$\begin{aligned}
 P(L) &= P(L \text{ and } M^C) + P(L \text{ and } M) && \text{Law of total probability} \\
 &= \frac{40}{100} + P(L \text{ and } M) && \text{From part a} \\
 &= \frac{40}{100} + P(M|L)P(L) && \text{Chain rule} \\
 P(L) - P(M|L)P(L) &= \frac{40}{100} && \text{The rest is algebra} \\
 P(L) \cdot [1 - P(M|L)] &= \frac{40}{100} \\
 P(L) \cdot \frac{4}{5} &= \frac{40}{100} \\
 P(L) &= \frac{40}{100} \cdot \frac{5}{4} \\
 P(L) &= \frac{1}{2}
 \end{aligned}$$

3 Taking Expectation: Breaking Vegas

Preamble: When a random variable fits neatly into a family we've seen before (e.g. Binomial), we get its expectation for free. When it does not, we have to use the definition of expectation.

Problem: If you bet on "Red" in Roulette, there is $p = 18/38$ that you will win \$Y and a $(1 - p)$ probability that you lose \$Y. Consider this algorithm for a series of bets:

Let $Y = \$1$. First you bet Y. If you win, then stop. If you lose, then set Y to be 2Y and repeat.

What are your expected winnings when you stop? It will help to recall that the sum of a geometric series $a^0 + a^1 + a^2 + \dots = \frac{1}{1-a}$ if $0 < a < 1$. Vegas breaks you: Why doesn't everyone do this?

Let X be the number of dollars that you earn.

The possible values of x are from the outcomes of: winning on your first bet, winning on your second bet, and so on.

$$\begin{aligned}
 E[X] &= \frac{18}{38} + \frac{20}{38} \frac{18}{38} (2 - 1) + \left(\frac{20}{38}\right)^2 \frac{18}{38} (4 - 2 - 1) + \dots \\
 &= \sum_{i=0}^{\infty} \left(\frac{20}{38}\right)^i \left(\frac{18}{38}\right) \left(2^i - \sum_{j=0}^{i-1} 2^j\right) \\
 &= \left(\frac{18}{38}\right) \sum_{i=0}^{\infty} \left(\frac{20}{38}\right)^i \\
 &= \left(\frac{18}{38}\right) \frac{1}{1 - \frac{20}{38}} = 1
 \end{aligned}$$

Real games have maximum bet amounts. You have finite money and casinos can kick you

out. But, if you had no betting limits and infinite money, then go for it! (and tell me which planet you are living on).

Here are some interesting numbers if you can play at most:

- 2 games, $E[X] = -0.108$
- 5 games, $E[X] = -0.292$
- 10 games, $E[X] = -0.67$
- 100 games, $E[X] = -168$
- 1000 games, $E[X] = -1.89 \cdot 10^{22}$

The code to test yourself is commented out beneath this in the LaTeX

4 Binomial Babies

Each child in a daycare has a 0.2 probability of having disease A, and has an independent 0.4 probability of having disease B. A child is sick if they have either disease A or disease B.

- a. What is the probability that a child is sick?
- b. If there are 10 children in a daycare, what is the probability that 3 or more are sick?

- a. Let A and B be the events that a child has disease A and disease B, respectively. A child is healthy if they have neither disease A nor disease B. So,

$$\begin{aligned}
 P(\text{sick}) &= 1 - P(\text{healthy}) \\
 &= 1 - P(A^C, B^C) \\
 &= 1 - P(A^C)P(B^C) && (A \perp B) \\
 &= 1 - (1 - P(A))(1 - P(B)) \\
 &= 1 - (0.8)(0.6) \\
 &= 1 - (0.48) \\
 &= 0.52
 \end{aligned}$$

Alternate solution: A child is considered sick if they have a disease. A sick child may have disease A, disease B, or both. Note that a sick child can have both diseases because disease A and disease B are independent events, which by definition means that disease A and B are not mutually exclusive. (Note: Independent events cannot be mutually exclusive, and mutually exclusive events cannot be independent. For example, when events A and B are independent, knowing that event A has happened

does not change our belief in B (i.e. $P(B|A) = P(B)$). On the other hand, when events A and B are mutually exclusive, knowing that event A happened does change our belief in B (i.e. $P(B|A) \neq P(B)$. If A has happened, we are now certain that event B has not happened because mutually exclusive events do not share any outcomes.)

Thus, we can use the Inclusion-Exclusion Principle to find $P(\text{sick})$. Because a sick child can have both diseases, the Inclusion-Exclusion Principle will correct for over-counted sick children who have both diseases:

$$\begin{aligned}
 P(\text{sick}) &= P(A) + P(B) - P(AB) \\
 &= P(A) + P(B) - (P(A) \cdot P(B)) && \text{Definition of Independence} \\
 &= (0.2) + (0.4) - (0.4 \cdot 0.2) \\
 &= (0.2) + (0.4) - (0.08) \\
 &= 0.52
 \end{aligned}$$

- b. Let Y be the number of children that are sick. We can write this as $Y \sim \text{Bin}(10, P(\text{sick}))$. Thus, we have

$$\begin{aligned}
 P(Y \geq 3) &= 1 - P(Y < 3) \\
 &= 1 - \sum_{k=0}^2 \binom{10}{k} (0.52)^k (1 - 0.52)^{10-k}.
 \end{aligned}$$