08: Poisson Distributions

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The Week So Far

$X \sim \text{Ber}(p)$

indicates whether a single trial succeeds or not

$$P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases} \qquad E[X] = p$$

$X \sim \text{Geo}(p)$

count **independent** trials until first success

$$P(X = k) = (1 - p)^{k-1} p \qquad E[X] = \frac{1}{p}$$

$Y \sim \text{Bin}(n, p)$

counts successes in n independent trials

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k} \qquad E[X] = np$$

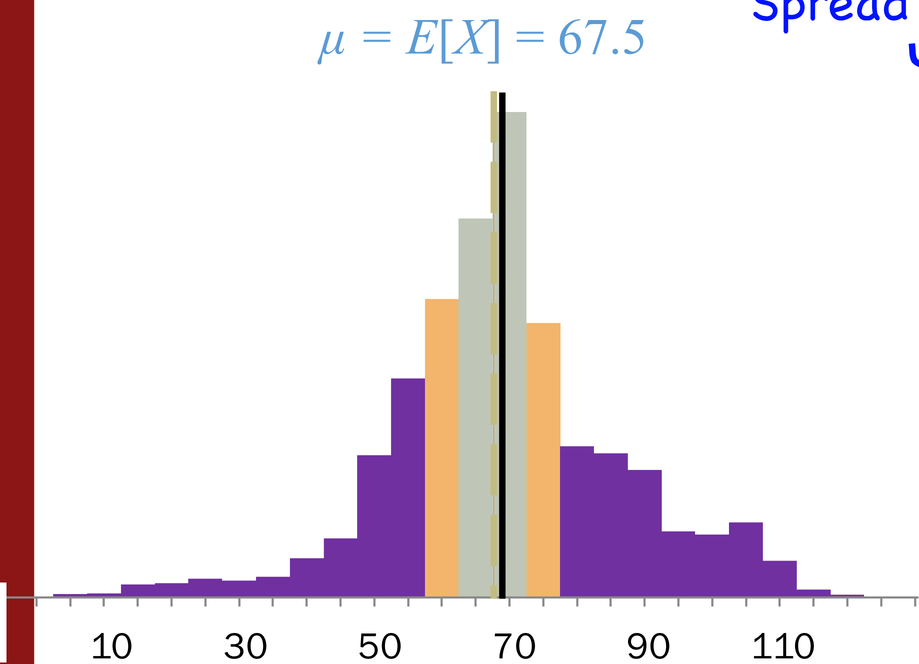
$Y \sim \text{NegBin}(r, p)$

count **independent** trials until r^{th} success

$$P(Y = k) = \binom{k-1}{r-1} (1 - p)^{k-r} p^r \qquad E[X] = \frac{r}{p}$$

How Should We Measure Spread? Variance

Let X be a random variable!



Spread stat. On average.. square of distance

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

The random variable X

The mean of X

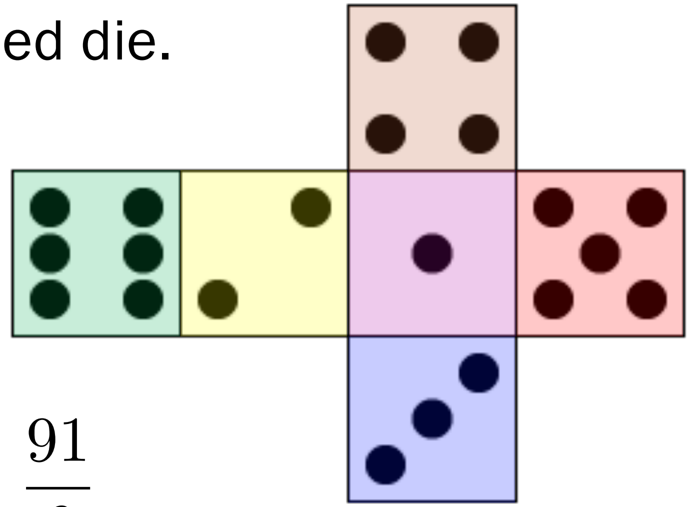
more commonly used

Property 1	$\mathrm{Var}(X) = E[X^2] - E[X]^2$
Property 2	$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Example: Variance of a Die Roll

Let X be the result of rolling a traditional six-sided die.

What is Var(X)?



$$E[X] = 3.5$$

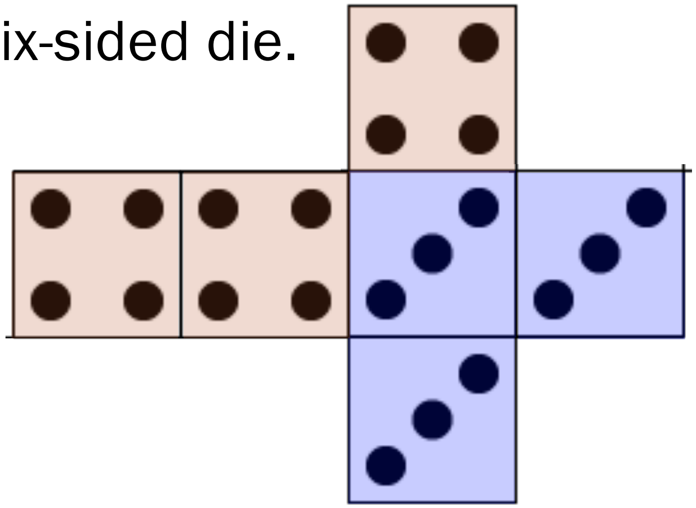
$$E[X^2] = 1^2\frac{1}{6} + 2^2\frac{1}{6} + 3^2\frac{1}{6} + 4^2\frac{1}{6} + 5^2\frac{1}{6} = \frac{91}{6}$$

$$\begin{aligned}\mathrm{Var}(X) &= E[X^2] - E[X]^2 \\ &= \frac{91}{6} - (3.5)^2 = 2.91\end{aligned}$$

Example: Variance of a Weird Die Roll

Let X be the result of rolling this very strange six-sided die.

What is Var(X) now?



$$E[X] = 3.5$$

$$E[X^2] = 3^2 \cdot \frac{3}{6} + 4^2 \cdot \frac{3}{6} = 12.5$$

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 \\ &= 12.5 - (3.5)^2 = 0.25 \end{aligned}$$

You Get So Much For Free!

Binomial Random Variable

Notation: $X \sim \text{Bin}(n, p)$

Description: Number of "successes" in n identical, independent experiments each with probability of success p.

Parameters: $n \in \{0, 1, \ldots\}$, the number of experiments.
$p \in [0, 1]$, the probability that a single experiment gives a "success".

Support: $x \in \{0, 1, \ldots, n\}$

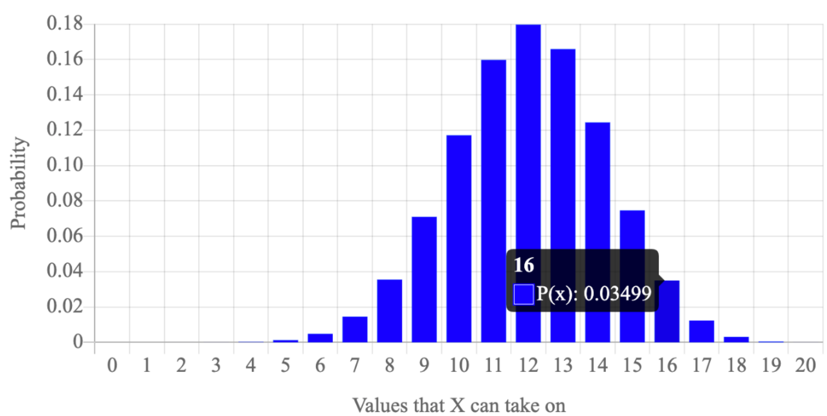
PMF equation: $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

Expectation: $E[X] = n \cdot p$

Variance: $\text{Var}(X) = n \cdot p \cdot (1-p)$

PMF graph:

Parameter n: 20 Parameter p: 0.60



Bernoulli Random Variable

Notation: $X \sim \text{Bern}(p)$

Description: A boolean variable that is 1 with probability p

Parameters: p, the probability that $X = 1$.

Support: x is either 0 or 1

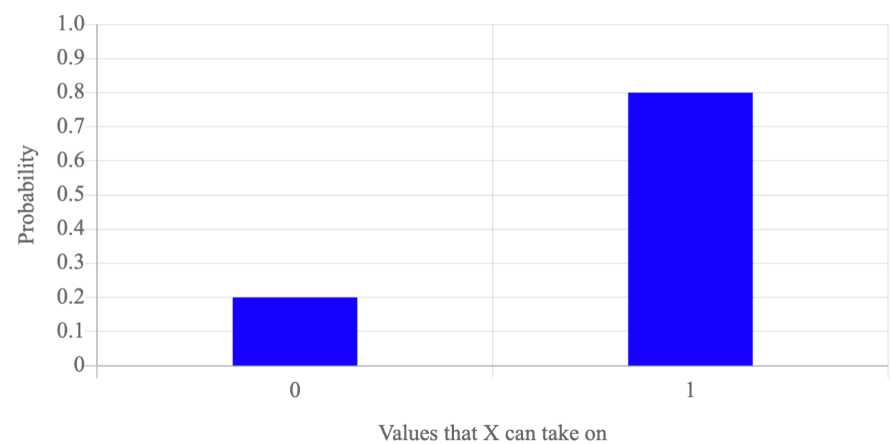
PMF equation: $\Pr(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$

Expectation: $E[X] = p$

Variance: $\text{Var}(X) = p(1-p)$

PMF graph:

Parameter p: 0.80



Curious? Proof of Variance for a Binomial

$E\left(X^2\right) = \sum_{k\geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k}$	Definition of Binomial Distribution: $p+q=1$
$= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k}$	Factors of Binomial Coefficient: $k\binom{n}{k} = n\binom{n-1}{k-1}$
$= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)}$	Change of limit: term is zero when $k-1=0$
$= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j}$	putting $j=k-1, m=n-1$
$= np \left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right)$	splitting sum up into two
$= np \left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right)$	Factors of Binomial Coefficient: $j\binom{m}{j} = m\binom{m-1}{j-1}$
$= np \left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right)$	Change of limit: term is zero when $j-1=0$
$= np\left((n-1)p(p+q)^{m-1} + (p+q)^m\right)$	Binomial Theorem
$= np\left((n-1)p+1\right)$	as $p+q=1$
$= n^2p^2 + np(1-p)$	by algebra

Happy Family of Random Variables

$$X \sim \text{Ber}(p)$$

$$P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

$$E[X] = p, \text{Var}(X) = p(1-p)$$

$$X \sim \text{Geo}(p)$$

$$P(X = k) = (1-p)^{k-1}p$$

$$E[X] = \frac{1}{p}, \text{Var}(X) = \frac{1-p}{p^2}$$

$$Y \sim \text{Bin}(n, p)$$

$$P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

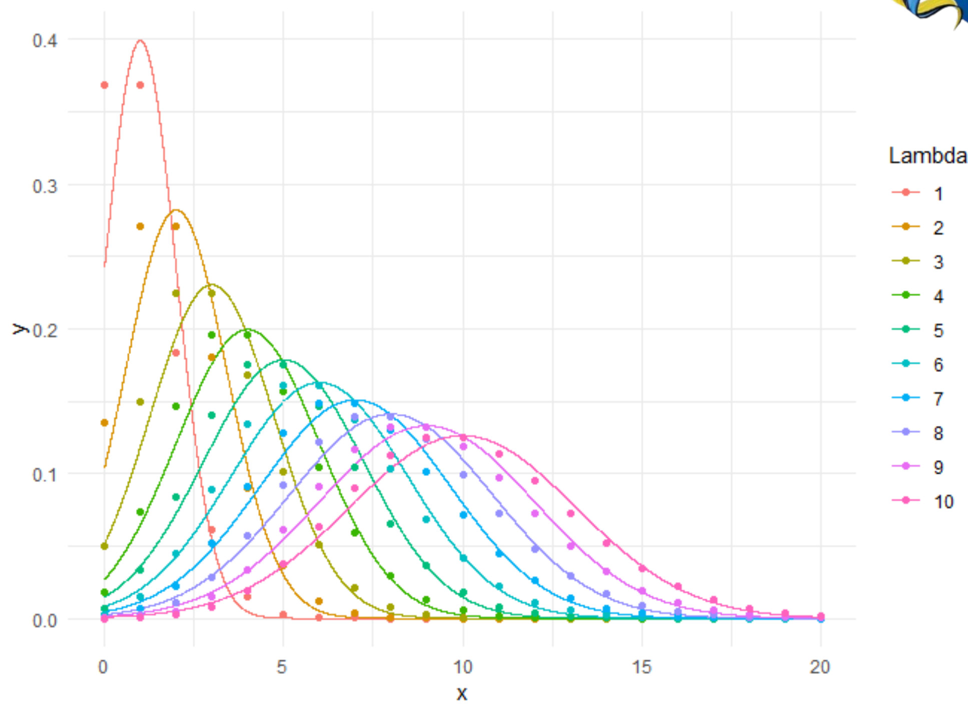
$$E[X] = np, \text{Var}(X) = np(1-p)$$

$$Y \sim \text{NegBin}(r, p)$$

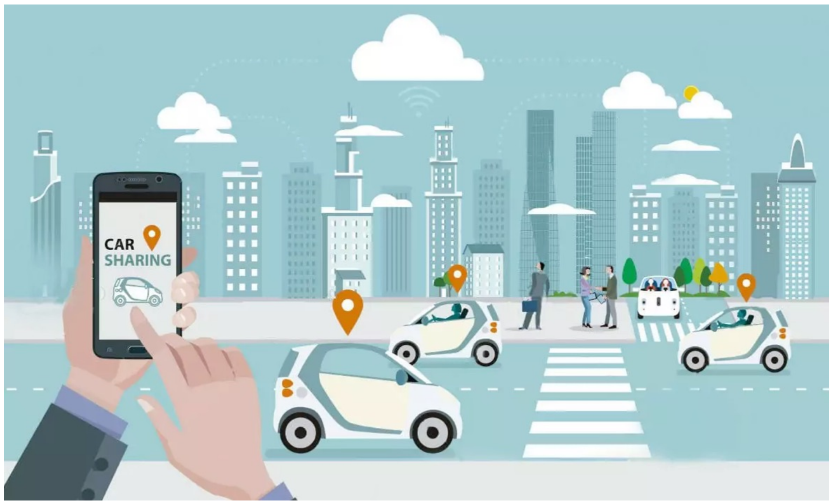
$$P(Y = k) = \binom{k-1}{r-1} (1-p)^{k-r} p^r$$

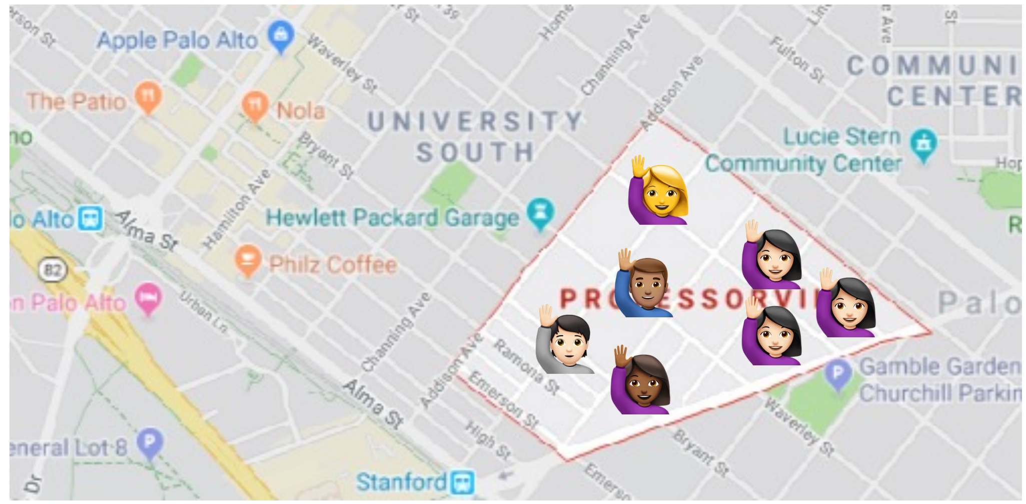
$$E[X] = \frac{r}{p}, \text{Var}(X) = \frac{r(1-p)}{p^2}$$

Poisson



Algorithmic ride sharing





What's the probability of getting exactly k requests in the next minute?

On average, $\lambda = 5$ requests per minute

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Question: Why did we choose $p = 5/60$?

Break a minute down into 60 seconds:

0	0	1	0	1	…	0	0	0	0	1
1	2	3	4	5						60

At each second:

- Independent Bernoulli trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n = 60,\ p = 5/60)$$

$$P(X = k) = \binom{60}{k}\left(\frac{5}{60}\right)^k \left(1 - \frac{5}{60}\right)^{n-k}$$

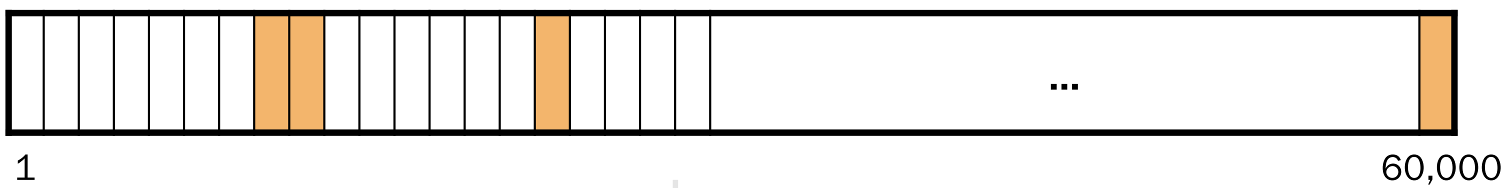
But what if there are *two* requests in the same second?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60,000 milliseconds:



At each millisecond:

- Independent Bernoulli trial
- You get a request (1) or you don't (0).

Let $X = \#$ of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n = 60000,\ p = 5/60000)$$

$$P(X = k) = \binom{60{,}000}{k}\left(\frac{5}{60{,}000}\right)^{k}\left(1 - \frac{5}{60{,}000}\right)^{60000-k}$$

But what if there are *two* requests in the same millisecond?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into infinitely small buckets:

omg so small

1 ∞

For each time bucket:

- Independent Bernoulli trial
- You get a request (1) or you don't (0).

Let $X = \#$ of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n,\ p = \lambda/n)$$

$$P(X = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Gnarly math incoming!

Binomial in the limit

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$P(X=k) = \lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

Expand

$$= \lim_{n\to\infty}\frac{n!}{k!(n-k)!}\frac{\lambda^k}{n^k}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Rearrange

$$= \lim_{n\to\infty}\frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Def natural exponent

$$= \lim_{n\to\infty}\frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{(n-k)!}{(n-k)!}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

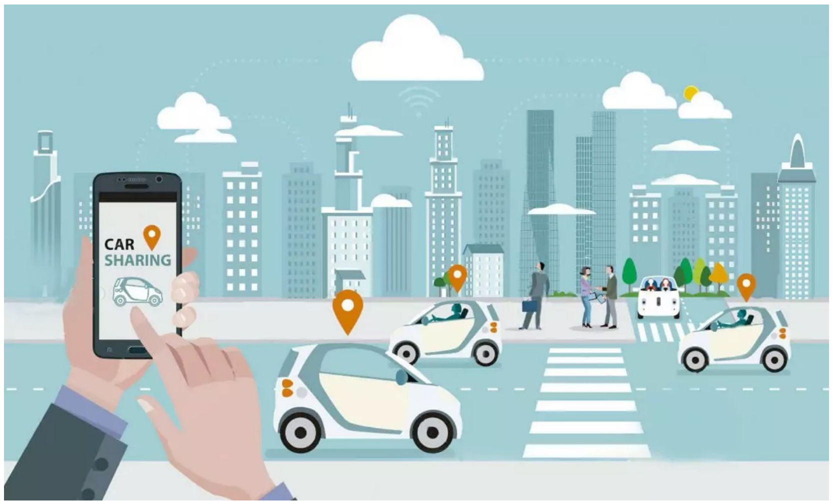
Limit analysis + cancel

$$= \lim_{n\to\infty}\frac{n^k}{n^k}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{1}$$

Simplify

$$= \frac{\lambda^k}{k!}e^{-\lambda}$$

Algorithmic ride sharing



What's the probability of getting exactly k requests in the next minute?

On average, $\lambda = 5$ requests per minute

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson distribution

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of successes over the experiment duration, assuming **the time that each success occurs is independent** and the average # of requests over time is constant.

$X \sim \text{Poi}(\lambda)$

Support: $\{0, 1, 2, \ldots\}$

PMF $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$

Expectation $E[X] = \lambda$

Variance $\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Yes, expectation == variance for Poisson RV!

Earthquakes

$X \sim \text{Poi}(\lambda)$
$E[X] = \lambda$
$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$

There are an average of 2.79 major earthquakes in the world each year, and major earthquakes occur independently.

What's the probability there are exactly 3 major earthquakes next year?

1. Define RVs

X: # major earthquakes in a year

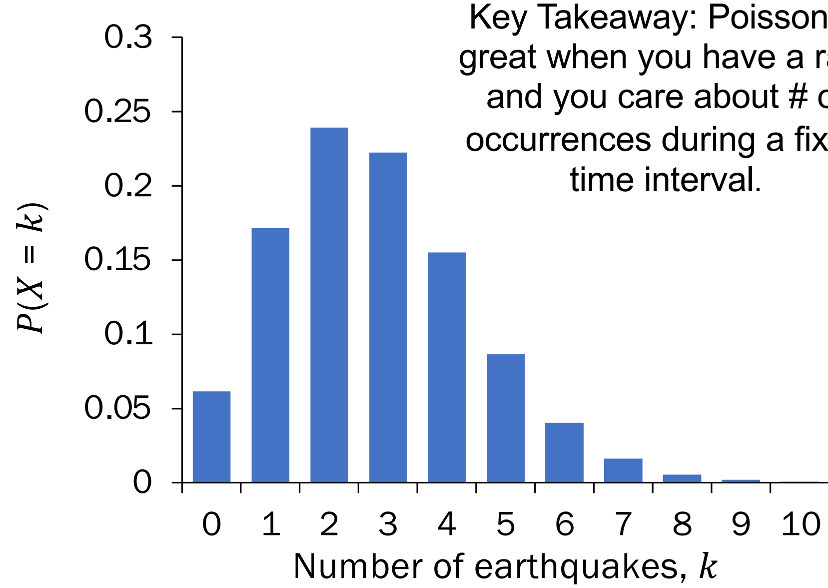
$$X \sim \text{Poi}(\lambda = 2.79)$$

2. Solve

Want $P(X = 3)$

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$P(X = 3) = e^{-2.79} \frac{2.79^3}{3!} \approx 0.2223$$



Key Takeaway: Poisson is great when you have a rate and you care about # of occurrences during a fixed time interval.

Exercises

The hardest part of is almost always deciding what you're modeling and what random variable to use.

Kickboxing with RVs

Choose from:
A. $\text{Ber}(p)$
B. $\text{Bin}(n, p)$
C. $\text{Poi}(\lambda)$
D. $\text{Geo}(p)$
E. $\text{NegBin}(r, p)$

How might you model the following?

1. # of Reels you receive in a day — C. $\text{Poi}(\lambda)$
2. # of children born to the same parents until a baby with green eyes is born — D. $\text{Geo}(p)$ or E. $\text{NegBin}(1, p)$
3. If stock ended up higher (1) or lower (0) in any given day — A. $\text{Ber}(p)$ or B. $\text{Bin}(1, p)$
4. # of probability problems you complete until you get 5 correct (if randomly correct) — E. $\text{NegBin}(r = 5, p)$
5. # of years between now and 2050, inclusive, with 6+ Atlantic hurricanes — B. $\text{Bin}(n = 26, p)$, where $p = P(\geq 6 \text{ hurricanes in a year})$ calculated from C. $\text{Poi}(\lambda)$

These exercises are designed to build intuition. In a problem statement, you'll be provided with more detail.

Poisson Random Variable

Review

$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0, 1, 2, \dots\}$	Variance	$\text{Var}(X) = \lambda$

In CS109, a Poisson RV $X \sim \text{Poi}(\lambda)$ most often models

1. # of successes in a fixed interval of time, where successes are independent $\lambda = E[X]$, average success/interval

Web server load

$X \sim \text{Poi}(\lambda)$
$E[X] = \lambda$
$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$

Consider requests to a web server in 1 second.

- A server averages 10 hits/second (and requests arrive independently)
- Let X = # requests the server receives in a second

What is $P(X < 5)$?

Desktop — ssh -Y poohbear@myth.stanford.edu — 80×5

```
>>>
>>> lamb = 10
>>> sum([math.e ** (-lamb) * lamb ** k / math.factorial(k) for k in range(5)])
0.02925268807696109
>>>
```

Define RVs

X: # server hits in a second

$$X \sim \text{Poi}(\lambda = 10)$$

Want $P(X < 5)$

Solve

$$P(X < 5) = \sum_{k=0}^{4} P(X = k)$$

$$= e^{-10} \sum_{k=0}^{4} \frac{10^k}{k!} \approx 0.0293$$

Lollipops in Class!

Students ask on average **15 questions per class**. Each interrogator gets a lollipop!

Jerry only brought **10 lollipops**!

What is the probability he has enough lollipops?

* Assume: (a) question rate is constant (b) questions don't impact one another.

Let X be the number of questions asked in class. $X \sim \text{Poi}(\lambda = 15)$

$$P(X \leq 10) = \sum_{i=0}^{10} P(X = i)$$

$$= \sum_{i=0}^{10} \frac{\lambda^i e^{-\lambda}}{i!}$$ PMF of Poisson

$$= \sum_{i=0}^{10} \frac{15^i e^{-15}}{i!}$$ $\lambda = 15$

```
from scipy.stats import poisson
def main():
  lamb = int(input("Questions per class: "))
  num_lols = int(input("Number of lollipops: "))
  X = poisson(lamb)
  prob_enough = 0
  for i in range(0, num_lols + 1):
    pr_i_questions = X.pmf(i)
    prob_enough += pr_i_questions
  print(prob_enough)
```

Poisson in Python

```
from scipy import stats # great package
X = stats.poisson(2.5)  # X ~ Poi(λ = 2.5)
print(X.pmf(2))         # P(X = 2)
```

Function	Description
`X.pmf(k)`	$P(X = k)$
`X.cdf(k)`	$P(X \leq k)$
`X.mean()`	$E[X]$
`X.var()`	$\text{Var}(X)$
`X.std()`	$\text{Std}(X)$

Poisson Random Variable

$X \sim \text{Poi}(\lambda)$

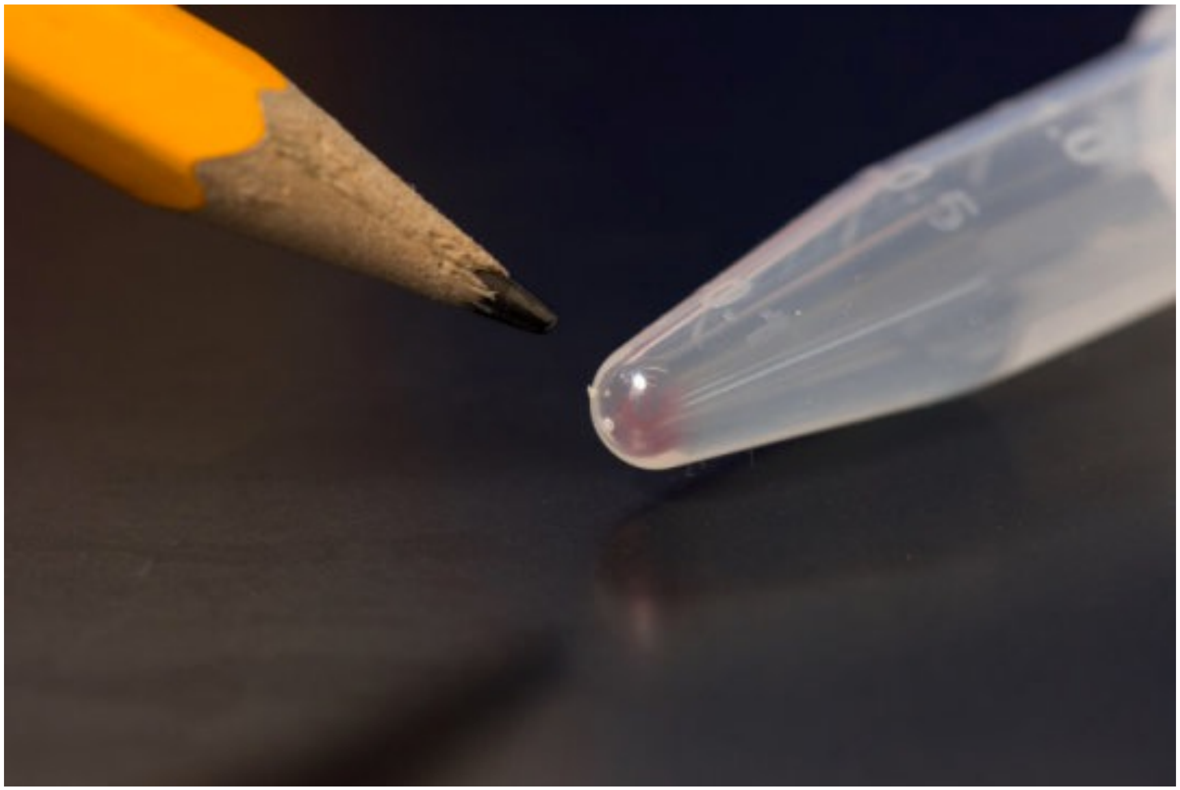
Support: $\{0, 1, 2, \ldots\}$

PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
Expectation	$E[X] = \lambda$
Variance	$\text{Var}(X) = \lambda$

In CS109, a Poisson RV $X \sim \text{Poi}(\lambda)$ most often models

1. # of successes in a fixed time interval, where successes are independent $\lambda = E[X]$, average success/interval
2. Approximation of $Y \sim \text{Bin}(n, p)$ where n is large and p is small.
   $\lambda = E[Y] = np$
   Approximation works well even when trials not entirely independent.

DNA



All the movies, images, emails and other digital data from more than 600 smartphones (10,000 GB) can be stored in the faint pink smear of DNA at the end of this test tube.

What is the probability that a DNA strand is uncorrupted?

DNA

What is the probability that DNA storage stays uncorrupted?

- In DNA (and real networks), we store large strings.
- Let string length be long, e.g., $n \approx 10^4$
- Probability of corruption of each base pair is very small, e.g., $p = 10^{-6}$
- Let $X =$ # of corruptions.

What is P(DNA storage is uncorrupted) = $P(X = 0)$?

1. Approach 1:

$$X \sim \text{Bin}(n = 10^4, p = 10^{-6})$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

unwieldy! ⚠️ $= \binom{10^4}{0} 10^{-6 \cdot 0} (1 - 10^{-6})^{10^4 - 0}$

≈ 0.990049829

2. Approach 2:

$$X \sim \text{Poi}(\lambda = 10^4 \cdot 10^{-6} = 0.01)$$

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} = e^{-0.01} \frac{0.01^0}{0!}$$

$$= e^{-0.01}$$

≈ 0.990049834 a good approximation! ✅

When is a Poisson approximation appropriate?

$$P(X = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \cdots$$

Def natural exponent

$$= \lim_{n\to\infty} \frac{n!}{n^k (n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1 - \frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(n-k)!}{(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1 - \frac{\lambda}{n}\right)^k}$$

Limit analysis

$$= \lim_{n\to\infty} \frac{n^k}{n^k} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{1}$$

Simplify

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

Under which conditions will $X \sim \text{Bin}(n, p)$ behave like $\text{Poi}(\lambda)$, where $\lambda = np$?

A. Large n, large p
B. Small n, small p
C. Large n, small p
D. Small n, large p
E. Other

Poisson approximation

$X \sim \text{Poi}(\lambda)$
$E[X] = \lambda$

$Y \sim \text{Bin}(n, p)$
$E[Y] = np$

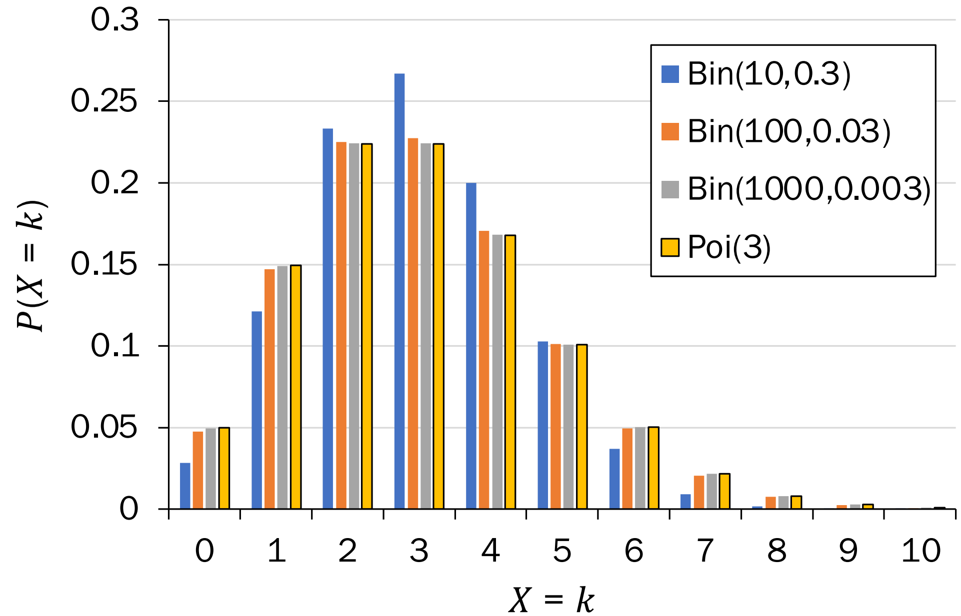
Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate".

Different interpretations of "moderate":

- $n > 20$ and $p < 0.05$
- $n > 100$ and $p < 0.1$

Poisson is Binomial in the limit:

- $\lambda = np$, where $n \to \infty, p \to 0$



Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of occurrences over the experiment duration.

$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0,1,2,\ldots\}$	Variance	$\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Time to show intuition for why expectation == variance!

Properties of Poi(λ) with the Poisson paradigm

Recall the Binomial:

$Y \sim \text{Bin}(n, p)$	Expectation	$E[Y] = np$
	Variance	$\text{Var}(Y) = np(1-p)$

Consider $X \sim \text{Poi}(\lambda)$, where $\lambda = np$ ($n \to \infty, p \to 0$):

$X \sim \text{Poi}(\lambda)$	Expectation	$E[X] = \lambda$
	Variance	$\text{Var}(X) = \lambda$

Proof:

$$E[X] = np = \lambda$$
$$\text{Var}(X) = np(1-p) \to \lambda(1-0) = \lambda$$

Can these Binomial RVs be approximated?

Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate".

Different interpretations of "moderate":

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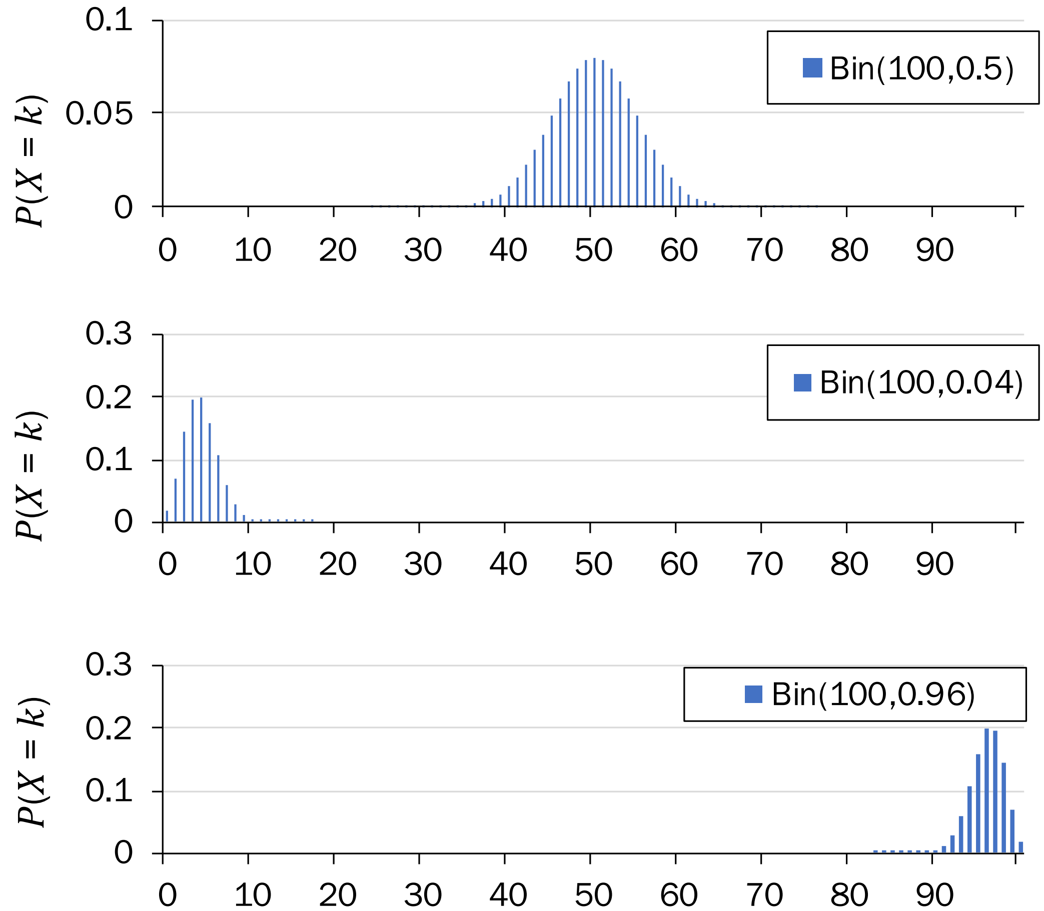

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