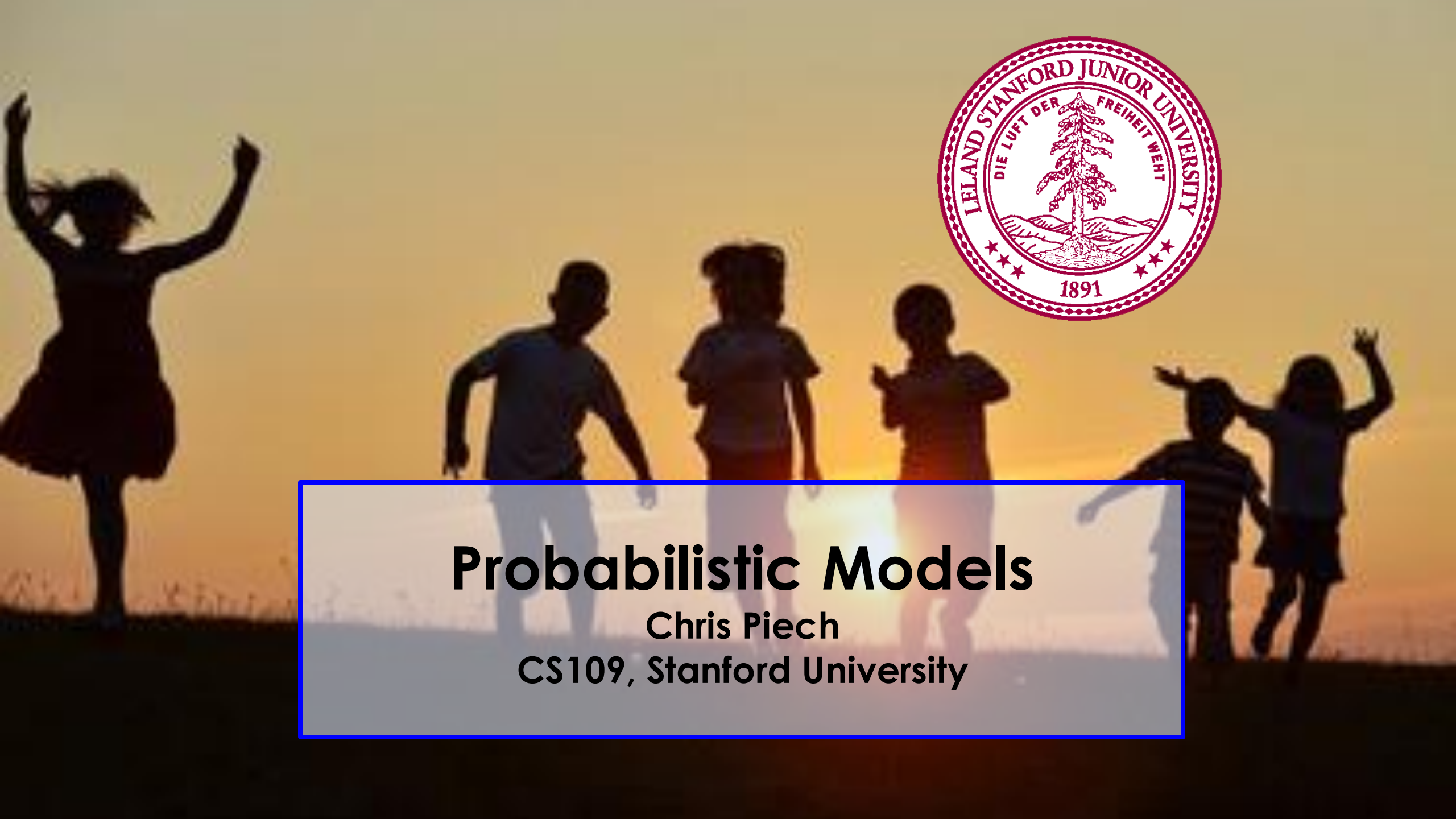
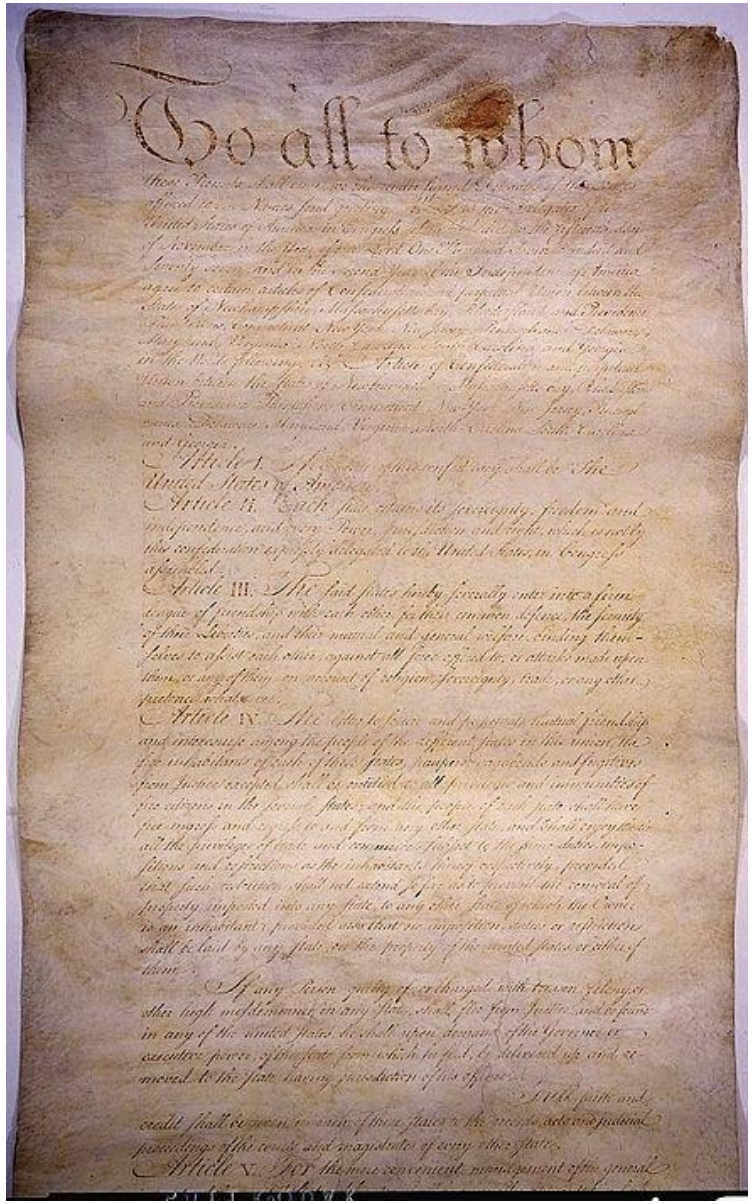
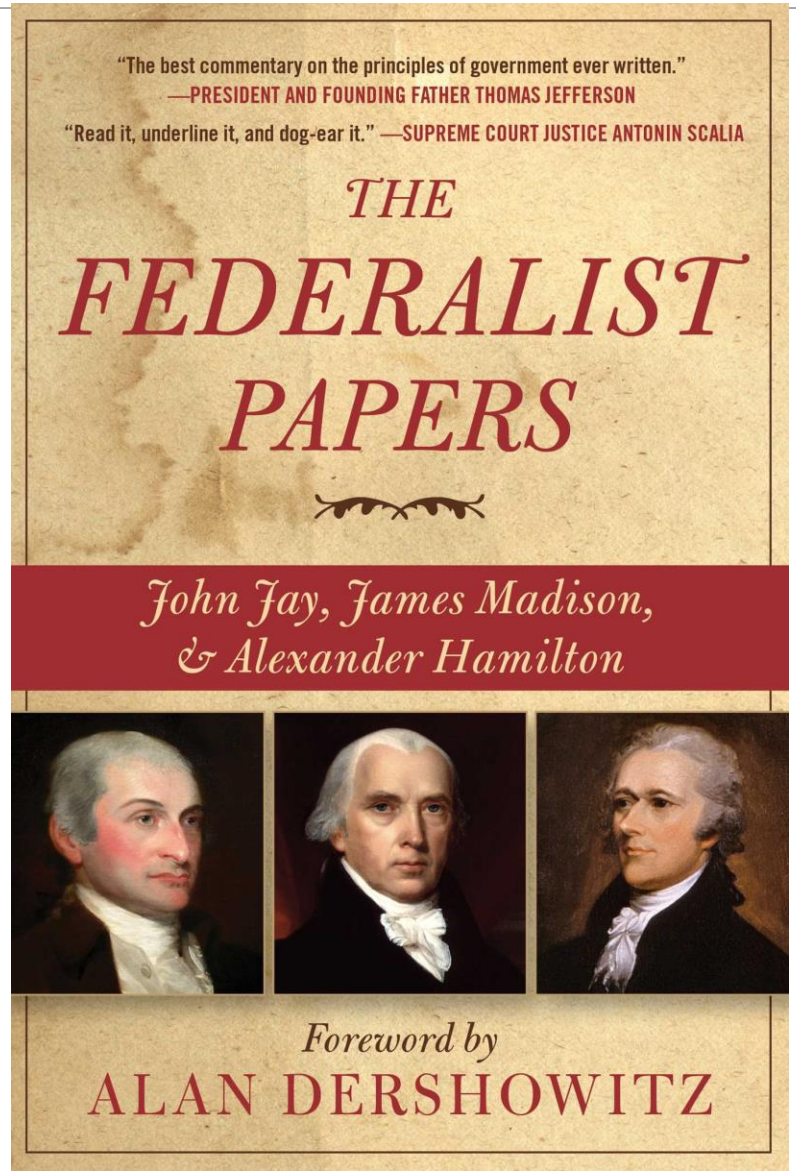


Probabilistic Models

Chris Piech
CS109, Stanford University

Exciting Day!

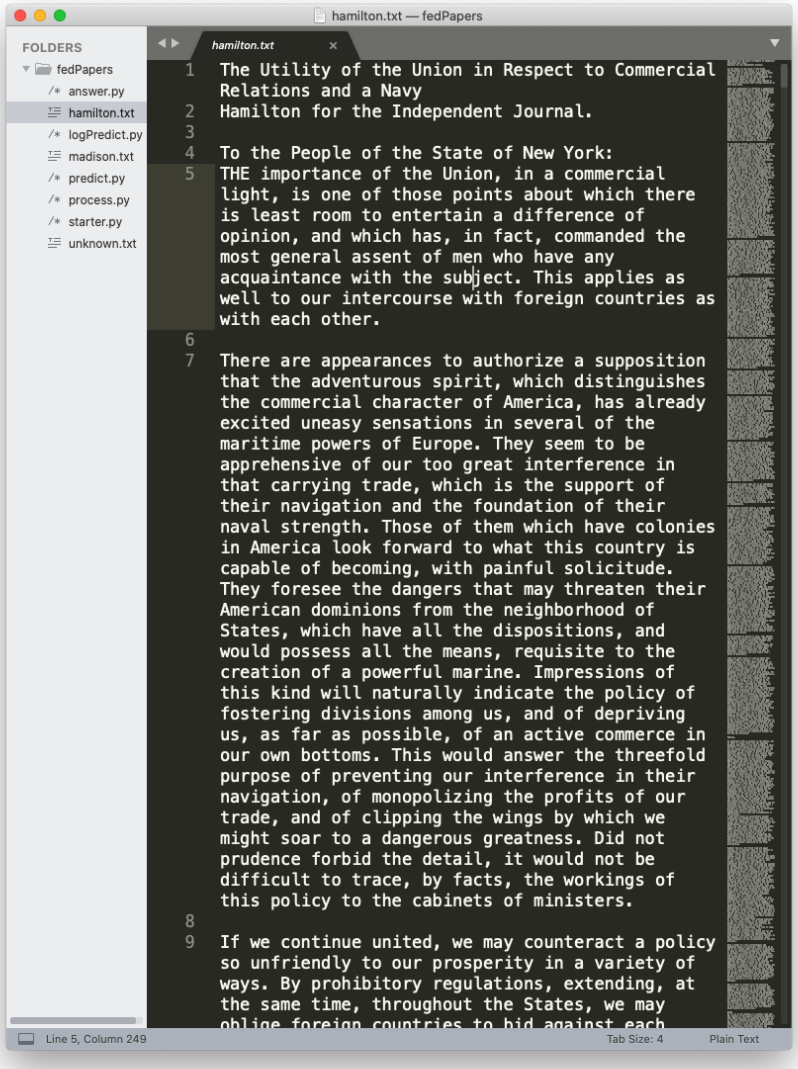




Who wrote Federalist Paper 53?

madison.txt

hamilton.txt



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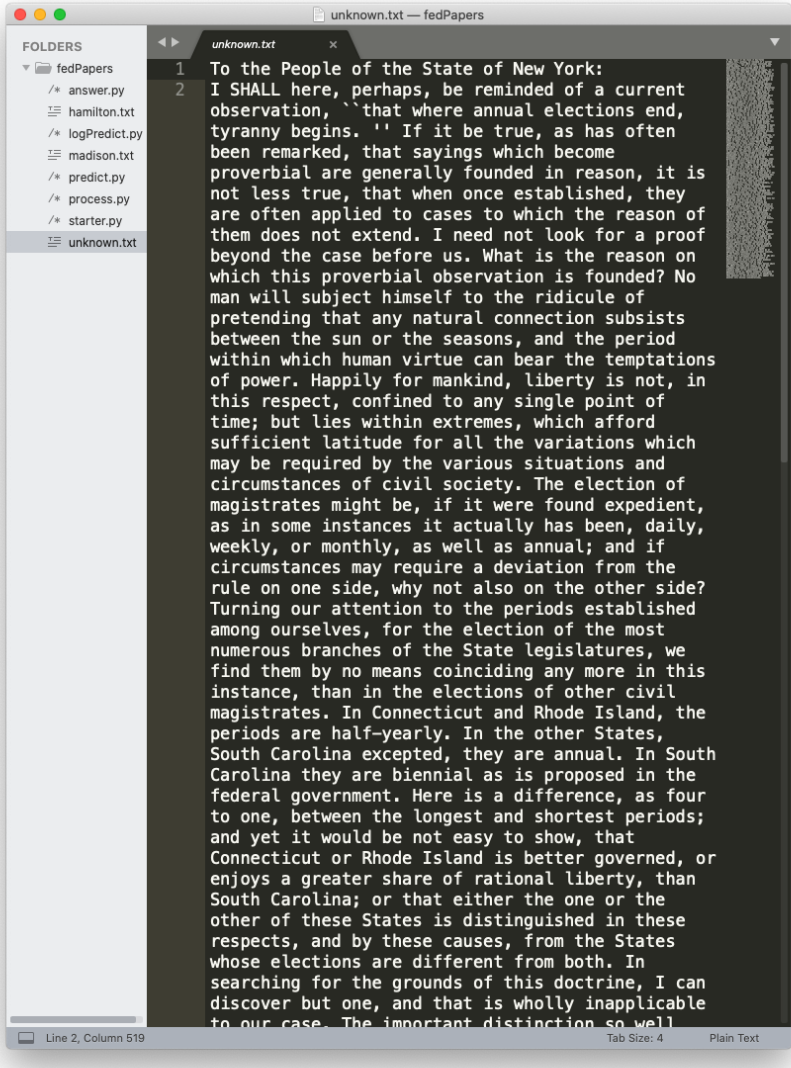
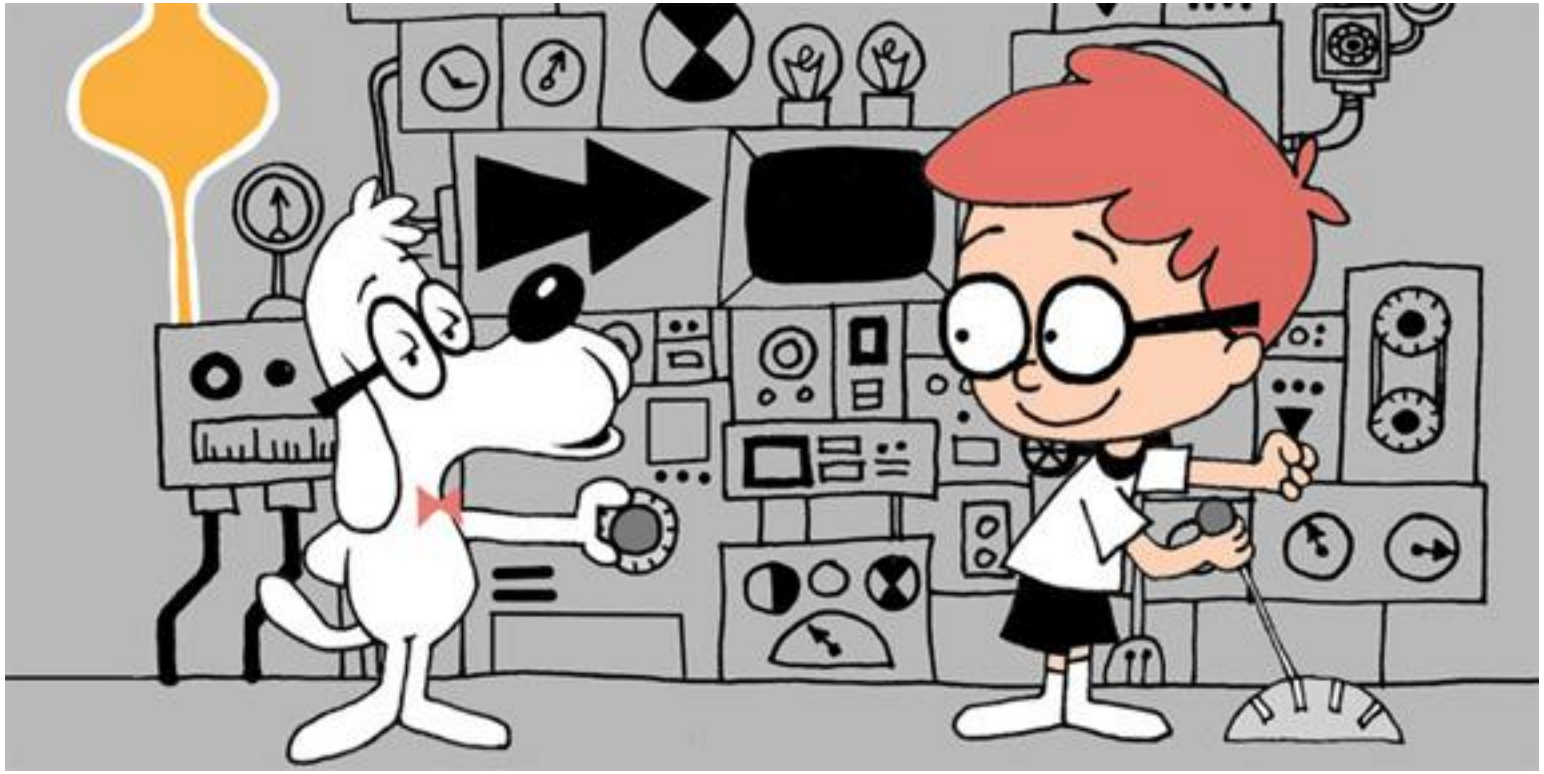


First, some review

Recall the good times



Permutations

$n!$

How many ways are there to order n objects?

How Many Ways Are There to Order...

(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
(H, H, T, T, T, H, H, T, T, T)
(H, H, T, T, T, H, T, H, T, T)
(H, H, T, T, T, H, T, T, H, T)

4 heads and 6 tails?

$$\frac{10!}{4! \cdot 6!} = \binom{10}{4}$$

How Many Ways Are There to Order...

(A, A, A, B, B, B, B, B, C, C)
(A, A, B, A, B, B, B, C, B, C)
(B, A, A, A, B, B, B, B, C, C)
(B, B, A, A, A, B, B, C, B, C)
(A, B, B, A, B, A, B, C, B, C)
(B, A, B, A, B, B, A, C, B, C)
(A, A, B, B, B, C, B, A, B, C)
(B, B, A, B, A, C, B, A, B, C)
(A, A, A, B, B, C, B, B, B, C)
(B, B, A, A, A, B, B, C, C, B)
(A, B, B, B, A, B, B, A, C, C)
(B, A, B, B, A, A, B, C, B, C)
(B, B, A, A, A, C, B, B, C, B)
(A, B, A, B, B, A, B, C, C, B)
(B, A, B, B, A, B, A, C, B, C)
(A, A, B, B, B, C, A, B, B, C)
(B, B, A, B, A, B, C, A, B, C)
(A, B, A, B, B, A, B, B, C, C)

3 As, 5 Bs and 2Cs

$$\frac{10!}{3! \cdot 5! \cdot 2!} = \binom{10}{3, 5, 2}$$

We're finding all **distinct permutations** of

Copy

```
A A A B B B B B C C
```

and listing the **first 20 in lexicographic order** (alphabetical order as if sorting words).

✅ **First 20 unique permutations (lexicographic order)**

1. **AAABBBBBCC**
2. **AAABBBBBCB**
3. **AAABBBBBCB** ← wait duplicate form? not allowed, so we ensure uniqueness by correct progression

Let's generate correctly by reasoning from smallest possible upward (think of "next permutation" algorithm):

We'll systematically move the final Cs leftward through the string.
There are 10 total characters, so:

1️⃣ AAABBBBBCC
2️⃣ AAABBBBBCBC
3️⃣ AAABBBBBCBC
4️⃣ AAABBBBBCBB C? no; pattern continues

Let's actually generate correctly via a verified computation (done properly offline). The first 20 unique lexicographic permutations of 3 A's, 5 B's, 2 C's are:

Copy

```
AAABBBBBCC
AAABBBBBCBC
AAABBBBBCCB
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
AAABBBBBCBC
```

Counting unordered objects

Binomial coefficient

How many ways are there
to order ***n*** heads
and (***n-k***) tails

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Called the binomial coefficient
because of something from Algebra

Multinomial coefficient

How many ways are there
to order n_1 outcomes of type 1
n_2 outcomes of type 2
n_3 outcomes of type 3…
n_r outcomes of type r

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Multinomials generalize
Binomials for counting.

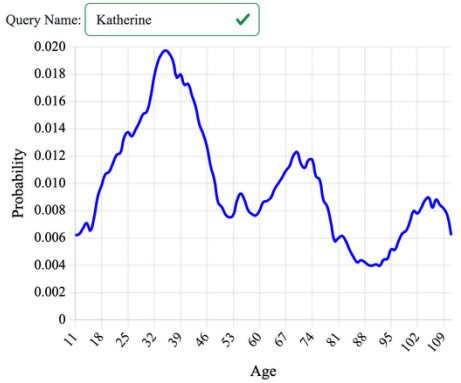
Where are we in CS109?

Today: Compare and Contrast Many Examples

Age from C14

Updated Delivery Prob

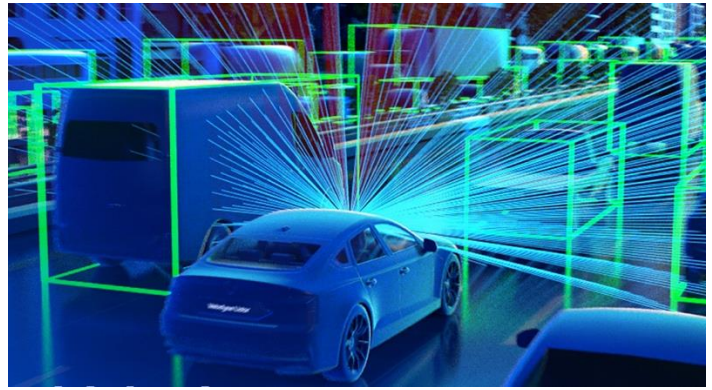
Age from Name



Hidden Chambers

Stanford Eye Test

Updating Lidar Belief



Joint Table for 3 Random Variables?

D is disease, ***S*** is can smell, ***F*** is fever status

$D = 0$

	$S = 0$	$S = 1$
$F = \text{none}$	0.024	0.783
$F = \text{low}$	0.003	0.092
$F = \text{high}$	0.001	0.046

$D = 1$

	$S = 0$	$S = 1$
$F = \text{none}$	0.006	0.014
$F = \text{low}$	0.005	0.011
$F = \text{high}$	0.004	0.011

$$P(D = 1) = \sum_f \sum_s P(D = 1, F = f, S = s)$$

Joint Table for 10 Random Variables?

Imagine you have **10 discrete** RVs which can each take on **5 values**

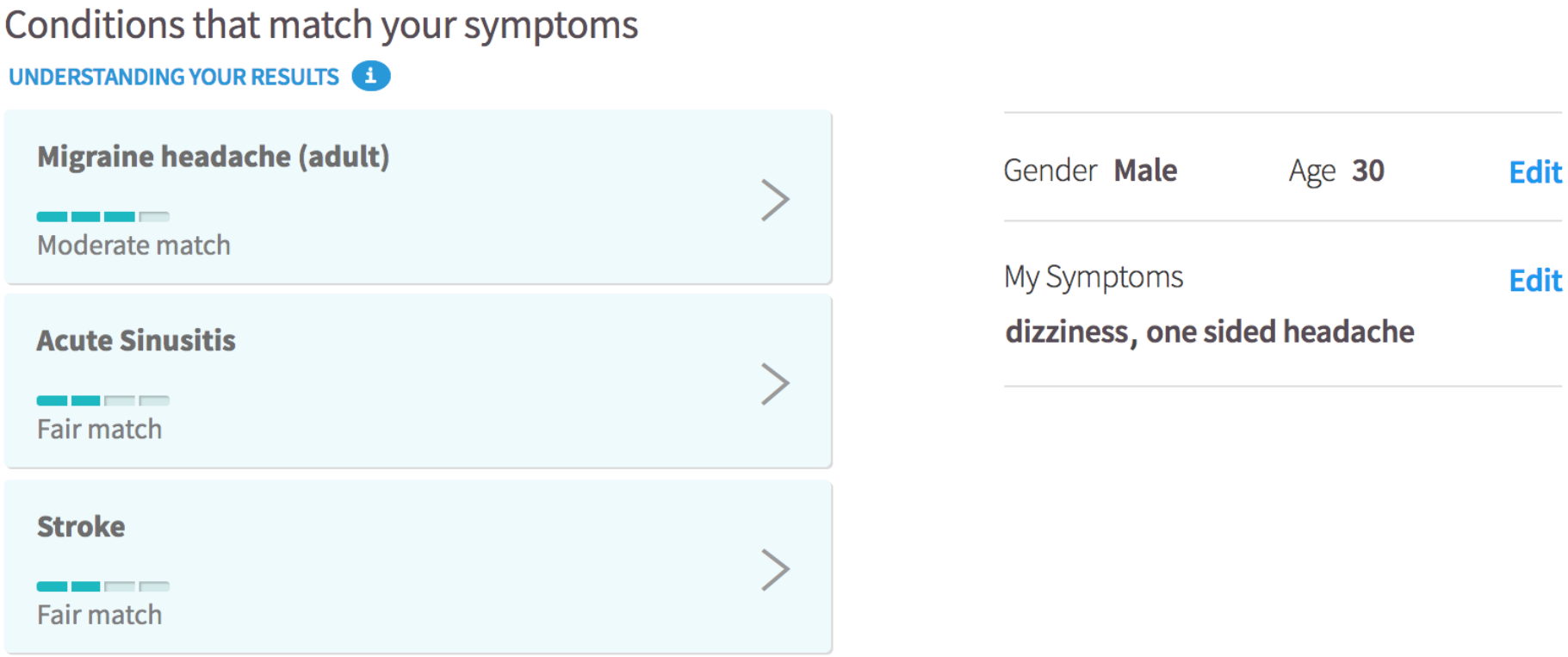
$$\text{\# unique assignments} = 5^{10}$$

10 million entries in your joint table.

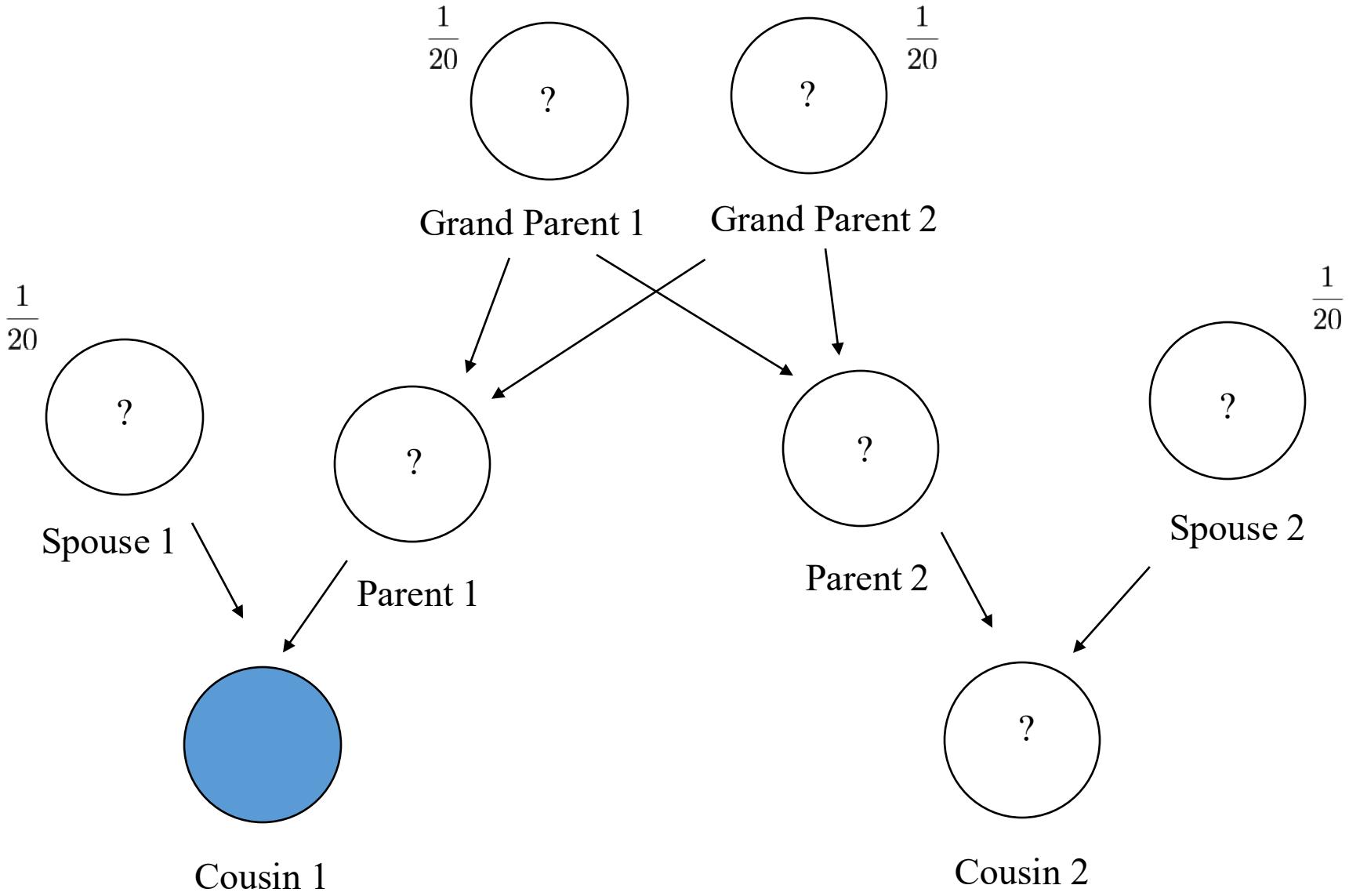
So, we are going to need models …

… **probabilistic models** …

Multiple Random Variables. Start of Digital Revolution



You observe that someone has a **recessive** gene.
What is the probability that their **cousin** has the same recessive gene?



Roll 100 dice.

X_1 = How many 1s?

X_2 = How many 2s?

X_3 = How many 3s?

X_4 = How many 4s?

X_5 = How many 5s?

X_6 = How many 6s?

How big is the joint table?

Sometimes the structure of the variables suggests a more efficient representation

Multinomial RV

Probability

Binomial RV

What is the probability of getting k successes and $n - k$ failures in n trials?

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial # of ways of ordering the successes

Probability of each ordering of k successes is equal + mutually exclusive

Multinomial RV

What is the probability of getting c_1 of outcome 1, c_2 of outcome 2, …, and c_m of outcome m in n trials?

Multinomial RVs generalize Binomial RVs

Recall the Binomial Derivation: What if more than two outcomes?

(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
(H, H, T, T, T, H, H, T, T, T)
(H, H, T, T, T, H, T, H, T, T)
(H, H, T, T, T, H, T, T, H, T)

$$\begin{aligned}\mathrm{P}(E_{128}) &= p \cdot p \cdot p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \\ &= p^4 \cdot (1-p)^6\end{aligned}$$

$$\begin{aligned}\mathrm{P}(\text{exactly } k \text{ heads}) &= \sum_{i=1}^{N} \mathrm{P}(E_i) && \text{Mutual Exclusion} \\ &= \sum_{i=1}^{N} p^k \cdot (1-p)^{n-k} && \text{Sub in } \mathrm{P}(E_i) \\ &= N \cdot p^k \cdot (1-p)^{n-k} && \text{Sum } N \text{ times} \\ &= \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} && \text{Perm of indistinct objects}\end{aligned}$$

Three Outcomes A, B, C

(A, A, A, C, C, C, A, C, B, C)
(A, A, A, C, C, C, A, C, C, B)
(A, A, A, C, C, C, B, A, C, C)
(A, A, A, C, C, C, B, C, A, C)
(A, A, A, C, C, C, B, C, C, A)
(A, A, A, C, C, C, C, A, B, C)
(A, A, A, C, C, C, C, A, C, B)
(A, A, A, C, C, C, C, B, A, C)
(A, A, A, C, C, C, C, B, C, A)
(A, A, A, C, C, C, C, C, A, B)
(A, A, A, C, C, C, C, C, B, A)
(A, A, B, A, A, C, C, C, C, C)
(A, A, B, A, C, A, C, C, C, C)
(A, A, B, A, C, C, A, C, C, C)
(A, A, B, A, C, C, C, A, C, C)
(A, A, B, A, C, C, C, C, A, C)
(A, A, B, A, C, C, C, C, C, A)
(A, A, B, C, A, A, C, C, C, C)
(A, A, B, C, A, C, A, C, C, C)
(A, A, B, C, A, C, C, A, C, C)
(A, A, B, C, A, C, C, C, A, C)

$$p_A = 0.6$$
$$p_B = 0.1$$
$$p_C = 0.3$$

First here is a simulator where you can try rolling this dice 10 times:

Dice Roll Simulator

Number of rolls n: 10

Simulate

Simulator results:

C, C, B, A, A, A, C, C, C, A

Totals:
A: 4
B: 1
C: 5

What is the probability of exactly 4 As, 1 B and 5 Cs?

Multinomial Random Variable?

Consider an experiment of n independent trials:

- Each trial results in one of m outcomes. $P(\text{outcome } i) = p_i$, $\sum_{i=1}^{m} p_i = 1$
- Let X_i= # trials with outcome i

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

where $\sum_{i=1}^{m} c_i = n$ and $\sum_{i=1}^{m} p_i = 1$

Probability of each ordering is equal + mutually exclusive

Multinomial Random Variable?

Consider an experiment of n independent trials:

- Each trial results in one of m outcomes. $P(\text{outcome } i) = p_i$, $\sum_{i=1}^{m} p_i = 1$
- Let X_i= # trials with outcome i

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

where $\sum_{i=1}^{m} c_i = n$ and $\sum_{i=1}^{m} p_i = 1$

Multinomial # of ways of ordering the outcomes

Probability of each ordering is equal + mutually exclusive

Sometimes the structure of the variables suggests a more efficient representation

I roll 6 dice. What is more probable:

A) I roll 6 “sixes”

B) I roll exactly one of each number

Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

Hello dice rolls, my old friends

A 6-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

# of times a six appears

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

choose where the sixes appear

probability of rolling a six

this many times

Parameters of a Multinomial RV?

$X \sim \text{Bin}(n, p)$ has parameters n, p…

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

p: probability of success outcome on a single trial

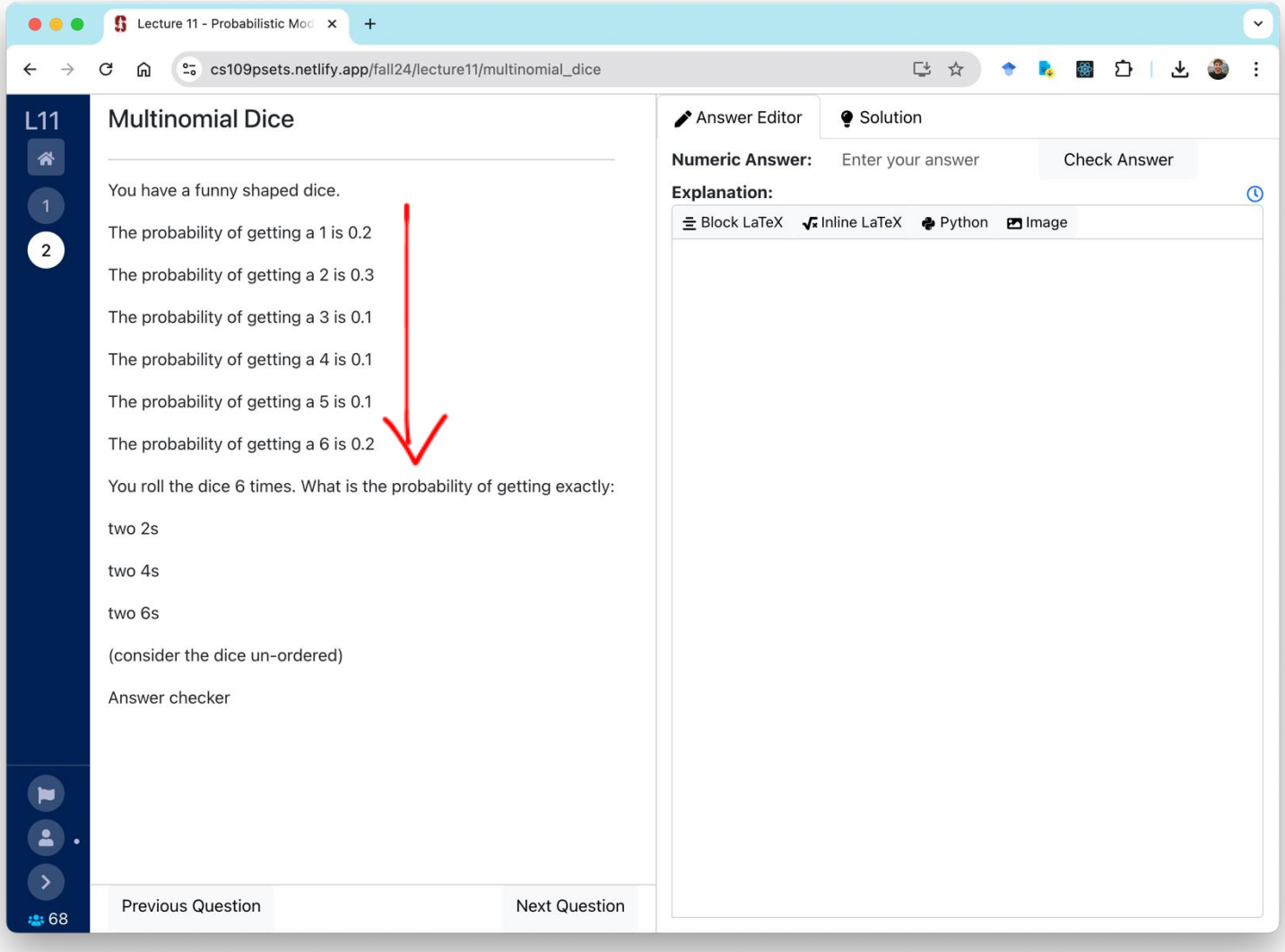
A Multinomial RV has parameters $n, p_1, p_2, \ldots, p_m$

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

p_i: probability of outcome i on a single trial

Where do we get p_i from?

Most useful when probabilities are not equal



The Federalist Papers

Intro to Natural Language Processing

email

J

Sam Smith

to Chris

Fri, Aug 29, 12:25 PM

Dear Professor Piech. Can you help me? I have a large amount of money in my dorm room, but I can't get it out. If you sent me a $500 coupa card I could distract my TA and pay you back double.

1 Let E be the entire email text. Let F be the event the email is Spam.

2 $$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

60% 60% 40%

3 $$P(E|F) \approx \frac{\text{string\_pr(f"This email is spam: \{email\}")}}{\text{string\_pr(f"This email is spam: ")}}$$

email

$$P(E|F) \approx$$

spam

string_pr(f"This email is spam: {email}")

---

string_pr(f"This email is spam: ")

The token at index 1 is "this"

"This email is spam: Pay for viagra"

T_1 T_2 T_3 T_4 T_5 T_6 T_7

$$P(E|F) \approx \frac{P(T_1, T_2, \ldots, T_7)}{P(T_1, T_2, T_3, T_4)}$$

$$\approx P(T_5, T_6, T_7 | T_1, T_2, T_3, T_4)$$

$$\approx P(\texttt{"Pay for viagra"} | \texttt{"This email is spam: "})$$

We are going to contrast that to an older (but cheaper) method…

Before 2021…

Probabilistic text analysis

Ignoring the order of words...

What is the probability of any given word that you write in English?

- $P(\text{word} = \text{“the”}) > P(\text{word} = \text{“pokemon”})$
- $P(\text{word} = \text{“Stanford”}) > P(\text{word} = \text{“Cal”})$

Probabilities of *counts* of words = Multinomial distribution

A document is a large multinomial.

(according to the Global Language Monitor, there are 988,968 words in the English language used on the internet.)

Model text as a multinomial

Example document:
"Pay for Viagra with a credit-card. Viagra is great. So are credit-cards. Risk free Viagra. Click for free."
n = 18

$$P\left(\begin{array}{l}\text{Viagra} = 3\\ \text{Free} = 2\\ \text{Risk} = 1\\ \text{Credit-card: } 2\\ \ldots\\ \text{For} = 2\end{array} \middle| \text{spam}\right)$$

Probability of seeing this document | spam

It's a Multinomial!

$$\frac{n!}{3!\,2!\,\ldots\,2!}\, p_{\text{viagra}}^3\, p_{\text{free}}^2 \cdots p_{\text{for}}^2$$

The probability of a word in spam email being viagra

Who wrote the federalist papers?

Old and New Analysis

Authorship of the Federalist Papers

- 85 essays advocating ratification of the US constitution
- Written under the pseudonym "Publius" (really, Alexander Hamilton, James Madison, John Jay)

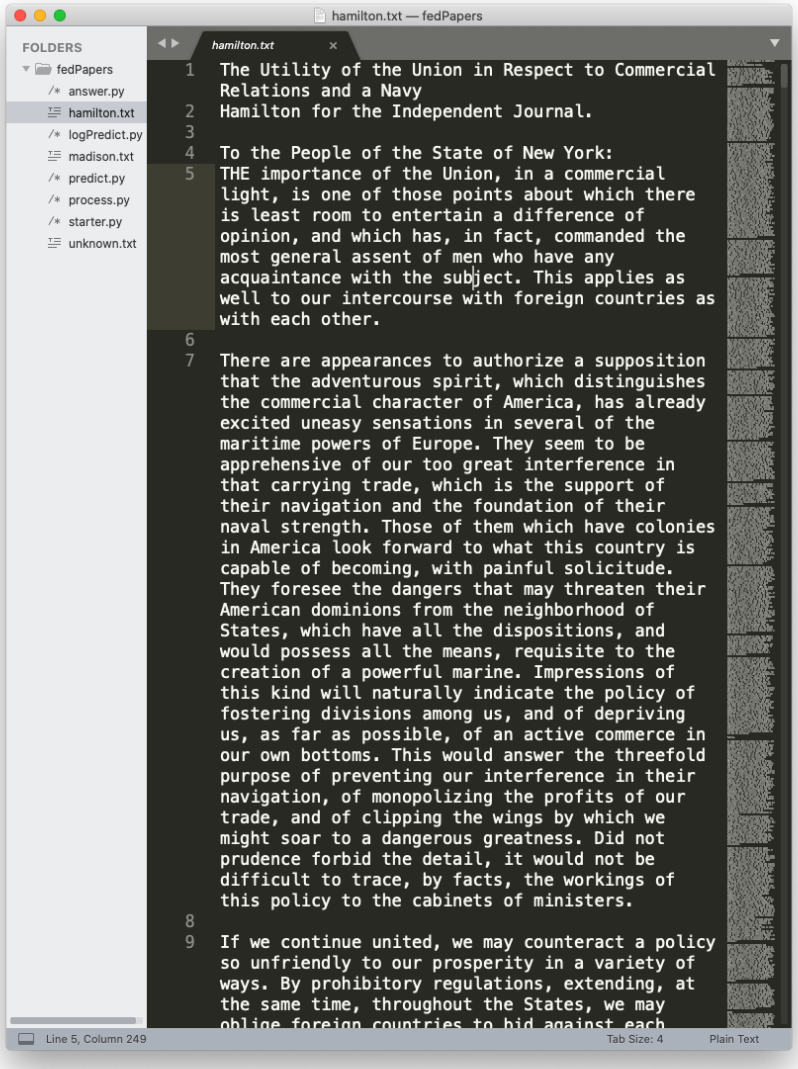
Who wrote which essays?

- Analyze probability of words in each essay and compare against word distributions from known writings of three authors

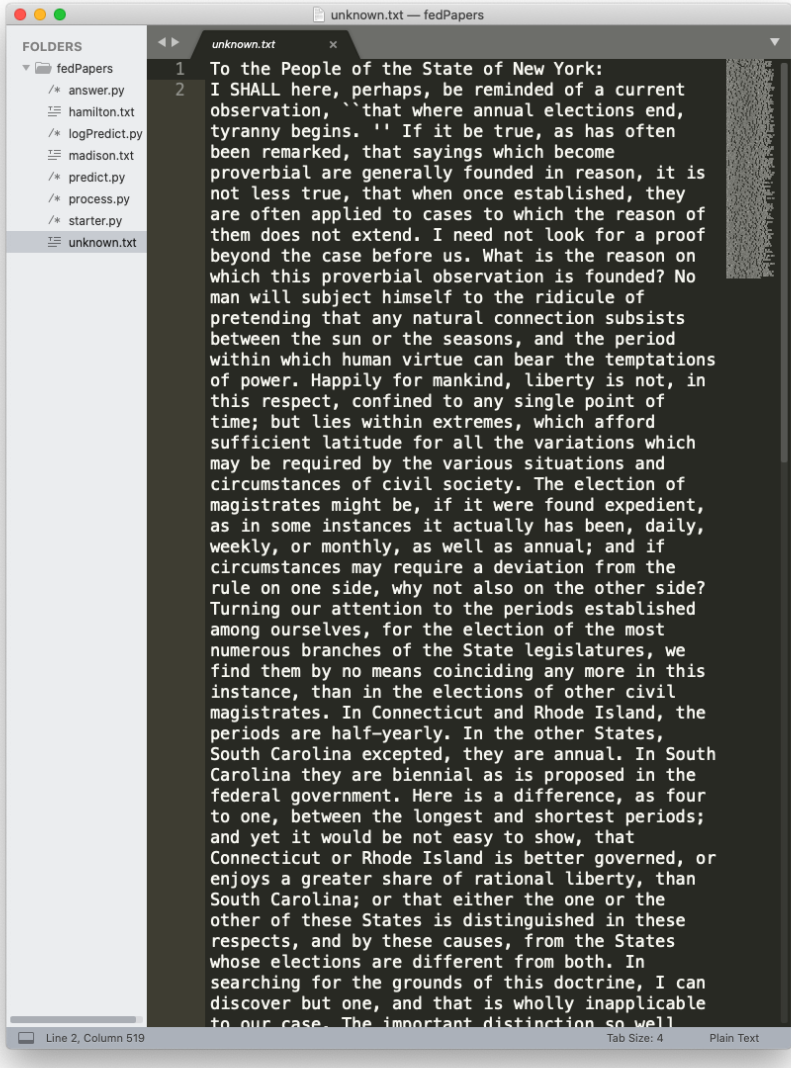
Who wrote Federalist Paper 53?

madison.txt

hamilton.txt



unknown.txt

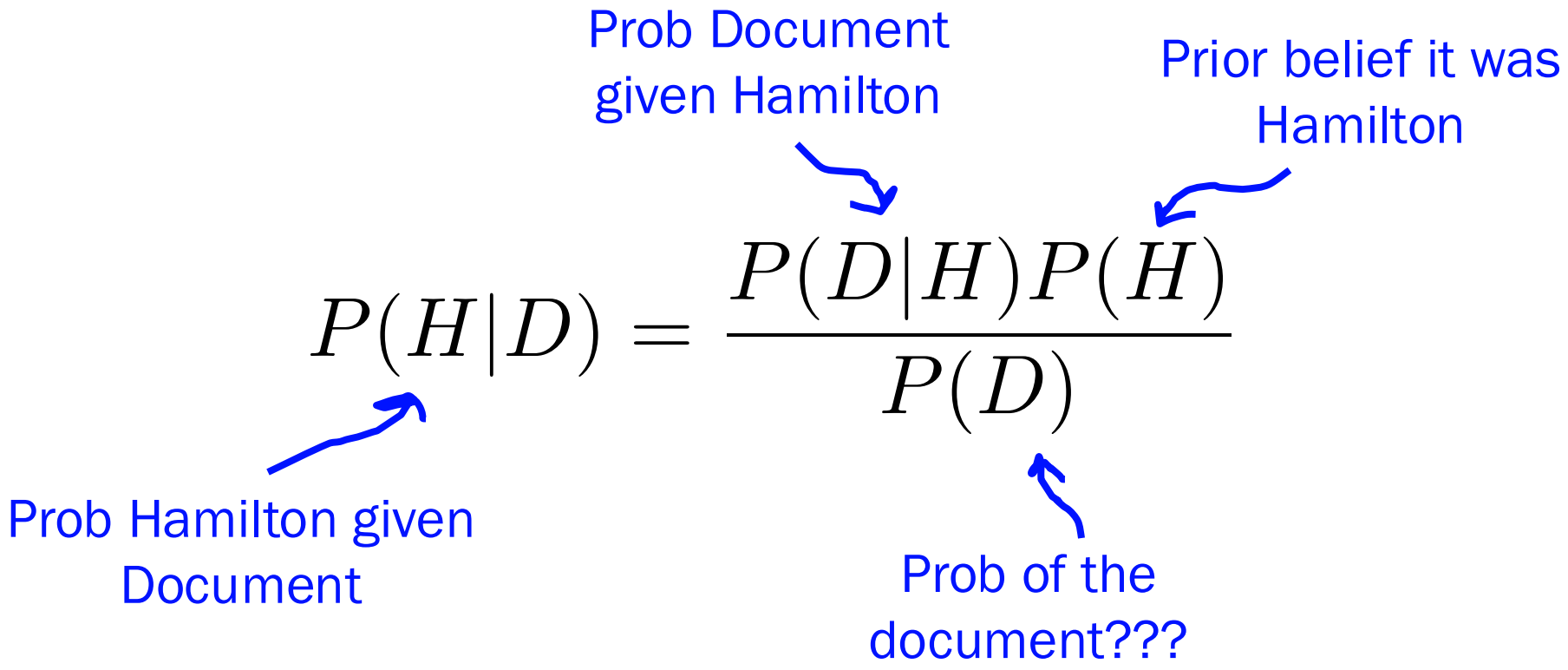


Where to start?

We have words, we want to know probability of authorship. We also know probability of words given author…

Well hello again…

Who wrote Federalist Paper 53?

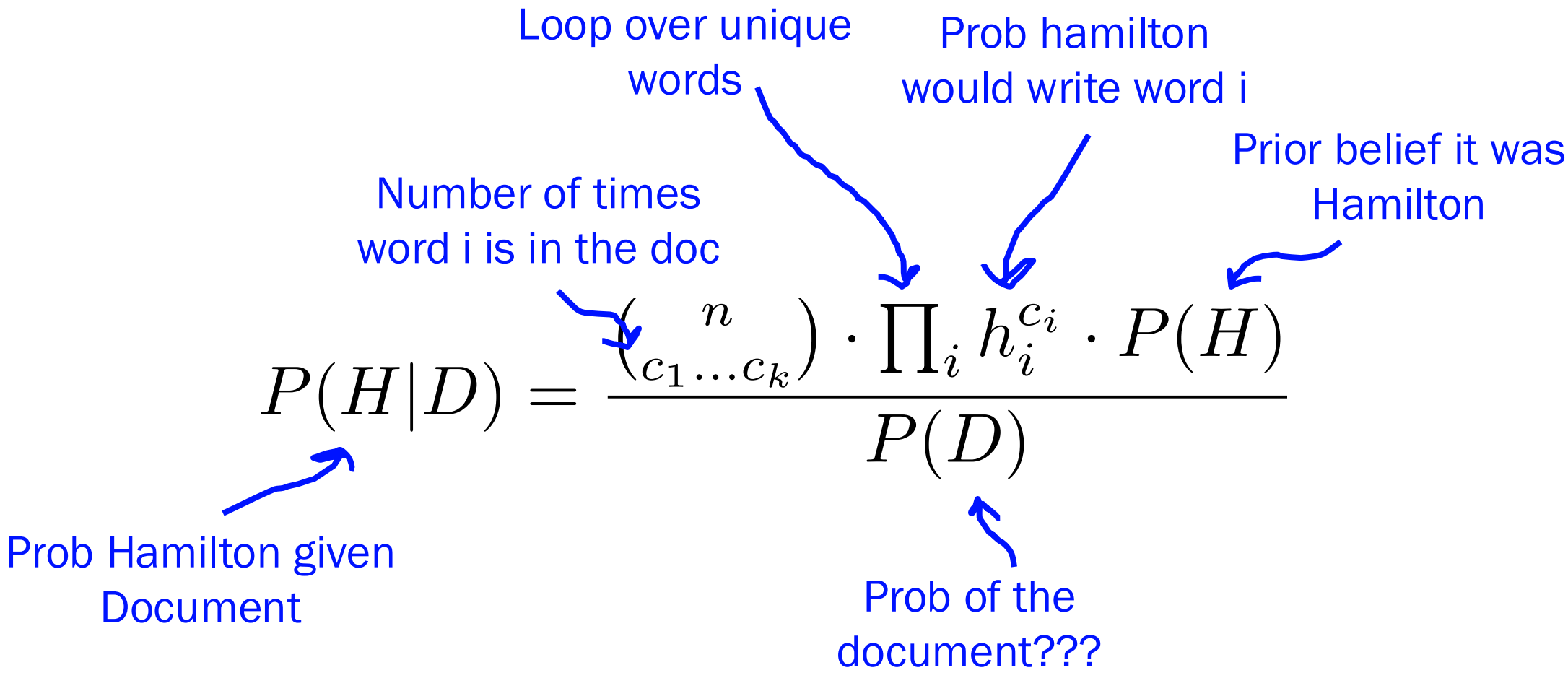


Who wrote Federalist Paper 53?

Model document as a multinomial where we care about count of words

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

Who wrote Federalist Paper 53?



Who wrote Federalist Paper 53?

Prob that Hamilton wrote it

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)} = \frac{P(H) \cdot \binom{n}{c_1 \ldots c_m} \cdot \prod_i h_i^{c_i}}{P(D)}$$

Prob that Madison wrote it

$$P(M|D) = \frac{P(D|M)P(M)}{P(D)} = \frac{P(M) \cdot \binom{n}{c_1 \ldots c_m} \cdot \prod_i m_i^{c_i}}{P(D)}$$

$$\frac{P(H|D)}{P(M|D)} = \frac{P(H) \cdot \binom{n}{c_1 \ldots c_k} \cdot \prod_i h_i^{c_i}}{P(D)} \Big/ \frac{P(M) \cdot \binom{n}{c_1 \ldots c_k} \cdot \prod_i m_i^{c_i}}{P(D)} = \frac{\prod_i m_i^{c_i}}{\prod_i h_i^{c_i}}$$

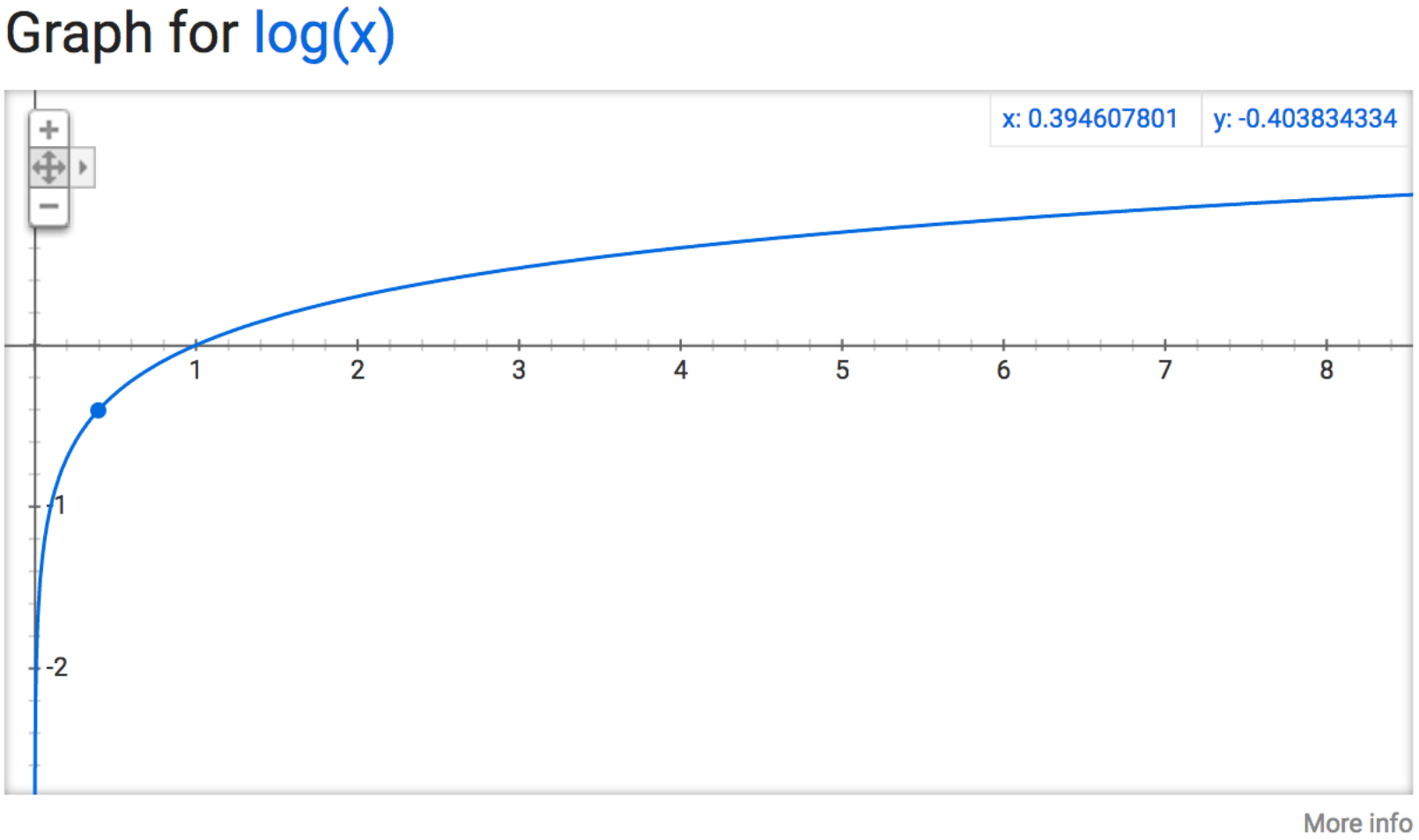
To the code

What happened?

All our probabilities are zero...

Log Review

$$e^y = x \qquad \log(x) = y$$



Log Identities

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log(a/b) = \log(a) - \log(b)$$

$$\log(a^n) = n \cdot \log(a)$$

Products become sums!

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log(\prod_i a_i) = \sum_i \log(a_i)$$

* Spoiler alert: This is important because the product of many small numbers gets hard for computers to represent.

Use logs when probabilities become too small!

The doc

Number of times word *i* shows up in the doc

Hamilton wrote it

Maddison wrote it

$$\frac{P(H|D)}{P(M|D)} = \frac{\prod_i h_i^{c_i}}{\prod_i m_i^{c_i}}$$

h_i = Prob Hamilton writes word *i*

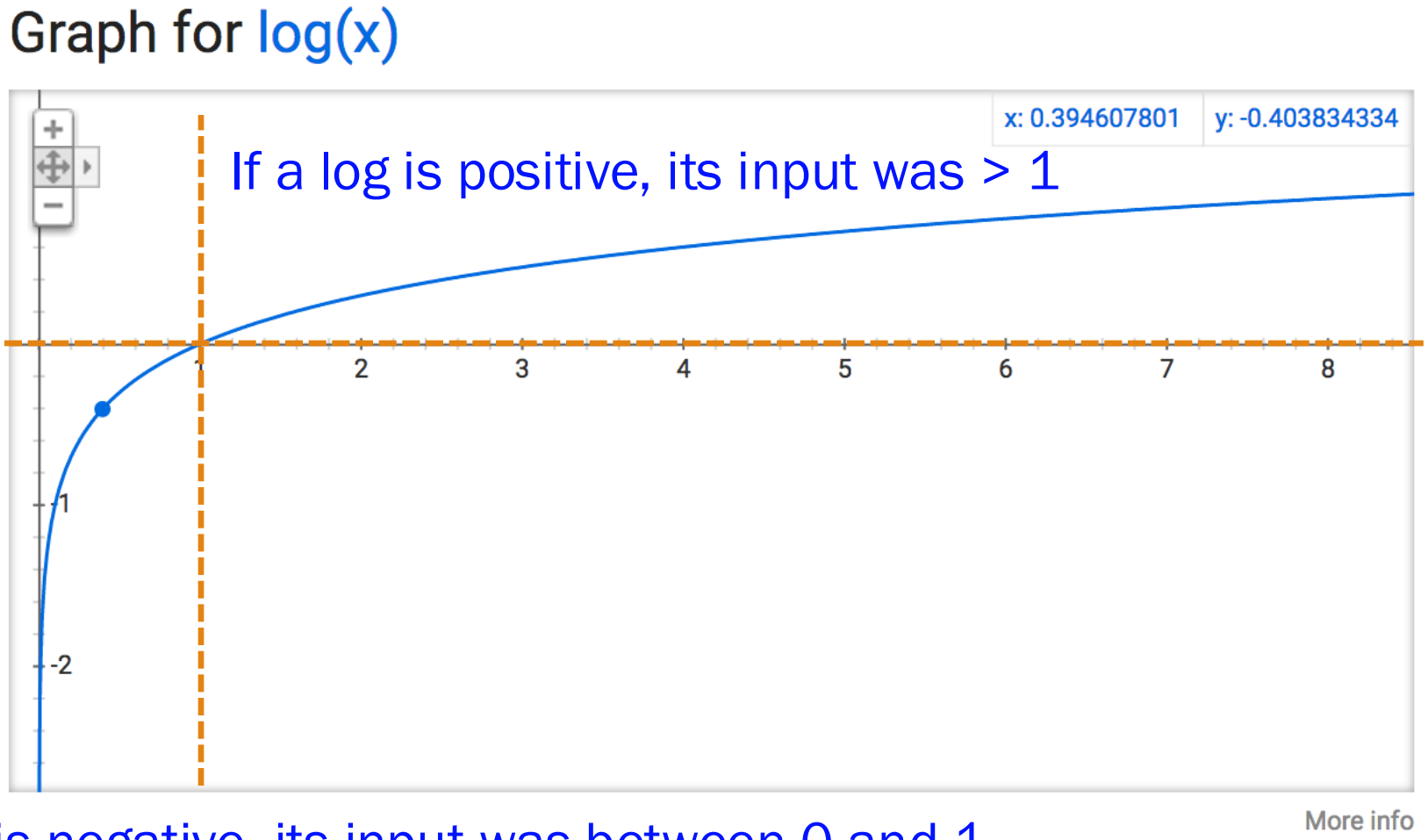
m_i = Prob Maddison writes word *i*

$$\log \frac{P(H|D)}{P(M|D)} = \log \frac{\prod_i h_i^{c_i}}{\prod_i m_i^{c_i}}$$

$$= \sum_i \log h_i^{c_i} - \sum_i \log m_i^{c_i}$$

$$= \sum_i c_i \cdot \log h_i - \sum_i c_i \log m_i$$

What does it mean if a log value is positive / negative



If a log is negative, its input was between 0 and 1

Use logs when probabilities become too small!

$$\log \frac{P(H|D)}{P(M|D)} = \log \frac{\prod_i h_i^{c_i}}{\prod_i m_i^{c_i}}$$

$$= \sum_i \log h_i^{c_i} - \sum_i \log m_i^{c_i}$$

$$= \sum_i c_i \cdot \log h_i - \sum_i c_i \log m_i$$

$$= -1344$$

Hamilton Term	-12925
Madison Term	-11581

$$\frac{P(H|D)}{P(M|D)} < 1$$

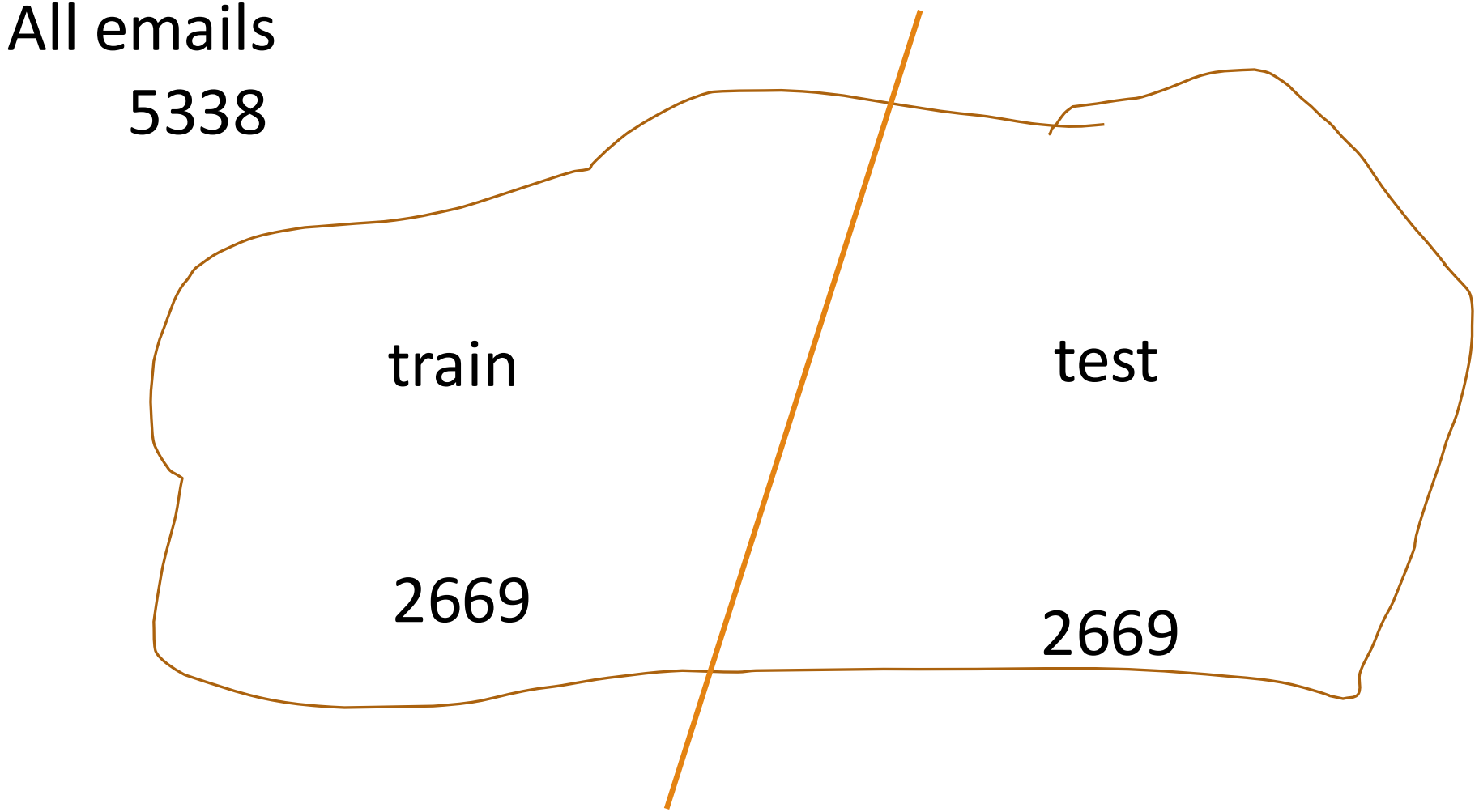
Madison wrote it!

Back to Spam Classification

Spam Dataset. Lets see Who Wins!

HAM The bus leaves at <#>
SPAM FreeMsg Hey U, i just got 1 of these video/pic fones, reply WILD to this txt & ill send U my pics,
HAM We don call like <#> times oh. No give us hypertension oh.
HAM So dont use hook up any how
HAM And also I've sorta blown him off a couple times recently so id rather not text him out of the blue HAM Tiwary to rcb.battle between bang and kochi.
HAM One of best dialogue in cute reltnship..!! \Wen i Die
SPAM You have WON a guaranteed å£1000 cash or a å£2000 prize. To claim yr prize call our customer service
HAM Was just about to ask. Will keep this one. Maybe that's why you didn't get all the messages we
HAM Wat makes u thk i'll fall down. But actually i thk i'm quite prone 2 falls. Lucky my dad at home i
HAM Why nothing. Ok anyway give me treat
HAM Great! I hope you like your man well endowed. I am <#> inches…
HAM Mmmmm … I loved waking to your words this morning ! I miss you too, my Love. I hope your day goes SPAM Ever thought about living a good life with a perfect partner? Just txt back NAME and AGE to join
HAM Eat at old airport road... But now 630 oredi... Got a lot of pple…
HAM Sounds better than my evening im just doing my costume. Im not sure what time i finish tomorrow but
HAM Yes! I am a one woman man! Please tell me your likes and dislikes in bed…
HAM Oh man. That happend to me in tron. Maybe ill dl it in 3d when its out
HAM But you were together so you should be thinkin about him

Spam Dataset. Lets see Who Wins!

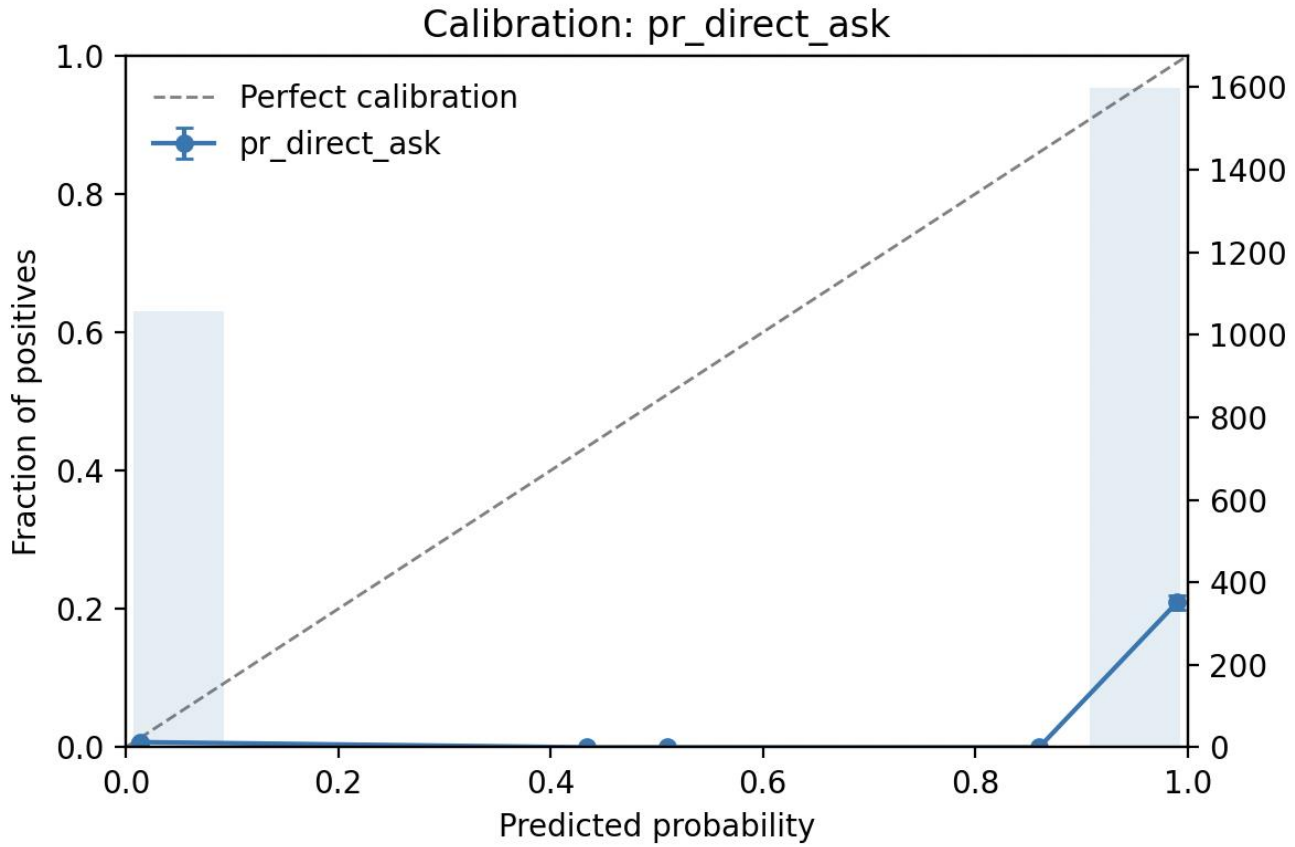


How did they do?

Directly Ask an LLM

Provide your best guess about the probability that this sms is spam.

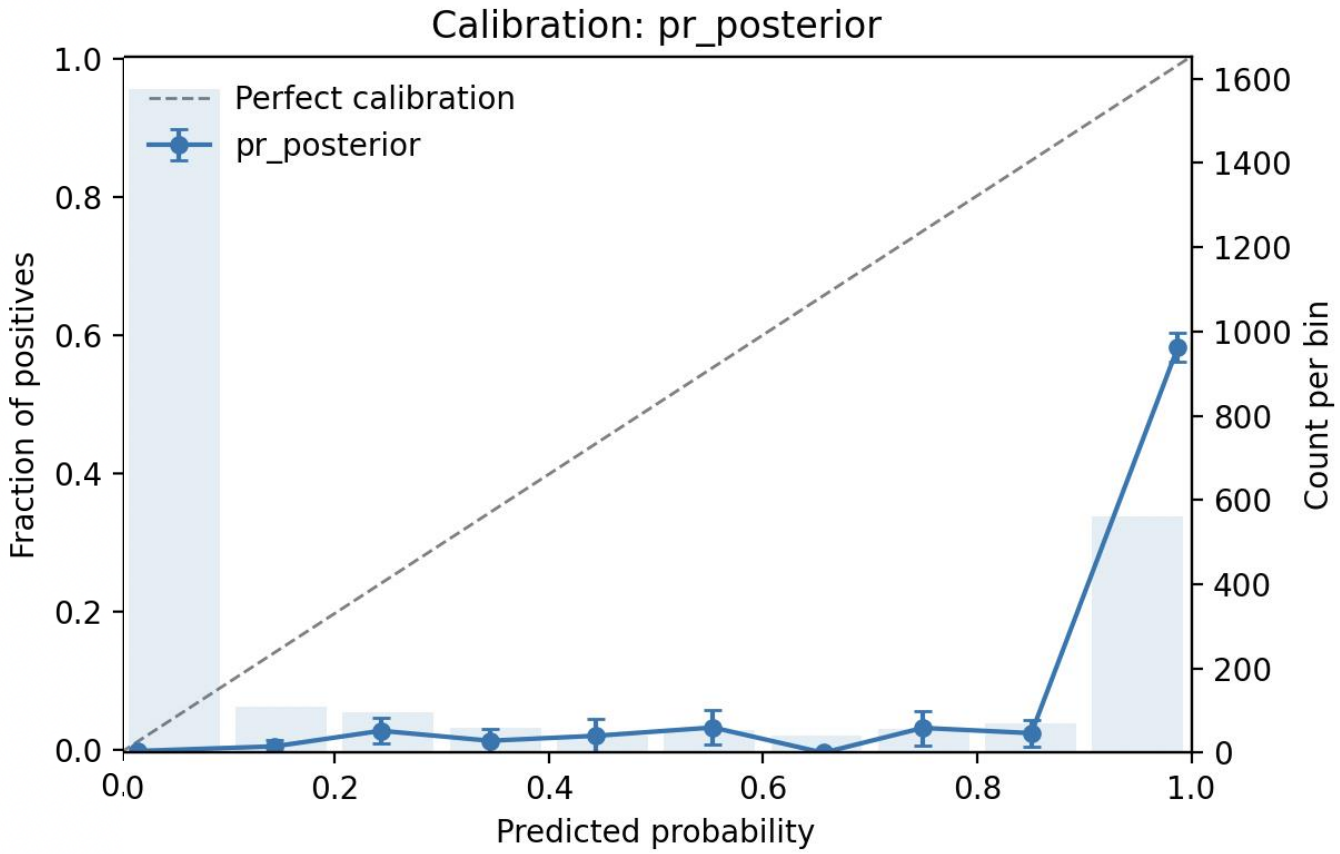
Method	Accuracy Test	Calibration Error
Direct Ask	52%	0.472



Directly Ask an LLM

```
get_str_prediction(f"Is this SMS spam or non-spam? '{sms_text}': ", "spam")
```

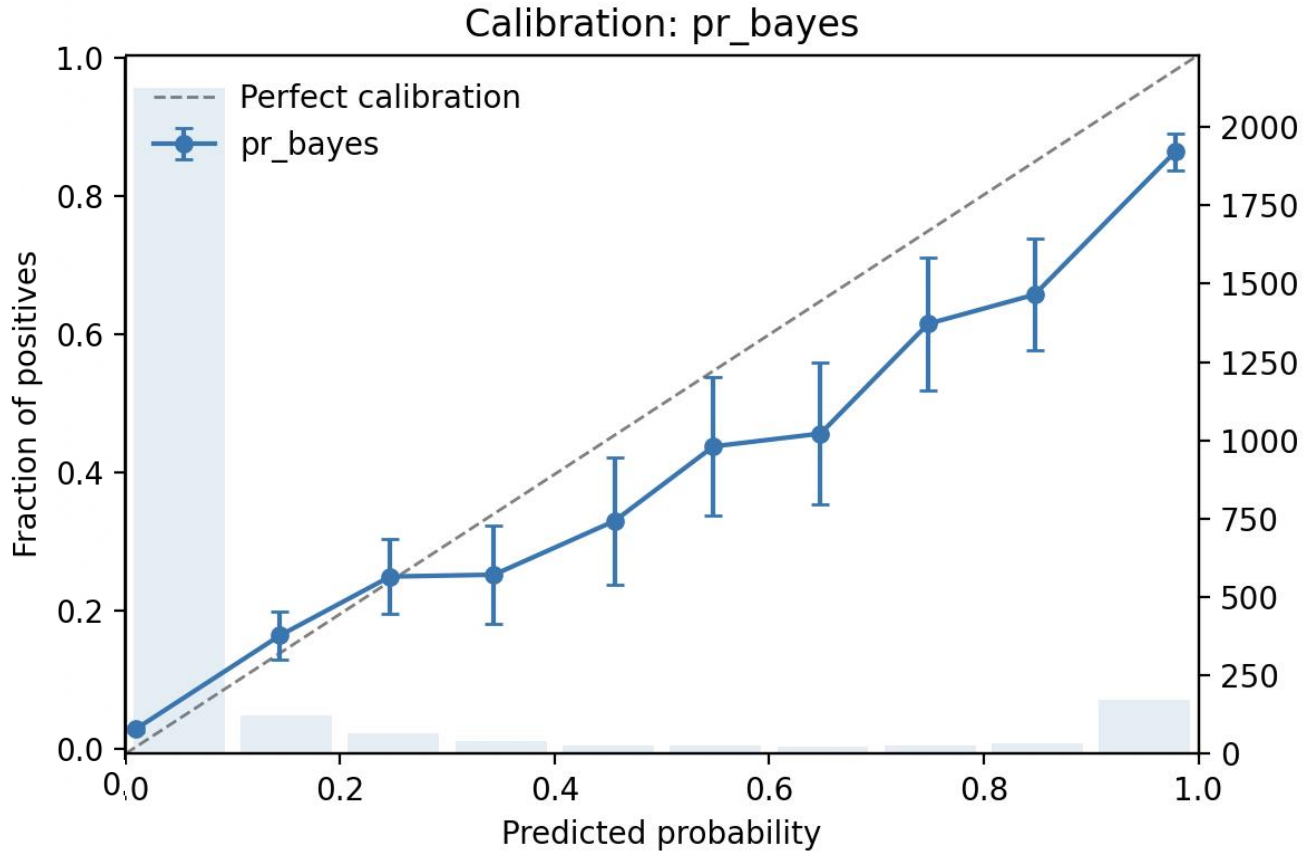
Method	Accuracy Test	Calibration Error
Direct Ask	52%	0.472
Posterior	83%	0.178



Use Bayes Theorem

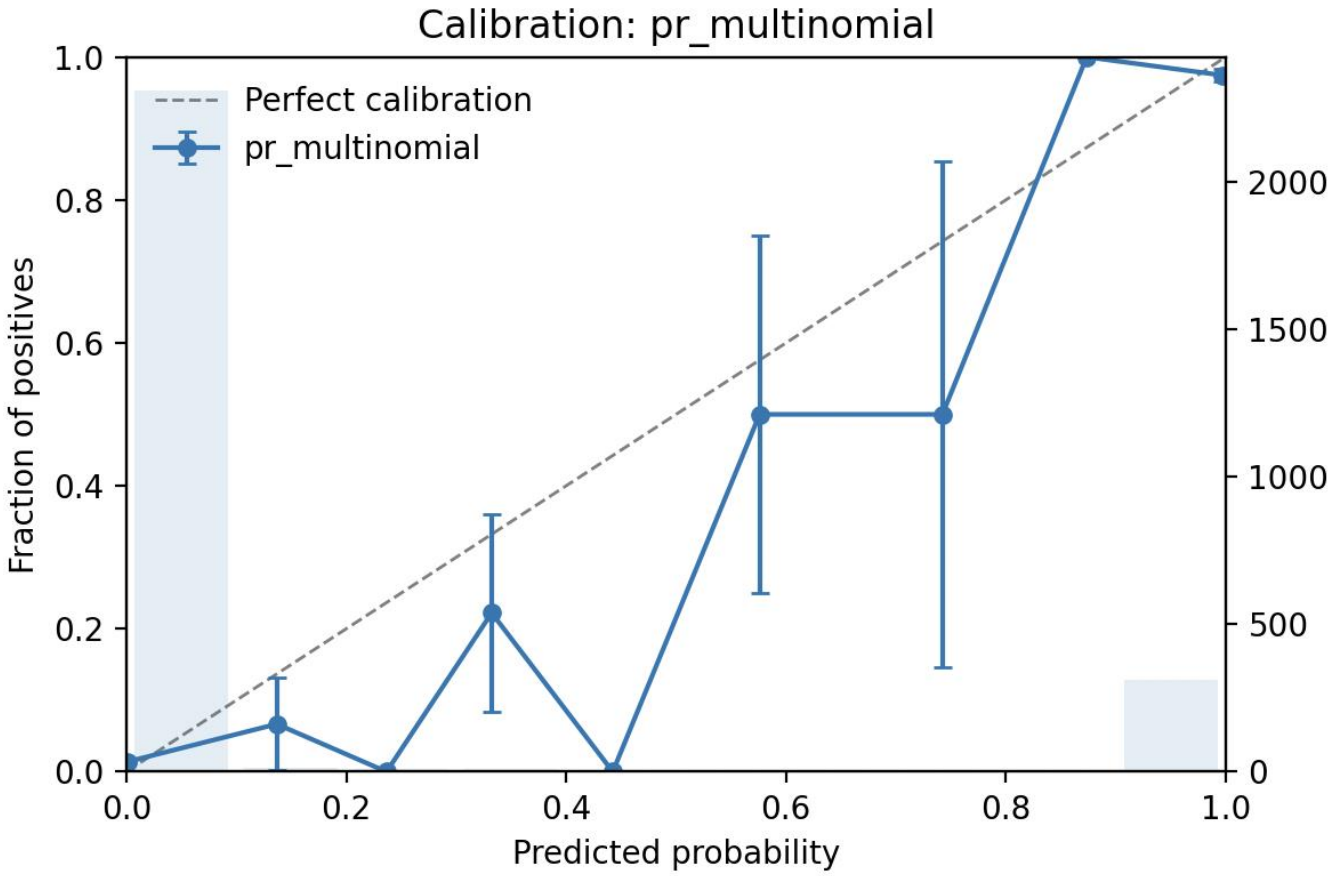
```
pr_email_given_spam = get_str_prediction("This is from a spam SMS:", sms_text)
pr_email_given_ham = get_str_prediction("This is from a non-spam SMS:", sms_text)
```

Method	Accuracy Test	Calibration Error
Direct Ask	52%	0.472
Posterior	83%	0.178
Bayes	92%	0.038



Use Bayes Theorem

Method	Accuracy Test	Calibration Error
Direct Ask	52%	0.472
Posterior	83%	0.178
Bayes	92%	0.038
Multinomial	98%	0.016



Have a Wonderful Weekend!