

Independence

Chris Piech, CS109

Today, start with a cool program

G_1

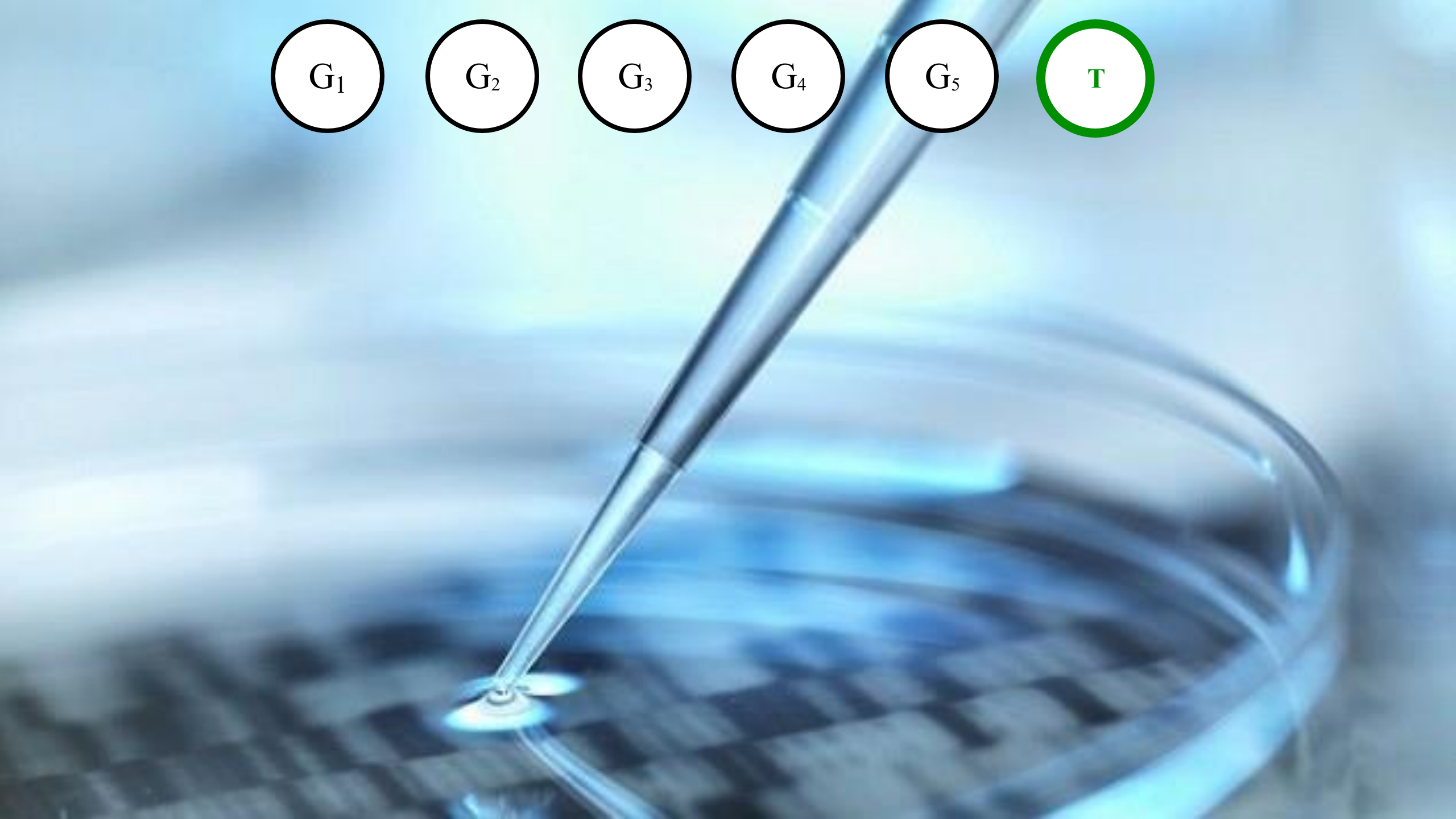
G_2

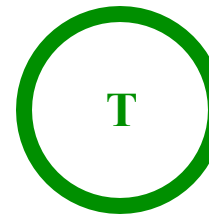
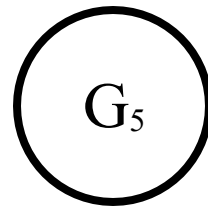
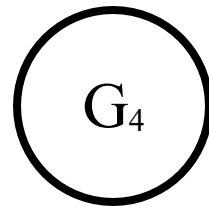
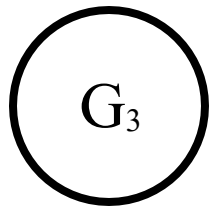
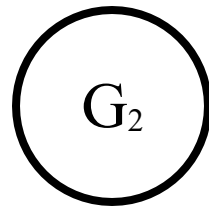
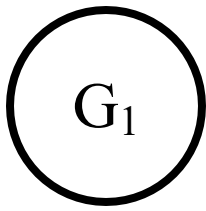
G_3

G_4

G_5

T





```
dna.txt — dna
dna.txt
1 False, True, False, False, True, False
2 True, True, False, True, True, False
3 True, True, False, True, True, True
4 False, True, False, True, True, False
5 False, True, False, False, True, False
6 True, True, False, True, True, True
7 False, False, True, False, False, False
8 False, False, True, False, True, False
9 True, False, False, True, False, False
10 False, True, False, True, True, False
11 True, False, False, True, False, False
12 True, False, True, True, False, False
13 False, True, False, False, True, False
14 False, False, True, True, False, False
15 True, True, False, False, True, True
16 True, False, True, True, False, False
17 True, True, True, True, True, True |
18 True, False, True, False, False, True
19 False, True, False, True, True, True
20 False, False, True, False, False, False
21 False, False, False, True, True, False
22 False, True, False, False, True, False
23 True, True, False, True, True, True
24 False, True, False, True, True, False
25 True, False, False, False, False, True
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27 False, False, False, True, False, False
28 False, True, True, False, False, True
29 False, True, False, False, True, True
30 False, False, False, False, False, True
31 False, True, False, True, True, False
32 True, False, False, True, False, False
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34 True, True, False, False, True, True
35 True, True, False, True, True, True
36 False, False, False, True, False, False
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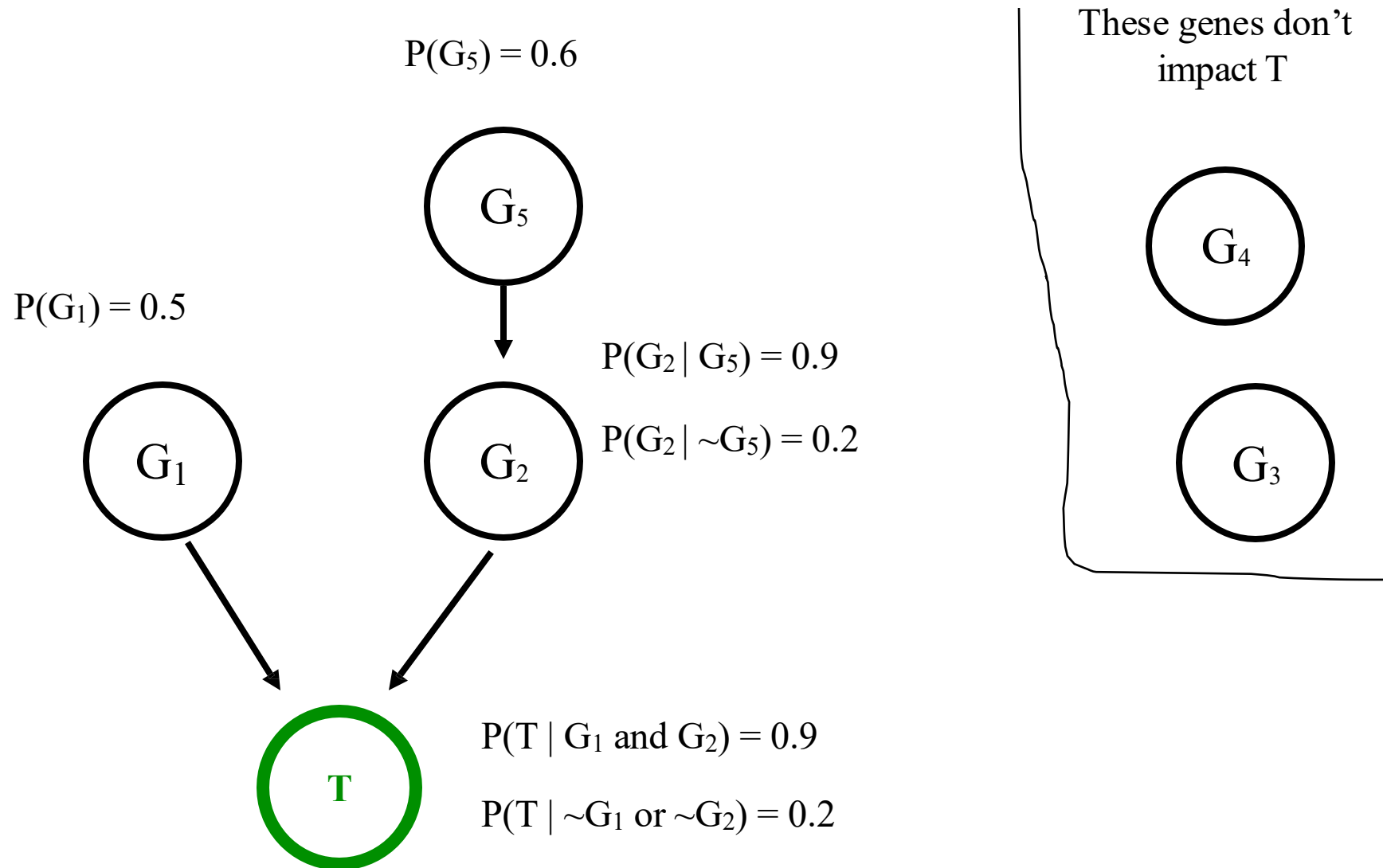


100,000 samples

6 observations per sample



Discovered Hypothesis



We've Gotten Ahead of Ourselves



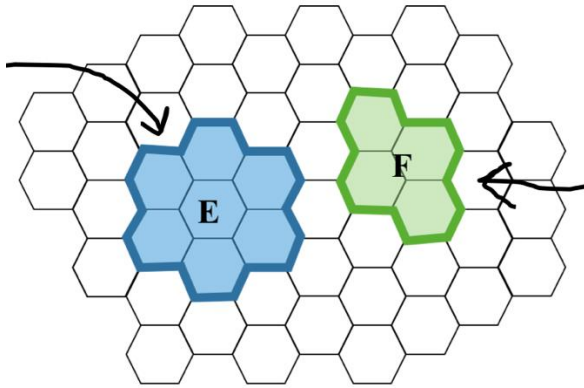
Source: The Hobbit

Start at the Beginning



Source: The Hobbit

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Review

Notation

And

Or

Given

$$P(E \text{ and } F)$$

$$P(E \text{ or } F)$$

$$P(E|F)$$

$$P(E, F)$$

$$P(E \cup F)$$

$$P(EF)$$

$$P(E \cap F)$$

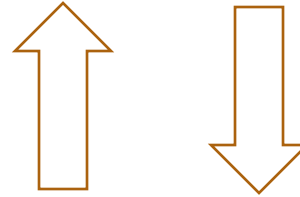


Relationship Between Probabilities



$$P(E \text{ and } F)$$

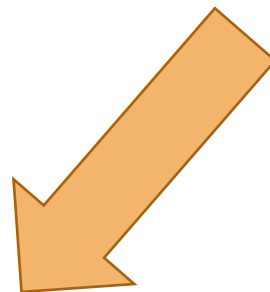
Chain rule
(Product rule)



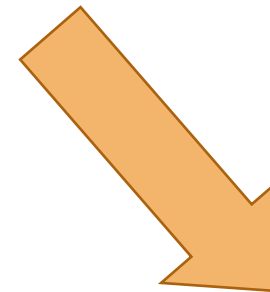
Definition of
conditional probability

$$P(E|F)$$

Law of Total
Probability



Bayes'
Theorem



$$P(E)$$

$$P(F|E)$$



Review: Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Bayes' Theorem

$$P(E|F) \Rightarrow P(F|E)$$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$



Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$



SARS Virus Testing

A test is 98% effective at detecting SARS

- However, test has a “false positive” rate of 1%
- 0.5% of US population has SARS
- Let E = you test positive for SARS with this test
- Let F = you actually have SARS
- What is $P(F | E)$?

$$P(E|F)$$

$$P(E|F^C)$$

$$P(F)$$

Solution:

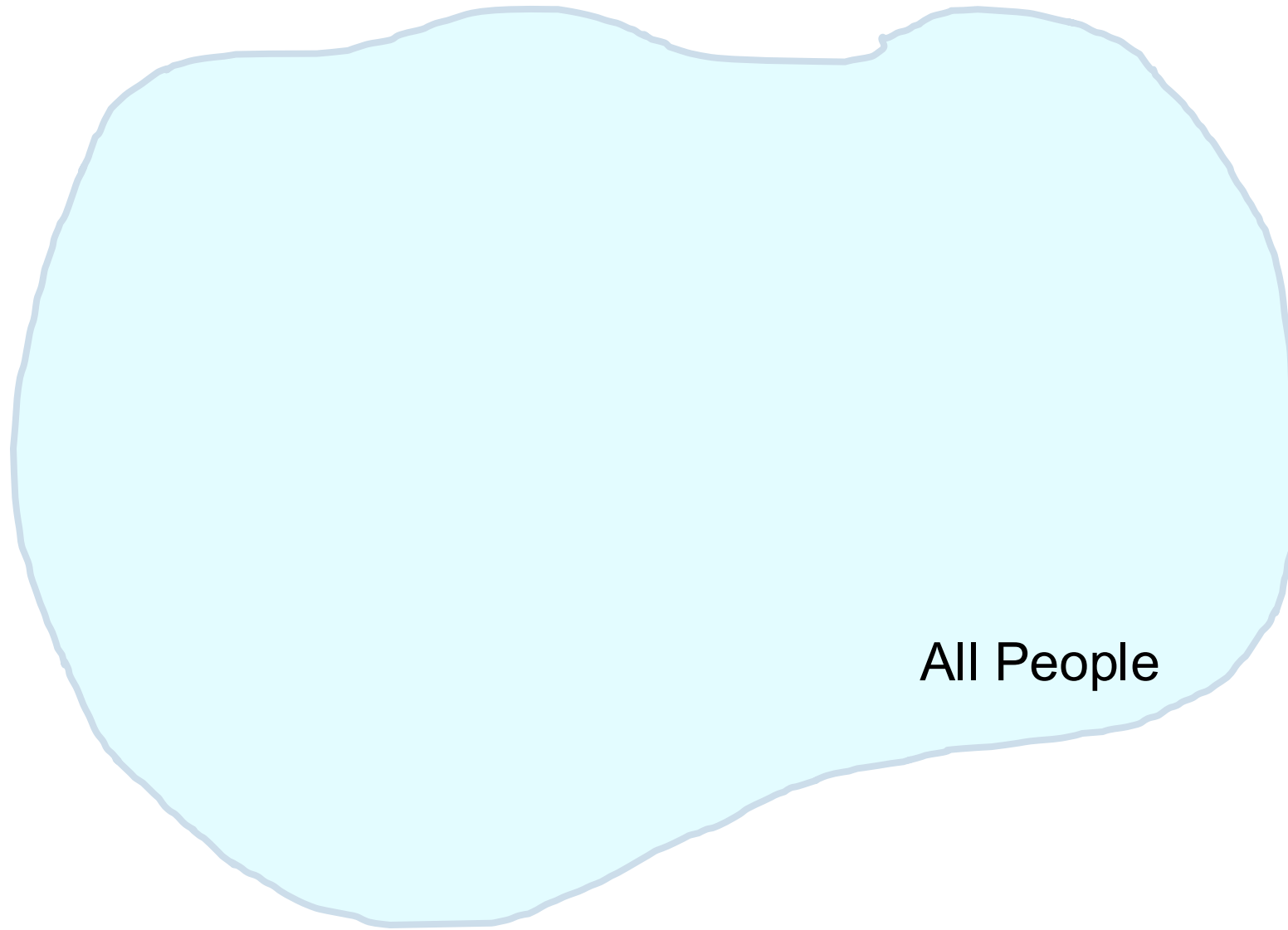
$$P(F | E) = \frac{P(E | F) P(F)}{P(E | F) P(F) + P(E | F^c) P(F^c)}$$

$$P(F | E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \approx \underline{0.330}$$

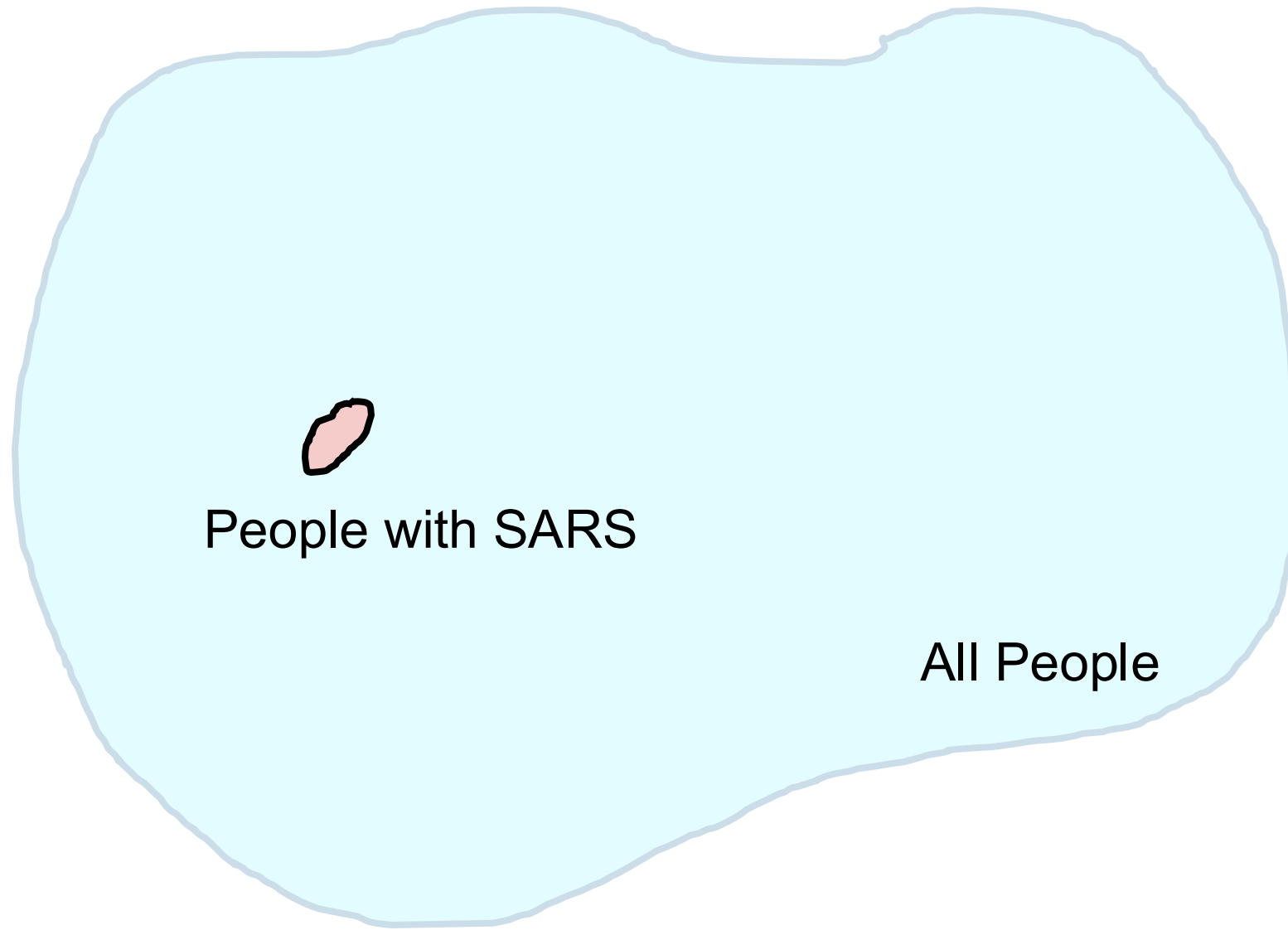


Intuition Time

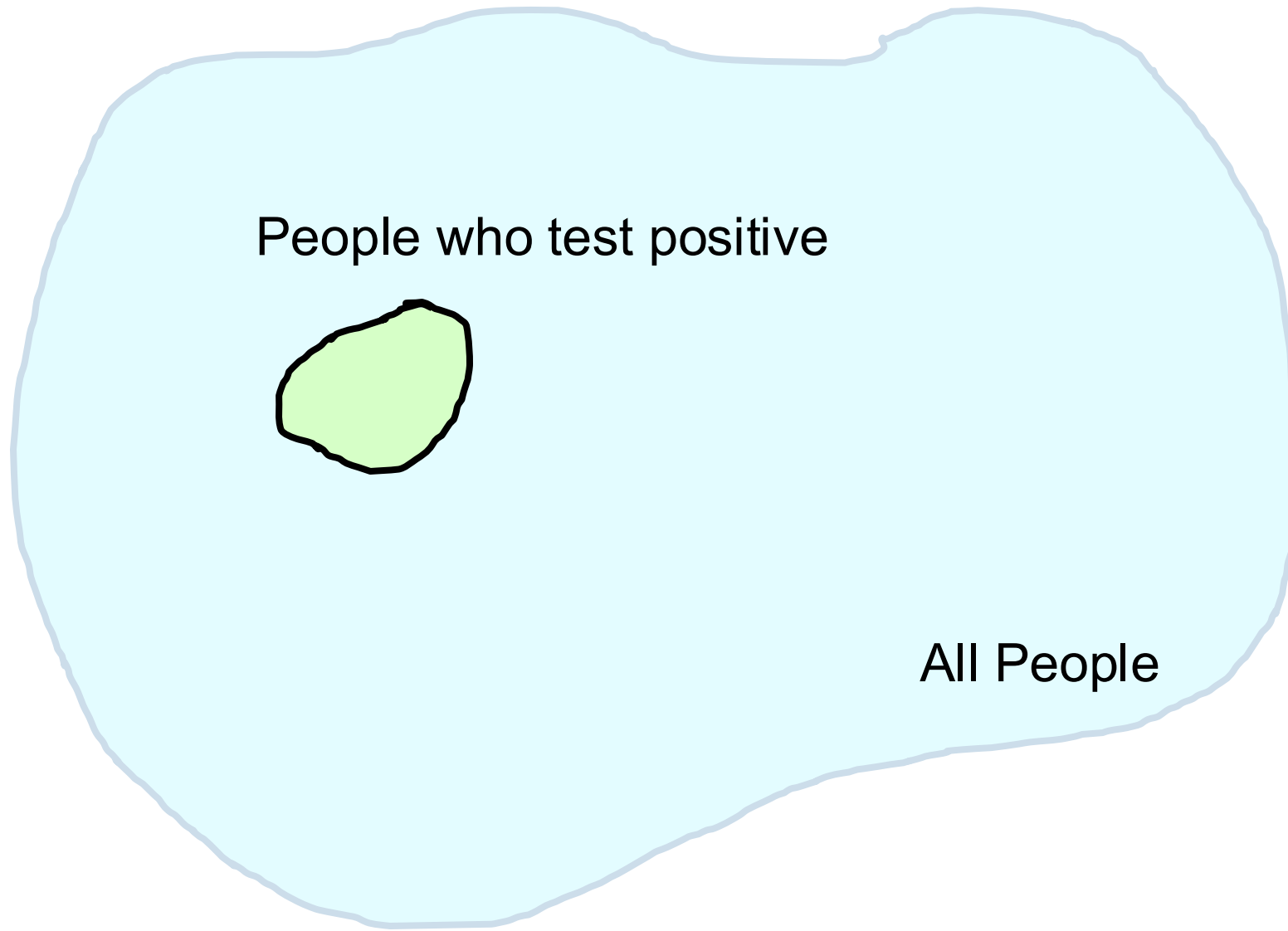
Bayes Theorem Intuition



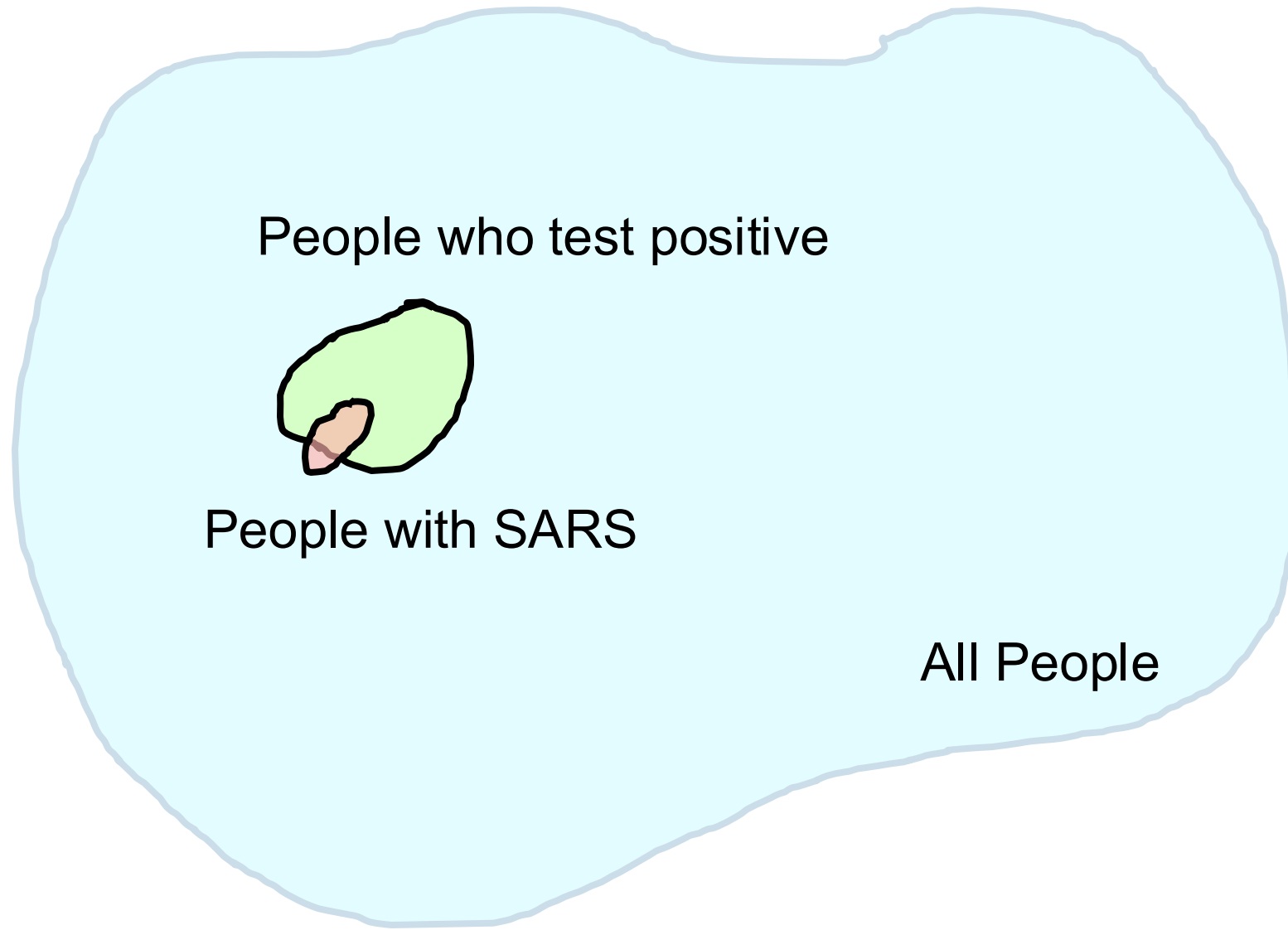
Bayes Theorem Intuition



Bayes Theorem Intuition

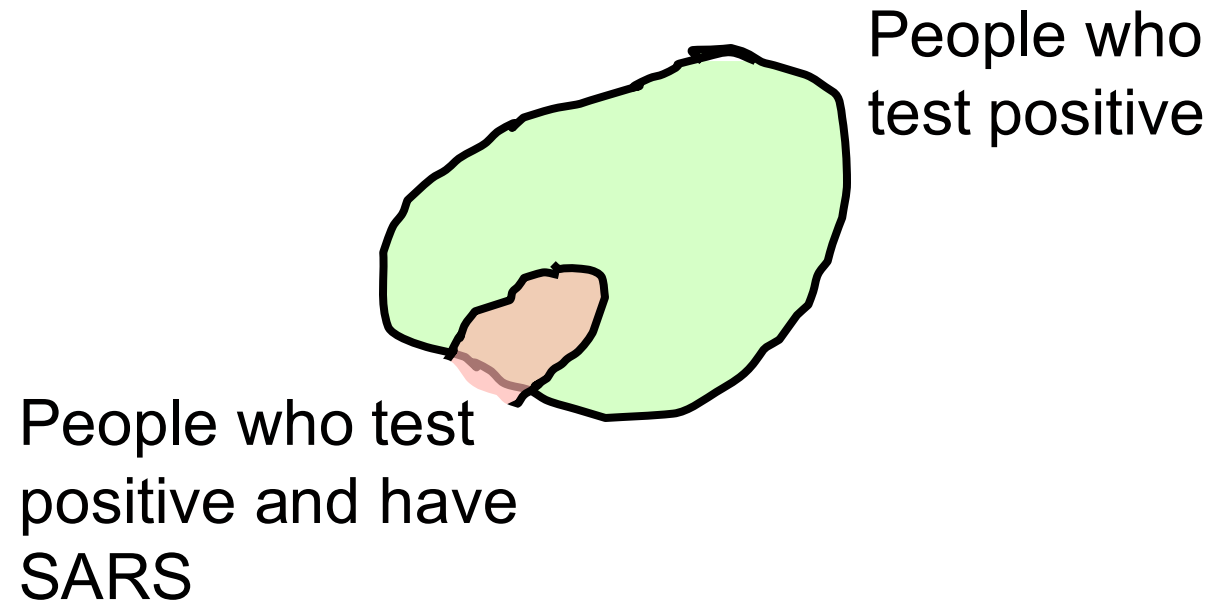


Bayes Theorem Intuition



Bayes Theorem Intuition

Conditioning on a positive result changes the sample space to this:



≈ 0.330

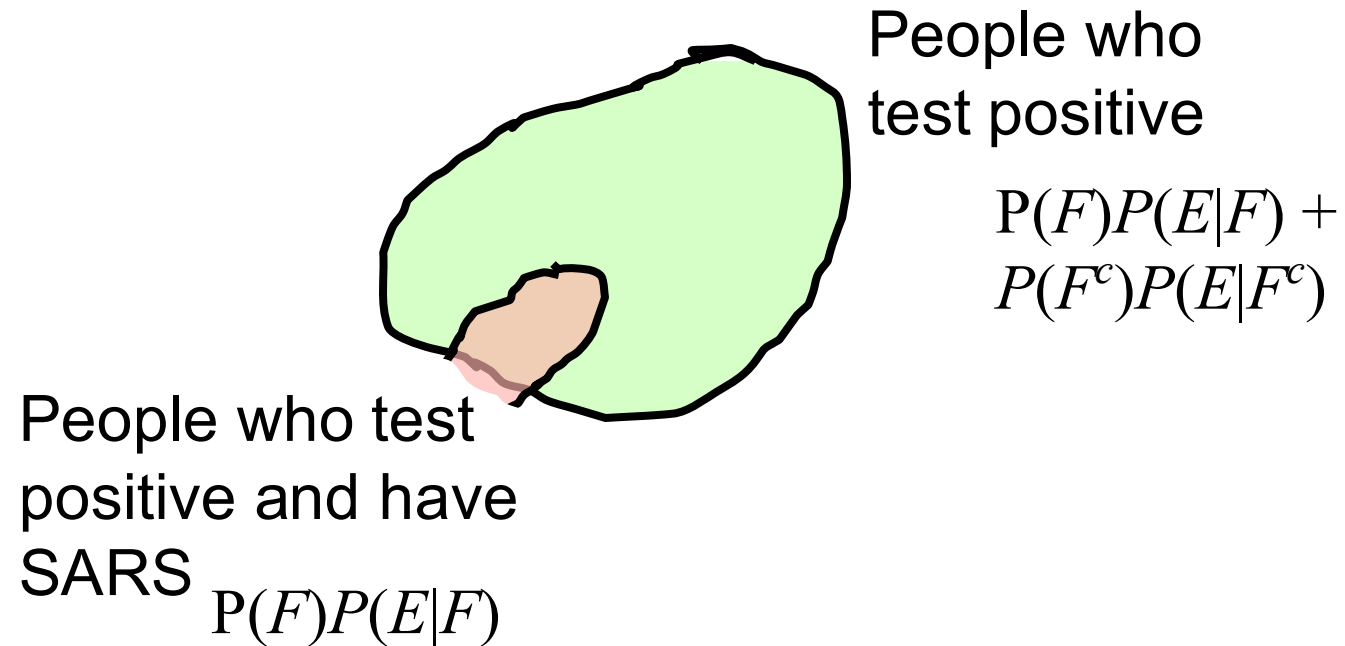


Bayes Theorem Intuition

Let E = you test positive for SARS with this test

Let F = you actually have SARS

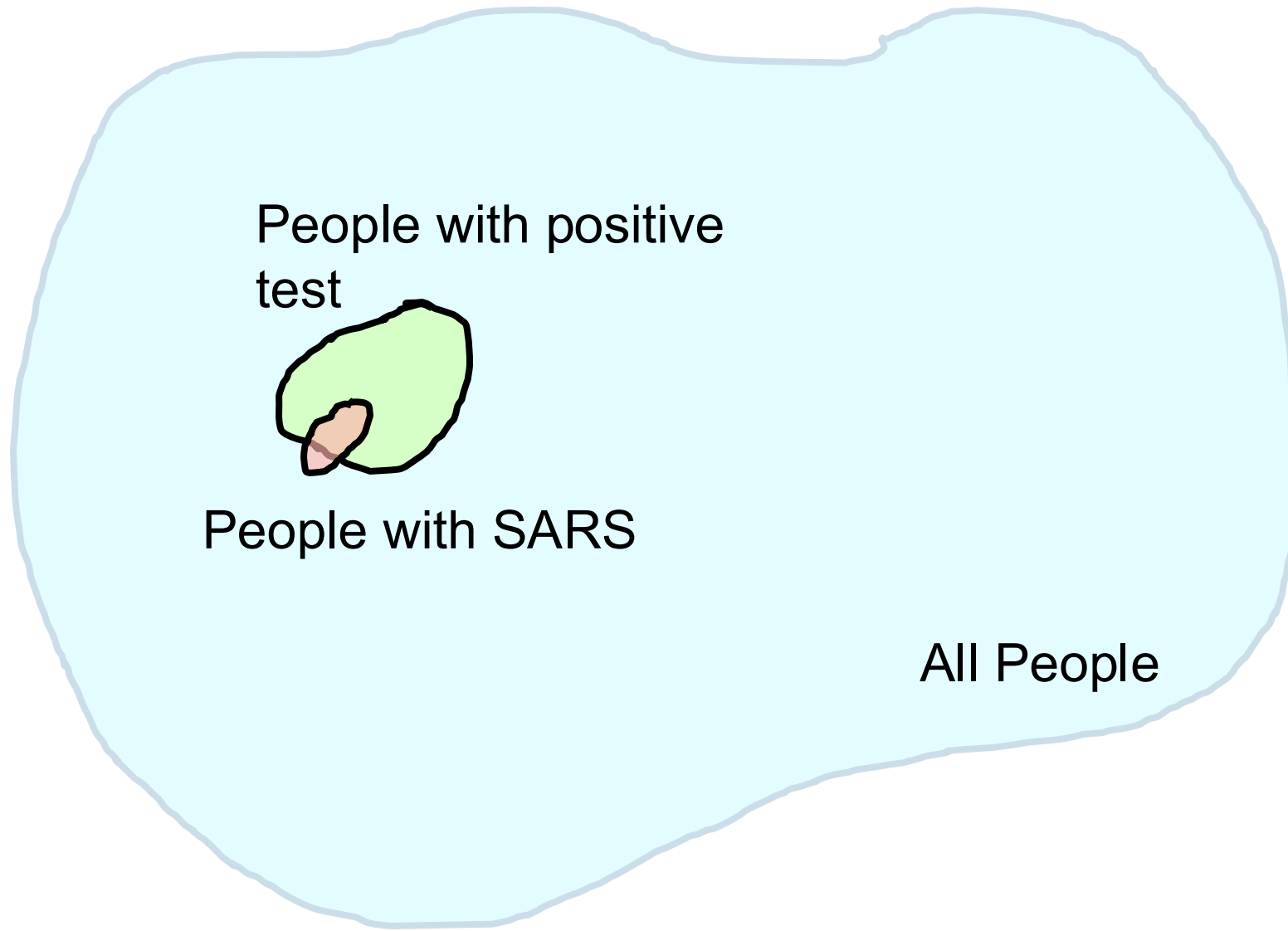
Conditioning on a positive result changes the sample space to this:



≈ 0.330



Bayes Theorem Intuition



Bayes Theorem Intuition

Say we have 1000 people:



5 have SARS and test positive, 985 **do not** have SARS and test negative.
10 **do not** have SARS and test positive. ≈ 0.333



Bayes Theorem Intuition

Conditioned on just those that test positive:



Notice that all the people with SARS are here,
but the group is still mainly folks without SARS

5 have SARS and test positive, 985 **do not** have SARS and test negative.
10 **do not** have SARS and test positive. ≈ 0.333



Why it is still good to get tested

	SARS + (F)	SARS - (F^c)
Test + (E)	0.98 = P(E F)	0.01 = P(E F ^c)
Test - (E^c)	0.02 = P(E ^c F)	0.99 = P(E ^c F ^c)

- Let E^c = you test **negative** for SARS with this test
- Let F = you actually have SARS
- What is P(F | E^c)?

$$P(F | E^c) = \frac{P(E^c | F) P(F)}{P(E^c | F) P(F) + P(E^c | F^c) P(F^c)}$$

$$P(F | E^c) = \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \approx 0.0001$$





Sam Smith

Fri, Aug 29, 12:25 PM



to Chris ▾

email



Dear Professor Piech. Can you help me? I have a large amount of money in my dorm room, but I can't get it out. If you sent me a \$500 coupa card I could distract my TA and pay you back double.

1

Let E be the entire email text. Let F be the event the email is Spam.

2

$$P(F|E) = \frac{P(E|F)P(F)^{60\%}}{P(E|F)P(F)^{60\%} + P(E|F^C)P(F^C)^{40\%}}$$

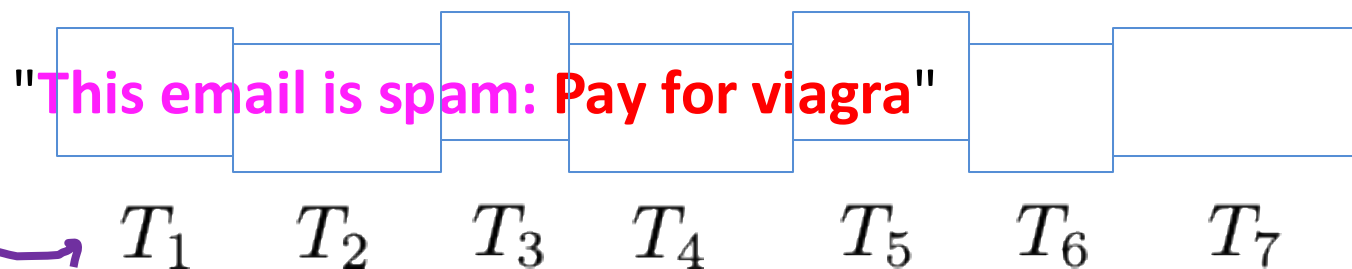
3

$$P(E|F) \approx \frac{\text{string_pr}(f\text{"This email is spam: \{email\}"})}{\text{string_pr}(f\text{"This email is spam: "})}$$

email $P(E|F) \approx \frac{\text{string_pr}(f\text{"This email is spam: \{email\}"})}{\text{string_pr}(f\text{"This email is spam: "})}$

spam

The token at index 1 is "this"



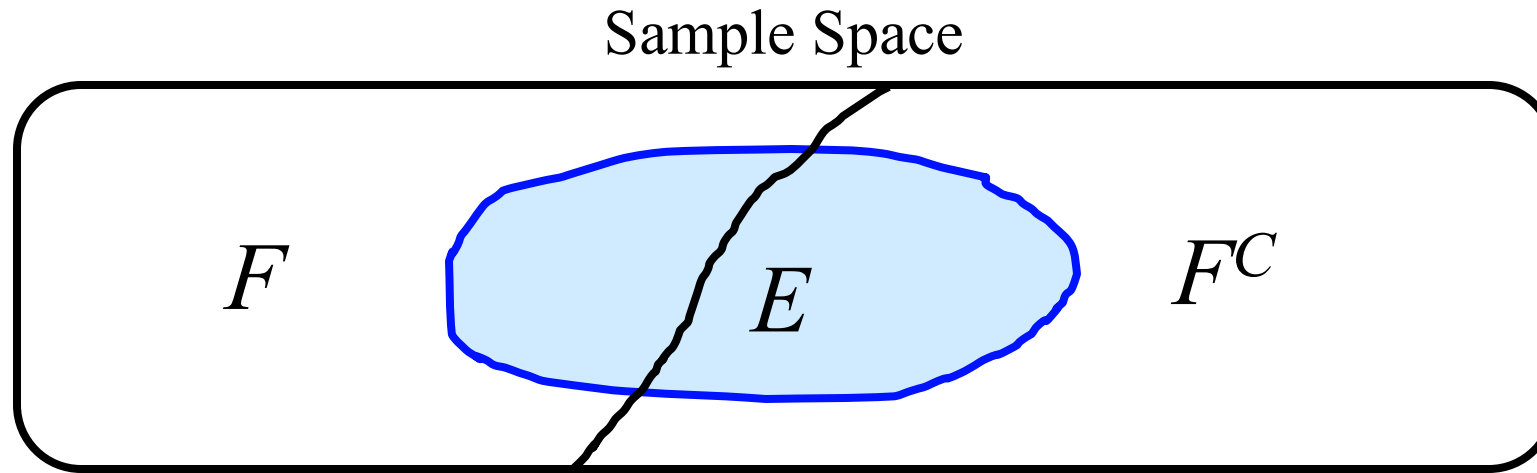
$$P(E|F) \approx \frac{P(T_1, T_2, \dots, T_7)}{P(T_1, T_2, T_3, T_4)}$$

$$\approx P(T_5, T_6, T_7 | T_1, T_2, T_3, T_4)$$

$$\approx P(\text{"Pay for viagra"} | \text{"This email is spam: "})$$

End Review

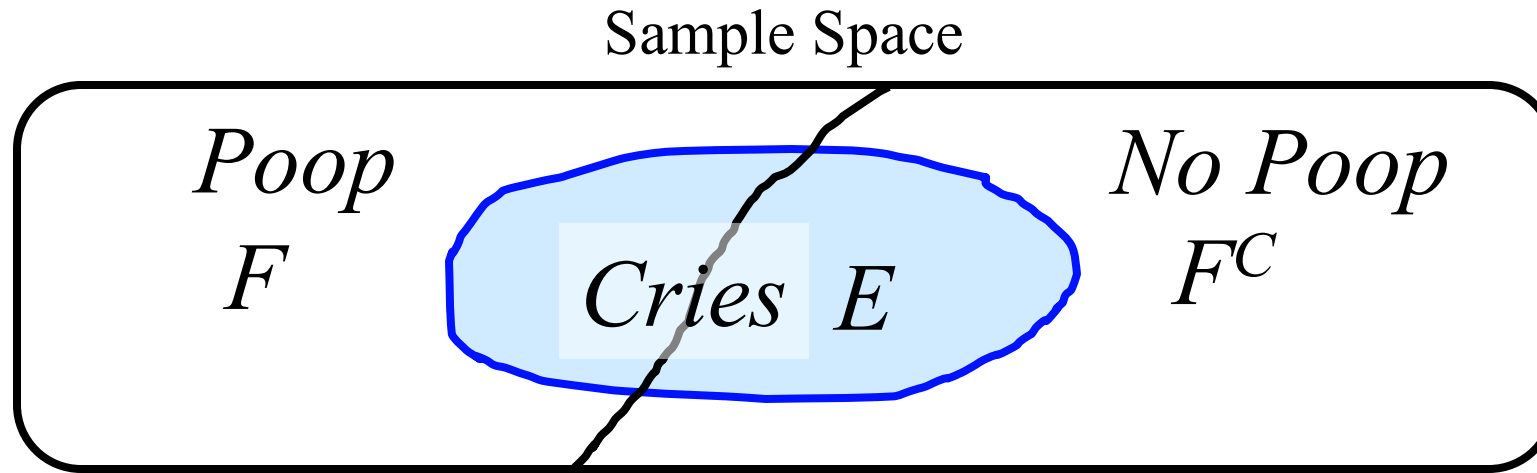
Law of Total Probability



$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E|F)P(F) + P(E|F^C)P(F^C) \end{aligned}$$



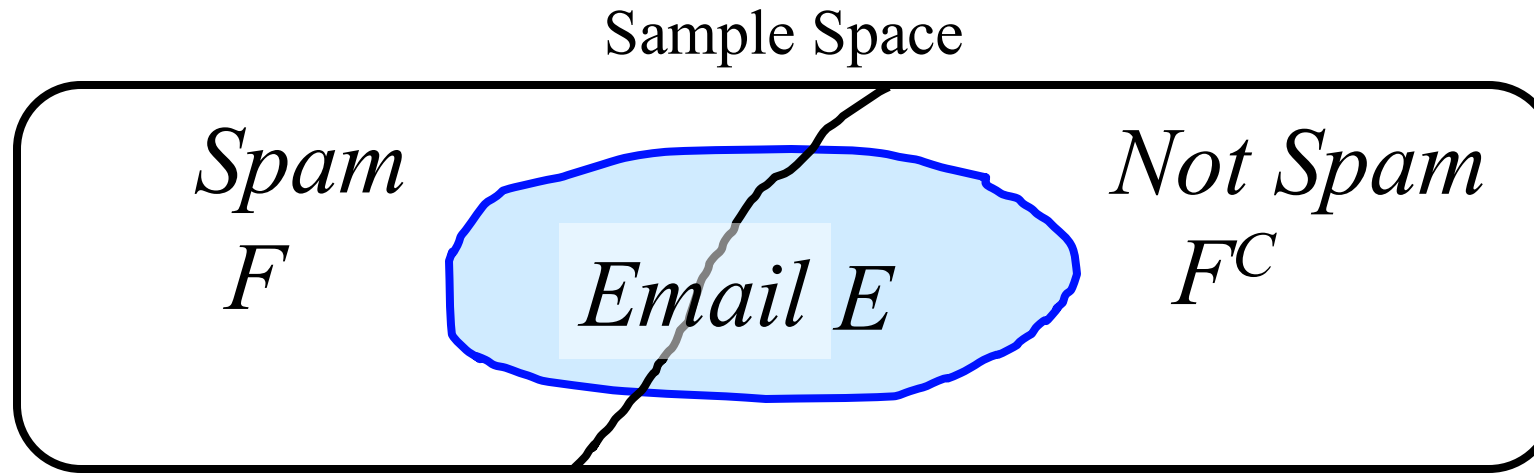
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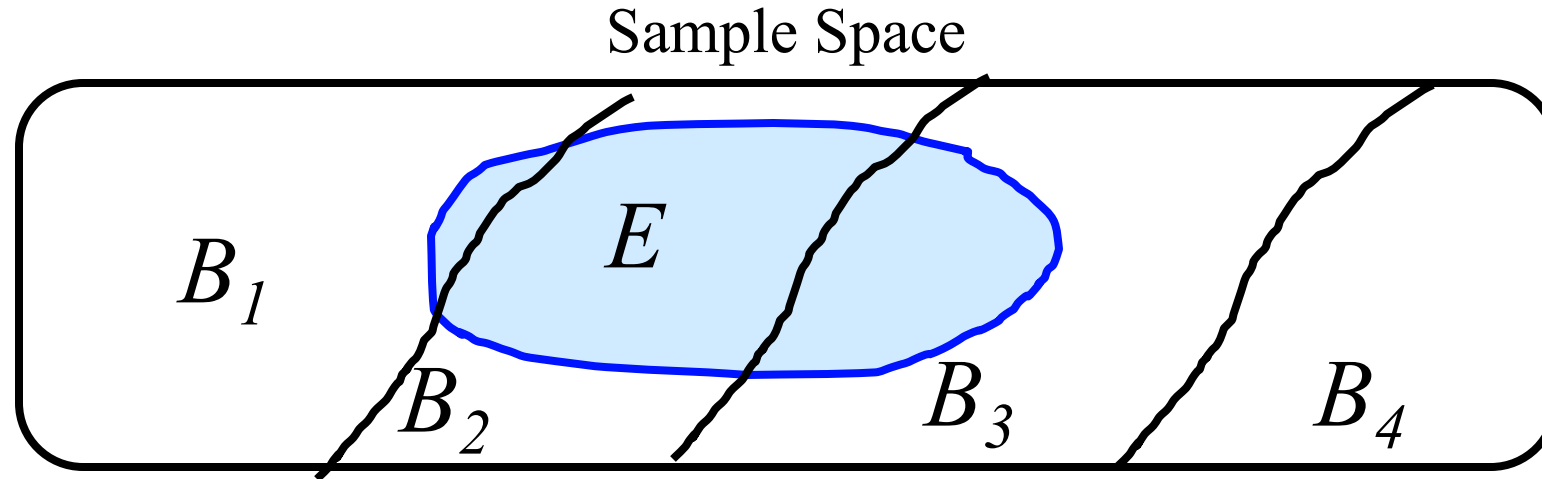
Law of Total Probability



$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E|F)P(F) + P(E|F^C)P(F^C) \end{aligned}$$



Law of Total Probability



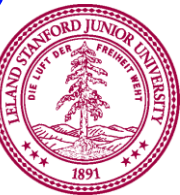
Thm For **partition events** B_1, B_2, \dots, B_n

$$\begin{aligned} P(E) &= \sum_i P(B_i \text{ and } E) \\ &= \sum_i P(E|B_i)P(B_i) \end{aligned}$$

partition events:

B_1, B_2, \dots, B_n
are mutually exclusive

B_1 or B_2 or ... or $B_n = S$



Results for San Francisco, CA



49 °F | °C

Precipitation: 90%
Humidity: 74%
Wind: 8 mph

Weather

Monday

Rain showers

Background event.
Where is the person in
San Francisco?

Results for San Francisco, CA



49 °F | °C

Precipitation: 90%

Humidity: 74%

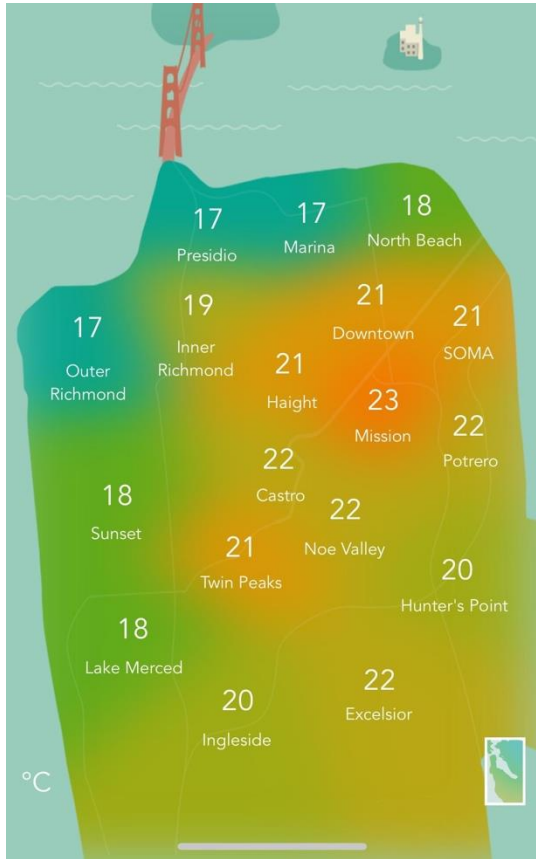
Wind: 8 mph

Weather

Monday

Rain showers

Background event.
Where is the person in San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

It rains tomorrow

$$P(R|D_i)$$

Person is in district i

$$P(D_i)$$

Want:

$$P(R)$$

Results for San Francisco, CA



49 °F | °C

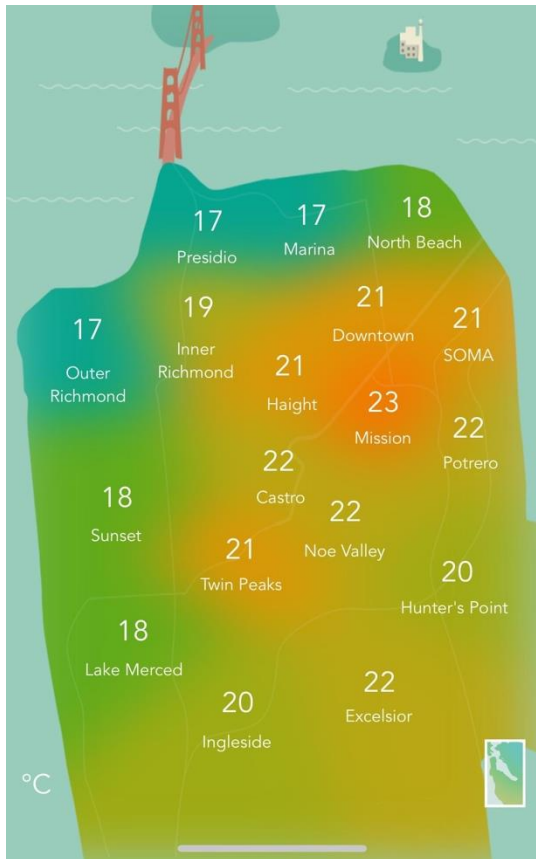
Precipitation: 90%
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Weather

Monday

Rain showers

Background event.
Where is the person in
San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

	Mission District	Presidio	...	SOMA
$P(R D_i)$	0.23	0.84	...	0.52
$P(D_i)$	0.15	0.02		0.24

Want:

$$P(R)$$

Results for San Francisco, CA



49 °F | °C

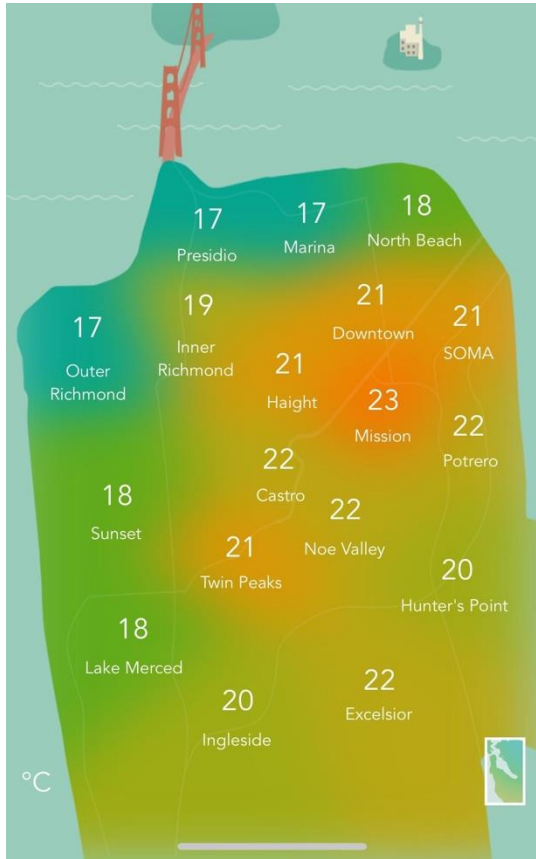
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Know:

	Mission District	Presidio	...	SOMA
$P(R D_i)$	0.23	0.84	...	0.52
$P(D_i)$	0.15	0.02		0.24

Want:

$$P(R) = \sum_{\text{district } i} P(R \text{ and } D_i) = \sum_{\text{district } i} P(R|D_i) \cdot P(D_i)$$

Monty Hall Problem

Monty Hall Problem



Monty Hall Problem from Let's Make a Deal

Behind one door is a prize (equally likely to be any door).

Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Should we switch?



Doors A,B,C

Note: If we don't switch,

$$P(\text{Win}) = 1/3$$



In the world where we switch

Without loss of generality, say we pick A (out of Doors A,B,C).

1/3



1/3



1/3



A: Prize in Door A

- Host opens B or C
- We switch
- We always lose

$$P(\text{Win} | A) = 0$$

B: Prize in Door B

- Host must open C
- We switch to B
- We always win

$$P(\text{Win} | B) = 1$$

C: Prize in Door C

- Host must open B
- We switch to C
- We always win

$$P(\text{Win} | C) = 1$$

$$\begin{aligned} P(\text{Win}) &= P(\text{Win}|A)P(A) + P(\text{Win}|B)P(B) + P(\text{Win}|C)P(C) \\ &= \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

You should switch!



Marilyn Vos Savant



Ask Marilyn™

BY MARILYN VOS SAVANT



Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

$$\left\{ \begin{array}{l} \frac{1}{1000} = P(\text{envelope is prize}) \\ \frac{999}{1000} = P(\text{other 999 envelopes have prize}) \end{array} \right.$$

2. I open 998 of remaining 999 (showing they are empty).

3. Should you switch?

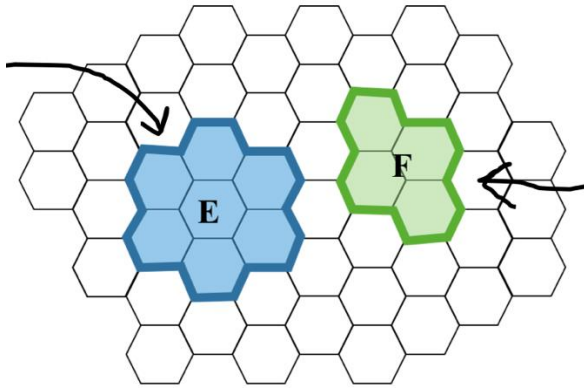
No: $P(\text{win without switching}) =$

$$\frac{1}{\text{original \# envelopes}}$$

Yes: $P(\text{win with new knowledge}) =$

$$\frac{\text{original \# envelopes} - 1}{\text{original \# envelopes}}$$

Learning Goals for Rest of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

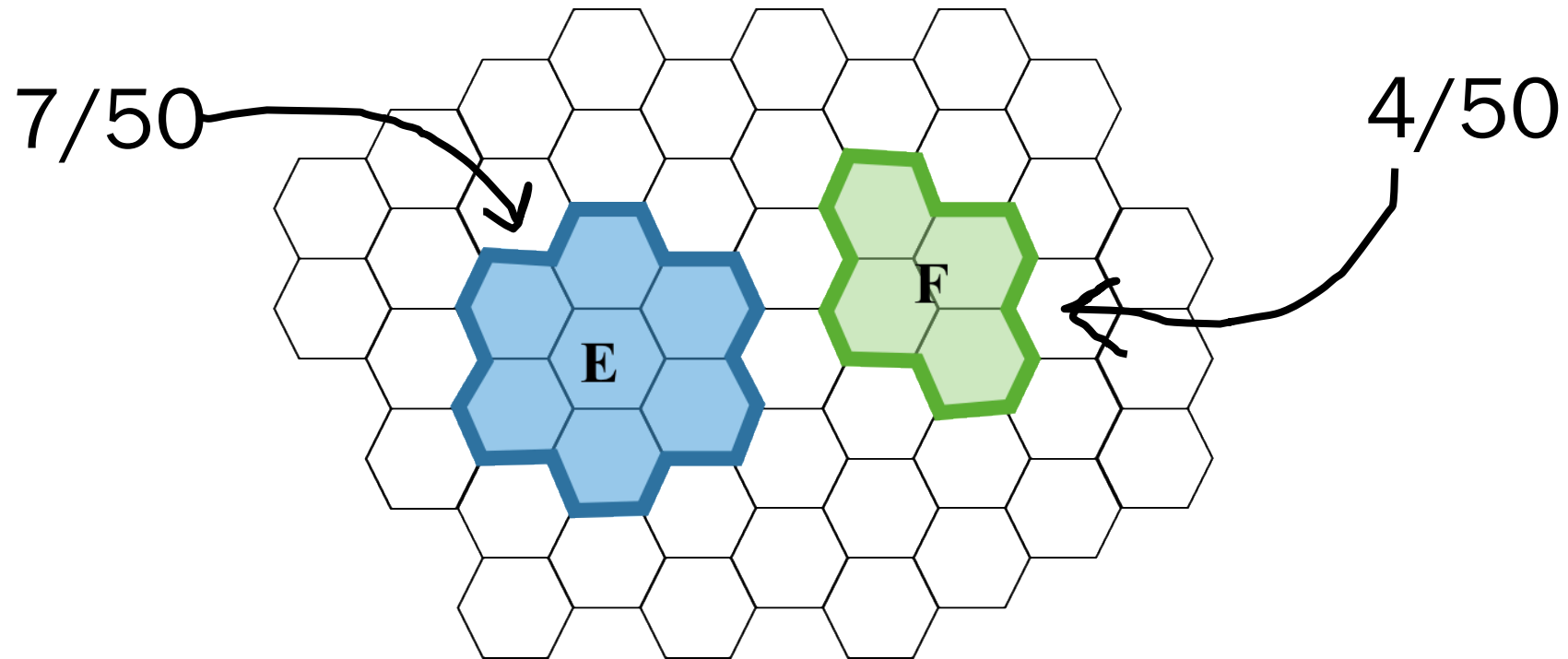
$$P(A \text{ and } B) = P(A) \cdot P(B)$$



Pedagogical Pause

Probability of “OR”

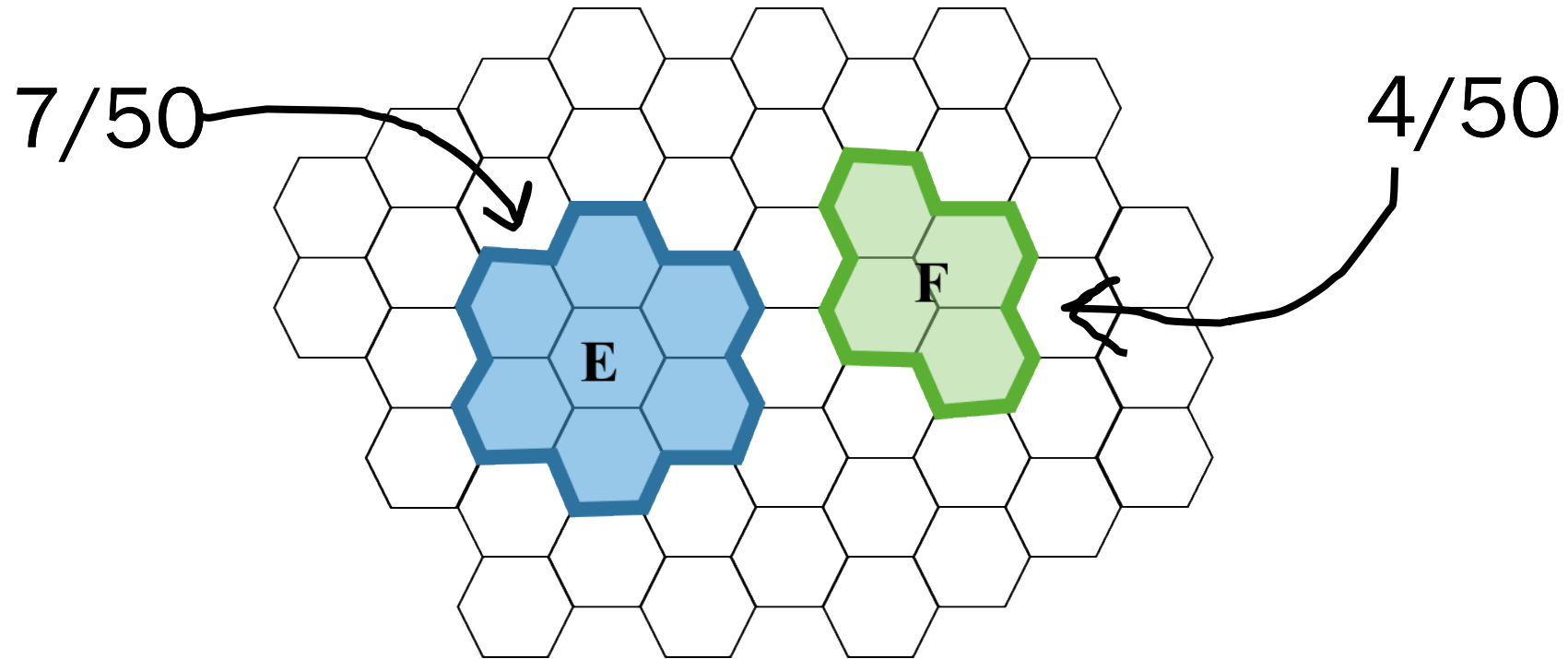
Review: OR with Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \text{ or } F) = P(E) + P(F)$$

Review: OR with Mutually Exclusive Events

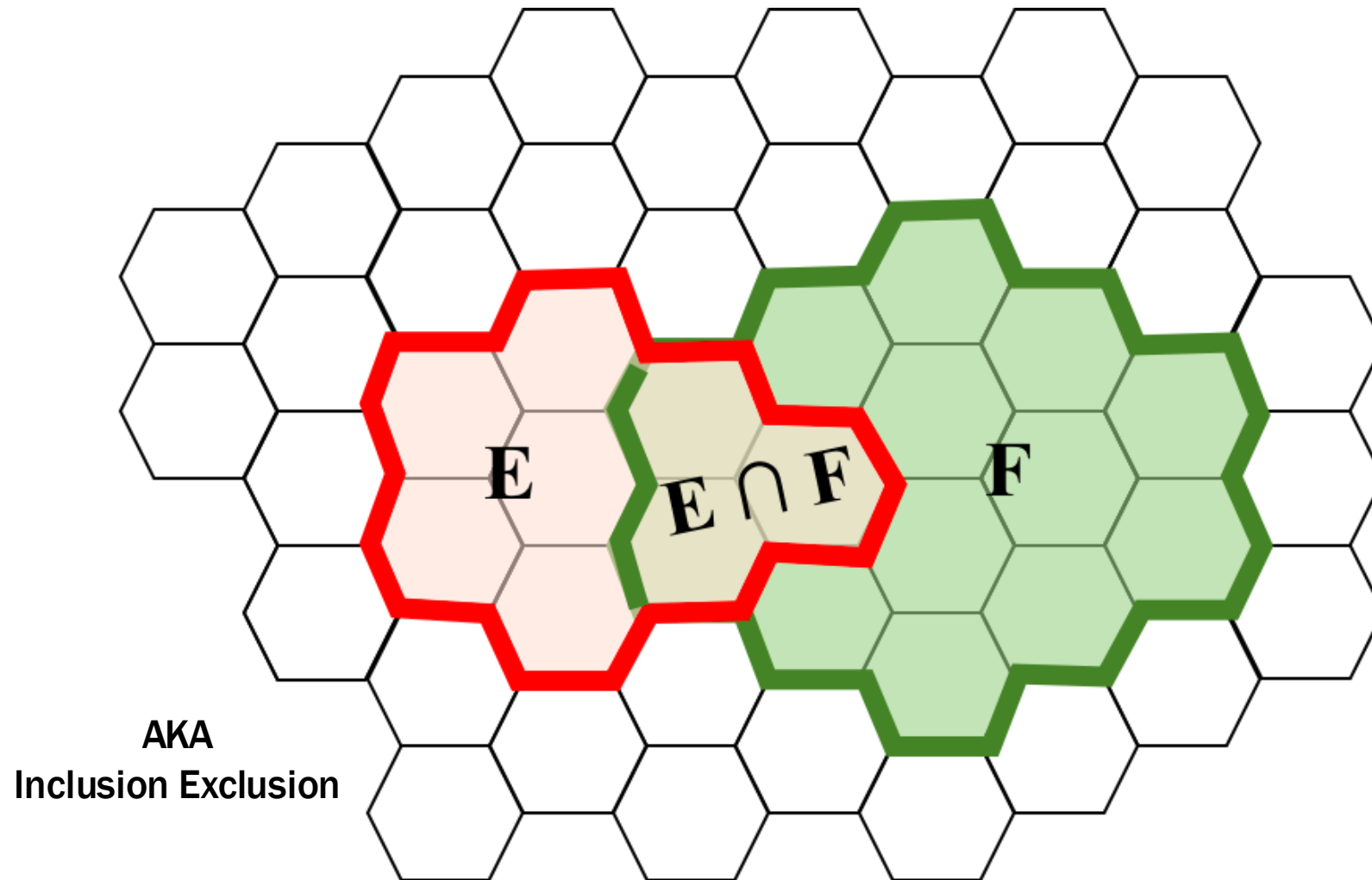


If events are mutually exclusive, probability of OR is simple:

$$P(E \text{ or } F) = \frac{7}{50} + \frac{4}{50} = \frac{11}{50}$$

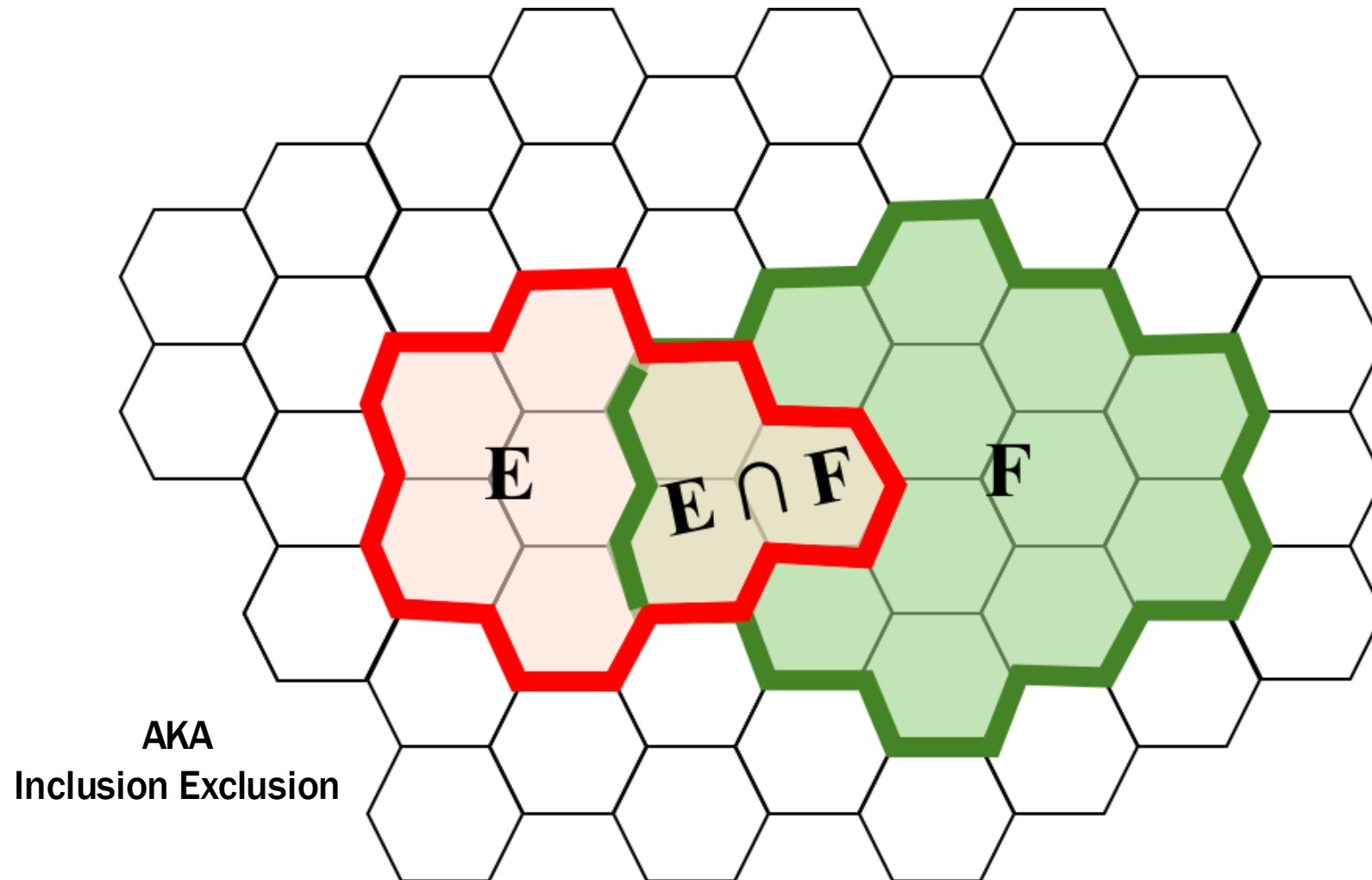
What about when they are not
Mutually exclusive?

OR *without* Mutually Exclusive Events



$$P(E \text{ or } F) = P(E) + P(F) - P(EF)$$

OR *without* Mutually Exclusive Events

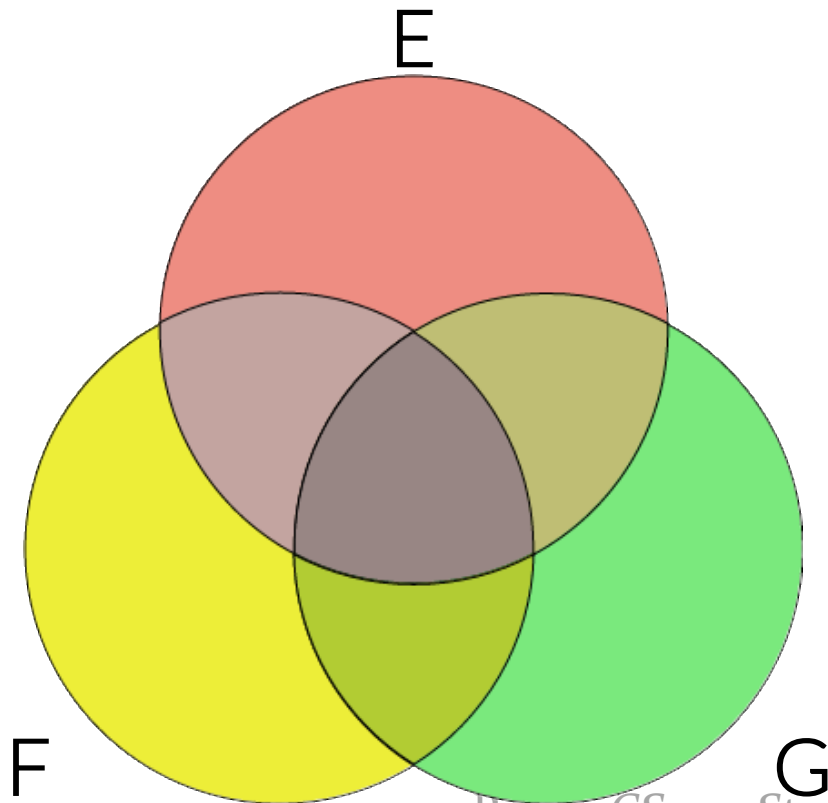


$$P(E \text{ or } F) = \frac{8}{50} + \frac{14}{50} - \frac{3}{50}$$

More than two sets?

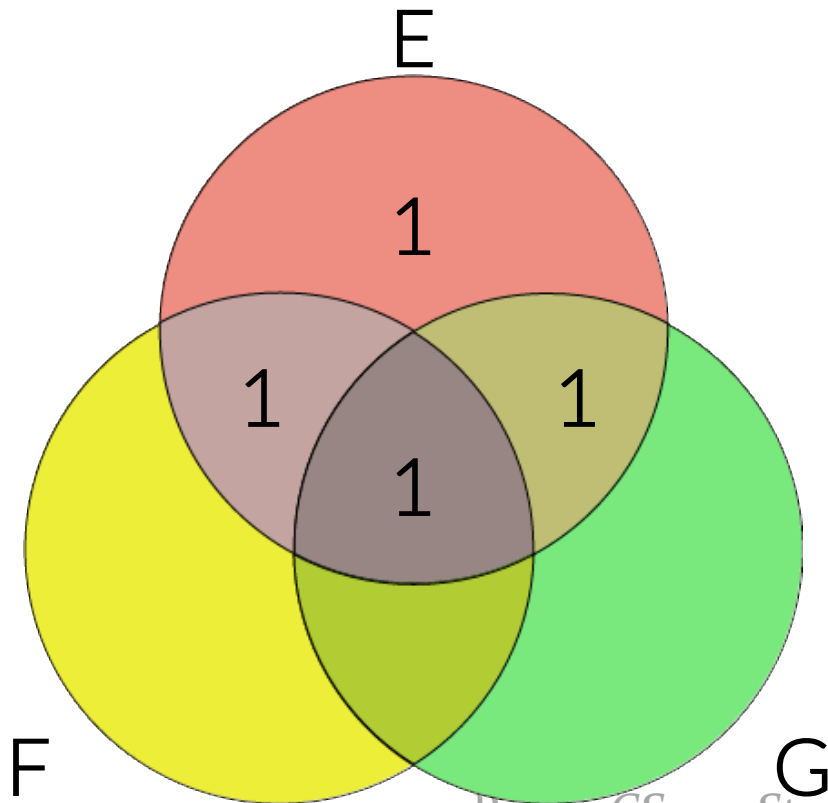
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) =$$



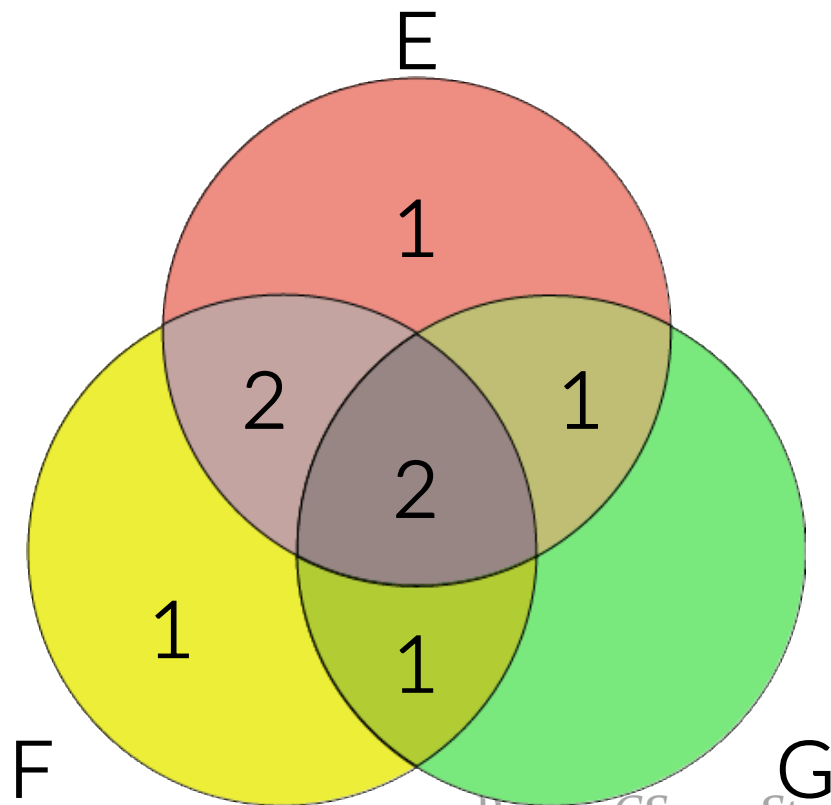
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E)$$



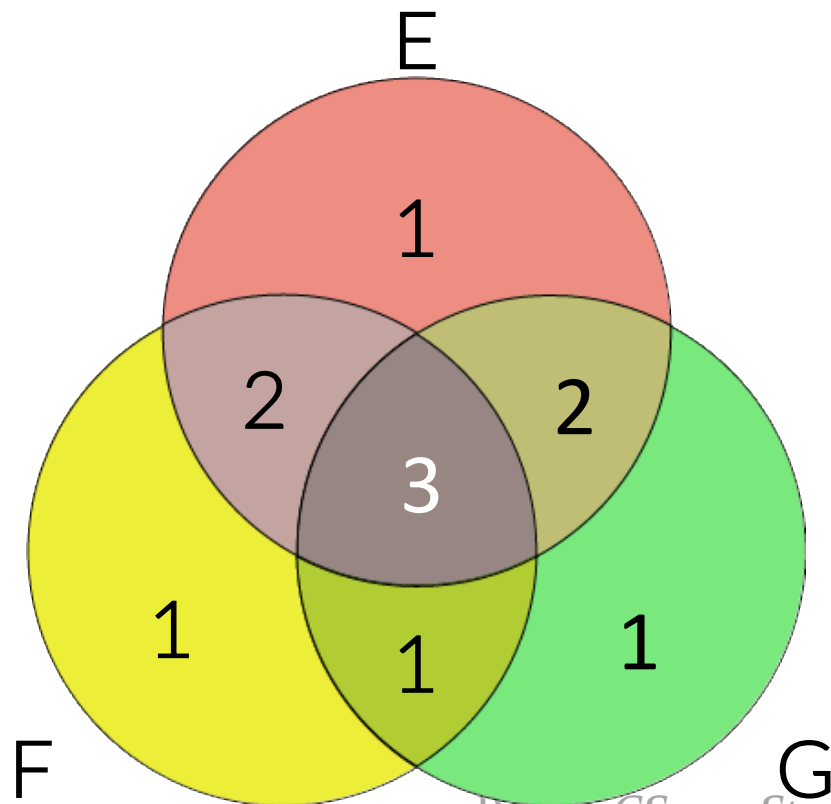
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$$



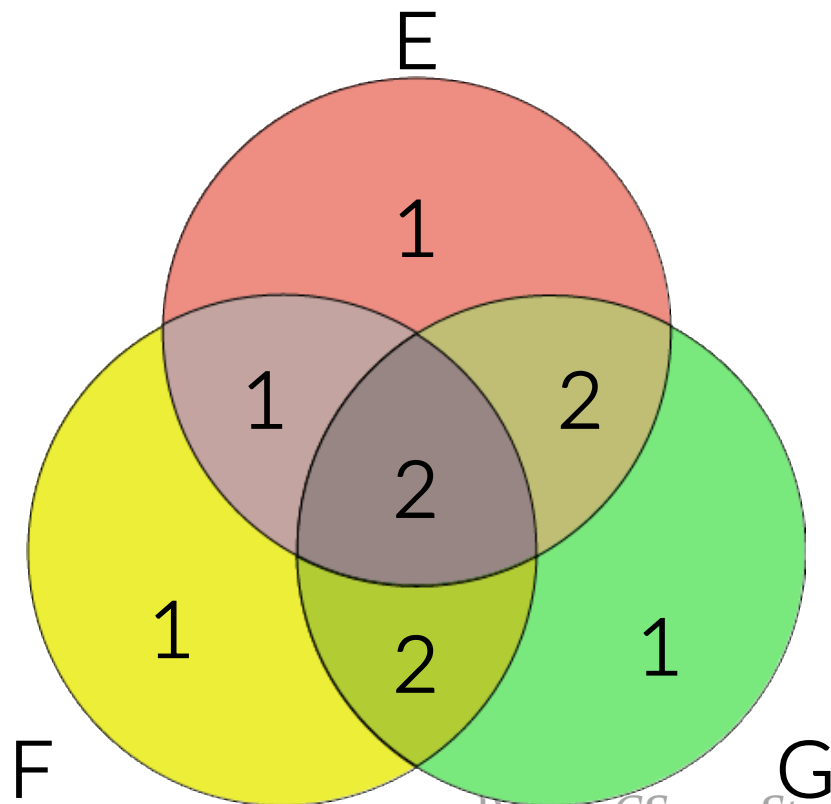
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G)$$



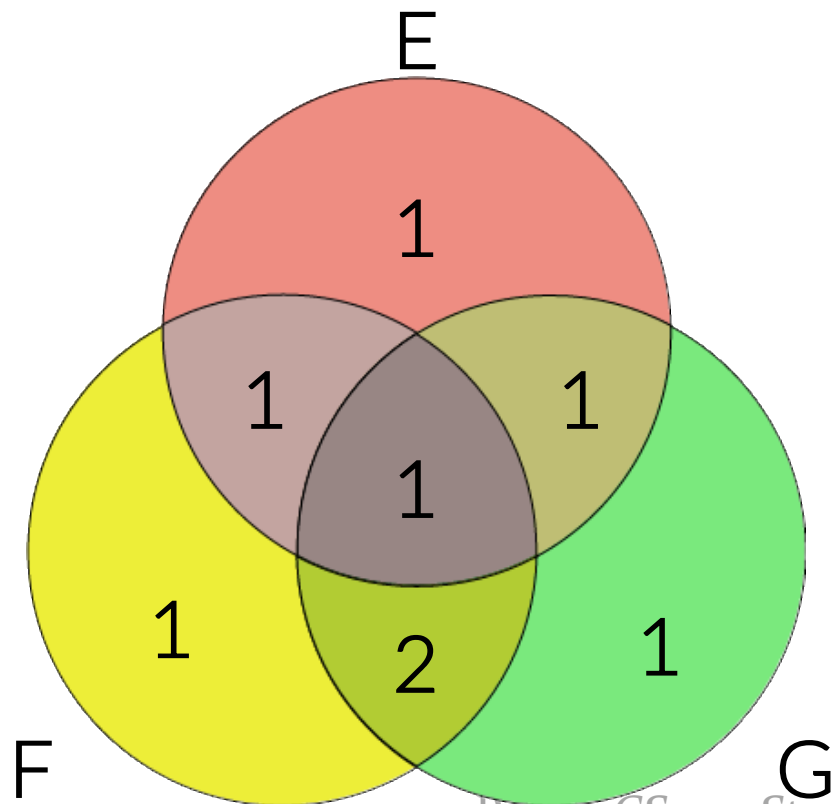
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) - P(EF)$$



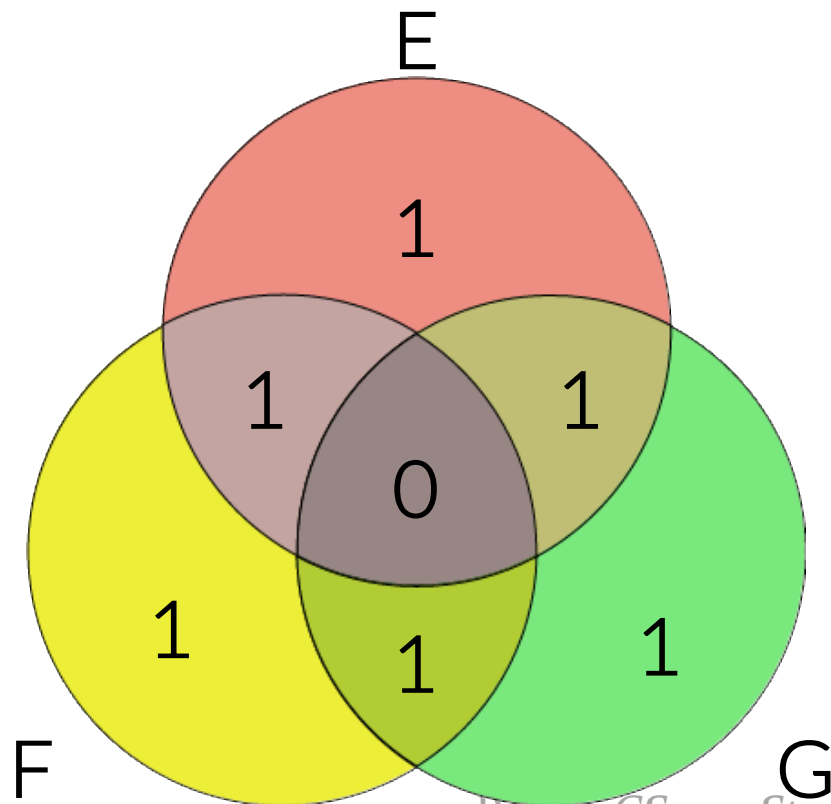
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) \\ - P(EF) - P(EG)$$



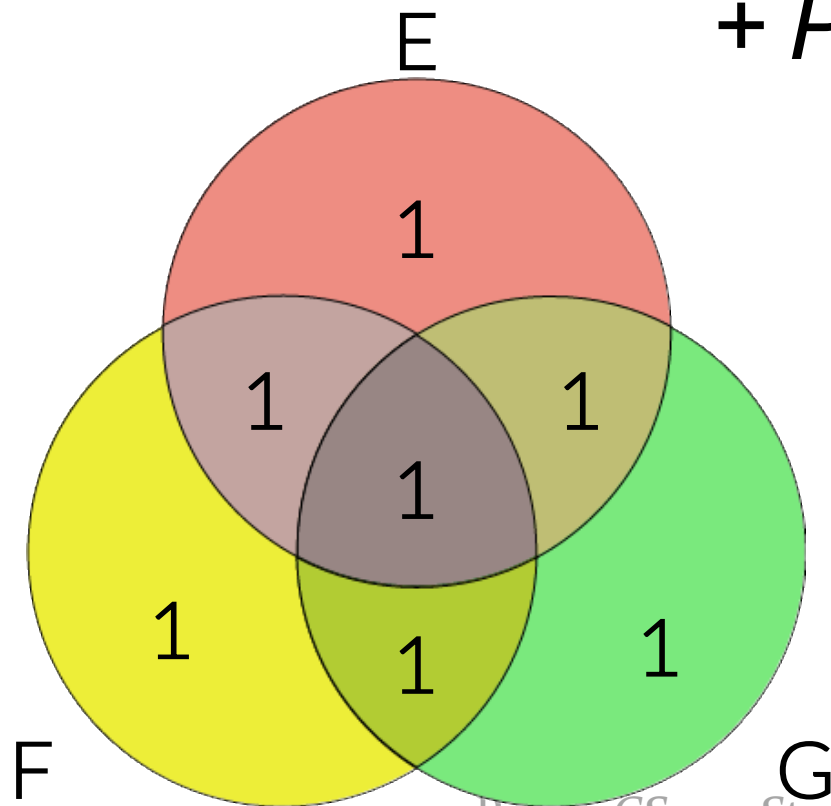
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) \\ - P(EF) - P(EG) - P(FG)$$



Inclusion / Exclusion with Three Events

$$\begin{aligned} P(E \text{ or } F \text{ or } G) &= P(E) + P(F) + P(G) \\ &\quad - P(EF) - P(EG) - P(FG) \\ &\quad + P(EFG) \end{aligned}$$



Inclusion / Exclusion with 3 Events

$$\begin{aligned} P(E_1 \text{ or } E_2 \text{ or } E_3) = & \\ & + P(E_1) \\ & + P(E_2) \\ & + P(E_3) \\ & - P(E_1 \text{ and } E_2) \\ & - P(E_1 \text{ and } E_3) \\ & - P(E_2 \text{ and } E_3) \\ & + P(E_1 \text{ and } E_2 \text{ and } E_3) \end{aligned}$$

Inclusion / Exclusion with 4 Events

$$\begin{aligned} P(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } E_4) = & \\ & + P(E_1) \\ & + P(E_2) \\ & + P(E_3) \\ & + P(E_4) \\ & - P(E_1 \text{ and } E_2) \\ & - P(E_1 \text{ and } E_3) \\ & - P(E_1 \text{ and } E_4) \\ & - P(E_2 \text{ and } E_3) \\ & - P(E_2 \text{ and } E_4) \\ & - P(E_3 \text{ and } E_4) \\ & + P(E_1 \text{ and } E_2 \text{ and } E_3) \\ & + P(E_1 \text{ and } E_2 \text{ and } E_4) \\ & + P(E_1 \text{ and } E_3 \text{ and } E_4) \\ & + P(E_2 \text{ and } E_3 \text{ and } E_4) \\ & - P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } E_4) \end{aligned}$$

General Inclusion / Exclusion

$$P(E_1 \text{ or } E_2 \text{ or } \dots \text{ or } E_n) = \sum_{r=1}^n (-1)^{r+1} Y_r$$

Y_1 = Sum of all events on their own

$$\sum_i P(E_i)$$

Y_2 = Sum of all pairs of events

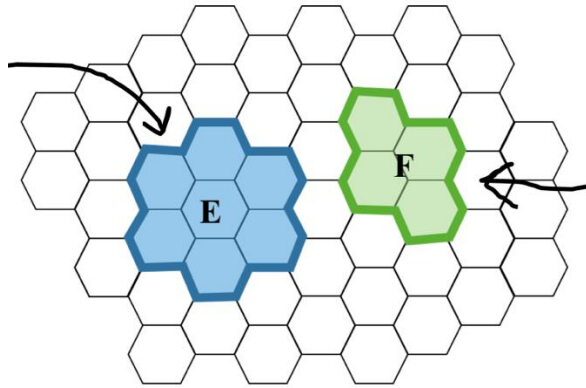
$$\sum_{i,j} P(E_i \text{ and } E_j) \quad \text{s.t. } i \neq j$$

Y_3 = Sum of all triples of events

$$\sum_{i,j,k} P(E_i \text{ and } E_j \text{ and } E_k) \quad \text{s.t. } i \neq j, j \neq k, i \neq k$$

Where Y_r is the sum, for all combinations of r events, of the probability of the intersection of those events

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Probability of “AND”



We the People of the United States, in order to form a more perfect Union, to establish Justice, to insure domestic Tranquility, provide for the common defence, and our Posterity, do ordain and establish this Constitution.

Article I
Section 1
All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Independence

Two events A and B are called **independent** if:

$$P(A) = P(A|B)$$

Knowing that event B happened, doesn't change our belief that A will happen.

Otherwise, they are called **dependent** events

Alternative Definition of Independence

Notation for *and*

$$\begin{aligned} P(A, B) &= P(A) \cdot P(B|A) \\ &= P(A) \cdot P(B) \end{aligned}$$

Chain rule

Since B is independent of A

If you show this is true, you have proved the two events are independent!



If events are *independent*
probability of AND is easy!

*You will need to use this “trick” with high probability

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 1$
- Let G be event: $D_1 + D_2 = 5$

What is $P(E)$, $P(F)$, and $P(E F)$?

- $P(E) = 1/6$, $P(F) = 1/6$, $P(E F) = 1/36$
- $P(E F) = P(E) P(F) \rightarrow E$ and F independent

What is $P(E)$, $P(G)$, and $P(E G)$? $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$

- $P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(E G) = 1/36$
- $P(E G) \neq P(E) P(G) \rightarrow E$ and G dependent

Intuition through proofs:

Independence is reciprocal

If A is independent of B, then B is independent of A

$$P(A) = P(A|B)$$

$$P(B|A) = P(B)$$

Proof:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' Thm.

$$= \frac{P(A)P(B)}{P(A)}$$

Because A is independent of B

$$= P(B)$$

Independence of a complement

Given independent events A and B , prove that A and B^C are independent

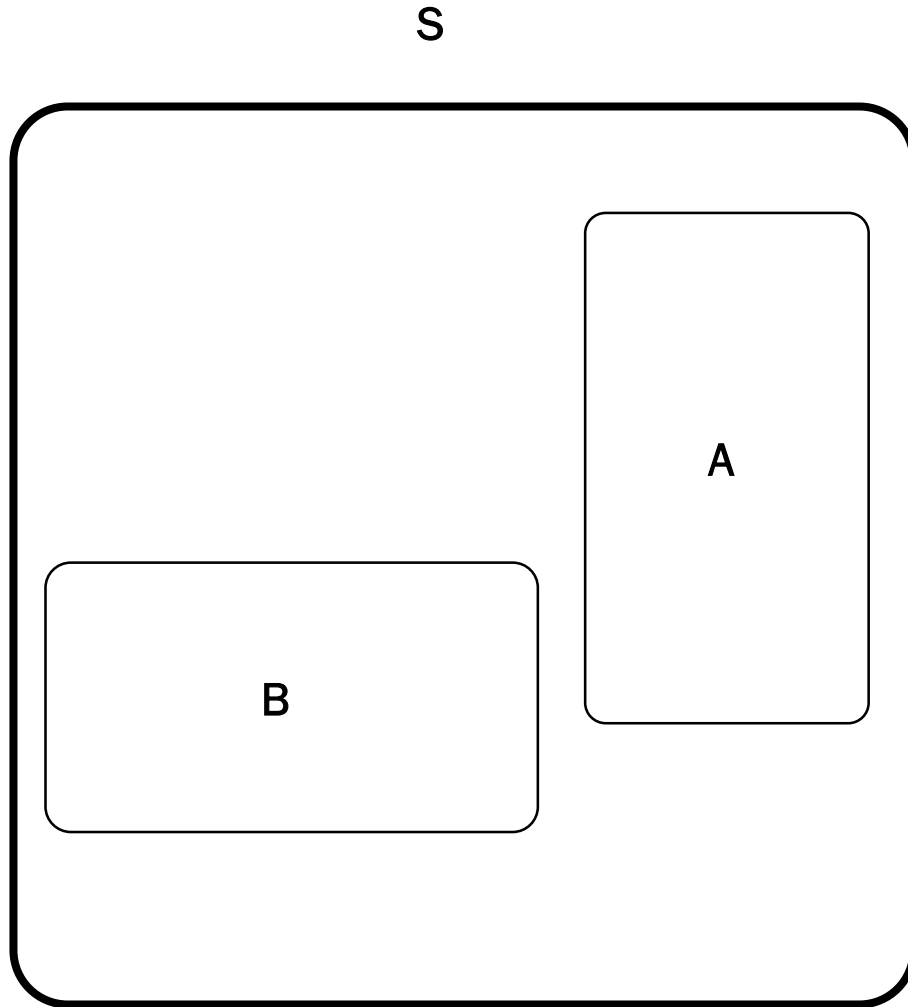
We want to show that $P(AB^C) = P(A)P(B^C)$

$$\begin{aligned}P(AB^C) &= P(A) - P(AB) && \text{By Total Law of Prob.} \\ &= P(A) - P(A)P(B) && \text{By independence} \\ &= P(A)[1 - P(B)] && \text{Factoring} \\ &= P(A)P(B^C) && \text{Since } P(B) + P(B^C) = 1\end{aligned}$$

So if A and B are independent A and B^C are also independent

What does independence look like?

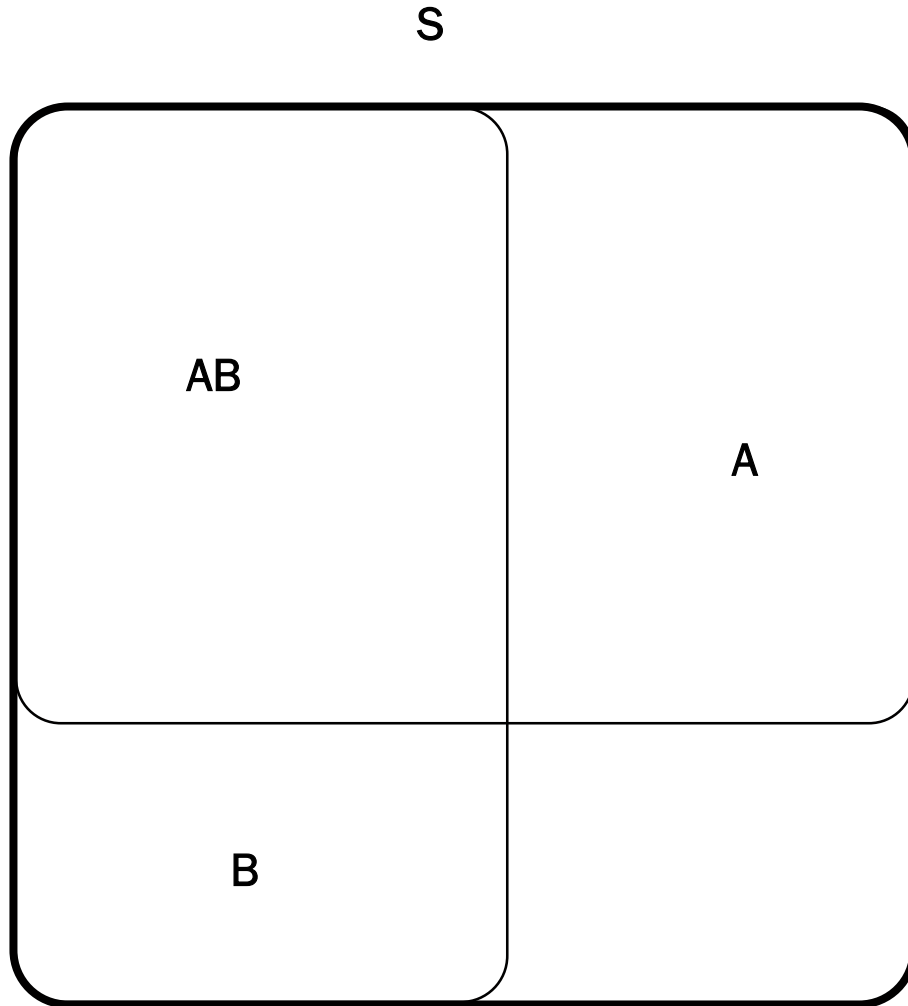
Independence



Independence Definition 1:

$$P(A \cap B) = P(A)P(B)$$
$$\frac{|A \cap B|}{|S|} = \frac{|A|}{|S|} \cdot \frac{|B|}{|S|}$$

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \not\rightarrow \frac{|B|}{|S|}$$

Independence Definition 2:

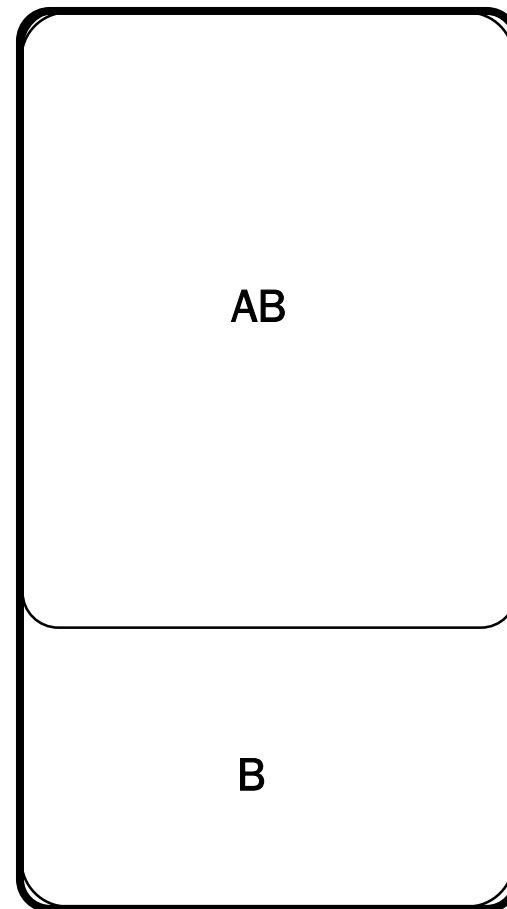
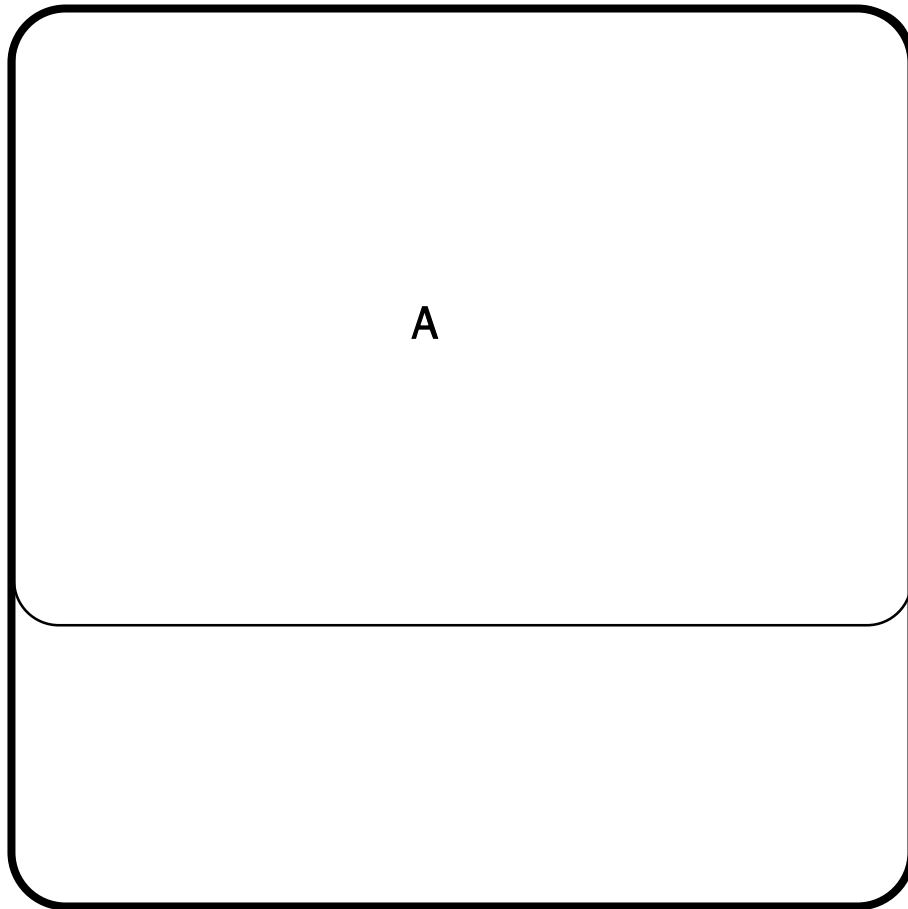
$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Independence

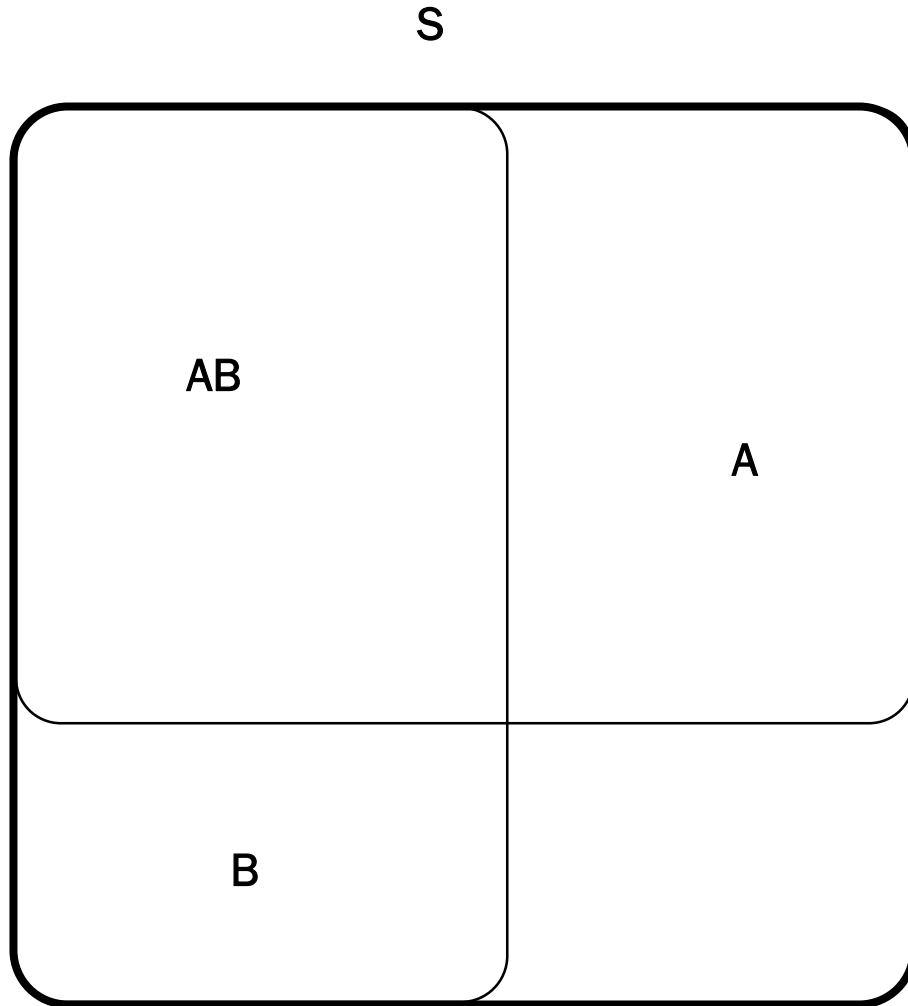
This ratio, $P(A)$...

... is the same as this one, $P(A|B)$



S

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

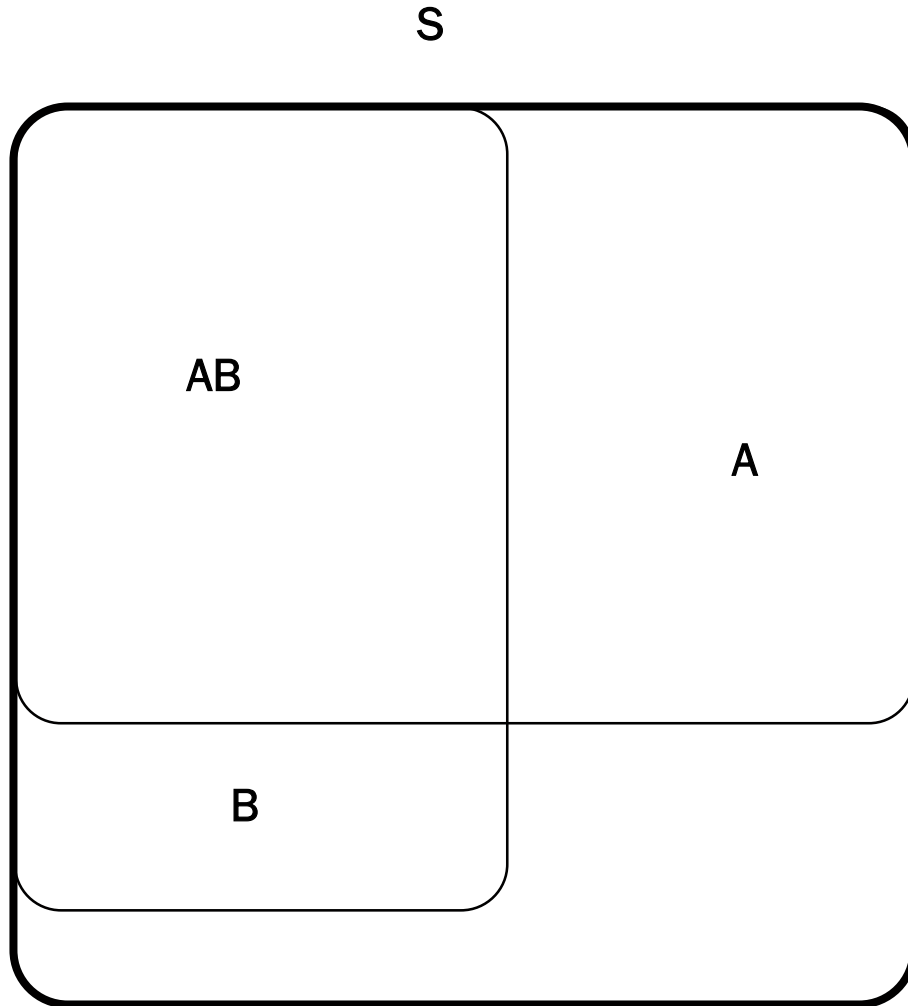
$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \not\rightarrow \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Dependence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \not\rightarrow \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Generalized Independence

General definition of Independence:

Events E_1, E_2, \dots, E_n are independent if **for every subset** with r elements (where $r \leq n$) it holds that:

$$P(E_1, E_2, E_3, \dots, E_r) = \prod_i P(E_i)$$

Example: outcomes of n separate flips of a coin are all independent of one another

- Each flip in this case is called a “trial” of the experiment

Math > Intuition



Two Dice

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 6$
- Let G be event: $D_1 + D_2 = 7$

1. Are E and F independent? Yes!

2. Are E and G independent? Yes!

- $P(E) = 1/6$, $P(G) = 1/6$, $P(E \ G) = 1/36$ [roll (1, 6)]

3. Are F and G independent? Yes!

- $P(F) = 1/6$, $P(G) = 1/6$, $P(F \ G) = 1/36$ [roll (1, 6)]

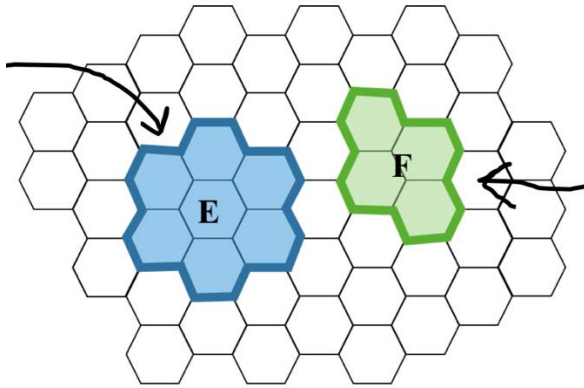
4. Are E , F and G independent? No!

- $P(E \ F \ G) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

New Ability



Properties of Pairs of Events



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

also:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

also:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Story: Ultimate Probability



Ultimate Probability

3,290 views • 1 Dec 2018

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Maika Isogawa
21 subscribers

SUBSCRIBE

<https://www.maikaisogawa.com/ultimate-frisbee-probability/>

<https://www.youtube.com/watch?v=H2lfTwGisOg>



Practice: Lets do the Frisbee Problem.

You flip **two frisbees**. For each frisbee, the probability that it lands “**heads**” is **0.6**. The two frisbees are considered “**even**” if both frisbees are heads **or** both frisbees are tails.

What is the probability that the frisbees are even?



Same Problem!

Let p be the probability of a 1 from `unknown_random`.
What is the probability of a True from `fair_random` if $p = .45$?

```
def fair_random():  
    """  
    There are four outcomes for assignments to r1 and r2:  
    [0, 0], [0, 1], [1, 0], [1, 1]. Return 1 if the  
    outcomes are [0, 0] or [1, 1]  
    """  
    r1 = unknown_random()  
    r2 = unknown_random()  
    return r1 == r2
```

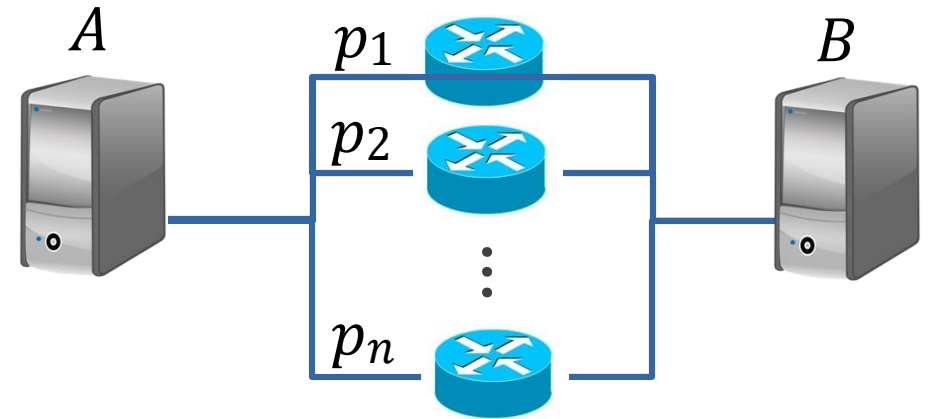


Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- $E =$ functional path from A to B exists.

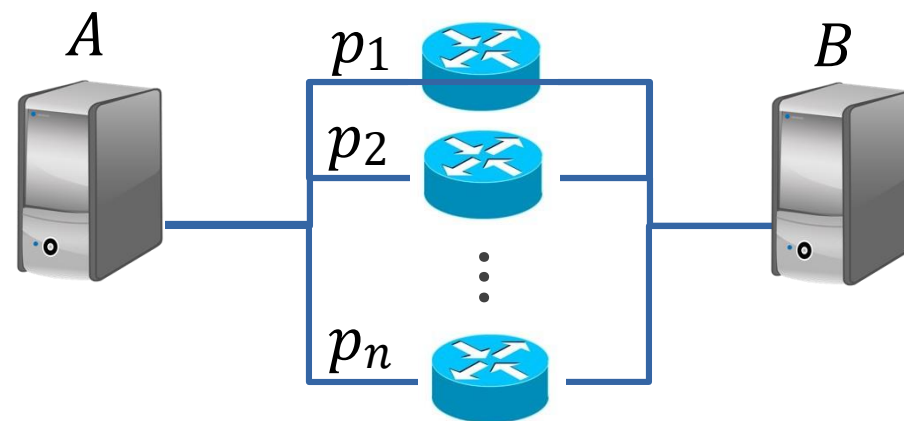
What is $P(E)$?



Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.



What is $P(E)$?

$$\begin{aligned} P(E) &= P(\geq 1 \text{ one router works}) \\ &= 1 - P(\text{all routers fail}) \\ &= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n) \\ &= 1 - \prod_{i=1}^n (1 - p_i) \end{aligned}$$

The Most Important Core Probability Question

The screenshot shows a web browser window with the title 'Many Coin Flips'. The address bar shows the URL 'chrispiech.github.io/probabilityForComputerScientists/en/example...'. The page content includes a title 'Many Coin Flips', an introductory paragraph about probability theory, a 'Coin Flip Simulator' with input fields for 'Number of flips n: 10' and 'Probability of heads p: 0.60', and a 'New simulation' button. Below the simulator, the results are displayed as 'T, H, T, H, T, H, H, H, T, H' and 'Total number of heads: 6'. A section titled '1. Warmups' asks 'What is the probability that all n flips are heads?'. Below this, it says 'Lets say $n = 10$ this question is asking what is the probability of getting:' followed by 'H, H, H, H, H, H, H, H, H, H'. A blue button labeled 'I'm Curious' is visible at the bottom right of the simulator area.

Many Coin Flips

chrispiech.github.io/probabilityForComputerScientists/en/example...

Many Coin Flips

In this section we are going to consider the number of heads on n coin flips. This thought experiment is going to be a basis for much probability theory! It goes far beyond coin flips.

Say a coin comes up heads with probability p . Most coins are fair and as such come up heads with probability $p = 0.5$. There are many events for which coin flips are a great analogy that have different values of p so lets leave p as a variable. You can try simulating coins here. Note that **H** is short for Heads and **T** is short for Tails. We think of each coin as distinct:

Coin Flip Simulator

Number of flips n : Probability of heads p : New simulation

Simulator results:

T, H, T, H, T, H, H, H, T, H

Total number of heads: 6

Let's explore a few probability questions in this domain.

1. Warmups

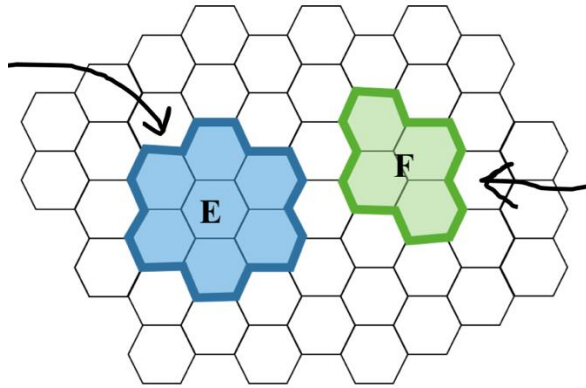
What is the probability that all n flips are heads?

Lets say $n = 10$ this question is asking what is the probability of getting:

H, H, H, H, H, H, H, H, H, H

Each coin flip is independent so we can use the rule for [probability of and with independent events](#). I'm Curious

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Here we are



Source: The Hobbit

G_1

G_2

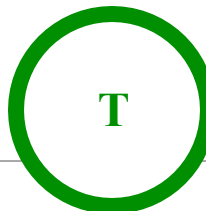
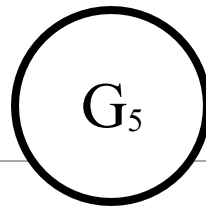
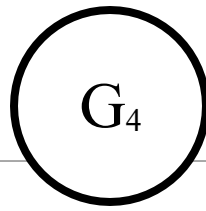
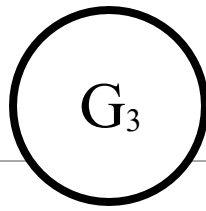
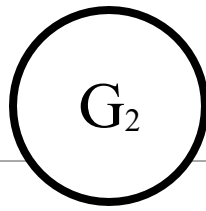
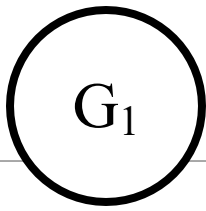
G_3

G_4

G_5

T





```

dna.txt
1 False,True,False,False,True,False
2 True,True,False,True,True,False
3 True,True,False,True,True,True
4 False,True,False,True,True,False
5 False,True,False,False,True,False
6 True,True,False,True,True,True
7 False,False,True,False,False,False
8 False,False,True,False,True,False
9 True,False,False,True,False,False
10 False,True,False,True,True,False
11 True,False,False,True,False,False
12 True,False,True,True,False,False
13 False,True,False,False,True,False
14 False,False,True,True,False,False
15 True,True,False,False,True,True
16 True,False,True,True,False,False
17 True,True,True,True,True,True |
18 True,False,True,False,False,True
19 False,True,False,True,True,True
20 False,False,True,False,False,False
21 False,False,False,True,True,False
22 False,True,False,False,True,False
23 True,True,False,True,True,True
24 False,True,False,True,True,False
25 True,False,False,False,False,True
26 False,False,True,True,False,True
27 False,False,False,True,False,False
28 False,True,True,False,False,True
29 False,True,False,False,True,True
30 False,False,False,False,False,True
31 False,True,False,True,True,False
32 True,False,False,True,False,False
33 True,True,False,True,True,True
34 True,True,False,False,True,True
35 True,True,False,True,True,True
36 False,False,False,True,False,False
--

```



100,000 samples

6 observations per sample



Discovered Pattern

```
[Piech-2:dna piech$ python findStructure.py
size data = 100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■ ■ ■

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

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```

■ ■ ■

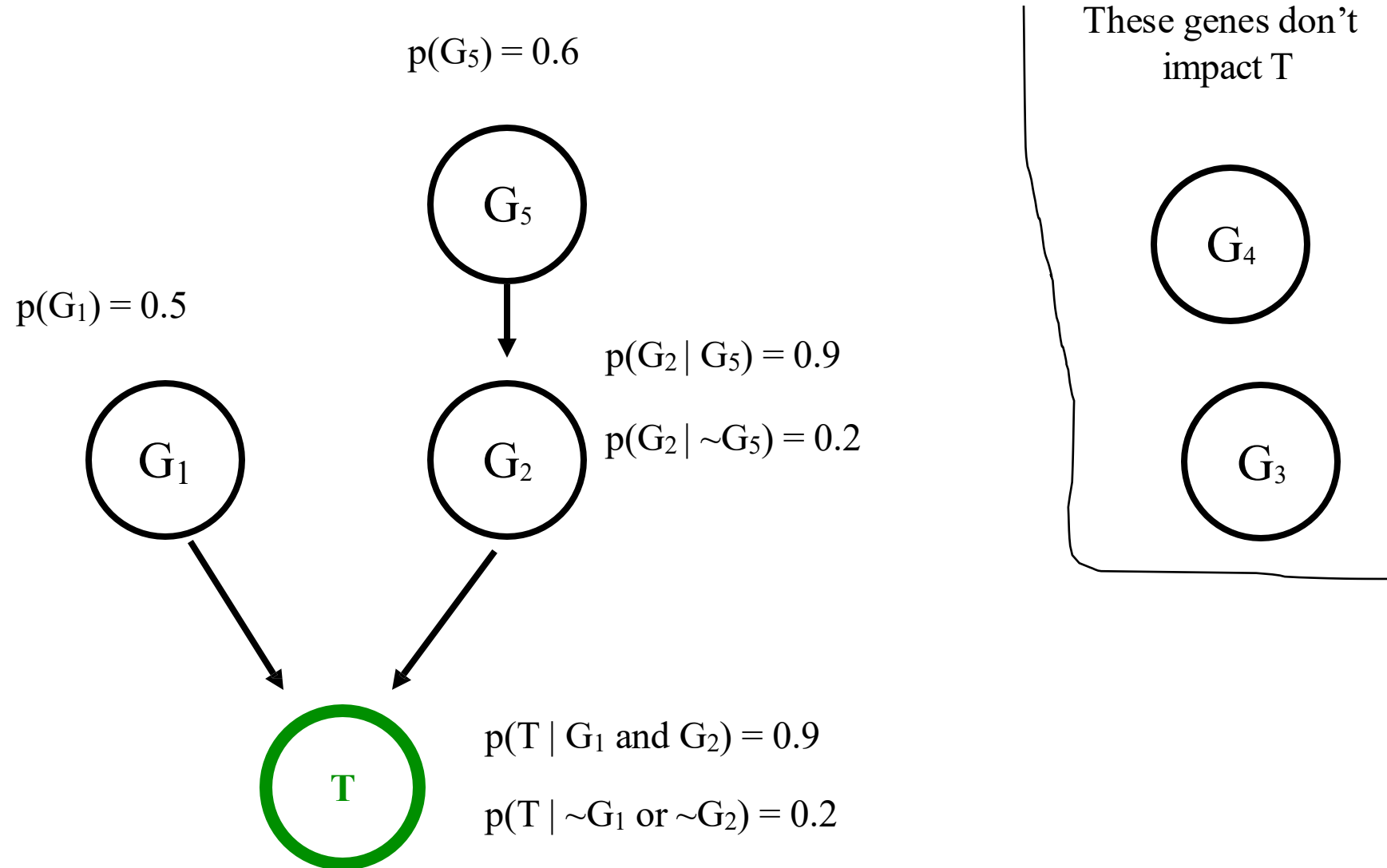
$p(T \text{ and } G5 \mid G2) = 0.450$
 $p(T \mid G2)p(G5 \mid G2) = 0.450$



Independence
relationships can change
with conditioning.

If E and F are independent, that does not mean they will still be independent given another event G.

Only Causal Structure That Fits



See you Monday