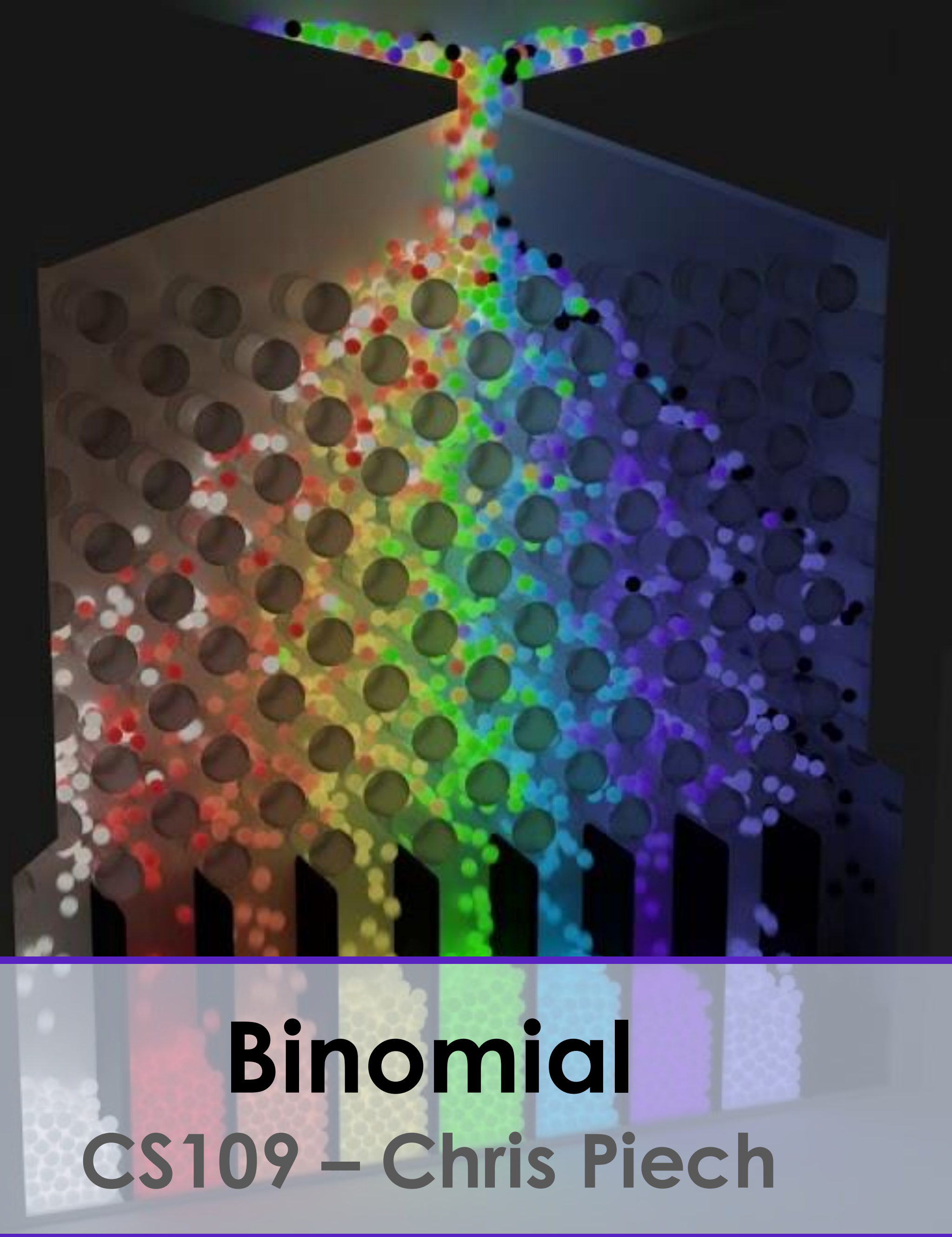
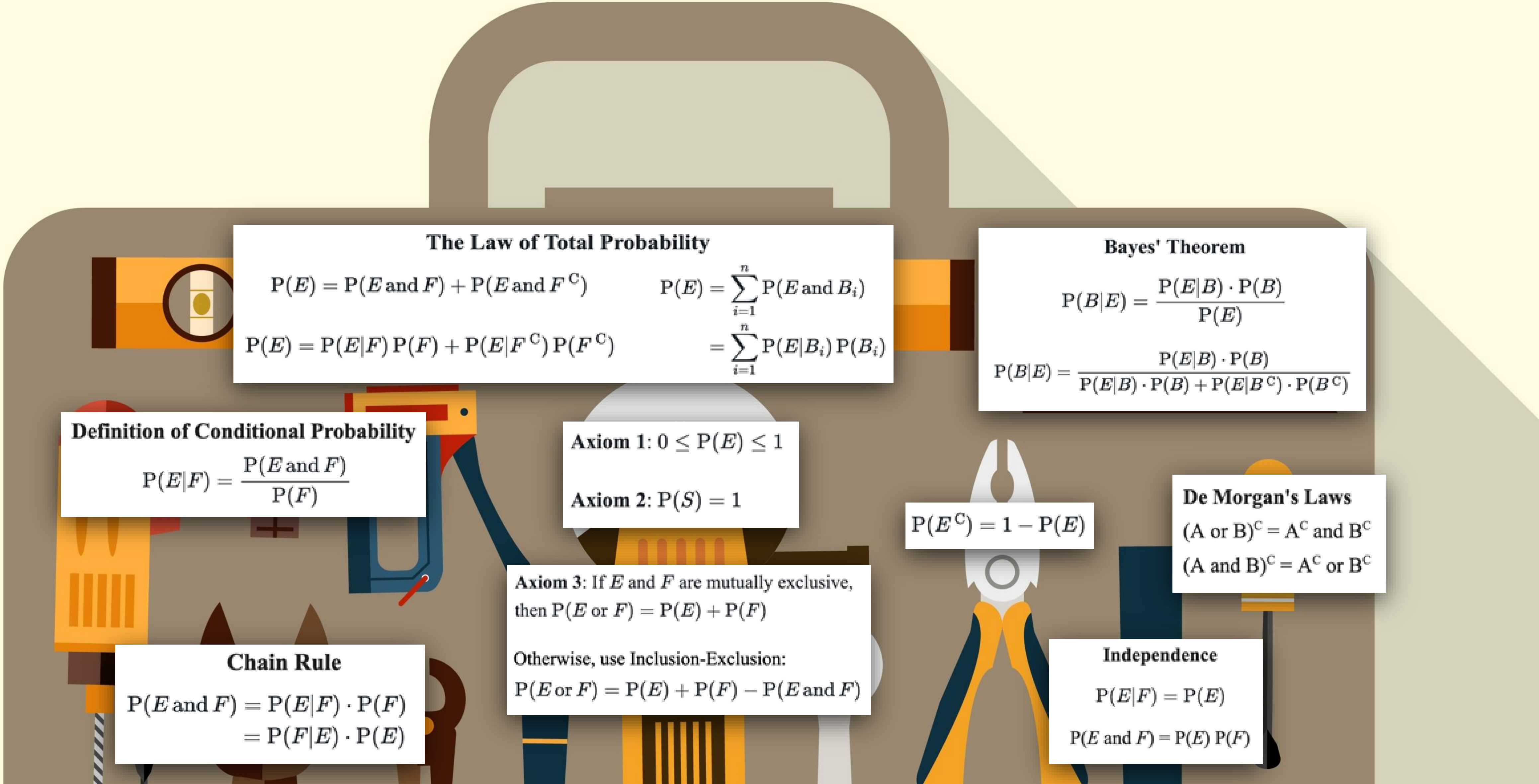


Binomial

CS109 – Chris Piech

Review

The Core Probability Toolkit



The Law of Total Probability

$$P(E) = P(E \text{ and } F) + P(E \text{ and } F^C) \qquad P(E) = \sum_{i=1}^{n} P(E \text{ and } B_i)$$

$$P(E) = P(E|F)\,P(F) + P(E|F^C)\,P(F^C) \qquad = \sum_{i=1}^{n} P(E|B_i)\,P(B_i)$$

Bayes' Theorem

$$P(B|E) = \frac{P(E|B) \cdot P(B)}{P(E)}$$

$$P(B|E) = \frac{P(E|B) \cdot P(B)}{P(E|B) \cdot P(B) + P(E|B^C) \cdot P(B^C)}$$

Definition of Conditional Probability

$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}$$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

$P(E^C) = 1 - P(E)$

De Morgan's Laws

$(A \text{ or } B)^C = A^C \text{ and } B^C$

$(A \text{ and } B)^C = A^C \text{ or } B^C$

Axiom 3: If E and F are mutually exclusive, then $P(E \text{ or } F) = P(E) + P(F)$

Otherwise, use Inclusion-Exclusion:

$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$

Chain Rule

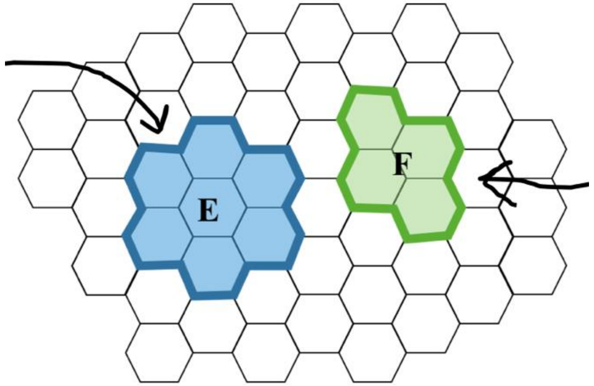
$$P(E \text{ and } F) = P(E|F) \cdot P(F)$$
$$= P(F|E) \cdot P(E)$$

Independence

$P(E|F) = P(E)$

$P(E \text{ and } F) = P(E)\,P(F)$

Review: Last Lecture



Mutually Exclusive Events

make ***OR*** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$

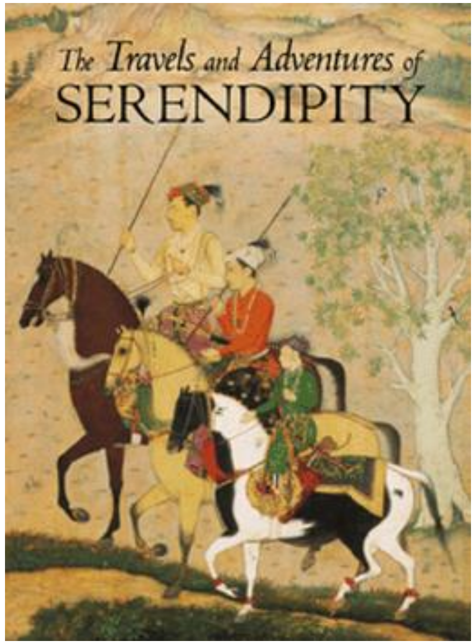
Independent Events

make ***AND*** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Serendipity

- Say the population of Stanford is 17,000 people
  - You are friends with 80 people.
  - Walk into a room, see 450 random people. Independently.
  - What is the probability that you see someone you know?
  - Assume you are equally likely to see each person at Stanford



End Review

Learning Goals for Today

A **random variable** is a number which takes on values probabilistically.

A discrete random variable is fully described by a **probability mass function**.

A **binomial** is a particular random variable which represents number of heads in n coin flips.

Counting!

Counting Rules

Counting operations on *n* **distinct** objects

Sort, order matters
{perms}

$$n!$$

```
# give me each possible permutation
itertools.permutations([1,2,3,4])
```

```
# calculate 20!
math.factorial(20)
```

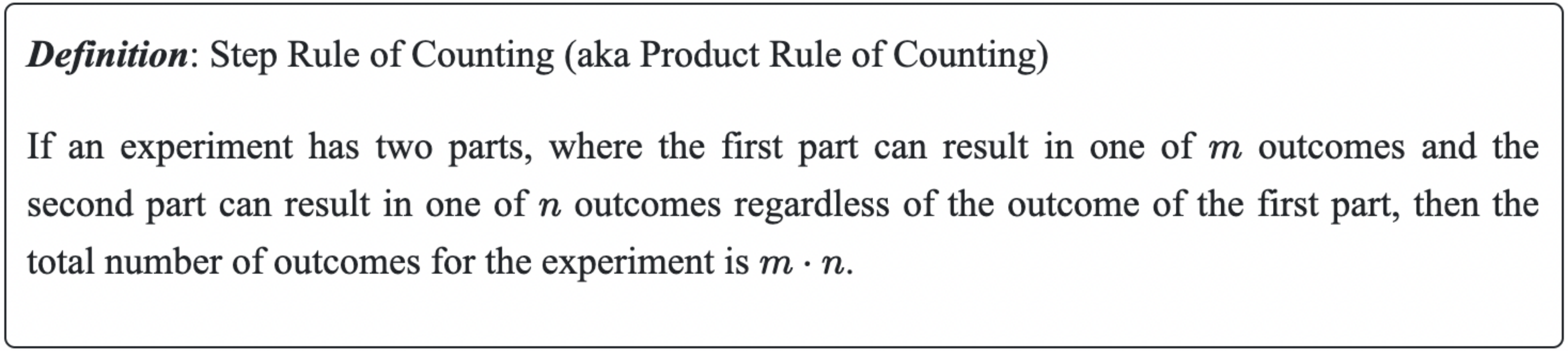
Choose *k*
{combinations}

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

```
# give me each possible combination
itertools.combinations([1,2,3,4,5], 3)
```

```
# calculate 10 choose 5
math.comb(10, 5)
```

Counting with Steps



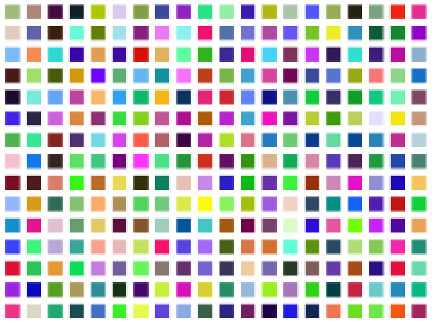
Definition: Step Rule of Counting (aka Product Rule of Counting)

If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is $m \cdot n$.

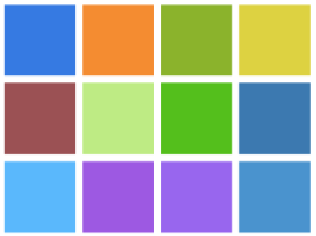
How Many Unique Images?

Each pixel can be one of 17 million distinct colors

(a) 12 million pixels



(b) 300 pixels



(c) 12 pixels

$$(17\ \text{million})^n$$

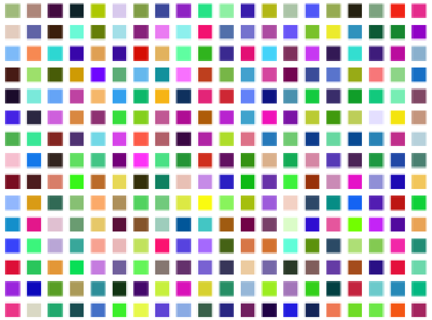
10^{80}

How Many Unique Images?

Each pixel can be one of 17 million distinct colors

(a) 12 million pixels

$\approx 10^{86696638}$



(b) 300 pixels

$\approx 10^{2167}$

$(17 \text{ million})^{n}$

(c) 12 pixels

$\approx 10^{86}$

Orderings of Letters

How many letter orderings are possible for the following string?

CHRIS

This is Jerry's dog, Doris. She puts her little Doris paw
up to her chin when she's thinking.

Orderings of Letters

chirs
chisr
chris
chrsi
chsir
chsri
cihrs
cihsr
cirhs
cirsh
cishr
cisrh
crhis
crhsi
crihs
crish
crshi
crsih
cshir
cshri
csihr
csirh
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shric
sichr
sicrh
sihcr
sihrc
sirch
sirhc
srchi
srcih
srhci
srhic
srich
srihc

Orderings of letters

Step 1:
Chose first letter

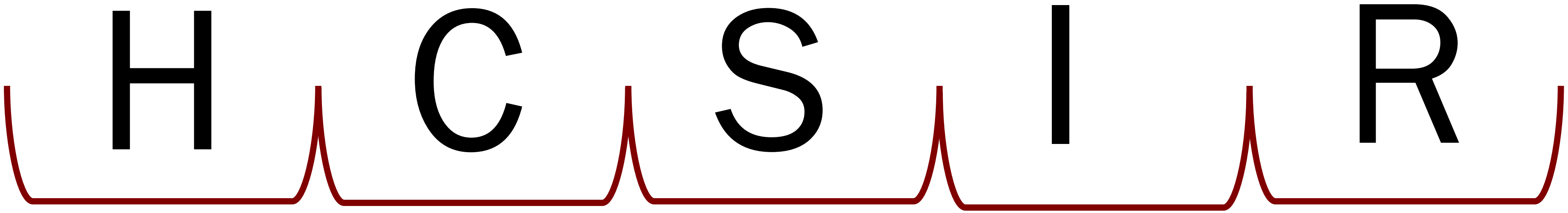
Step 2:
Chose 2nd letter

Step 3:
Chose 3rd letter

Step 4:
Chose 4th letter

Step 5:
Chose 5th letter

Orderings of letters



H C S I R

Step 1:
Chose first letter
(5 options)

Step 2:
Chose 2nd letter
(4 options)

Step 3:
Chose 3rd letter
(3 options)

Step 4:
Chose 4th letter
(2 options)

Step 5:
Chose 5th letter
(1 option)

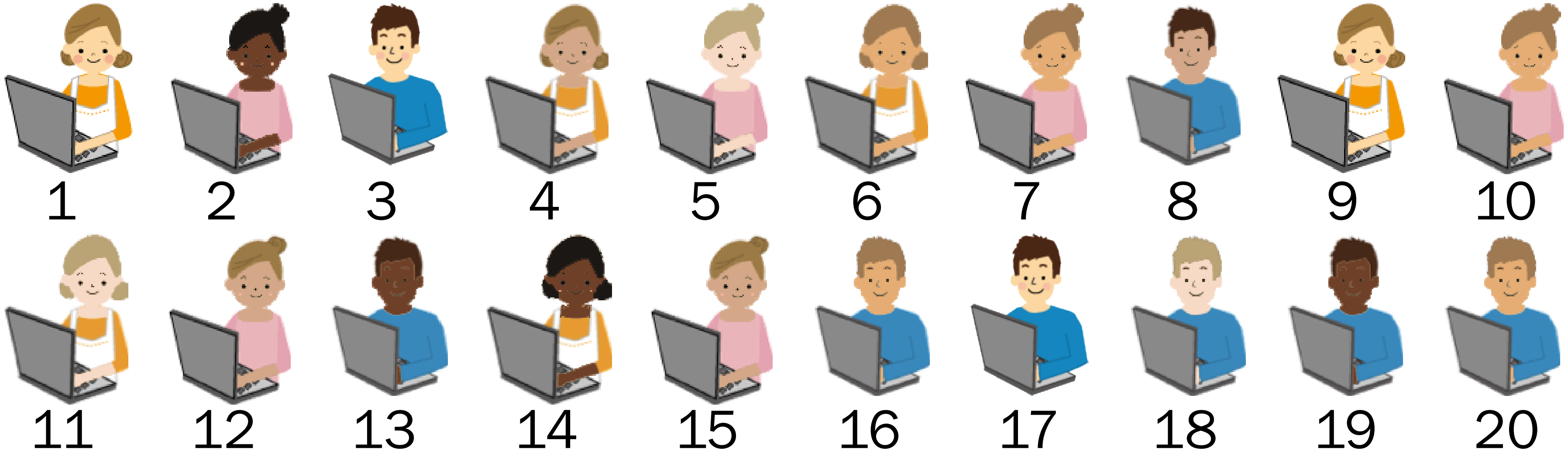
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

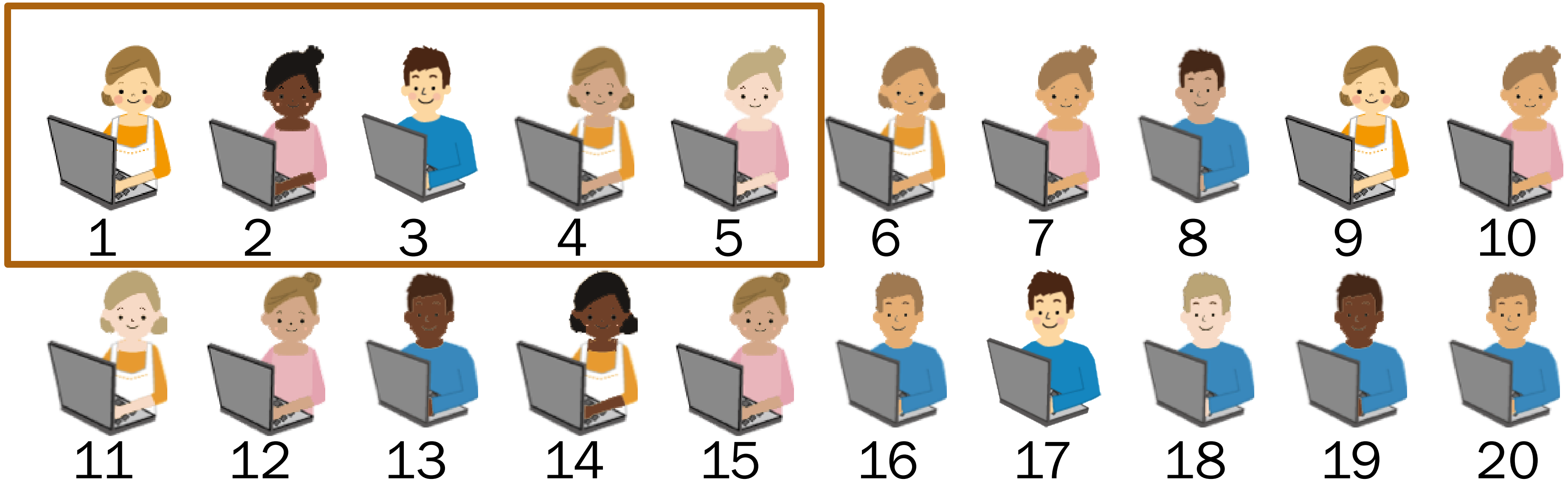


1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



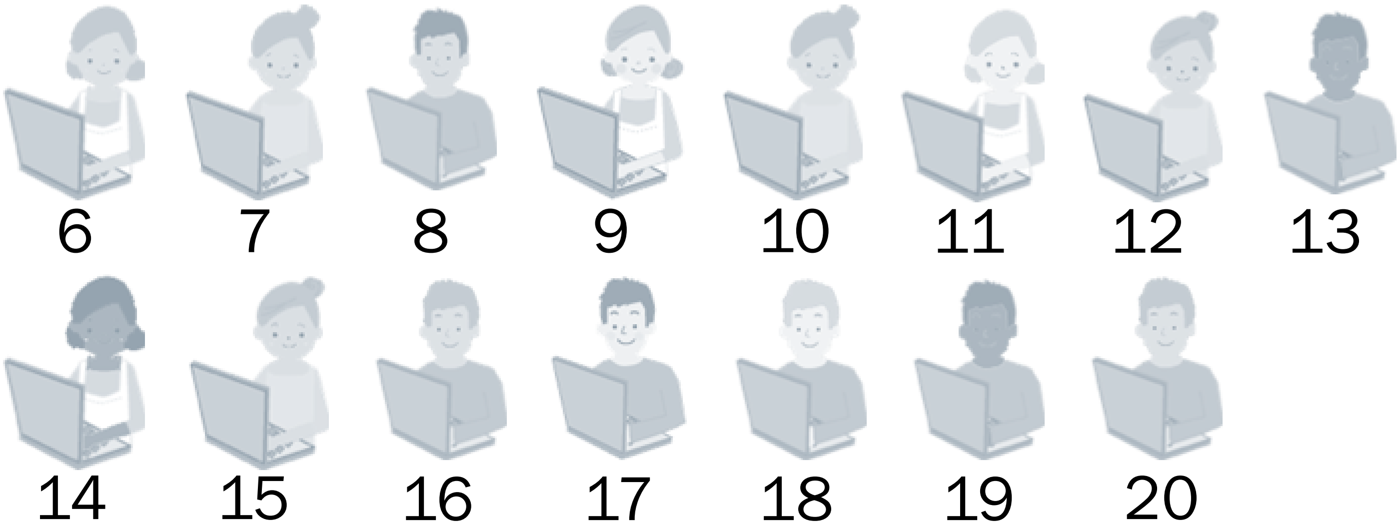
1. n people get in line
2. Put first k in cake room

$n!$ ways

1 way

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

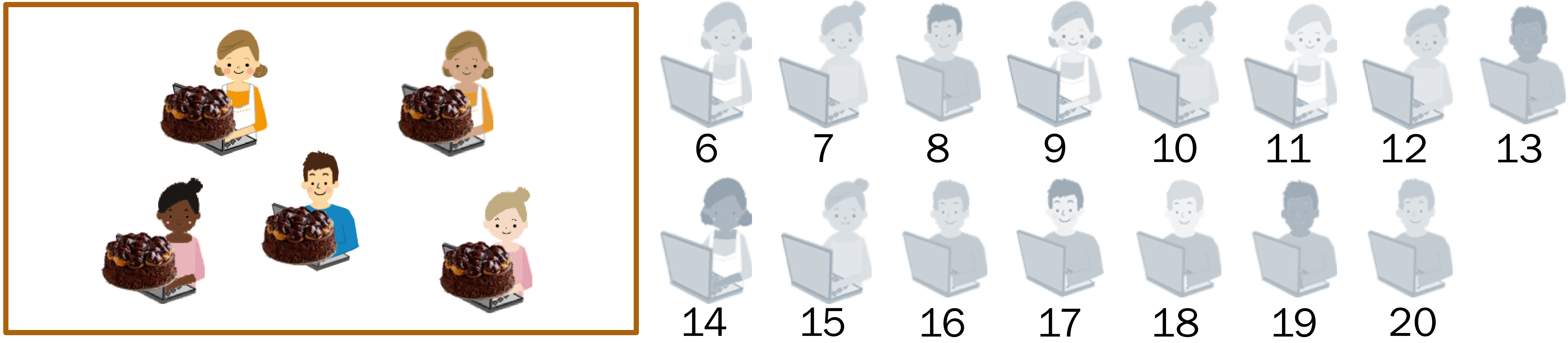
   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line
2. Put first k in cake room
3. Allow cake group to mingle

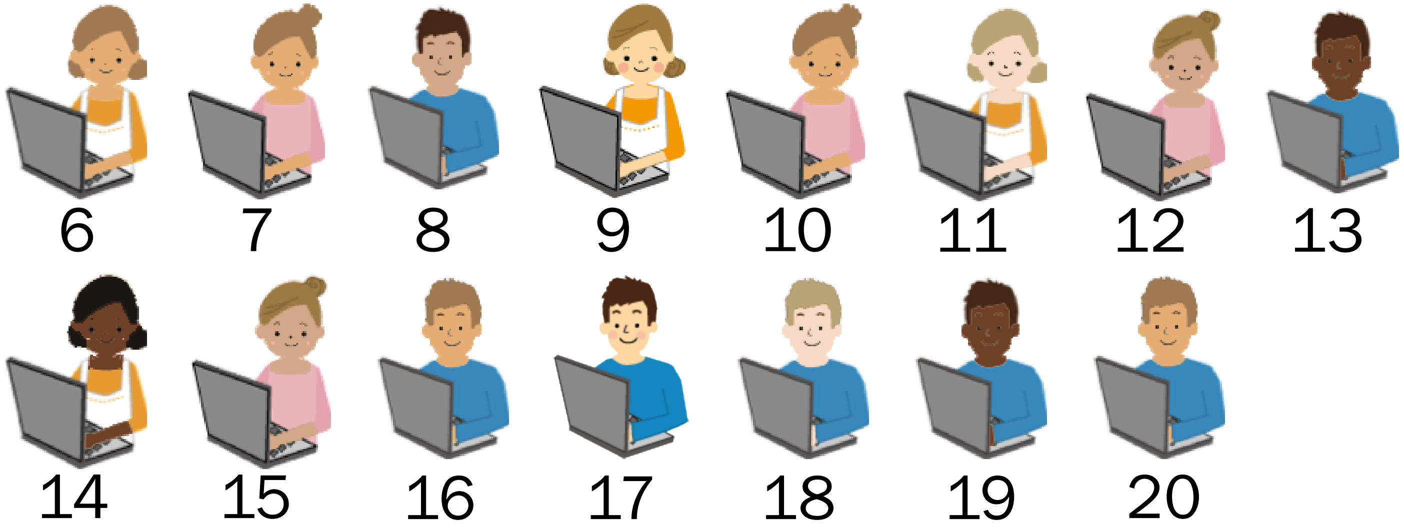
$n!$ ways

1 way

$k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

2. Put first k in cake room

1 way

3. Allow cake group to mingle

$k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

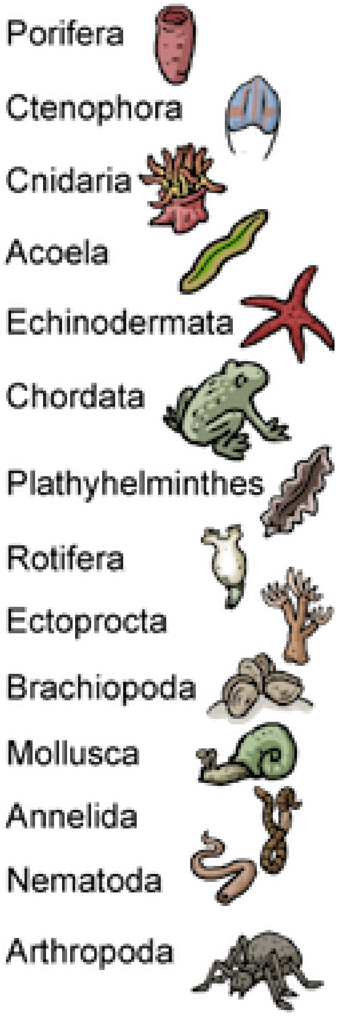
Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

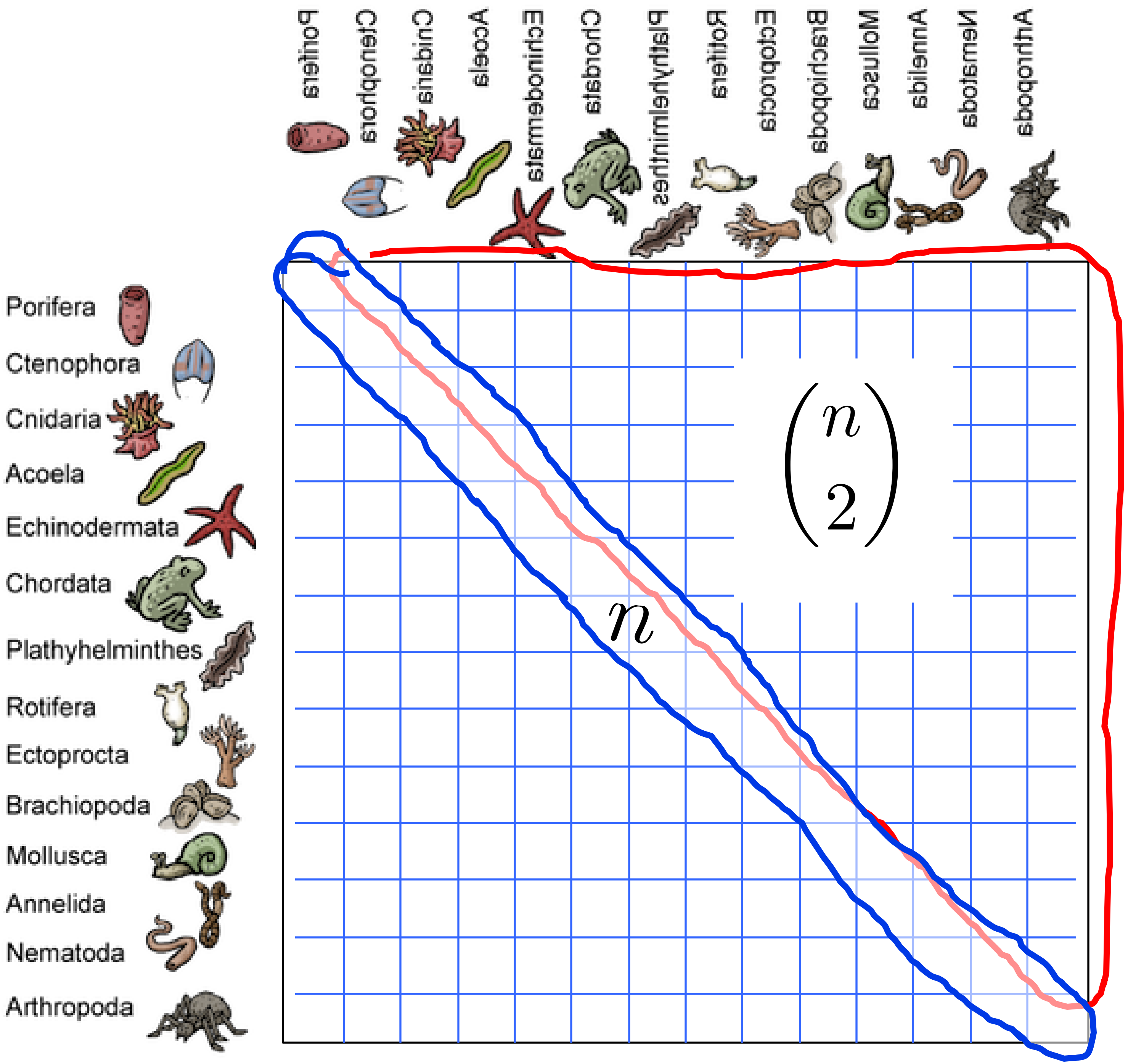
The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice



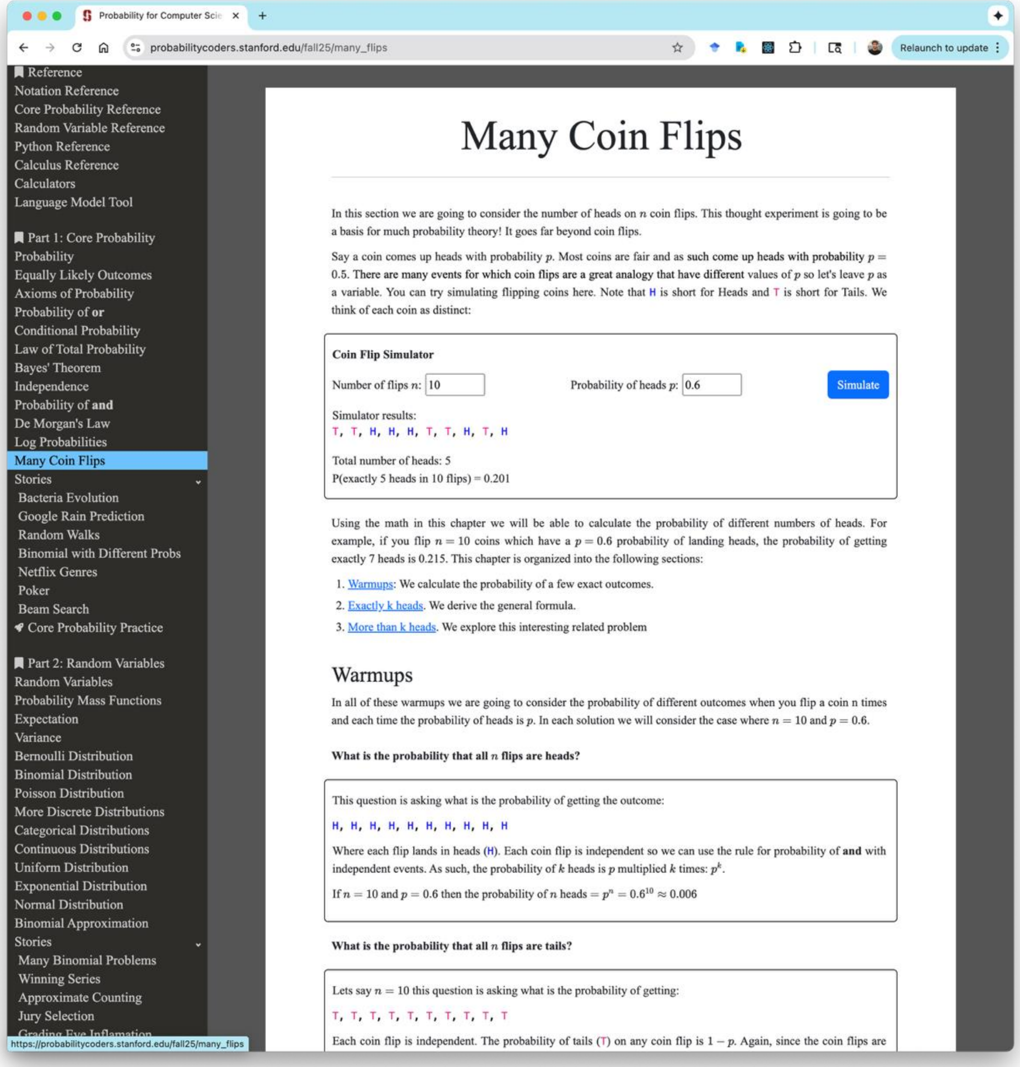
There are ***n*** animals.
How many **distinct pairs** of
animals are there?



There are *n* animals.
How many **distinct pairs** of
animals are there?

Super Critical Problem!

To the Course Reader!!



Many Coin Flips

In this section we are going to consider the number of heads on n coin flips. This thought experiment is going to be a basis for much probability theory! It goes far beyond coin flips.

Say a coin comes up heads with probability p. Most coins are fair and as such come up heads with probability $p = 0.5$. There are many events for which coin flips are a great analogy that have different values of p so let's leave p as a variable. You can try simulating flipping coins here. Note that `H` is short for Heads and `T` is short for Tails. We think of each coin as distinct:

Coin Flip Simulator

Number of flips n: 10

Probability of heads p: 0.6

Simulator results:
`T, T, H, H, H, T, T, H, T, H`

Total number of heads: 5
P(exactly 5 heads in 10 flips) = 0.201

Using the math in this chapter we will be able to calculate the probability of different numbers of heads. For example, if you flip $n = 10$ coins which have a $p = 0.6$ probability of landing heads, the probability of getting exactly 7 heads is 0.215. This chapter is organized into the following sections:

1. Warmups: We calculate the probability of a few exact outcomes.
2. Exactly k heads. We derive the general formula.
3. More than k heads. We explore this interesting related problem

Warmups

In all of these warmups we are going to consider the probability of different outcomes when you flip a coin n times and each time the probability of heads is p. In each solution we will consider the case where $n = 10$ and $p = 0.6$.

What is the probability that all n flips are heads?

This question is asking what is the probability of getting the outcome:

`H, H, H, H, H, H, H, H, H, H`

Where each flip lands in heads (`H`). Each coin flip is independent so we can use the rule for probability of **and** with independent events. As such, the probability of k heads is p multiplied k times: p^k.

If $n = 10$ and $p = 0.6$ then the probability of n heads $= p^n = 0.6^{10} \approx 0.006$

What is the probability that all n flips are tails?

Lets say $n = 10$ this question is asking what is the probability of getting:

`T, T, T, T, T, T, T, T, T, T`

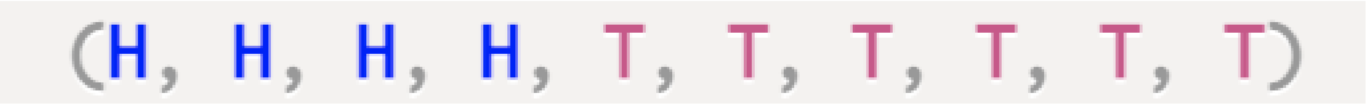
Each coin flip is independent. The probability of tails (`T`) on any coin flip is $1 - p$. Again, since the coin flips are

Probability of Exactly *k* Heads in *n* Coin Flips

We flip the coin 10 times. Probability of heads is p.

We want to know the probability of getting **4** heads.

What is the probability of the outcome below?



```
(H, H, H, H, T, T, T, T, T, T)
```

Probability of Exactly *k* Heads in *n* Coin Flips

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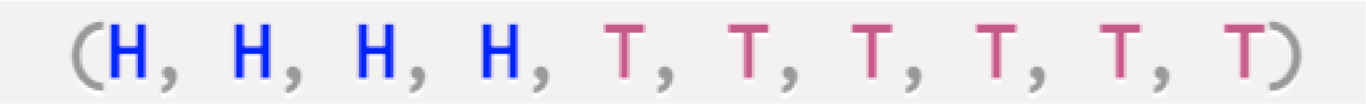


```
(H, H, H, H, T, T, T, T, T, T)
```

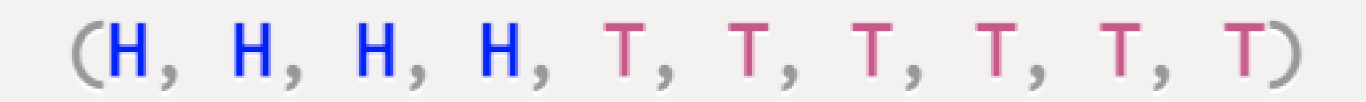
$$p^4(1-p)^6$$

Probability of Exactly *k* Heads in *n* Coin Flips

We flip the coin 10 times. Probability of heads is *p*.

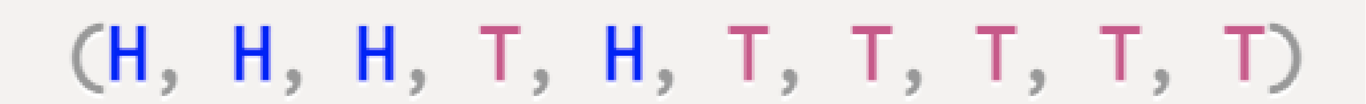
We want to know the probability of getting **4** heads.

What is the probability of the outcome below?



```
(H, H, H, H, T, T, T, T, T, T)
```

$$p^4(1-p)^6$$



```
(H, H, H, T, H, T, T, T, T, T)
```

Probability of Exactly *k* Heads in *n* Coin Flips

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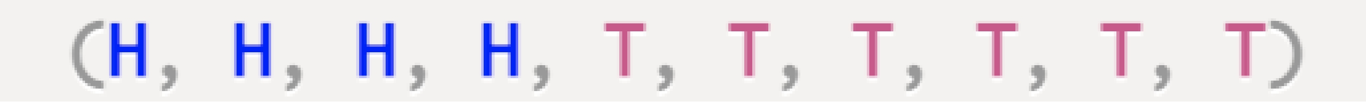
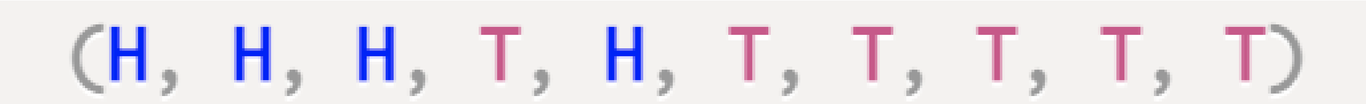
What is the probability of the outcome below?



```
(H, H, H, H, T, T, T, T, T, T)
```

$$p^4(1-p)^6$$

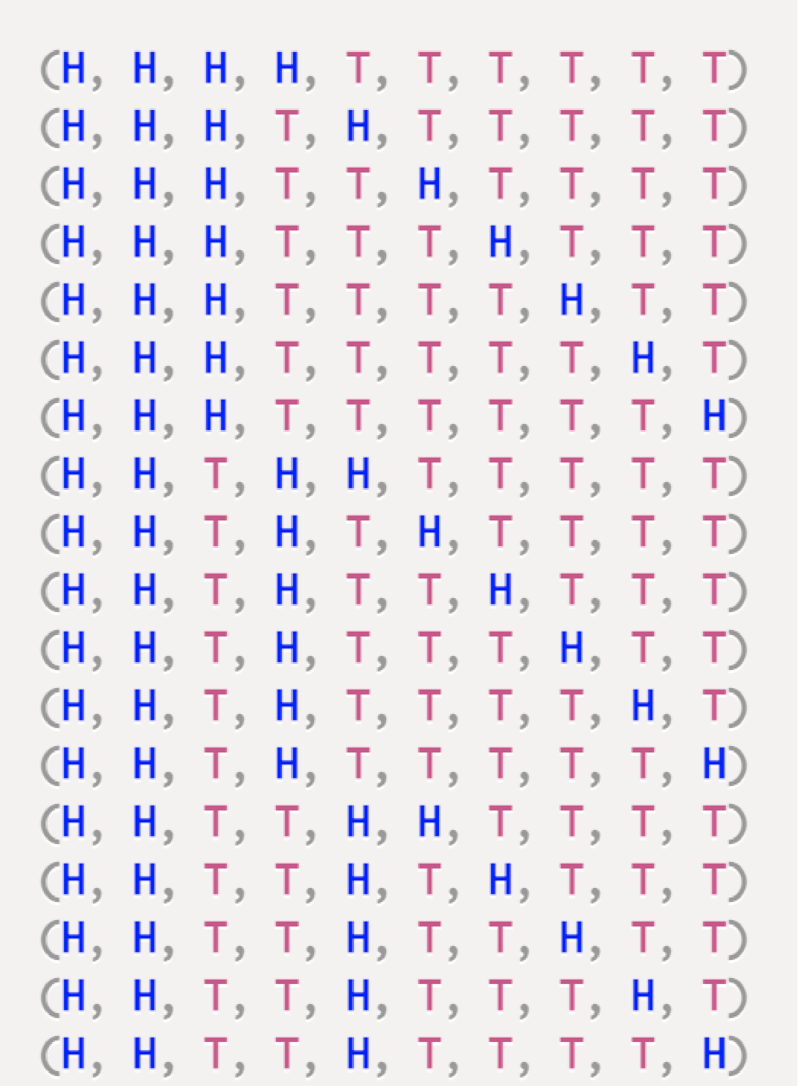


```
(H, H, H, T, H, T, T, T, T, T)
```

$$p^4(1-p)^6$$

All of the outcomes with exactly 4 heads have the same probability

Probability of Exactly *k* Heads in *n* Coin Flips



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(H, H, H, H, T, T, T, T, T, T)
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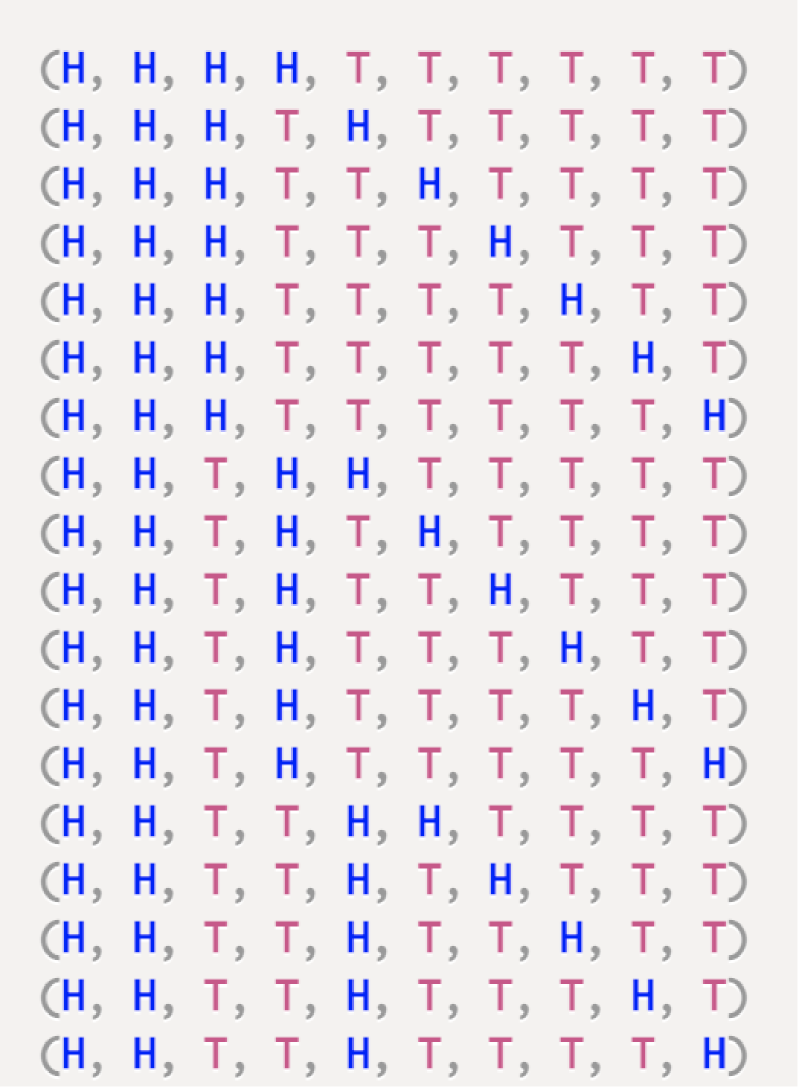
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

The probability of getting **4** heads, *in any ordering*, is the "**or**" of all these **mutually exclusive** cases

Each outcome has probability $p^4(1-p)^6$

How many cases are there?

Probability of Exactly *k* Heads in *n* Coin Flips



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(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```
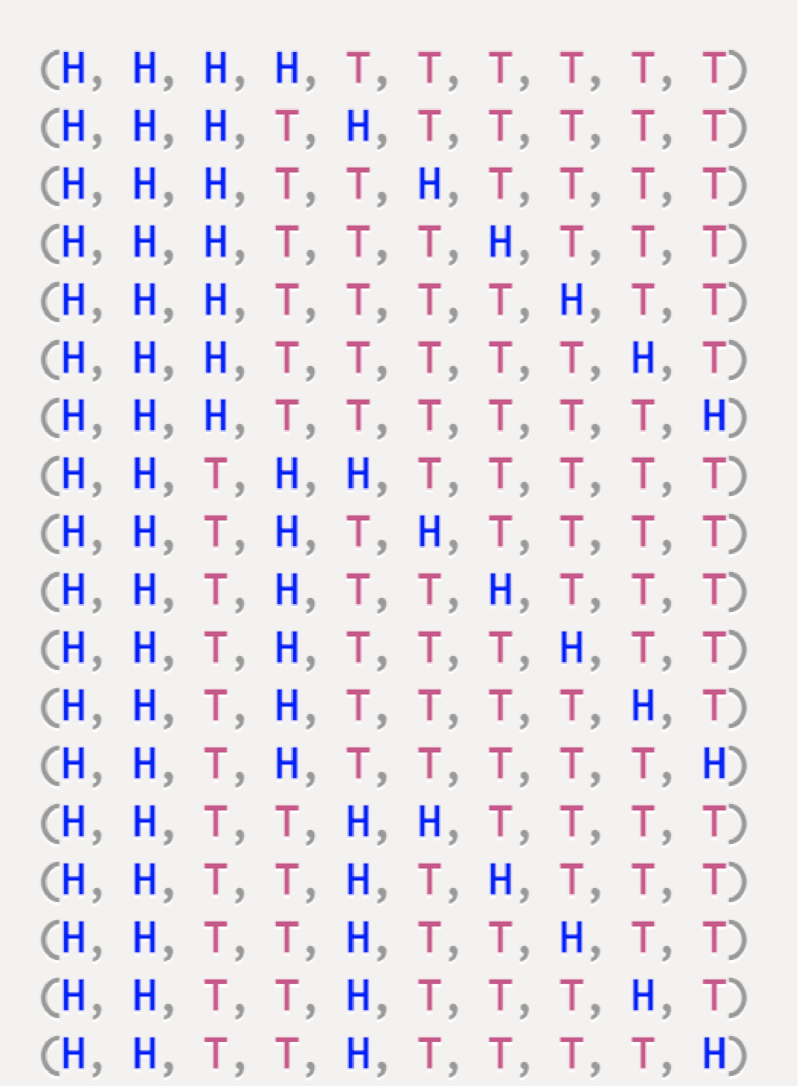

The probability of getting **4** heads,
in any ordering, is the "**or**" of all these
mutually exclusive cases

Each outcome has probability $p^4(1-p)^6$

How many cases are there? $\binom{10}{4}$

Probability of Exactly k Heads in n Coin Flips



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(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
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(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

←

The probability of getting **4** heads, *in any ordering*, is the "**or**" of all these **mutually exclusive** cases

Each outcome has probability $p^4(1-p)^6$

How many cases are there? $\binom{10}{4}$

$$P(4\ \text{heads}) = \binom{10}{4} p^4(1-p)^6$$

How do we formalize this result?

It’s Time…

It’s Time…

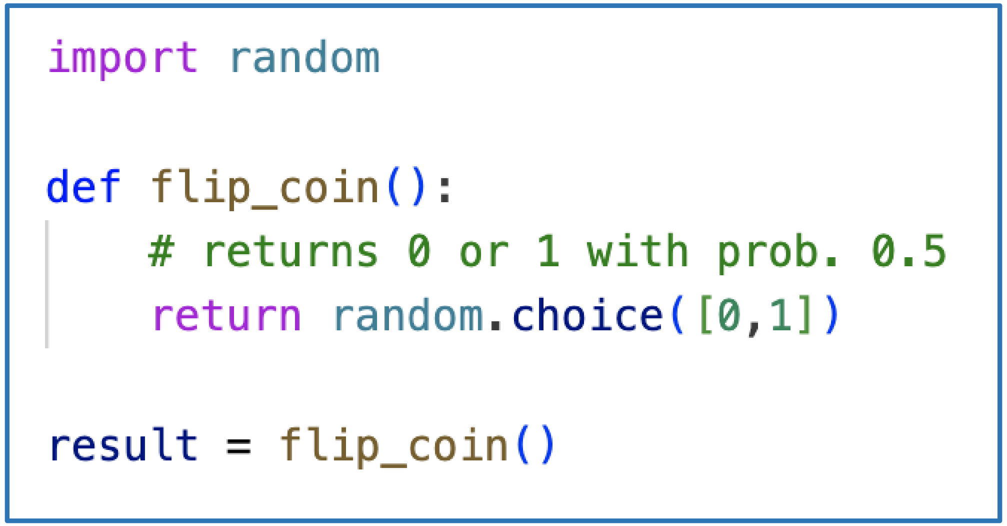
$$X$$

…For Random Variables

Random Variables Are Variables…That Are Random

Random Variables Are Variables…That Are Random

Check out the variable **`result`** in the code below.



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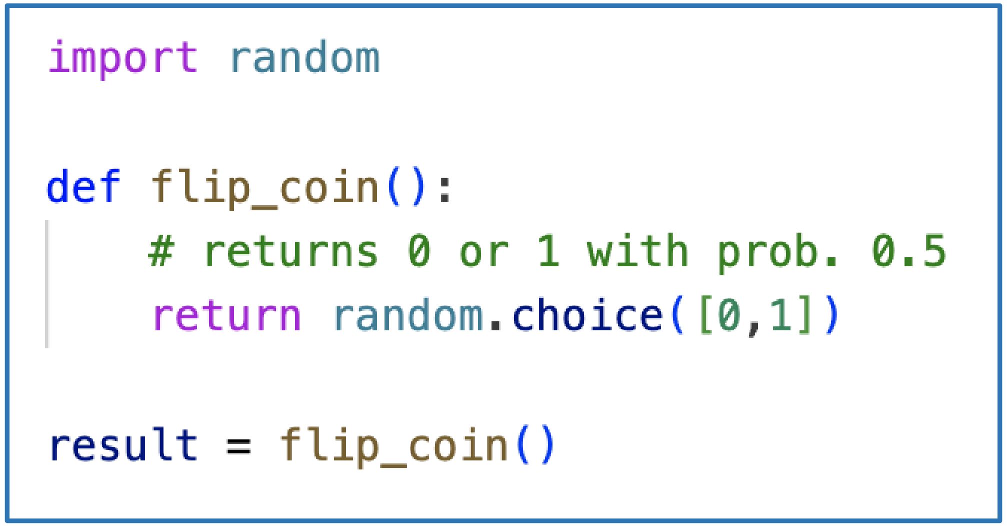

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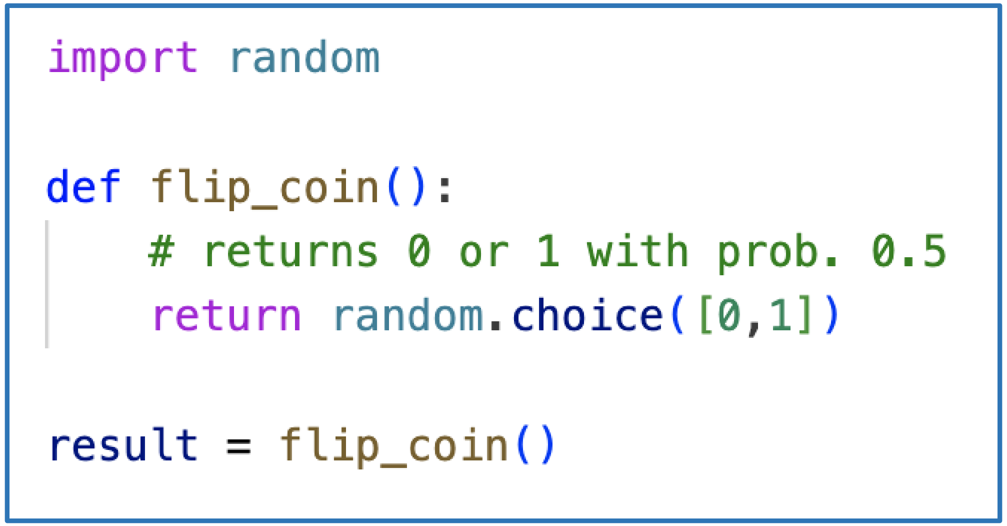

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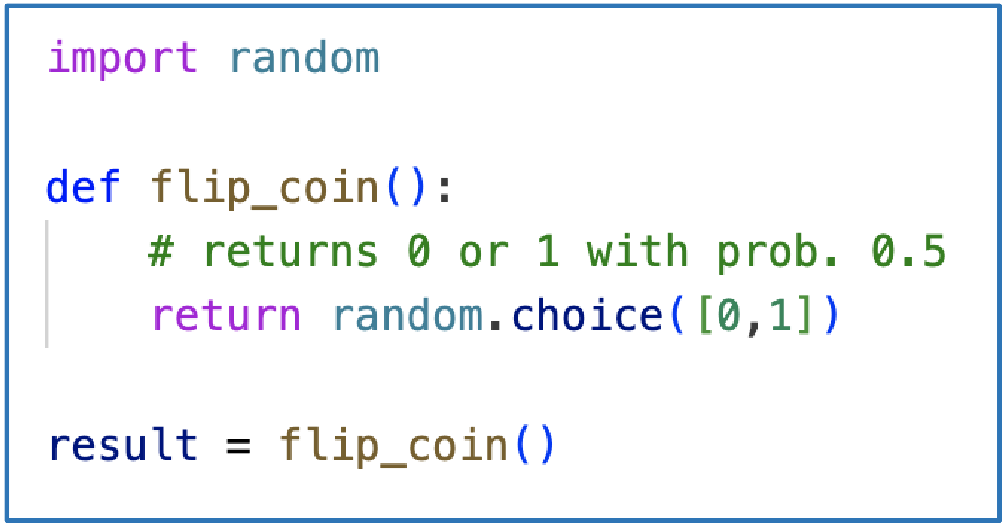

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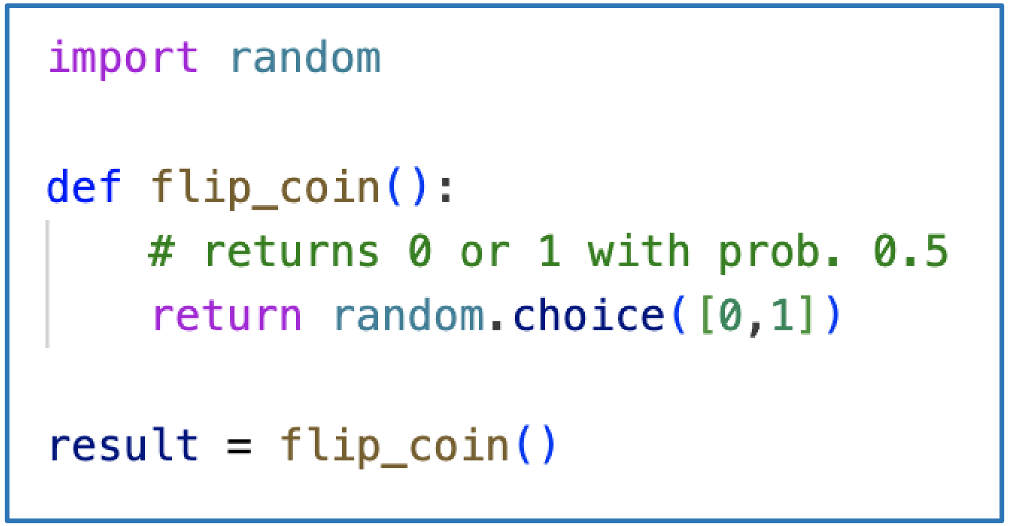
- Is the value of **`result`** the same every time we run the code? Nope!

Like **`result`**, a random variable is a variable whose value is uncertain.

Random Variables Are Variables…That Are Random

A **random variable** is a variable whose value is uncertain.



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```

"Let X be the result of flipping a coin."

$$P(X=0)=0.5$$
$$P(X=1)=0.5$$

Random variables are an abstraction on top of events

Random variables are *not* events

Random Variables vs. Events

$$X$$

Let X be a
random variable

Random Variables vs. Events

It is an event when
X takes on a value

$$X \qquad X = 2$$

Let X be a
random variable

Random Variables vs. Events

It is an event when
X takes on a value

$$X$$

$$X = 2$$

$$P(X = 2)$$

Let X be a
random variable

So we can still work with
probabilities of events

Examples of Random Variables

"Let X be the result of rolling a dice."

- $P(X = 1) = 1/6$
- $P(X = 2) = 1/6$
- $P(X = 3) = 1/6$
- $P(X = 4) = 1/6$
- $P(X = 5) = 1/6$
- $P(X = 6) = 1/6$

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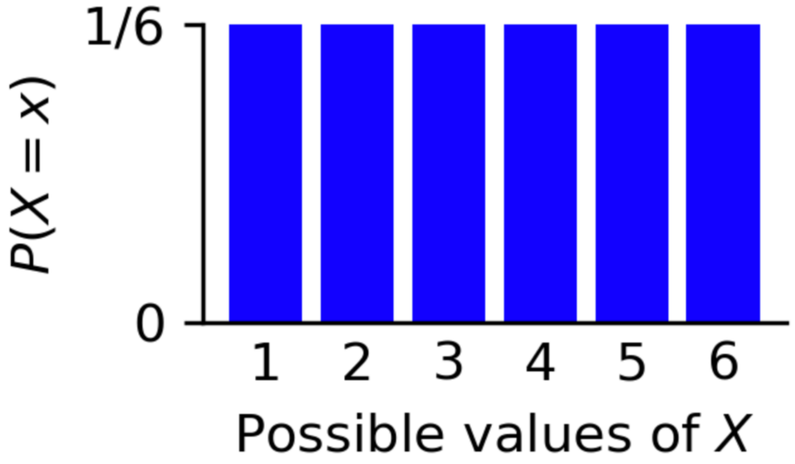

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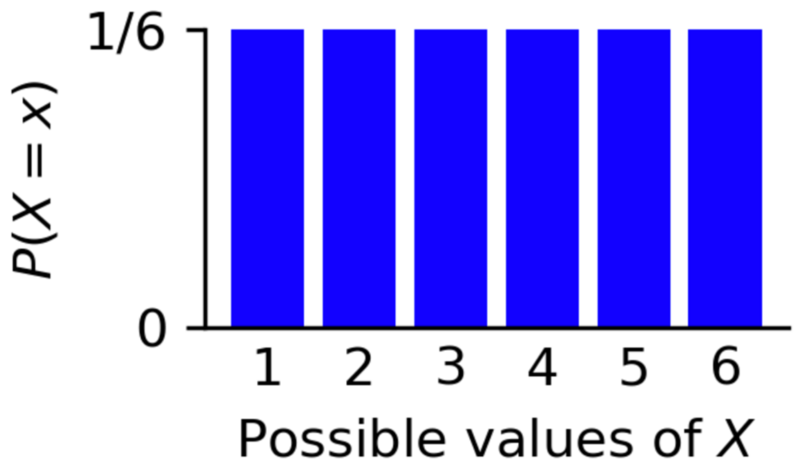

"Let Y be the number of heads seen in 2 coin flips."

$P(Y = 0) = 1/4$	(T, T)
$P(Y = 1) = 1/2$	(H, T), (T, H)
$P(Y = 2) = 1/4$	(H, H)

Examples of Random Variables

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- $P(X = 1) = 1/6$
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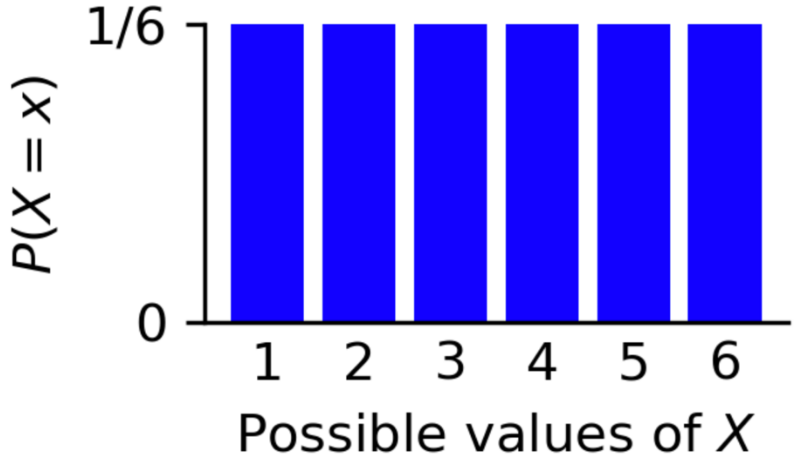

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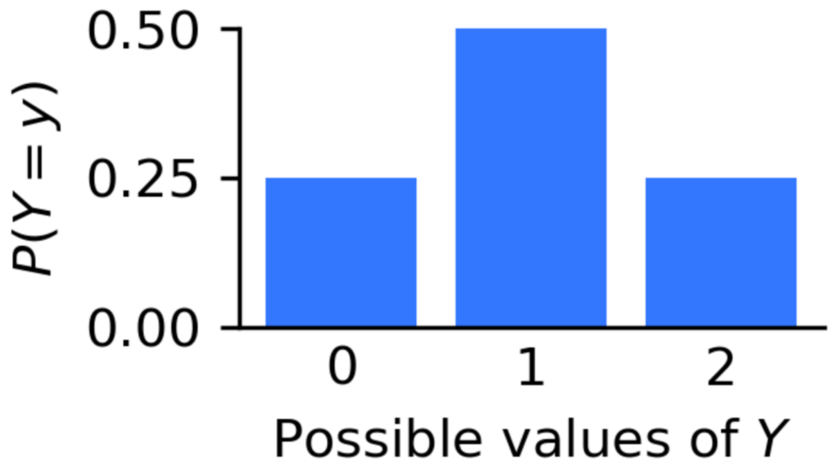
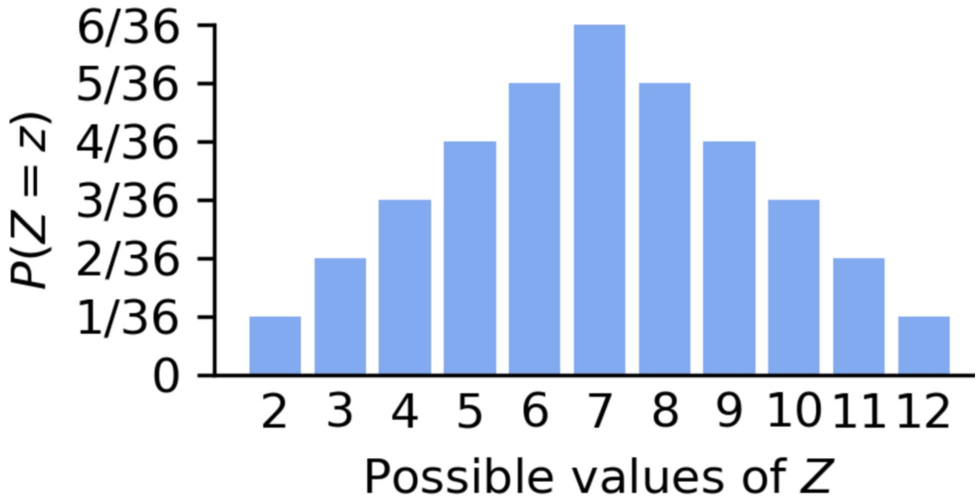


Examples of Random Variables

"Let Z be the sum of the result of rolling two dice."

- $P(Z = 2) = 1/36$
- $P(Z = 3) = 2/36$
- $P(Z = 4) = 3/36$
- $P(Z = 5) = 4/36$
- $P(Z = 6) = 5/36$
- $P(Z = 7) = 6/36$
- $P(Z = 8) = 5/36$
- $P(Z = 9) = 4/36$
- $P(Z = 10) = 3/36$
- $P(Z = 11) = 2/36$
- $P(Z = 12) = 1/36$



$$P(Z = z) = \begin{cases} \frac{z-1}{36} & z \in \mathbb{Z}, 1 \leq z \leq 6 \\ \frac{13-z}{36} & z \in \mathbb{Z}, 7 \leq z \leq 12 \\ 0 & \text{else} \end{cases}$$

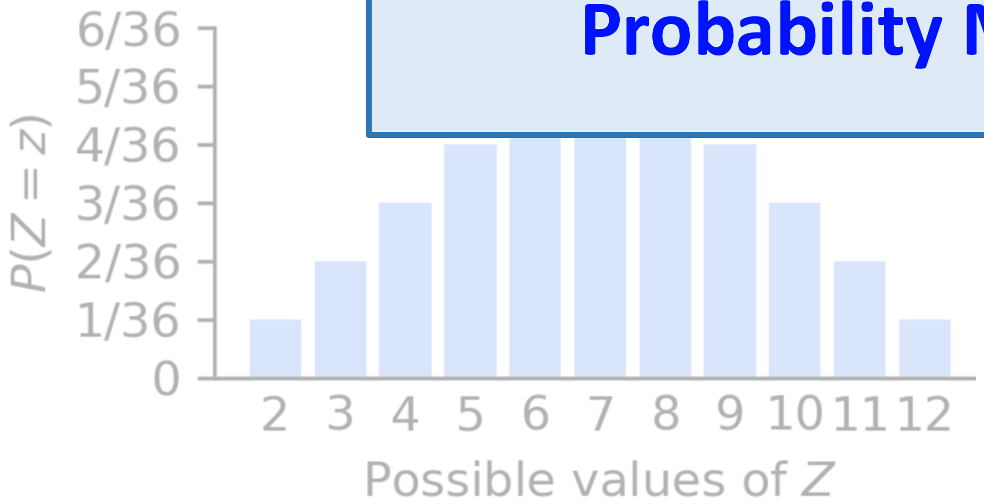
Examples of Random Variables

"Let Z be the sum of the result of rolling two dice."

- $P(Z = 2) = 1/36$
- $P(Z = 3) = 2/36$
- $P(Z =$
- $P(Z =$
- $P(Z = 6) = 5/36$
- $P(Z = 7) = 6/36$
- $P(Z = 10) = 3/36$
- $P(Z = 11) = 2/36$
- $/36$

There's a name for what we're describing, when we list out all possible outcomes + their probabilities:

Probability Mass Function (PMF)



$$\mathbb{Z}, 1 \leq z \leq 6$$

$$P(Z = z) = \begin{cases} \frac{13-z}{36} & z \in \mathbb{Z}, 7 \leq z \leq 12 \\ 0 & \text{else} \end{cases}$$

Probability Mass Functions

Random Variables & Functions

"Let Y be the number of heads seen in 2 coin flips."

If this is a number

$$P(Y = 2)$$

Then this is a number
(between 0 and 1)

Random Variables & Functions

"Let Y be the number of heads seen in 2 coin flips."

If this is a variable

$$\underbrace{P(Y = \overset{\downarrow}{k})}_{\text{Then this is a function}}$$

Random Variables & Functions

"Let Y be the number of heads seen in 2 coin flips."

We can put in different inputs...

$$k = 1$$

$$P(Y = k)$$

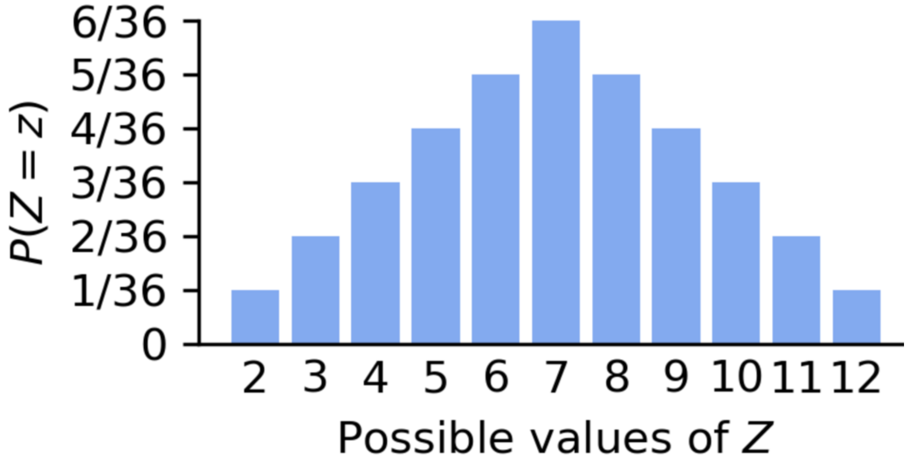
$$0.5$$

...and get out their probabilities!

The relationship between values a random variable can take on, and the corresponding probability, is a ***function***!

Probability Mass Function: Representations

$$P(Z = z) = \begin{cases} \frac{z-1}{36} & z \in \mathbb{Z}, 1 \leq z \leq 6 \\ \frac{13-z}{36} & z \in \mathbb{Z}, 7 \leq z \leq 12 \\ 0 & \text{else} \end{cases}$$



```
def event_probability(z):
    # probability mass function of Z
    if not z.is_integer() or z > 12 or z < 1:
        return 0

    if z < 7:
        return (z – 1) / 36
    else:
        return (13 - z) / 36
```

All of these are different ways we can represent probability mass functions!

Can We Formalize Many Coin Flips?

X = number of heads on n coin flips. Each flip is independent and has prob p of being heads.

Number of flips

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Probability that our variable takes on the value k

Probability of heads

Quick Understanding Check

$$\sum_{\text{all } k} P(Y = k) \overset{?}{=}$$

What is the sum of the probabilities of all possible outcomes for Y?

Quick Understanding Check

$$\sum_{\text{all } k} P(Y = k) = 1$$

What is the sum of the probabilities of all possible outcomes for Y?
1

Can You Calculate A PMF From Data? Yes

Say this is your dataset, a list of observations of random variable Y:

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4, 7, 7, 5, 7, 9, 12, 6, 7, 5, 5, 10, 7, 5, 4, 7, 6, 6, 5, 5, 8, 9, 7, 7, 7,
9, 9, 8, 9, 11, 11, 10, 5, 3, 8, 10, 9, 7, 11, 6, 12, 6, 3, 8, 6, 3, 11,
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12, 9, 8, 10, 4, 4, 5, 6, 4, 5, 6, 7, 3, 3, 11, 8, 9, 2, 8, 4, 8, 7, 8, 9,
10, 5, 10, 7, 9, 8, 8, 6, 7, 5, 6, 11, 2, 5, 3, 8, 4, 7, 7, 4, 7, 2, 7, 10,
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9, 12, 9, 8, 8, 8, 9, 6, 8, 5, 11, 6, 8, 6, 5, 8, 5, 8, 6, 11, 5, 8, 3, 7,
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7, 12, 9, 9, 11, 9, 4, 4, 10, 8, 8, 9, 8, 4, 4, 6, 2, 7, 5, 7, 7, 10, 4,
11, 5, 7, 8, 8, 2, 8, 6, 9, 8, 7, 8, 8, 10, 4, 7, 10, 10, 10, 4, 6, 12,
11, 4, 9, 12, 2, 3, 5, 3, 3, 11, 7, 8, 8, 5, 10, 8, 9, 4, 7, 7, 2, 5, 10,
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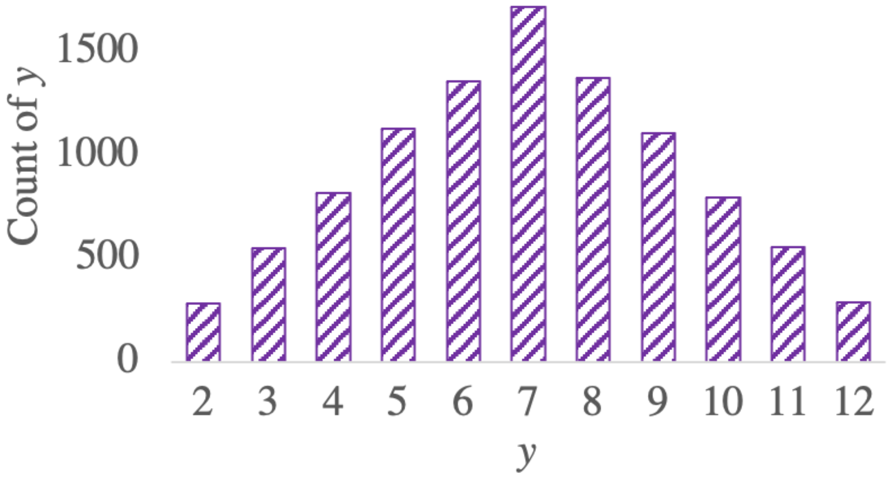
Just convert your data into counts:



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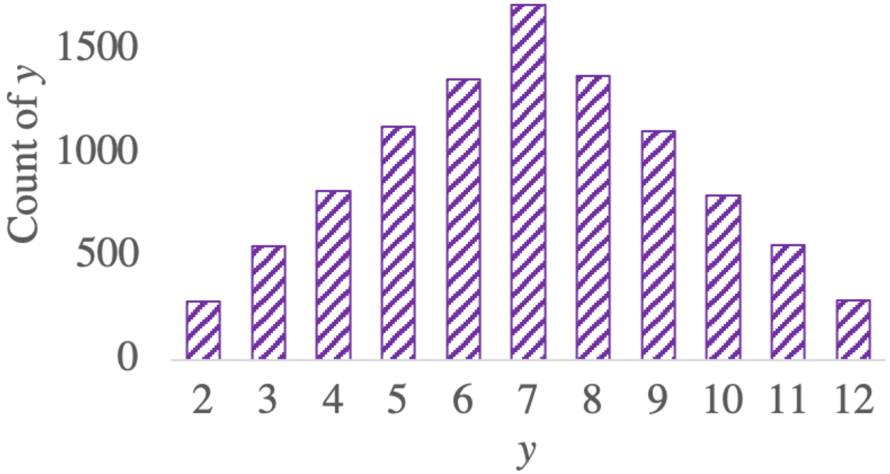


And then use those counts to calculate probabilities for each outcome:

$$\frac{\text{count}(Y = 3)}{n} = \frac{552}{10000} = 0.0552$$

You Can Use PMFs Other People Give You

Let X be the number of earthquakes that happen in California next year.

Here's the PMF for X:

$$\boxed{P(X = x) = \frac{69^x e^{-69}}{x!}}$$

What is the probability that there are 60 earthquakes in California next year?

You Can Use PMFs Other People Give You

Let X be the number of earthquakes that happen in California next year.

Here's the PMF for X:
$$\boxed{P(X = x) = \frac{69^x e^{-69}}{x!}}$$

What is the probability that there are 60 earthquakes in California next year?

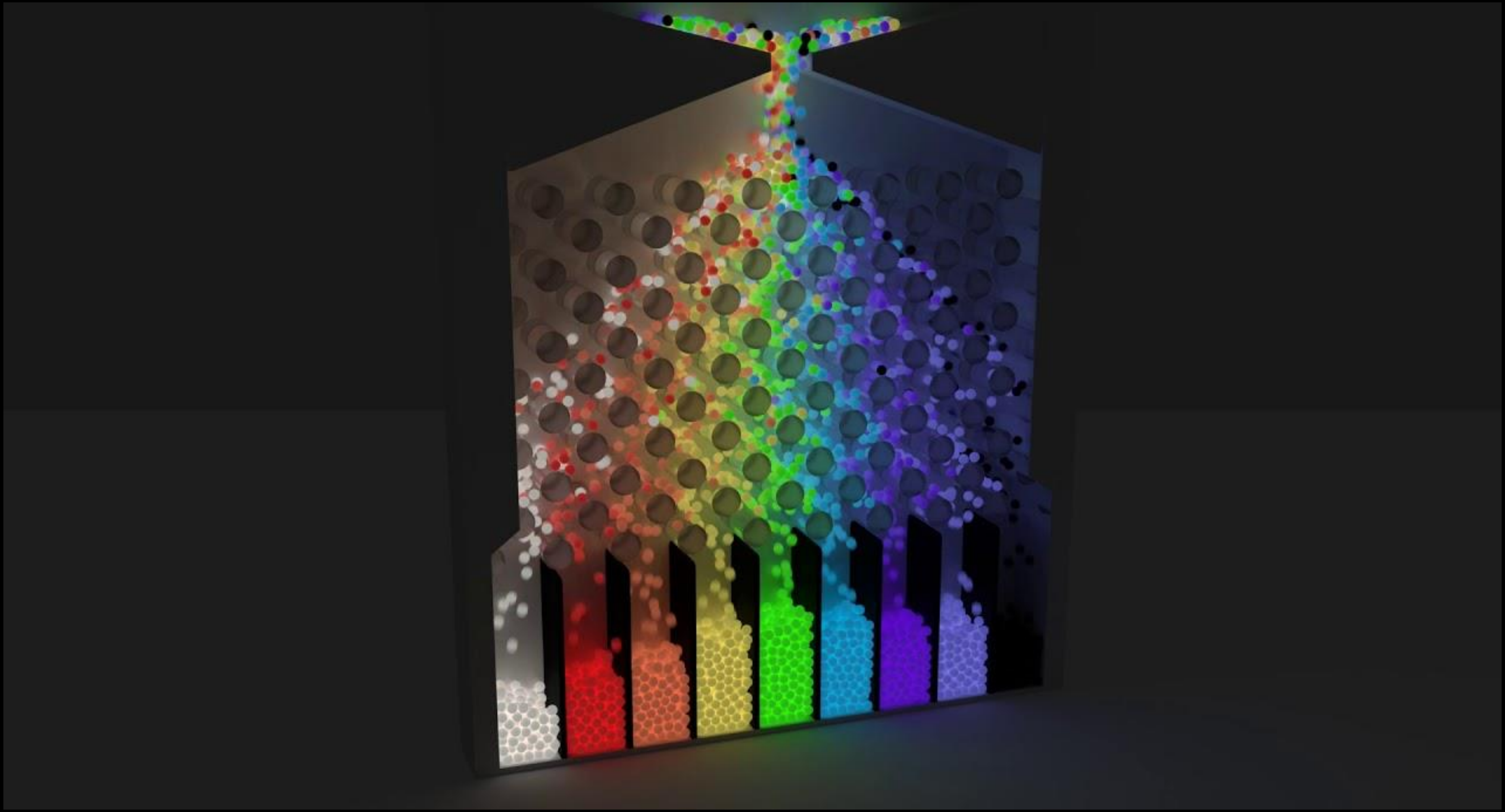
$$P(X = 60) = \frac{69^{60} e^{-69}}{60!} \approx 0.028$$

Just plug numbers in!

What is the most important thing to know about a discrete random variable?

Probability Mass Function

Random Variables Are Awesome 💪



Some Random Variables Are “Classics”

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with p = 0.01 (otherwise it’s ignored).

What is the probability of this ad getting 10 clicks?

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with p = 0.01 (otherwise it's ignored).

What is the probability of this ad getting 10 clicks?

Same as asking: "What is the probability of getting 10 heads out of 1000 coin flips, when the probability of each head is 0.01"

We Have Discovered The Binomial

The binomial describes scenarios with:

1. ***n* independent trials** (coin flips)
2. A consistent **probability *p*** of success on each trial (heads)
3. What we want: what is the probability of **exactly *k* successes**?

Many, many scenarios fit this description:

- # of 1’s in randomly generated in length *n* bit string
- # of servers working in a large computer cluster
- # of votes for one of two candidates in an election
- # of jury members selected from a particular demographic
- # of CS109 students who decided to come to lecture today

We Have Discovered The Binomial

Here yee. This type of random variable is so common it needs a name so that I can talk about it generally.

I shall call it: the ***Binomial*** *Random Variable. Huzzah.*

Jacob "James" Bernoulli (1654-1705): Swiss mathematician

One of many mathematicians in the Bernoulli family

We Have Discovered The Binomia

Here yee.
variable is s
name so th

I shall call it:
Vari

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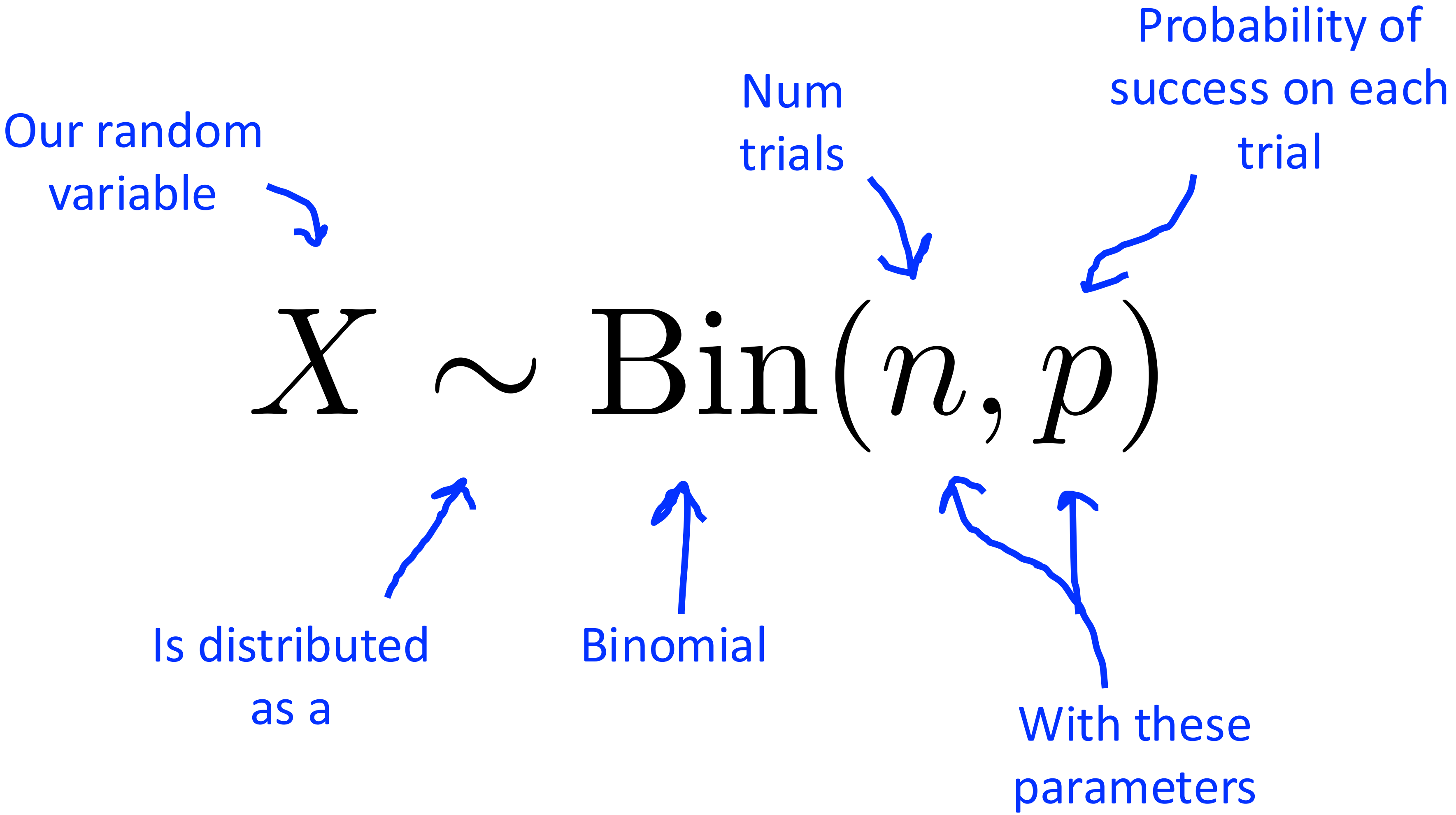
Jacob "James" Bernoulli (1654-1705): Swiss mathematician

One of many mathematicians in the Bernoulli family

Declaring a Random Variable to be Binomial



Our random variable

Num trials

Probability of success on each trial

$$X \sim \text{Bin}(n, p)$$

Is distributed as a

Binomial

With these parameters

Then We Automatically Know the PMF!

$$P(X = k) = \overbrace{\binom{n}{k} p^k (1-p)^{n-k}}^{\text{Probability Mass Function for a Binomial}}$$

Probability that our variable takes on the value k

New Recipe For Solving Problems!

1. Recognize a classic random variable type

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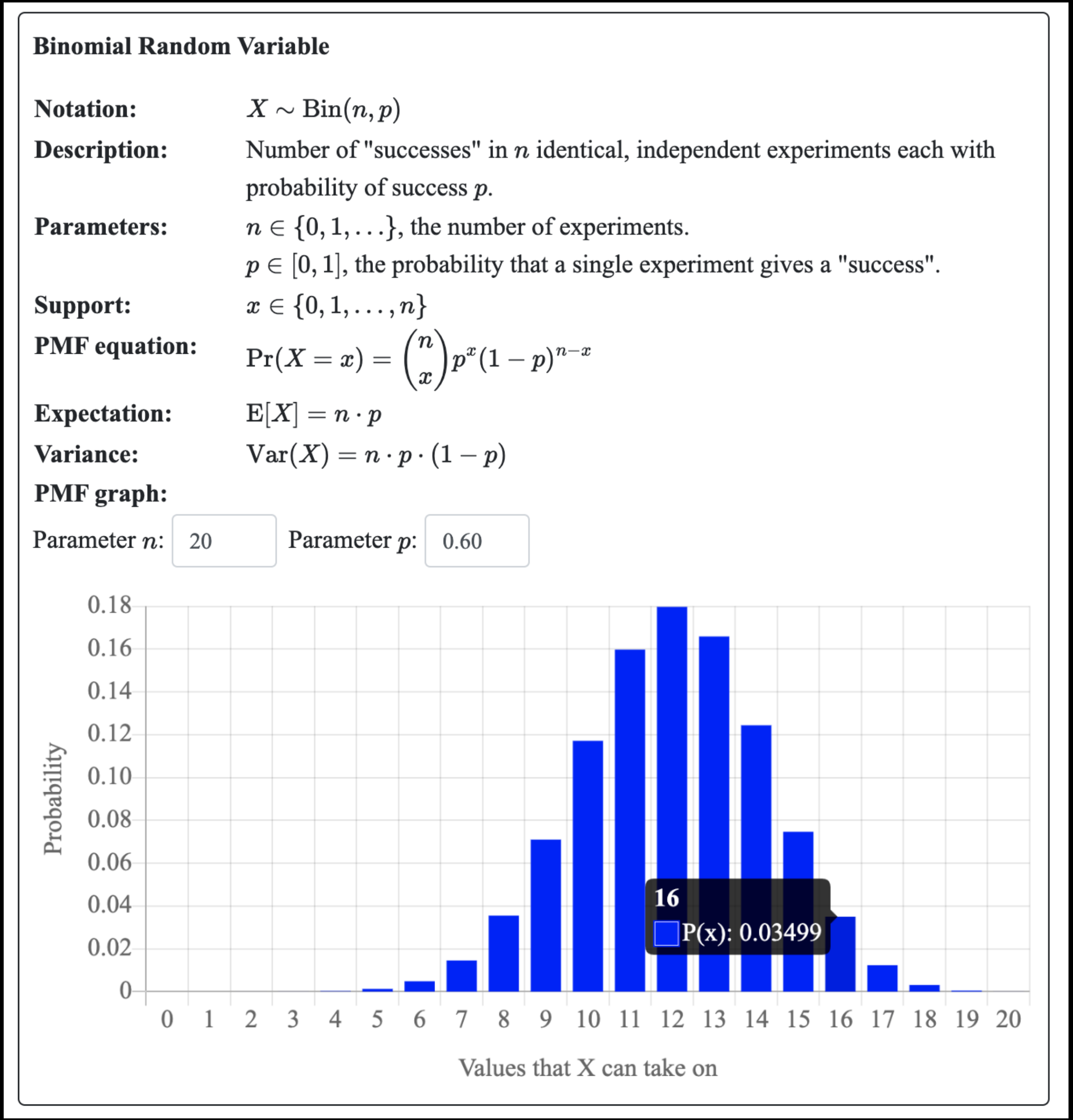
$$X \sim \text{Bin}(n, p)$$

3. Profit off the PMF

You Get So Much For Free!



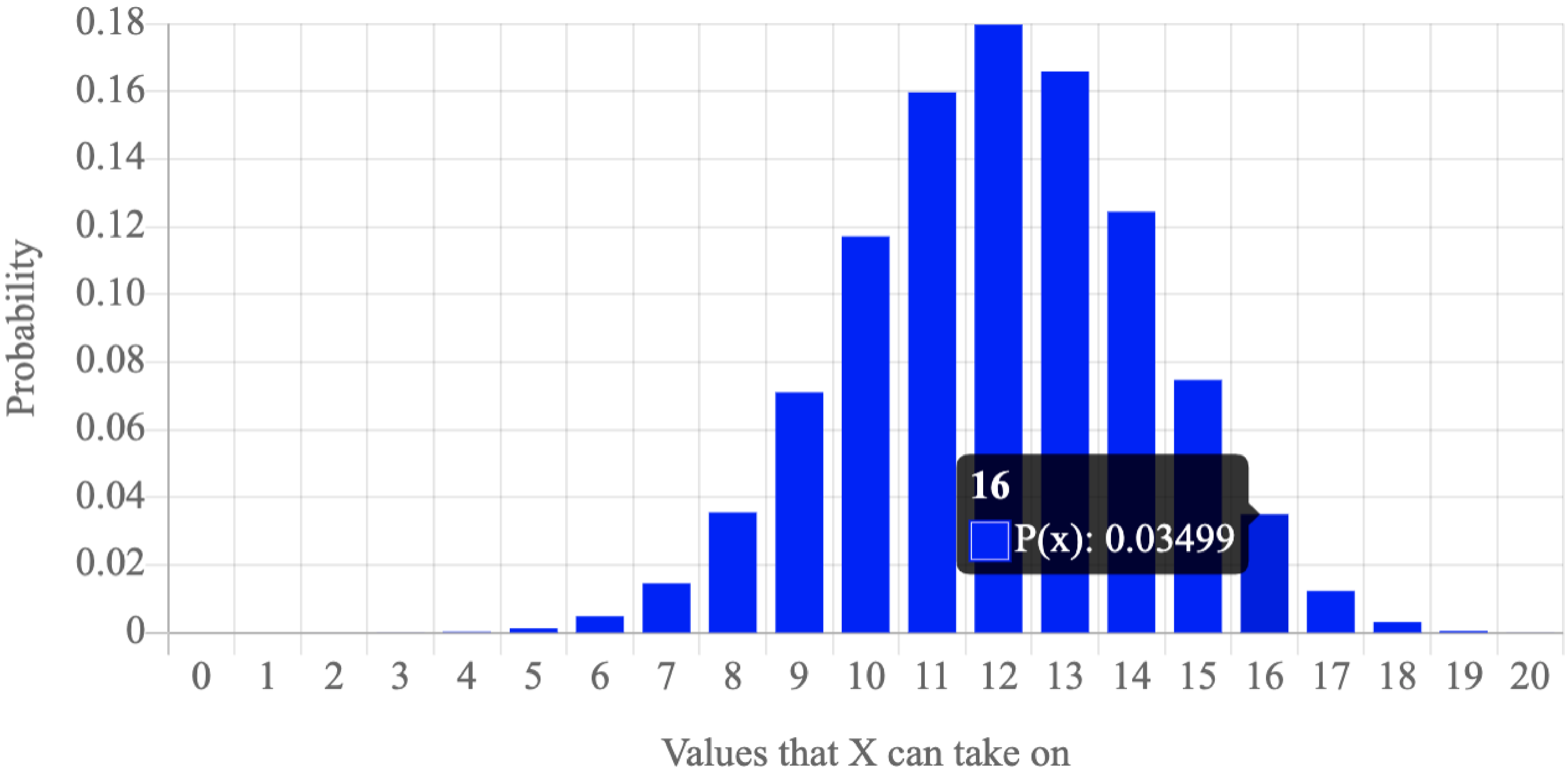
Binomial Random Variable

Notation:	$X \sim \text{Bin}(n, p)$
Description:	Number of "successes" in n identical, independent experiments each with probability of success p.
Parameters:	$n \in \{0, 1, \ldots\}$, the number of experiments. $p \in [0, 1]$, the probability that a single experiment gives a "success".
Support:	$x \in \{0, 1, \ldots, n\}$
PMF equation:	$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$
Expectation:	$\text{E}[X] = n \cdot p$
Variance:	$\text{Var}(X) = n \cdot p \cdot (1 - p)$
PMF graph:	

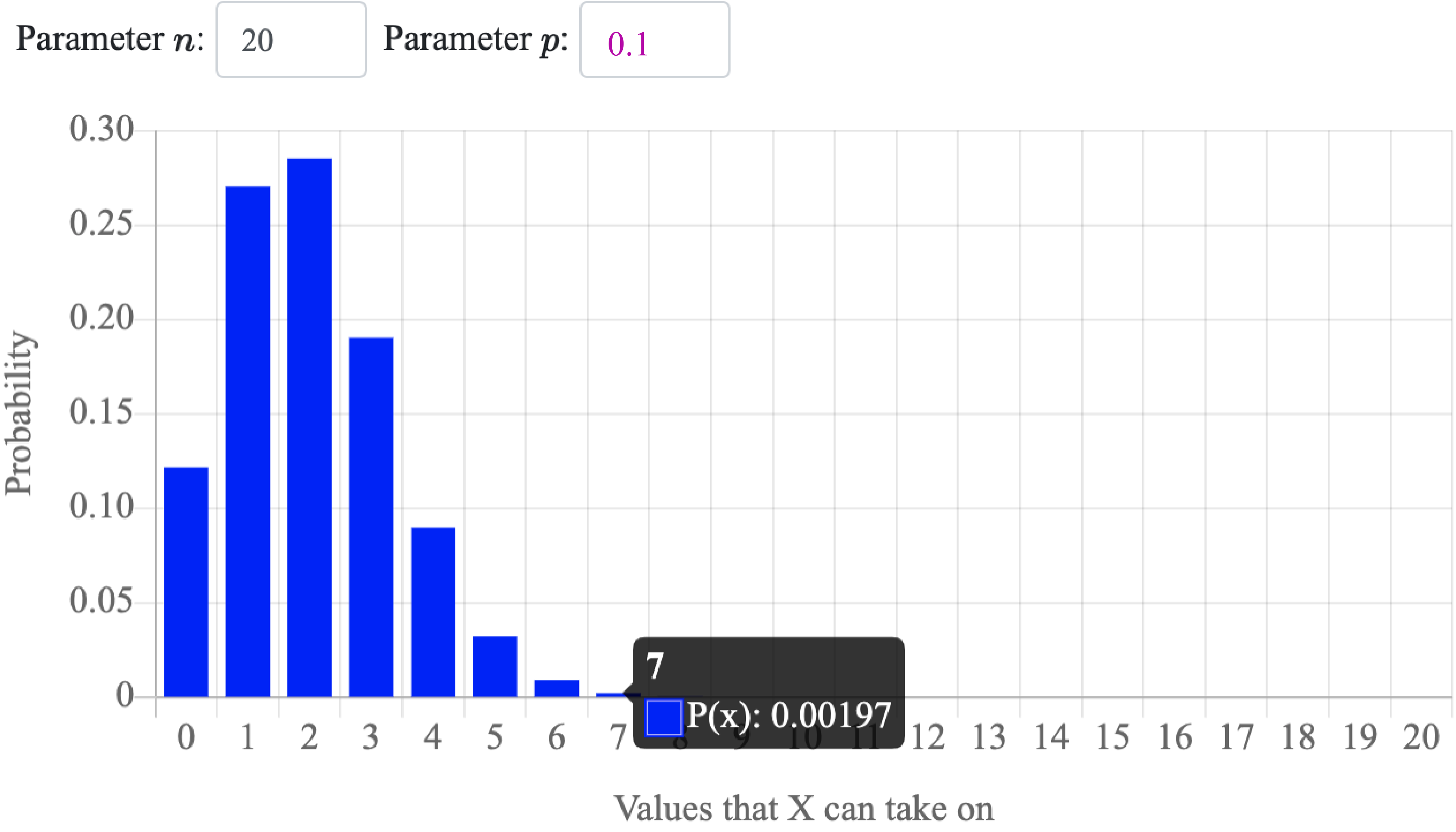
Parameter n: 20 Parameter p: 0.60

The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.6)$

Parameter n: 20 Parameter p: 0.60



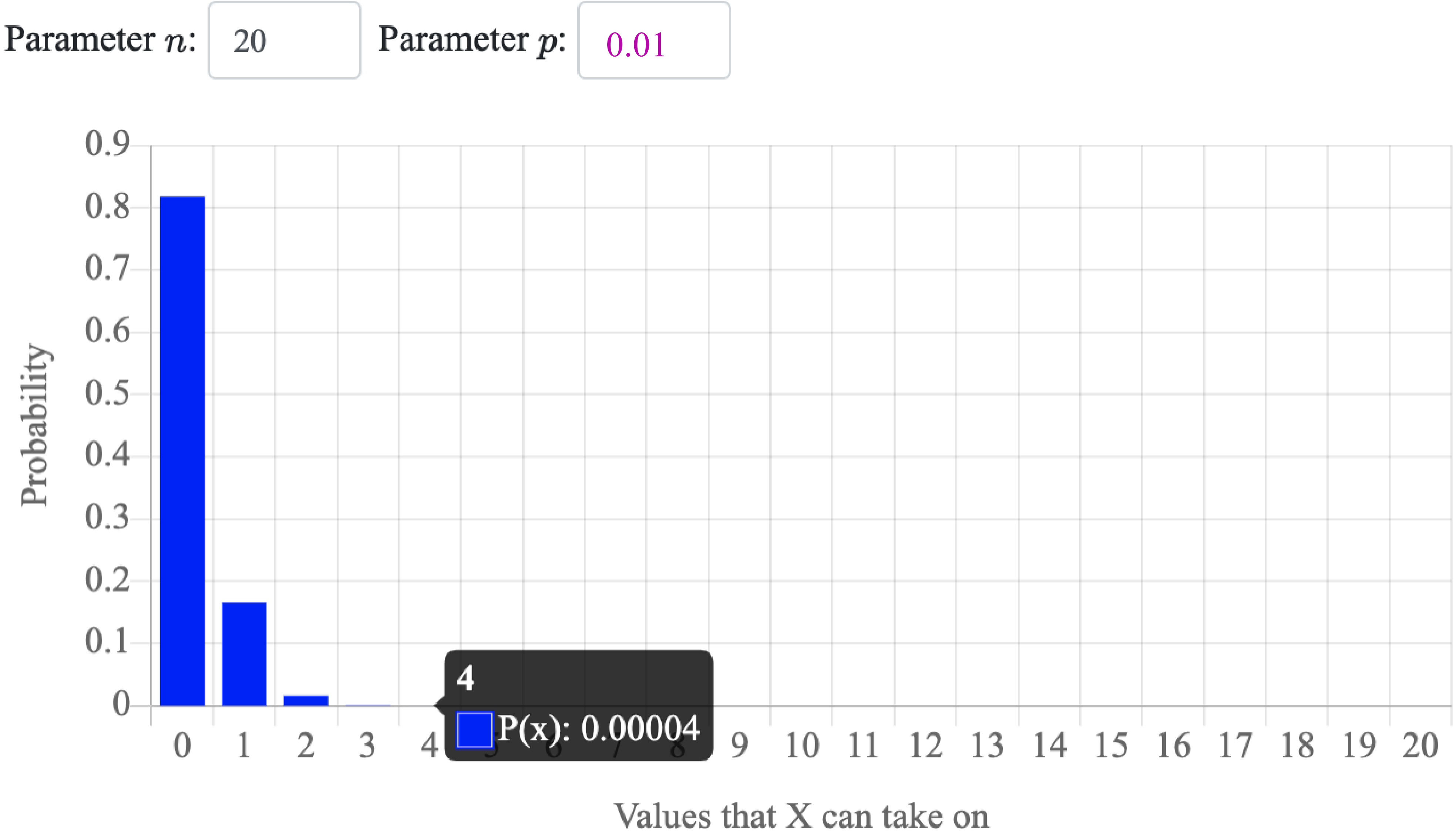
The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.1)$



Parameter n: 20

Parameter p: 0.1

The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.01)$



Parameter n: 20 Parameter p: 0.01

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

Each ad served is clicked with p = 0.01 (otherwise it's ignored).

What is the probability of this ad getting 10 clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

$$P(X = k) = \binom{1000}{k}(0.01)^k(0.99)^{1000-k}$$

$$P(X = 10) = \binom{1000}{10}(0.01)^{10}(0.99)^{990} \approx 0.125$$

Practice: Ad Clicks

Every day, Youtube shows a particular ad 1000 times.

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What is the probability of this ad getting **20** clicks?

Let X be the number of ad clicks. $X \sim \text{Bin}(n = 1000, p = 0.01)$.

$$P(X = k) = \binom{1000}{k}(0.01)^k(0.99)^{1000-k}$$

$$P(X = 20) = \binom{1000}{20}(0.01)^{20}(0.99)^{980} \approx 0.0018$$

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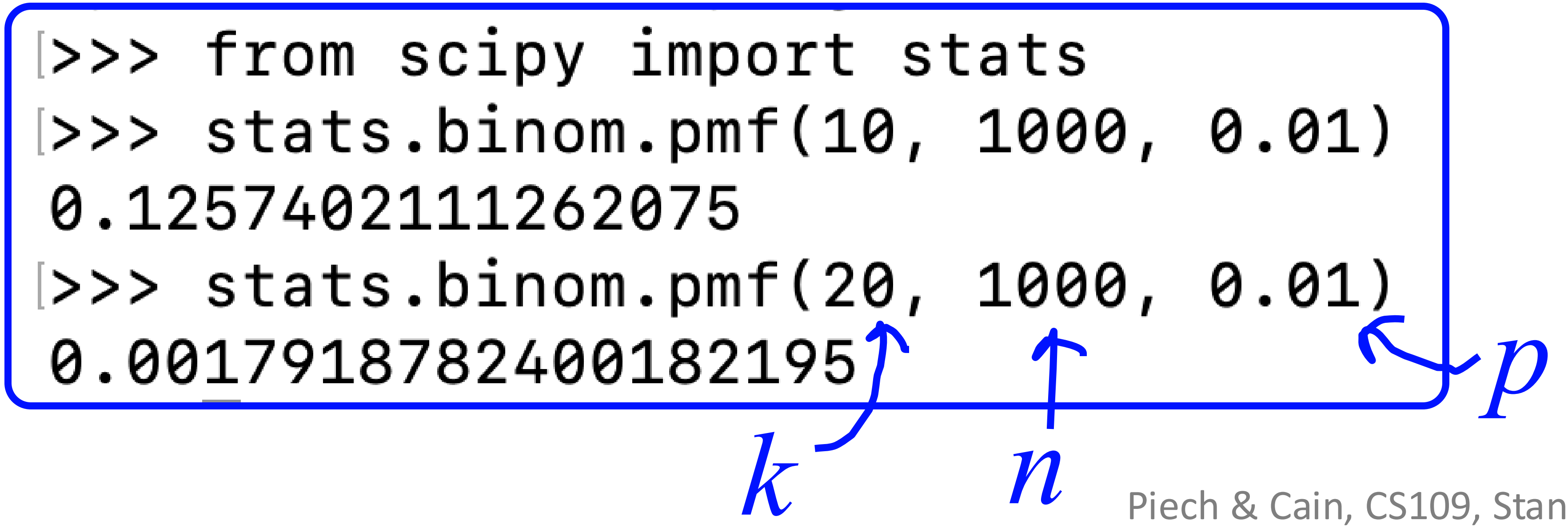

```
[>>> from scipy import stats
[>>> stats.binom.pmf(10, 1000, 0.01)
0.12574021112620 75
[>>> stats.binom.pmf(20, 1000, 0.01)
0.0017918782400182195
```

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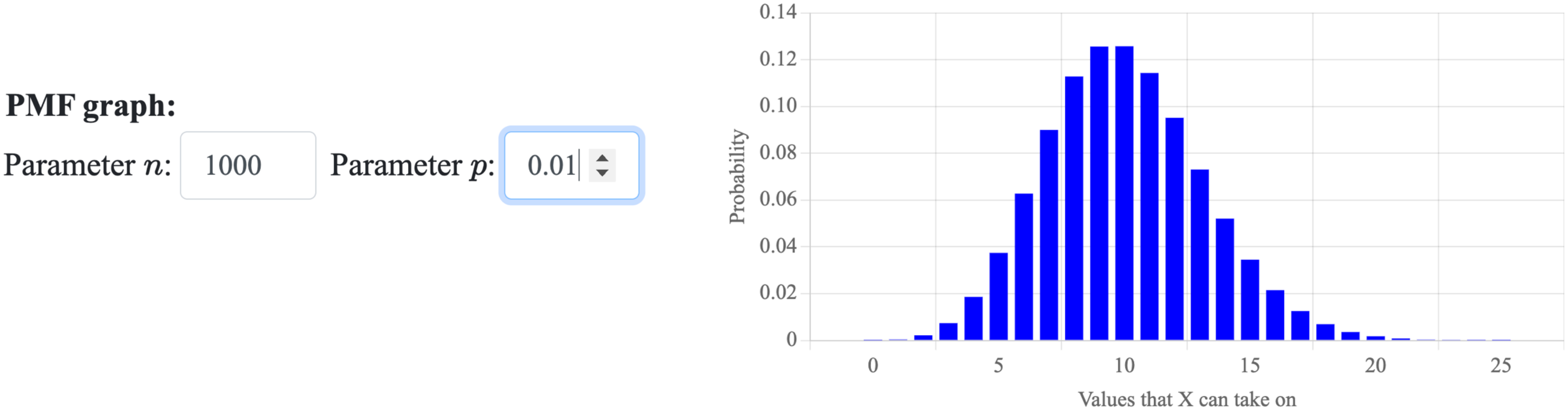

PMF graph:

Parameter n: 1000 Parameter p: 0.01

Server Redundancy

A network can remain functional as long as at least 2 out of 7 servers are alive.

The probability of any server working is 0.8.

What is the probability that less than 2 servers are alive?

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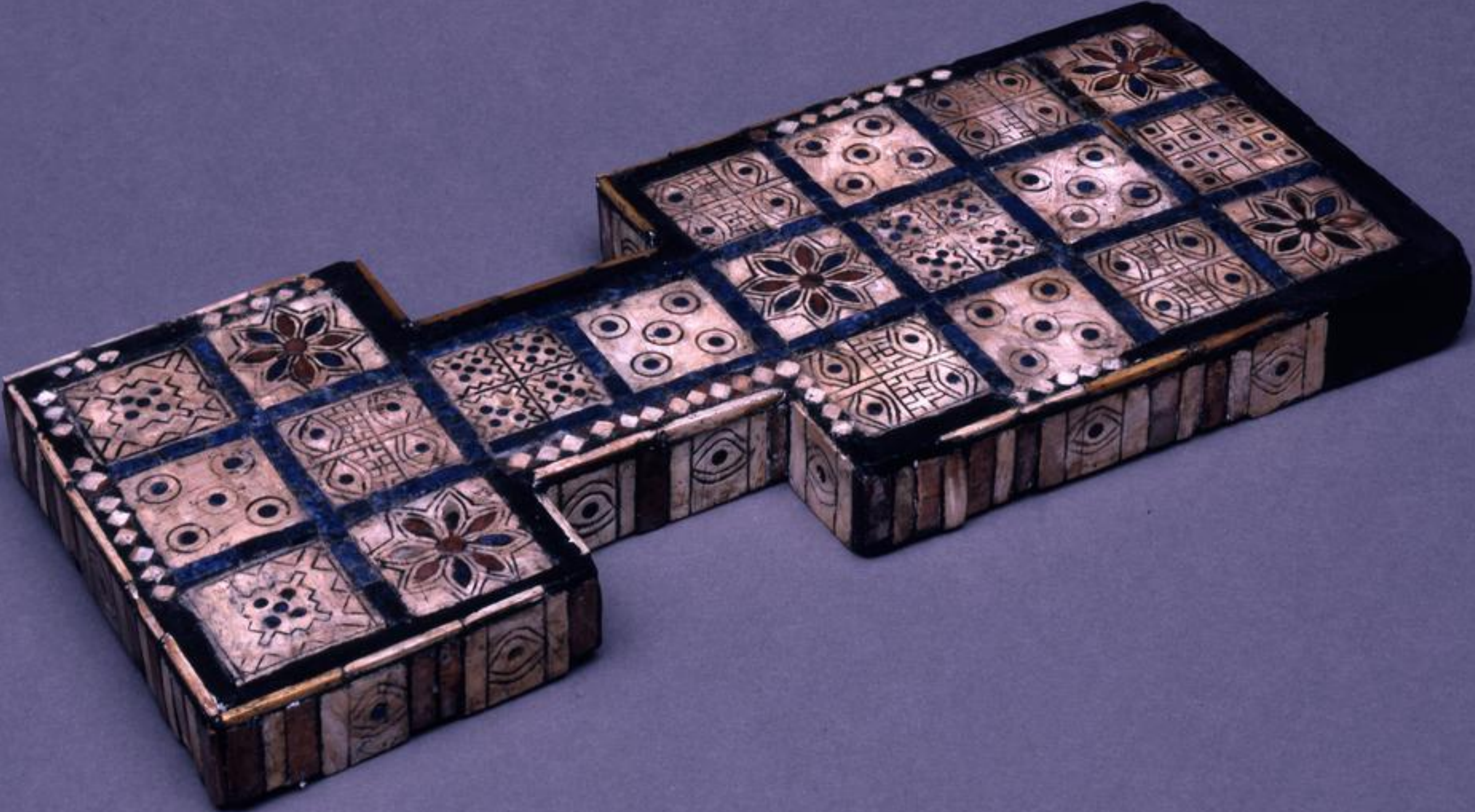
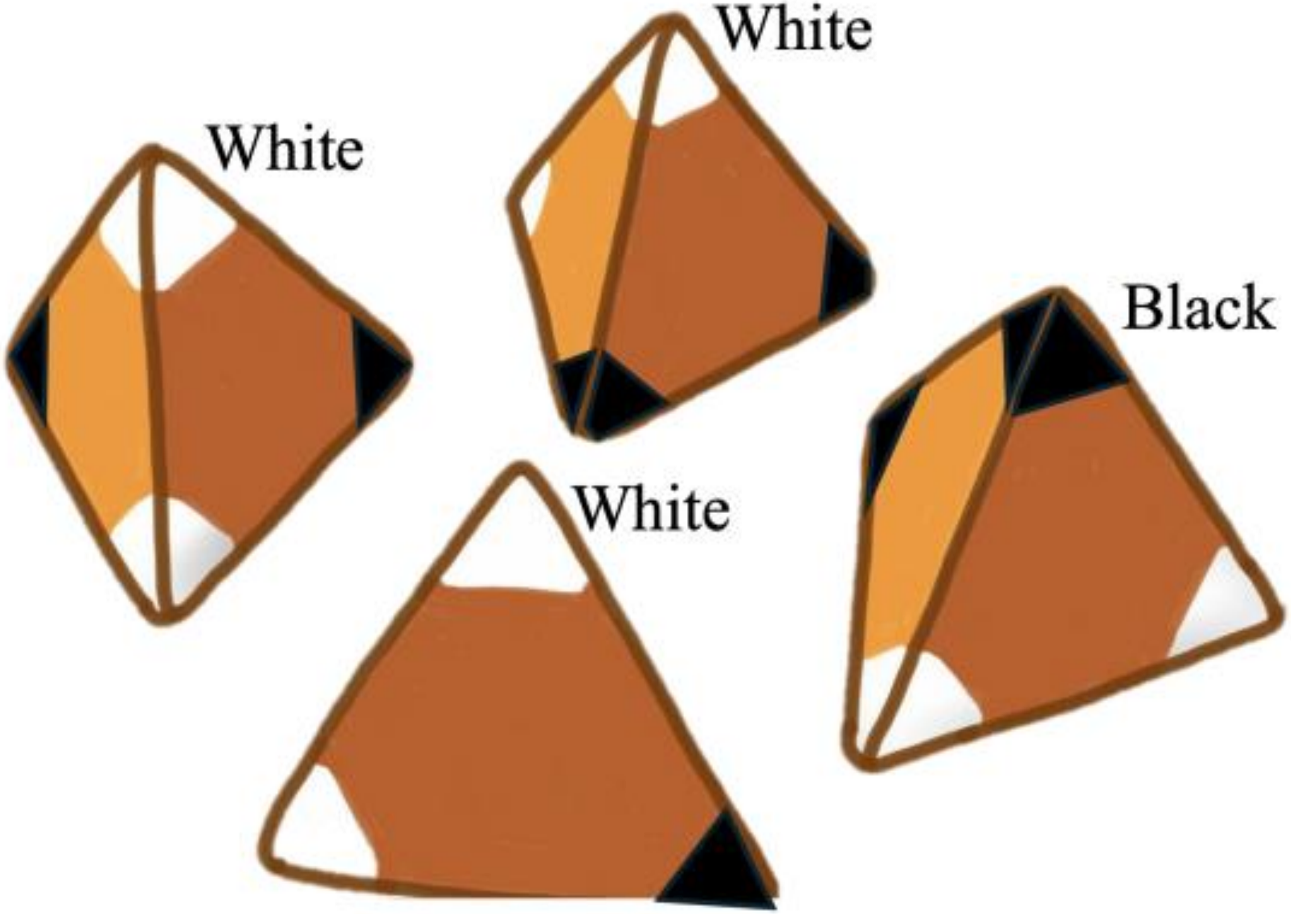
$$P(X = k) = \binom{7}{k}(0.8)^k(0.2)^{7-k}$$

$$P(X < 2) = P(X = 0) + P(X = 1) = \binom{7}{0}(0.8)^0(0.2)^{7-0} + \binom{7}{1}(0.8)^1(0.2)^{7-1} \approx 0.0004$$




White
White
Black
White

How many of these questions do you answer "yes"?

1. Would you rather have unlimited time or unlimited money?
2. Do you enjoy surprises?
3. Would you rather go to the beach than the mountains?
4. Do you think physical books are better than e-books?
5. Would you rather take a vacation in your home country than abroad?
6. Do you support the dictator?

50/50
questions

Let p be the probability that a person supports the dictator

What is the probability that someone would answer "yes" to 4 questions?

What is the most important thing to know about a discrete random variable?

Probability Mass Function


6
Rakuten
GOL
30

What is the probability of winning a 7 game series?

Warriors are going to play the Celtics in a best of 7 series during the 2050 NBA finals. What is the probability that the warriors win the series? Each game is **independent**. Each game, the warriors have a 0.55 probability of winning. Win series if you win at least 4 games.

Let X be the number of games won. $X \sim \text{Bin}(n = 7, p = 0.55)$. $\text{P}(X > 3)$?

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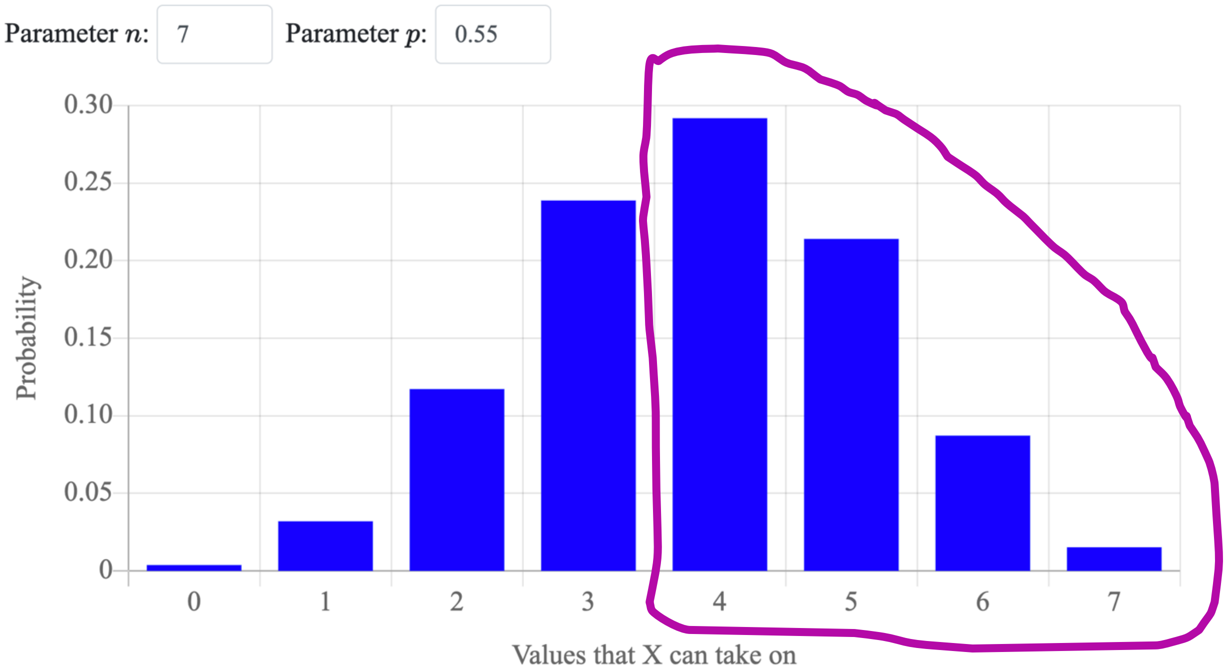

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Let X be the number of games won. $X \sim \text{Bin}(n = 7, p = 0.55)$. P($X > 3$)?

$$
\begin{aligned}
P(X \geq 4) &= \sum_{i=4}^{7} P(X = i) \\
&= \sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i} \\
&= \sum_{i=4}^{7} \binom{7}{i} 0.55^i (0.45)^{7-i}
\end{aligned}
$$

Debugging Probability

Debugging Probability

How to calculate the probability of at least k successes in n independent trials?

- X is number of successes in n trials each with probability p
- $P(X \geq k) =$

Chose slots for success, don't care about rest

$$\binom{n}{k}p^k$$

Probability that each is success

ways to choose slots for success

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Chose slots for success, don't care about rest

$$\binom{n}{k} p^k$$

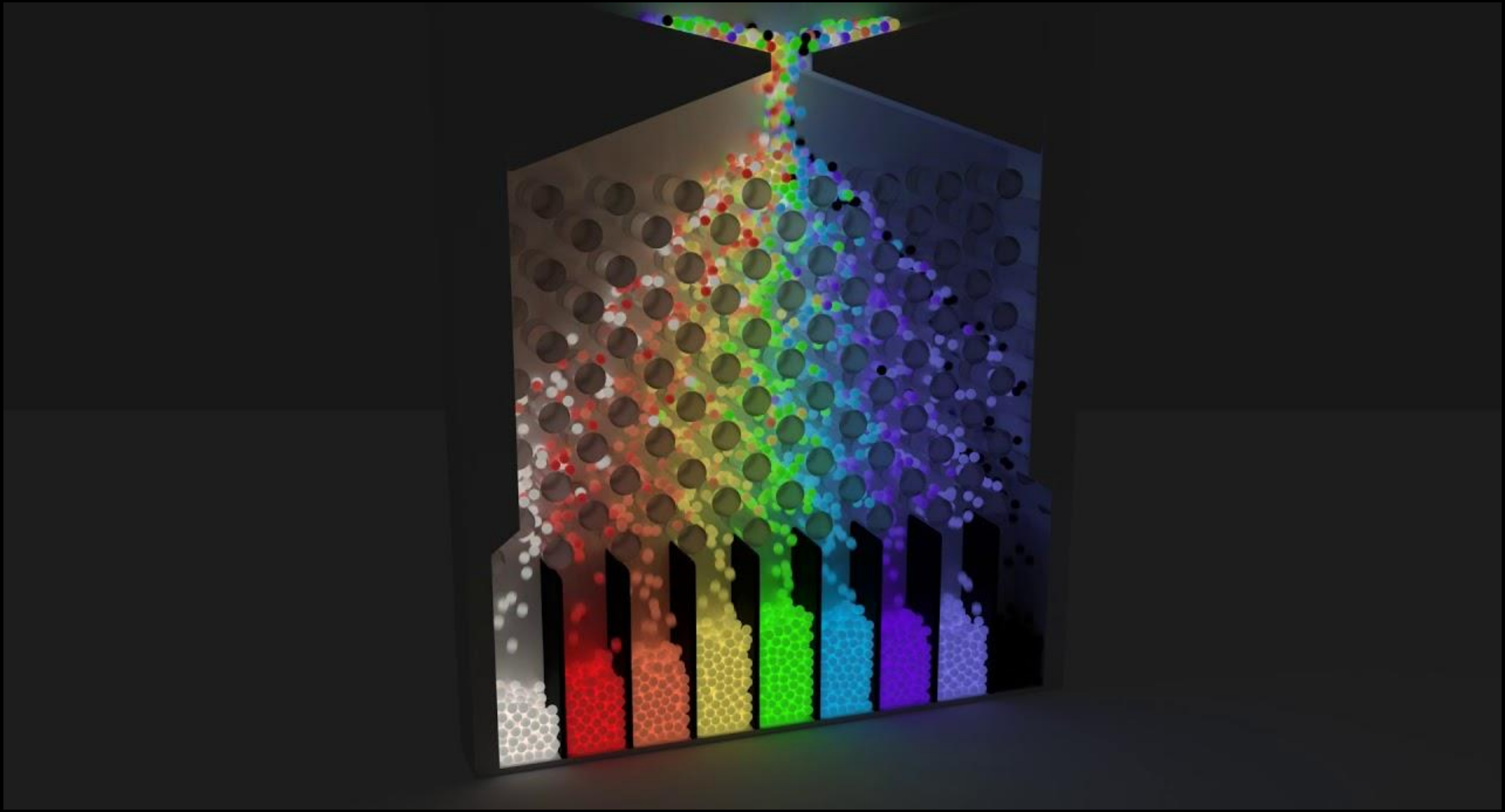
ways to choose slots for success

Probability that each is success

First clue that something is wrong. Think about $p = 1$

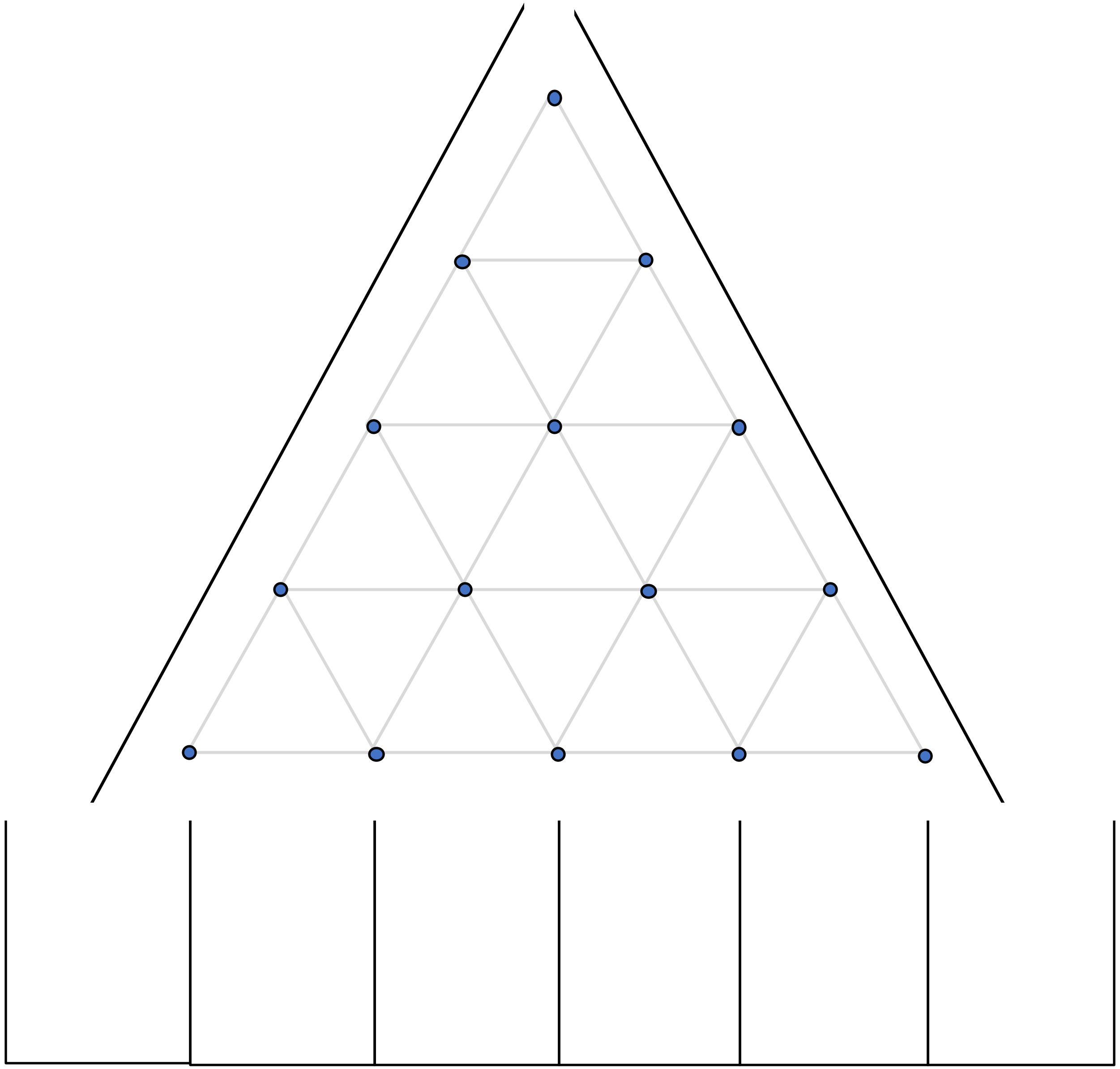
Not mutually exclusive…

Correct: $$P(X \geq k) = \sum_{i=k}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$



Galton Board Time!

Galton Board Fun



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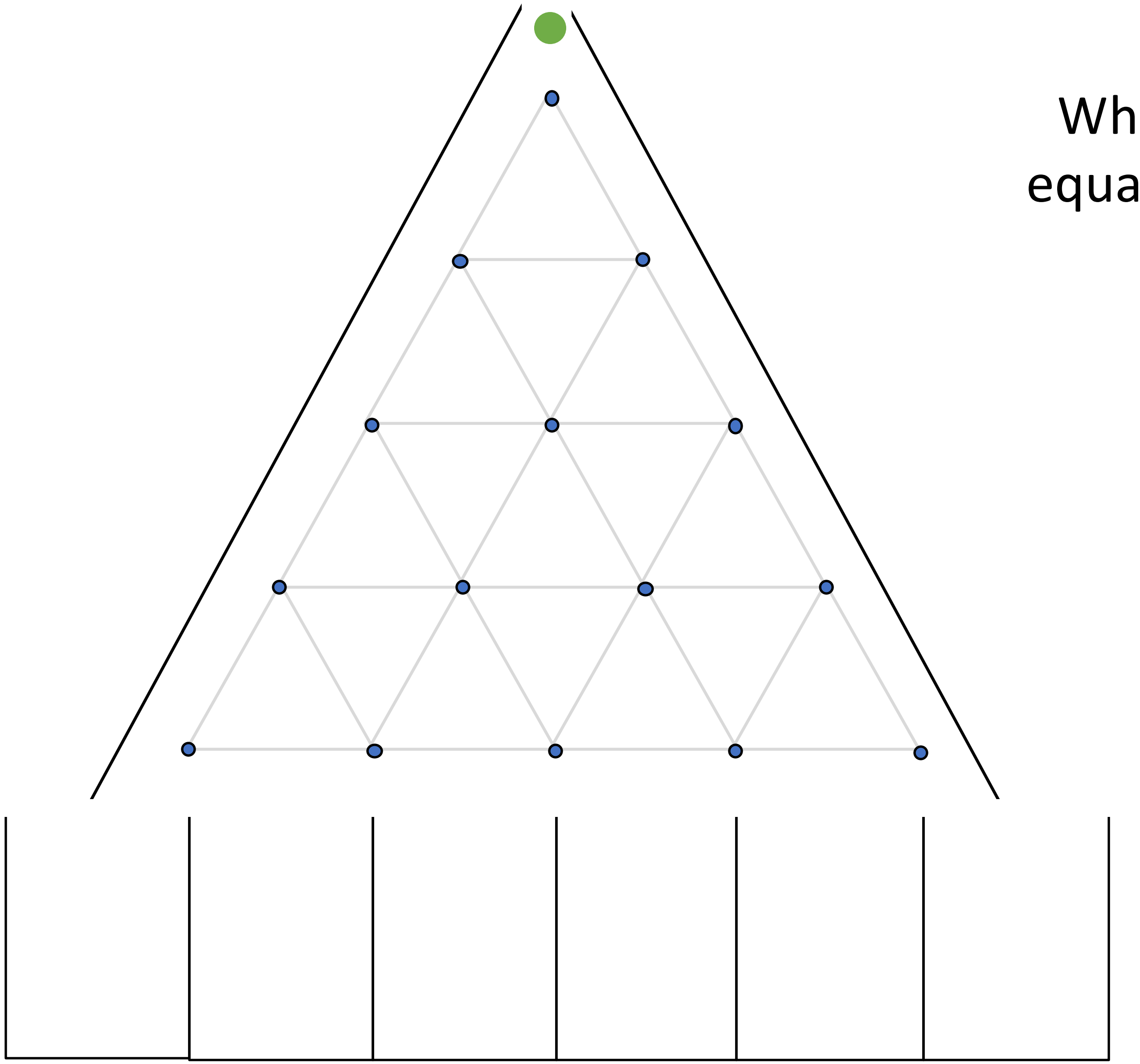
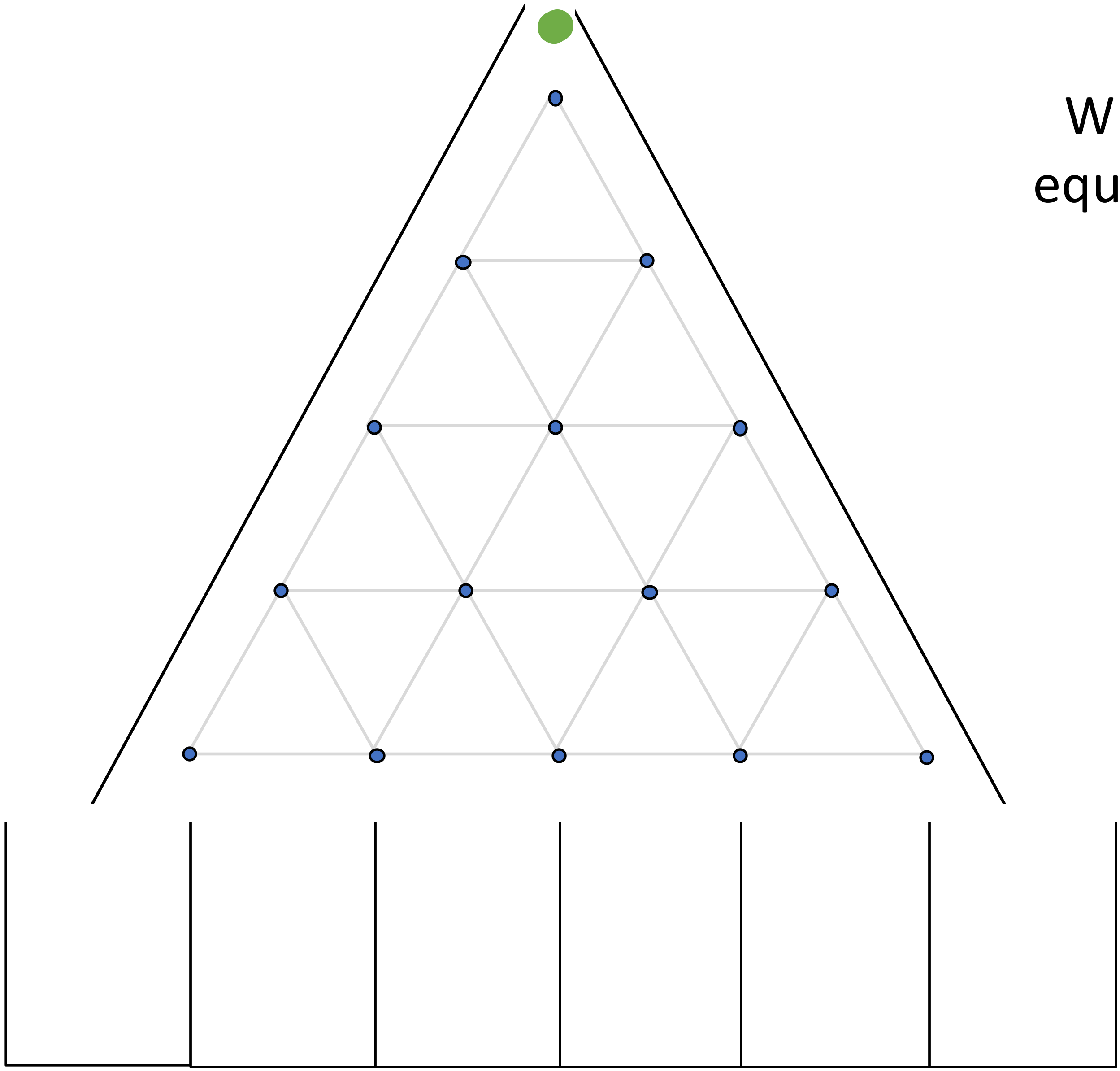


When a marble hits a pin, it has equal chance of going left or right.

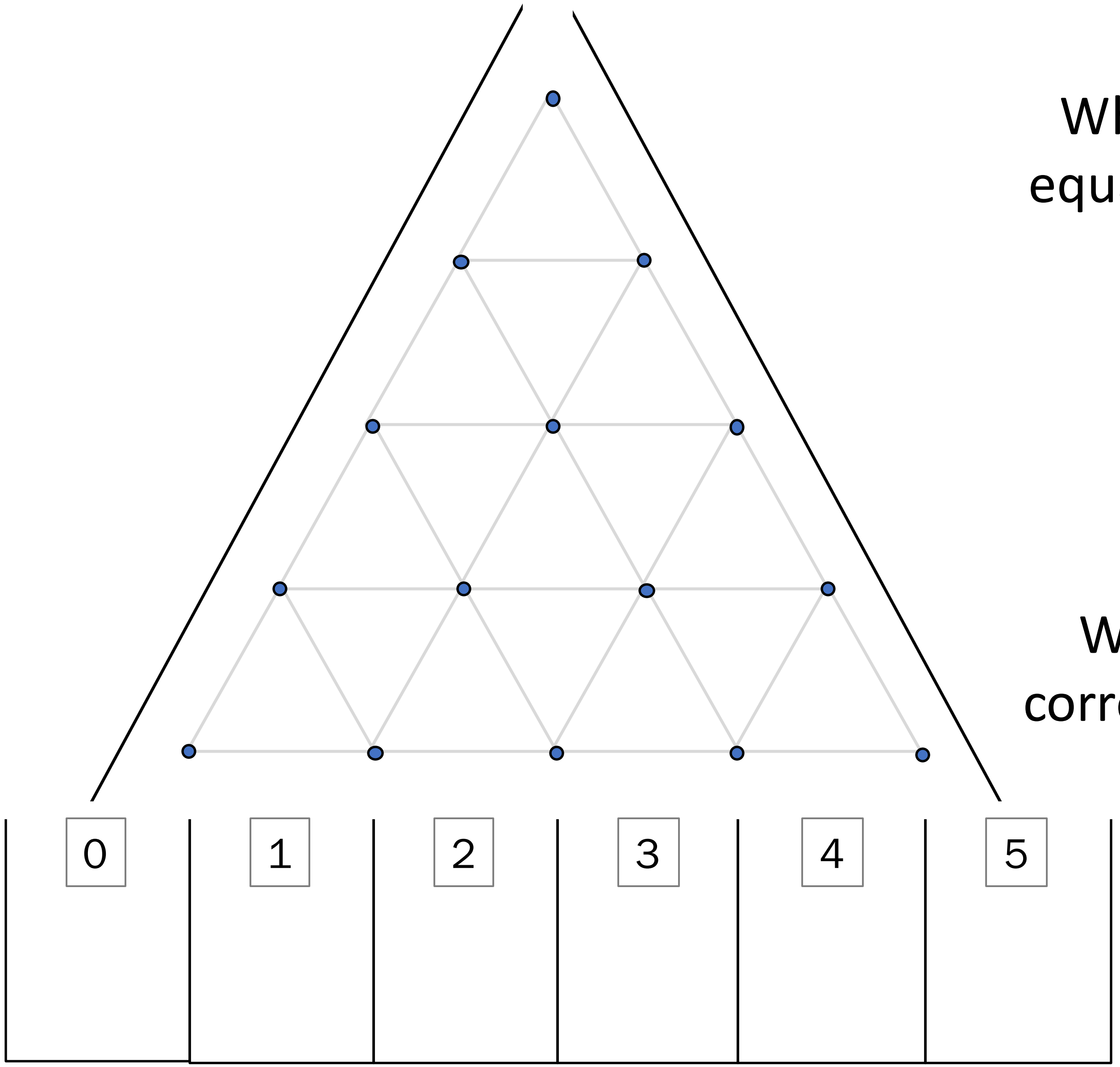
Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Galton Board Fun

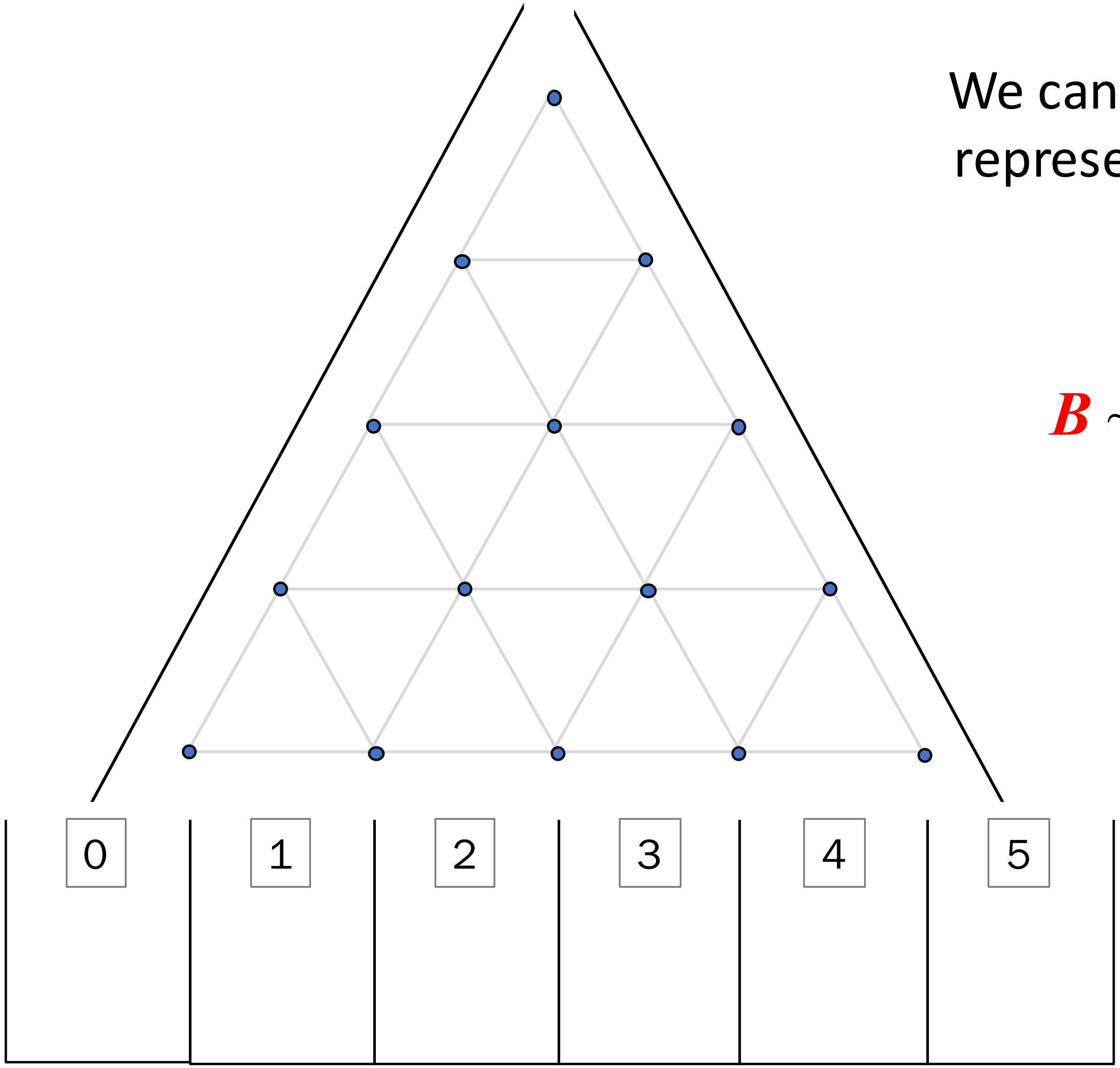


When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Which bucket a marble lands in corresponds to the number of times the marble went right.

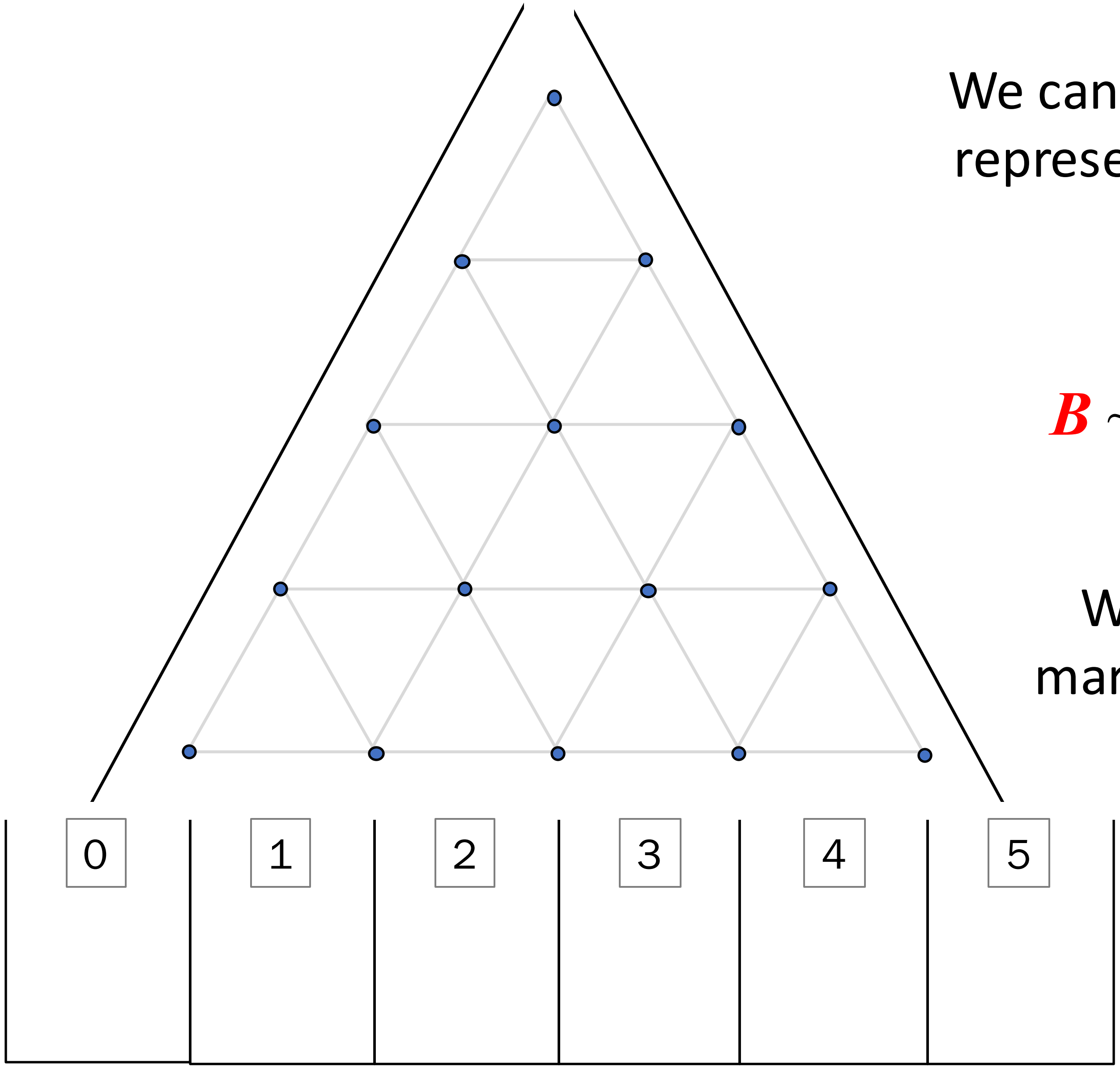
Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

Galton Board Fun



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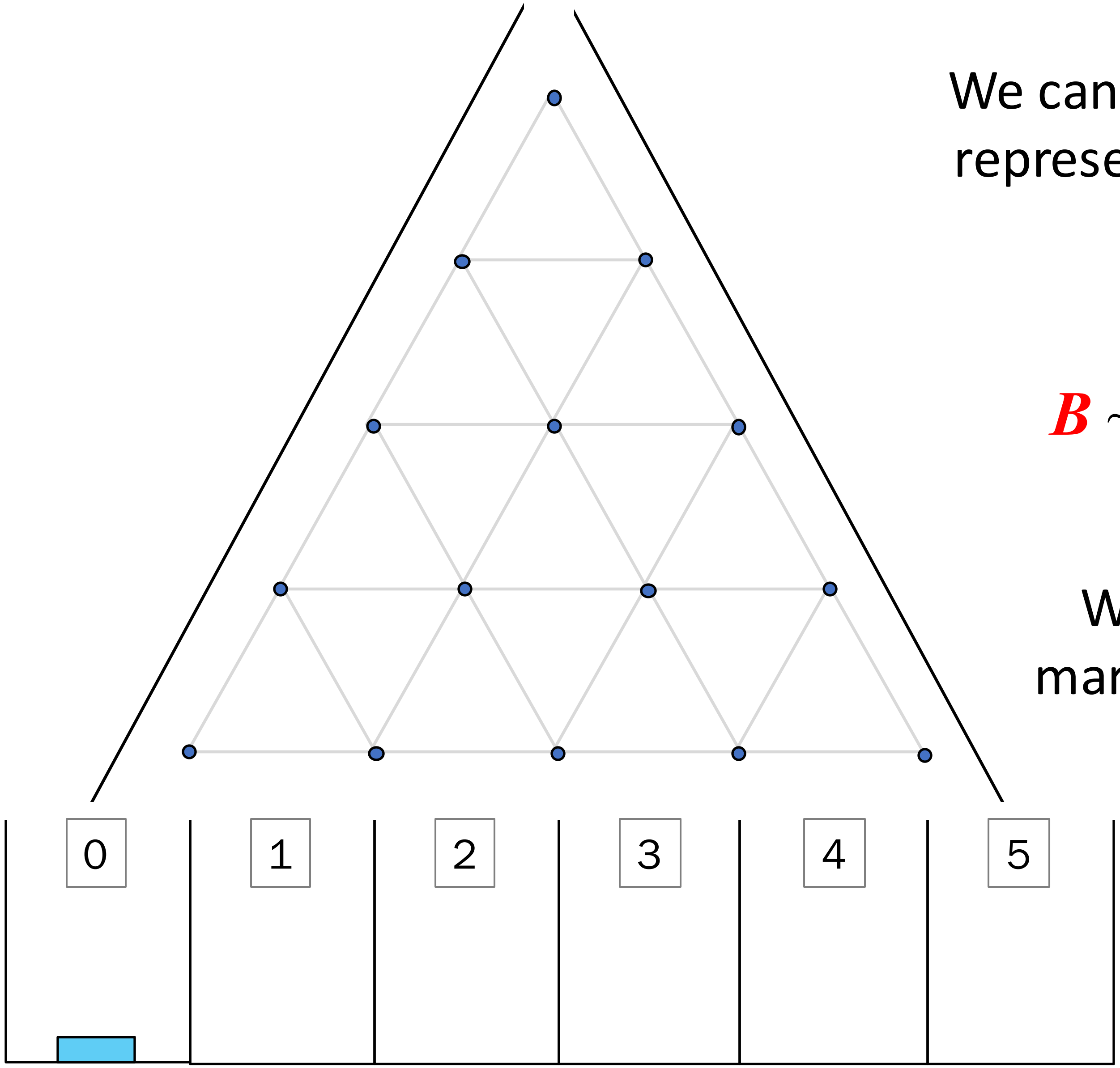
What is the probability of a marble landing in each bucket?

Galton Board Fun



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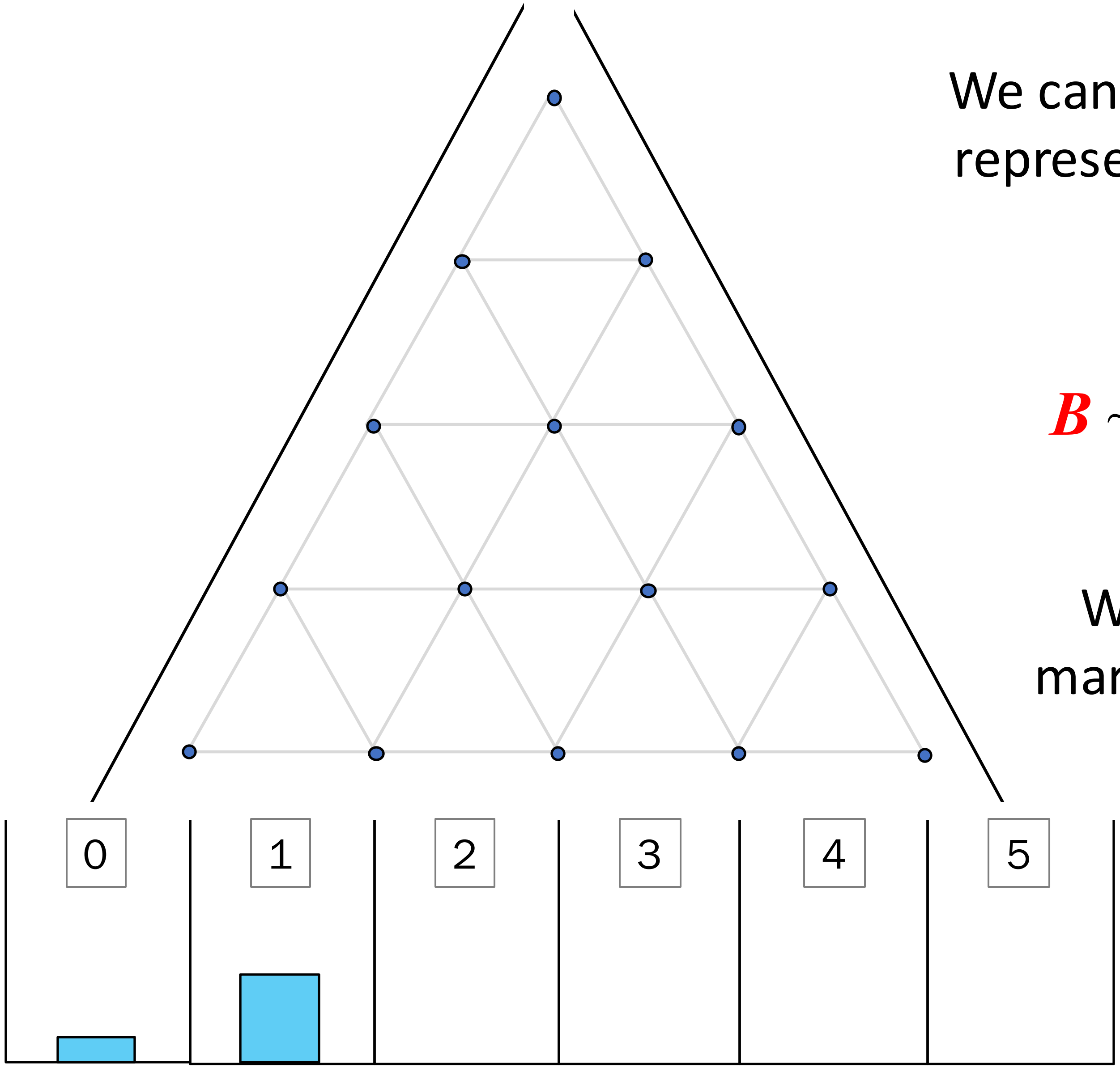
What is the probability of a marble landing in each bucket?

$$P(B = 0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

Galton Board Fun



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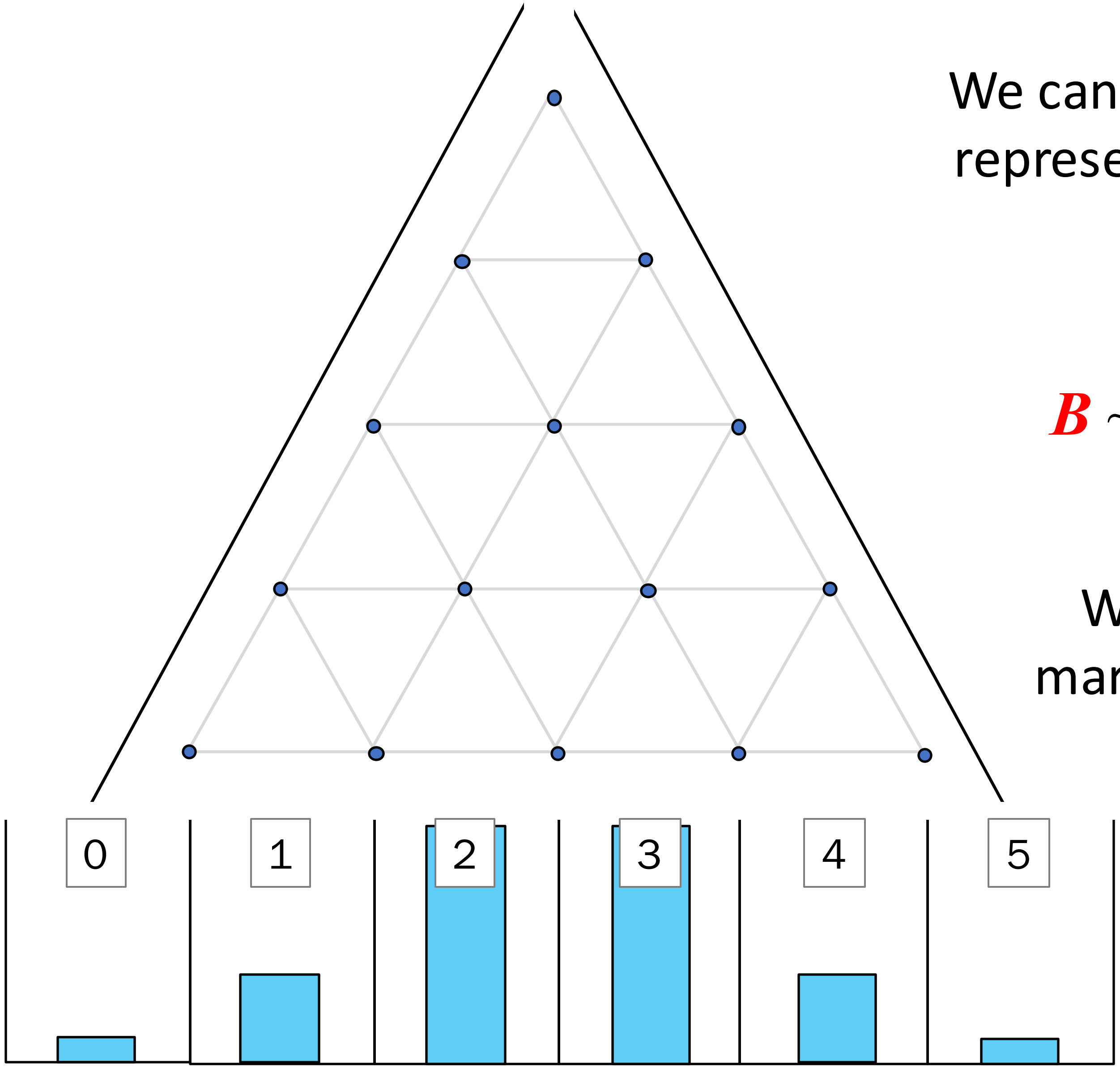
What is the probability of a marble landing in each bucket?

$$P(B=0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

$$P(B=1) = \binom{5}{1} \frac{1}{2}^5 \approx 0.16$$

Galton Board Fun



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What is the probability of a marble landing in each bucket?

This is the PMF of the binomial!

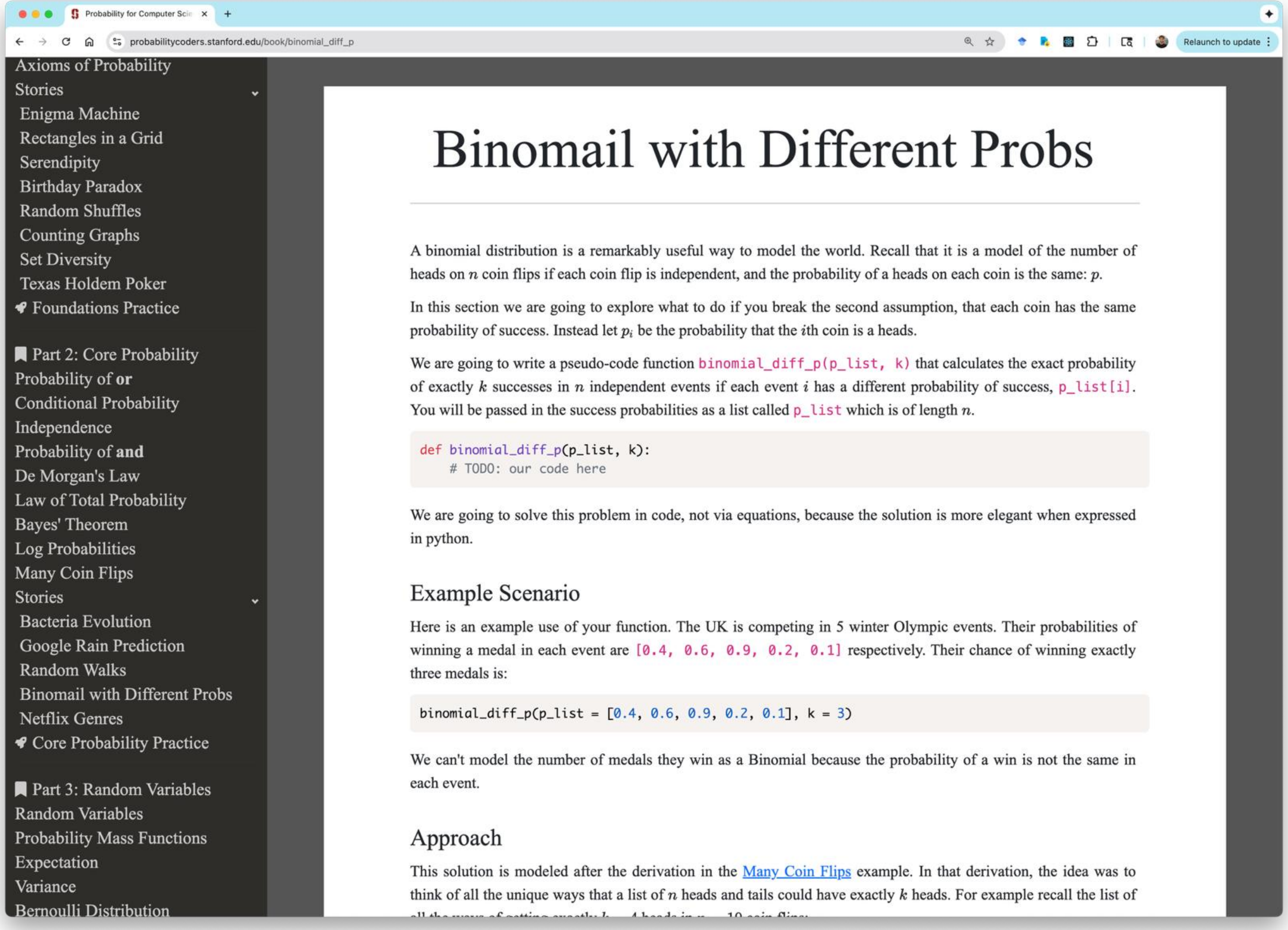
FROM CHAOS TO ORDER

Probability is *Everywhere*

Learning Goals for Today

- A **random variable** is a number which takes on values probabilistically.
- A discrete random variable is fully described by a **probability mass function**.
- A **binomial** is a particular random variable which represents number of heads in n coin flips.

Challenge!



Binomail with Different Probs

A binomial distribution is a remarkably useful way to model the world. Recall that it is a model of the number of heads on n coin flips if each coin flip is independent, and the probability of a heads on each coin is the same: p.

In this section we are going to explore what to do if you break the second assumption, that each coin has the same probability of success. Instead let p_i be the probability that the ith coin is a heads.

We are going to write a pseudo-code function `binomial_diff_p(p_list, k)` that calculates the exact probability of exactly k successes in n independent events if each event i has a different probability of success, `p_list[i]`. You will be passed in the success probabilities as a list called `p_list` which is of length n.

```
def binomial_diff_p(p_list, k):
    # TODO: our code here
```

We are going to solve this problem in code, not via equations, because the solution is more elegant when expressed in python.

Example Scenario

Here is an example use of your function. The UK is competing in 5 winter Olympic events. Their probabilities of winning a medal in each event are `[0.4, 0.6, 0.9, 0.2, 0.1]` respectively. Their chance of winning exactly three medals is:

```
binomial_diff_p(p_list = [0.4, 0.6, 0.9, 0.2, 0.1], k = 3)
```

We can't model the number of medals they win as a Binomial because the probability of a win is not the same in each event.

Approach

This solution is modeled after the derivation in the Many Coin Flips example. In that derivation, the idea was to think of all the unique ways that a list of n heads and tails could have exactly k heads. For example recall the list of

See you next Wed!