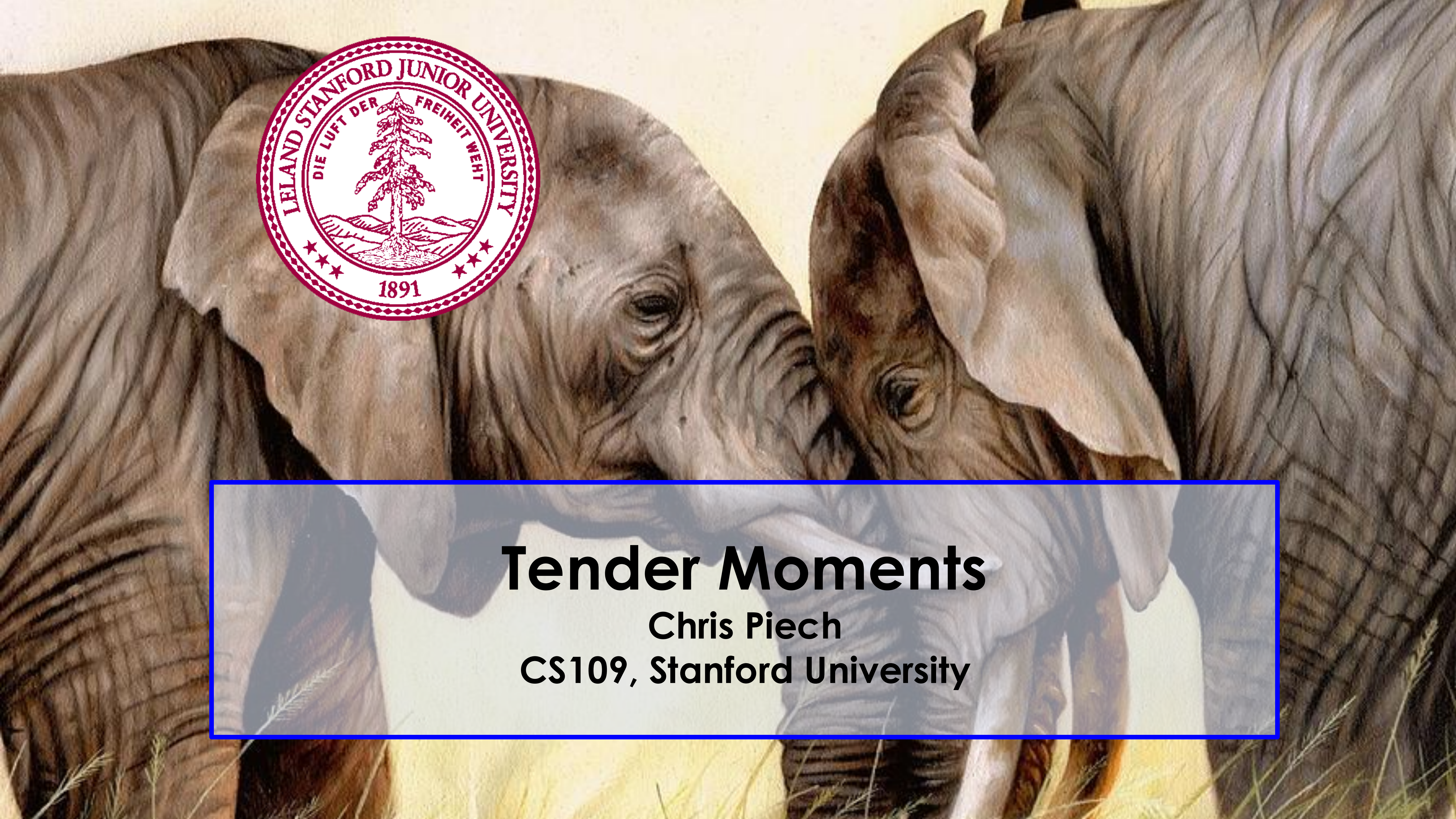

Tender Moments

Chris Piech
CS109, Stanford University

Learning Goals

1. Use New Random Variables!
2. Calculate Expectation of a Random Variable

Review

A **random variable** is a number which takes on values probabilistically.

A discrete random variable is fully described by a **probability mass function**.

Let Y be a random variable

$$Y$$

For example Y is the number of heads in 5 coin flips

Let Y be a random variable

$$Y = 2$$

*note: here equals means == in coding

It is an event when
Y takes on a value

For example Y is the number of heads in 5 coin flips

Let Y be a random variable

$$Y < 3$$

It is an event when
you ask any comparison question

For example Y is the number of heads in 5 coin flips

If this is a number

$$P(Y = 2)$$

Then this is a probability (between 0 and 1)

For example Y is the number of heads in 5 coin flips

If this is a variable

$$P(Y = k)$$

Then this is a function

For example Y is the number of heads in 5 coin flips

This is a function

$$P(Y = k)$$

$$k = 5$$

$$0.03125$$

For example Y is the number of heads in 5 coin flips

Random Variables are a big deal, because they allow other people to give you a PMF (and other helpful equations)

Classics

Declare a Random Variable to be Binomial

$$X \sim \text{Bin}(n, p)$$

Our random variable (X)

Is distributed as a ($\sim$)

Binomial (Bin)

Num trials (n)

Probability of success on each trial (p)

With these parameters (n, p)

Exactly *k* heads in *n* coin flips. Probability of exactly k heads:

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
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(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Automatically Know the PMF

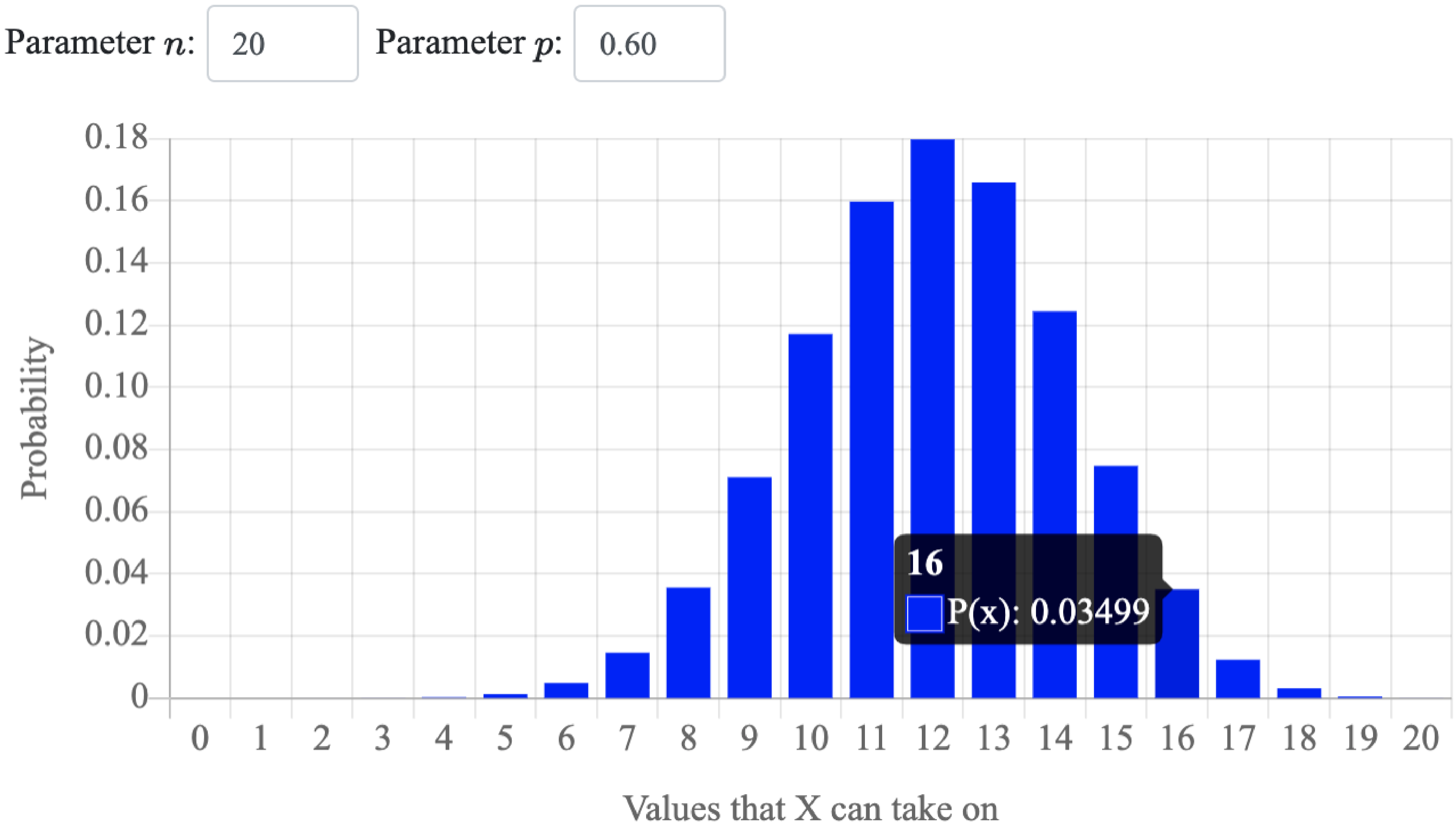
Probability Mass Function for a Binomial

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

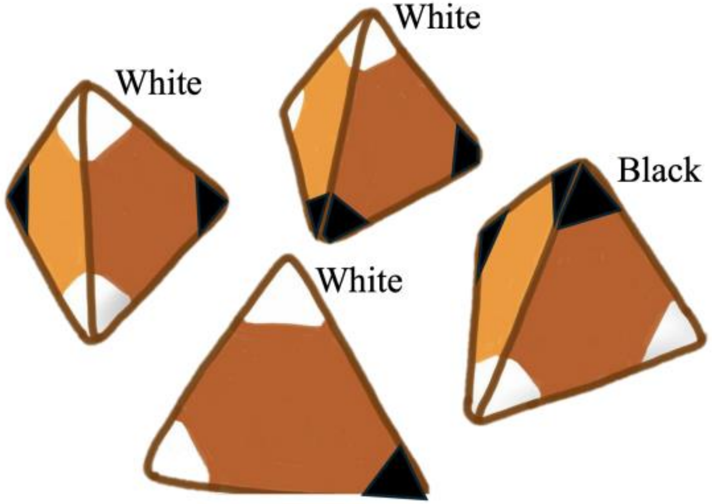
Probability that our variable takes on the value k

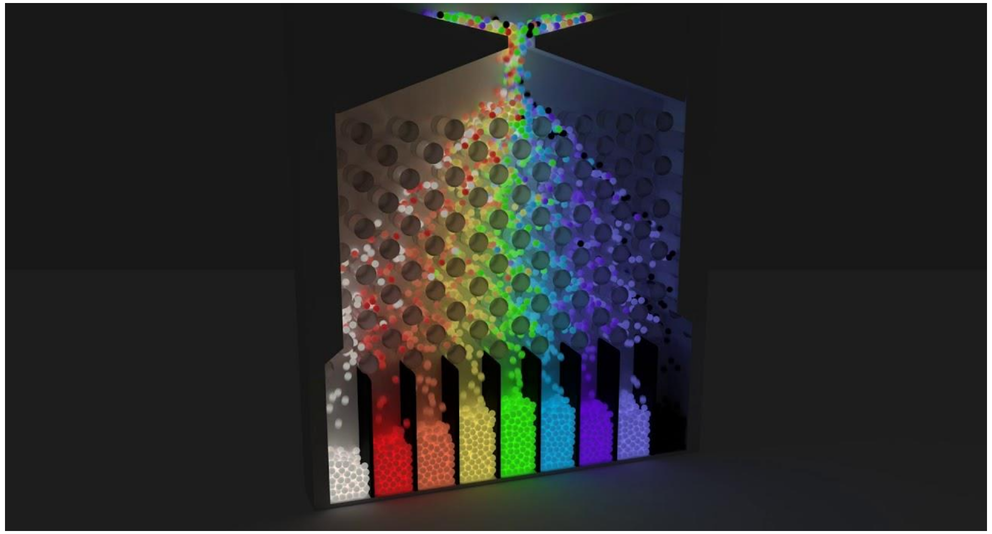
* This is also called the binomial term

The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.6)$

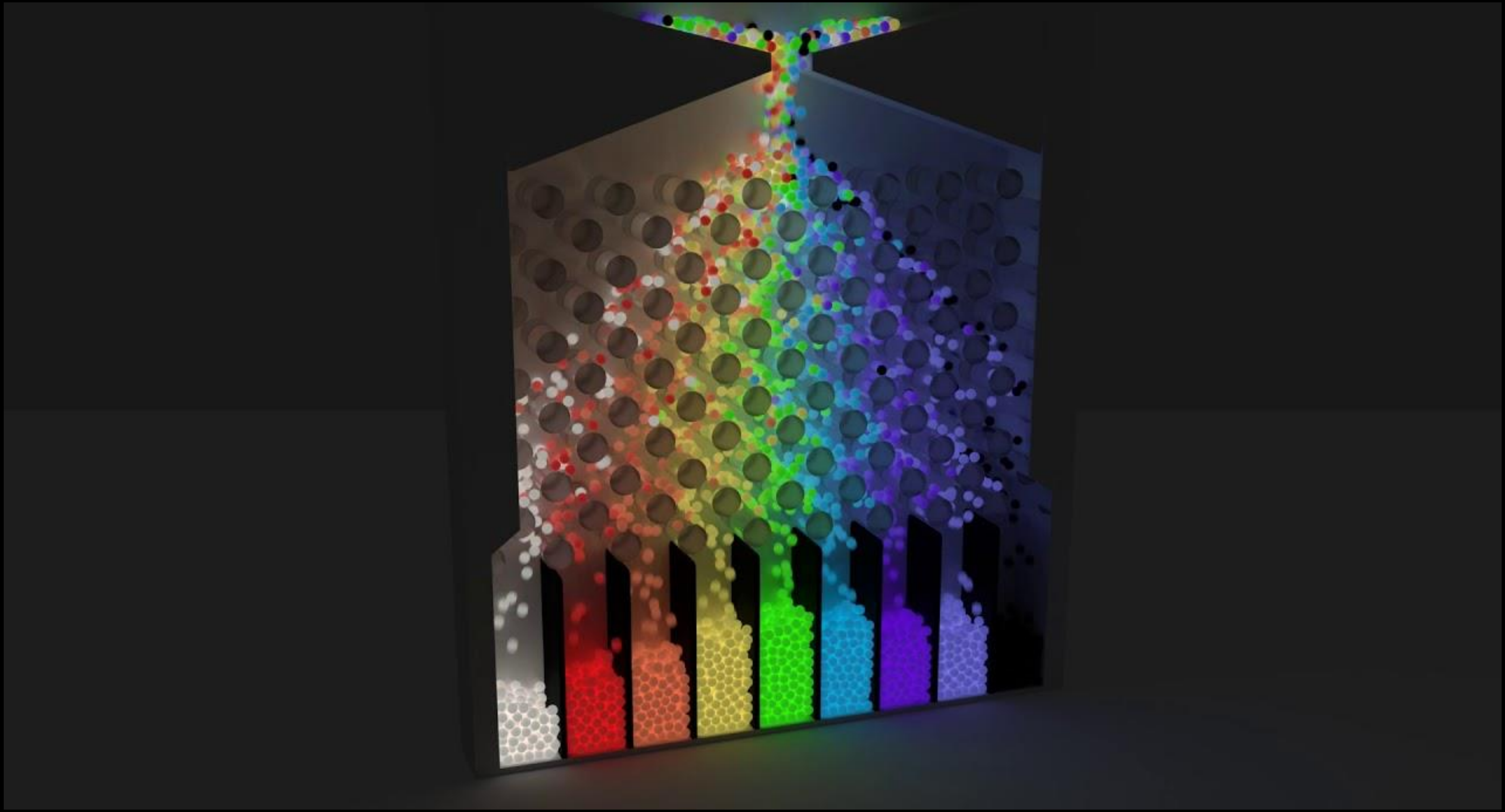


Many Stories fit the Binomial



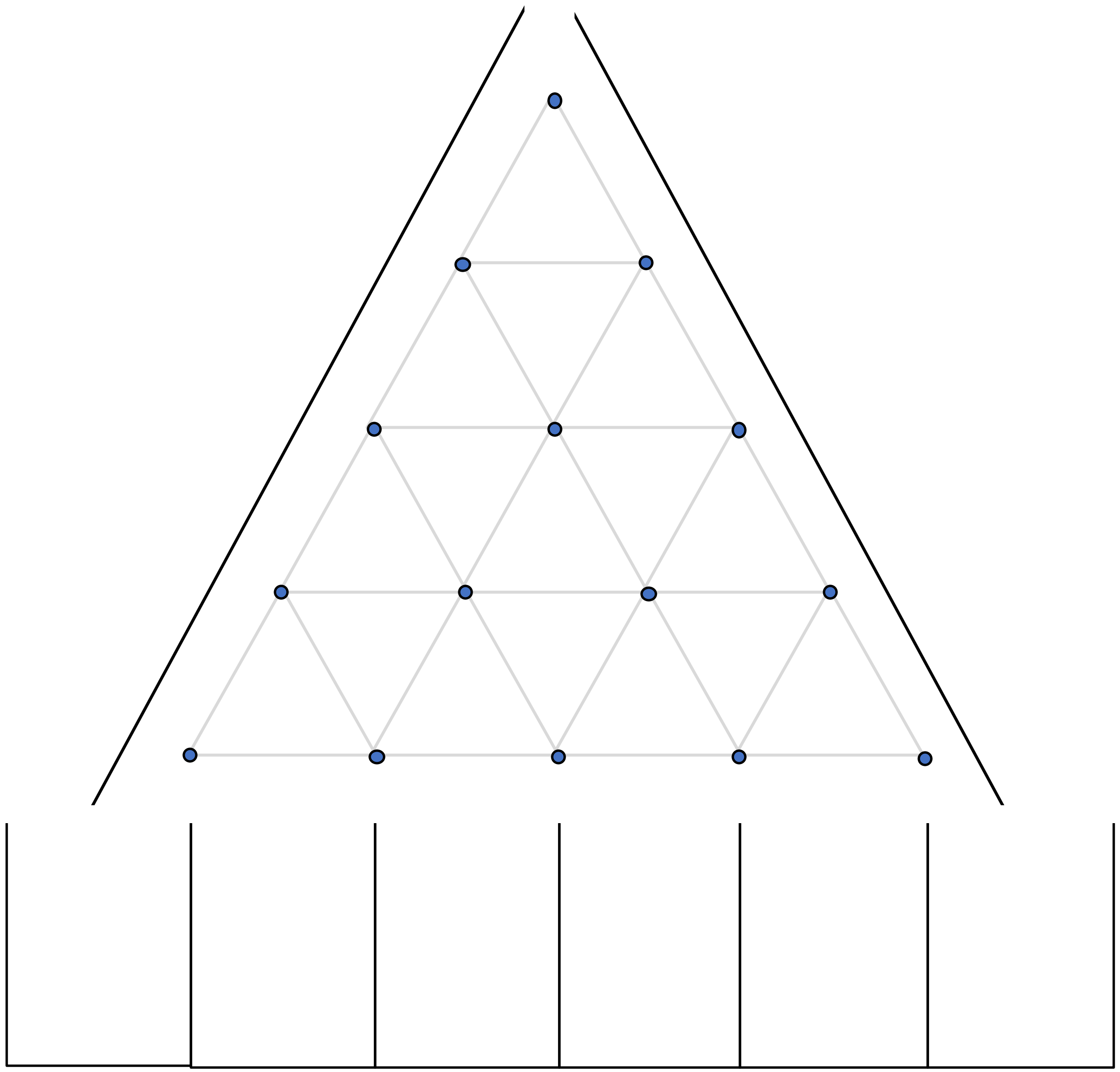


End Review

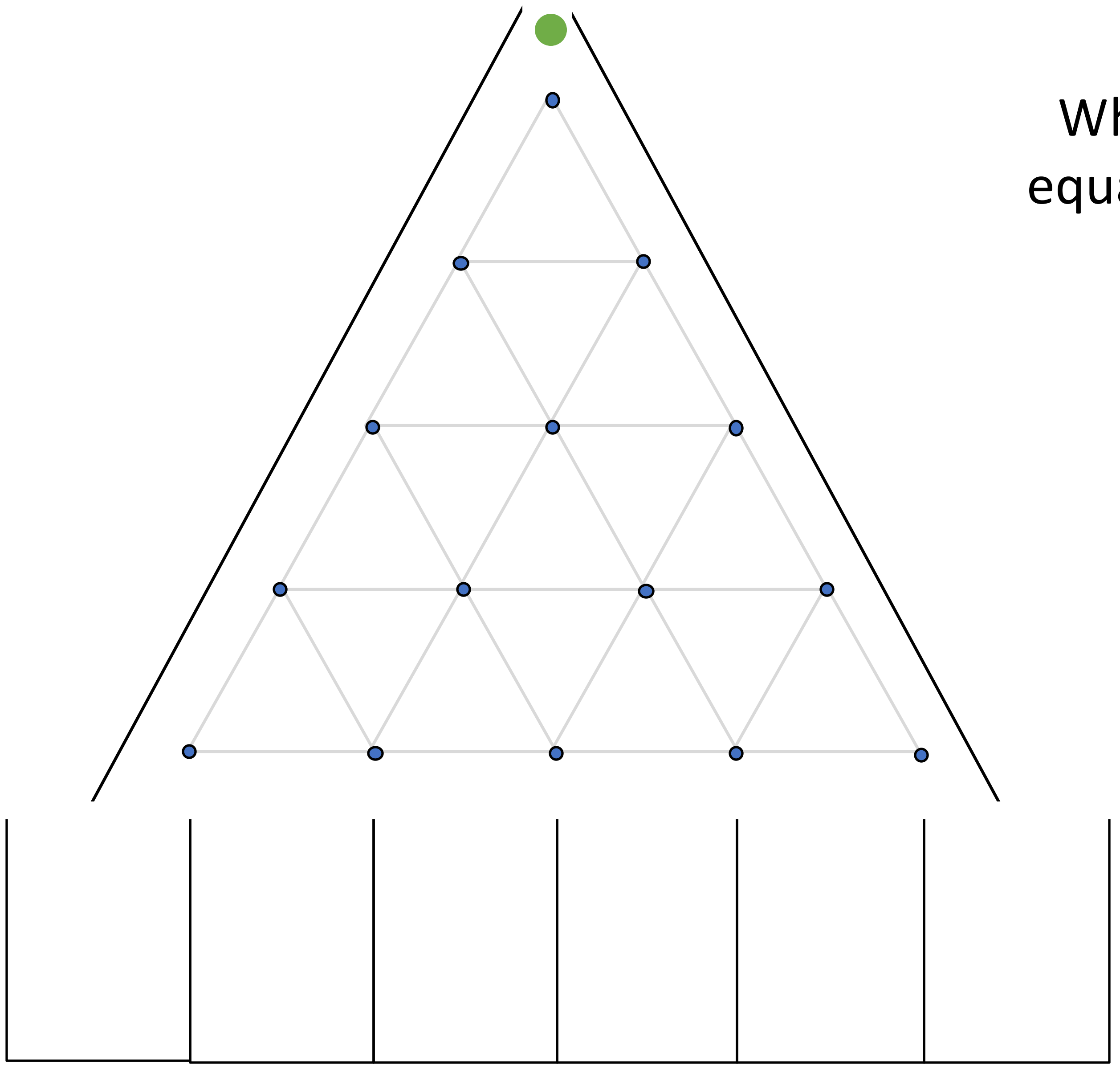


Galton Board Time!

Galton Board Fun

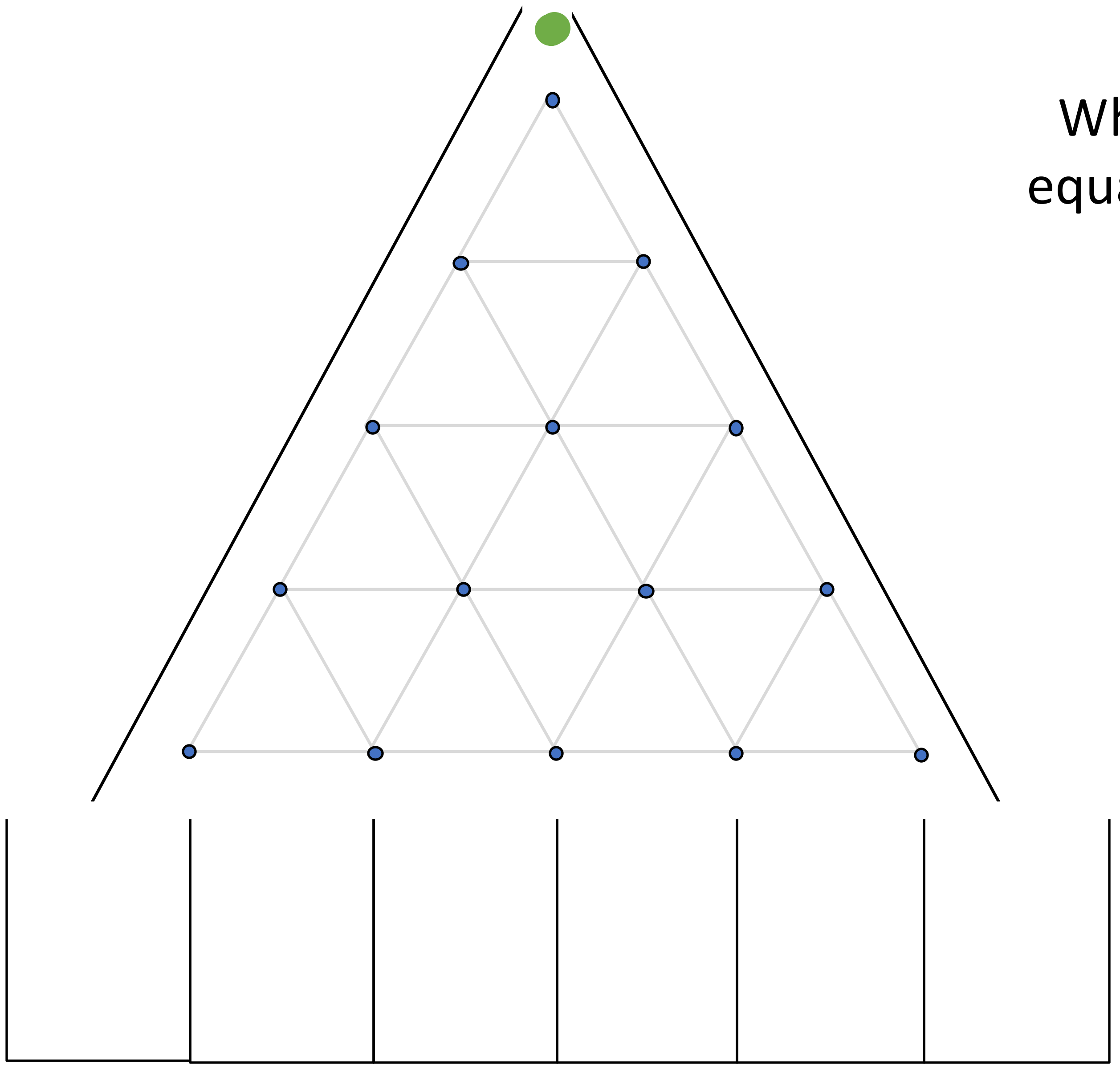


Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

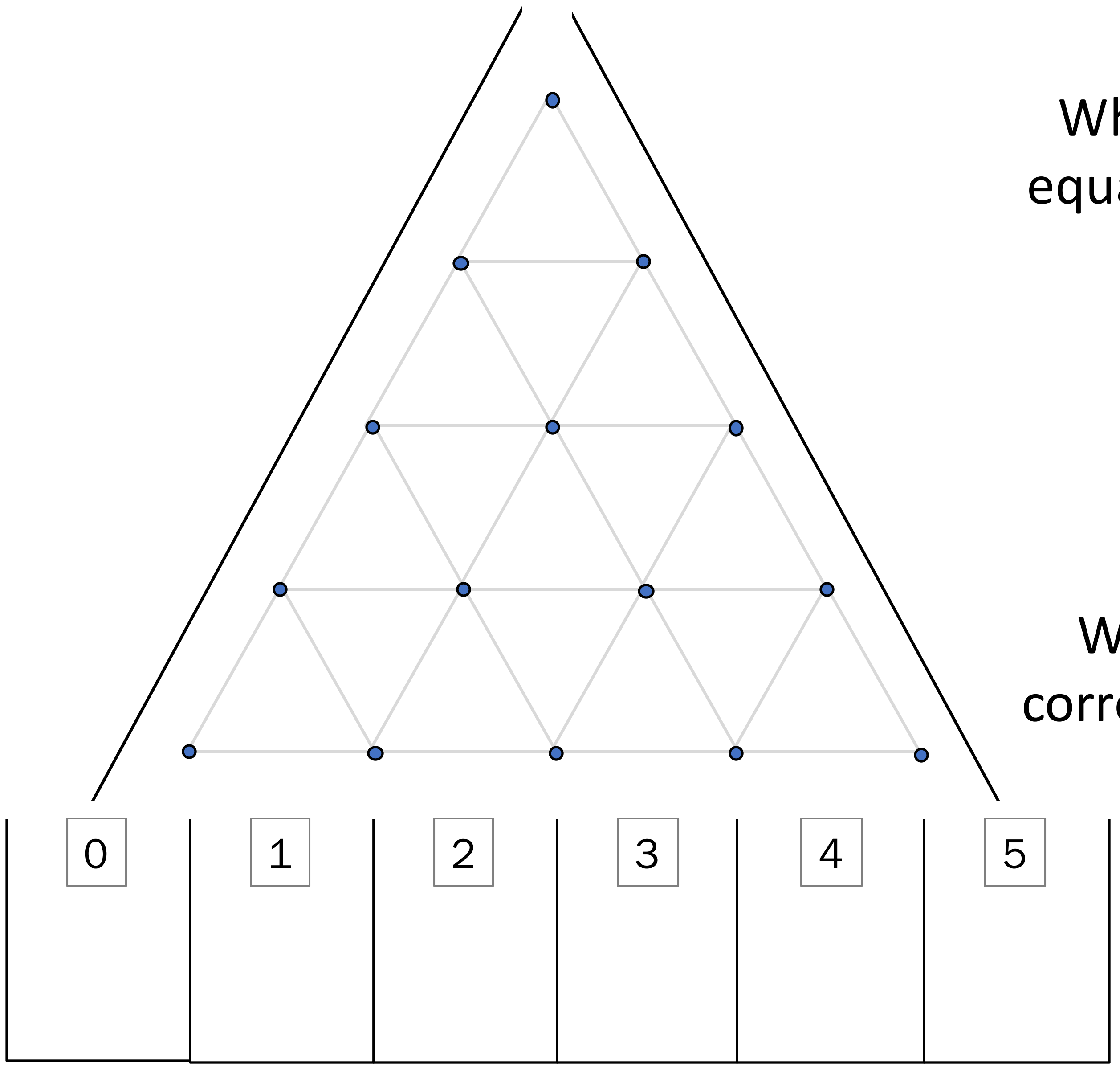
Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Galton Board Fun

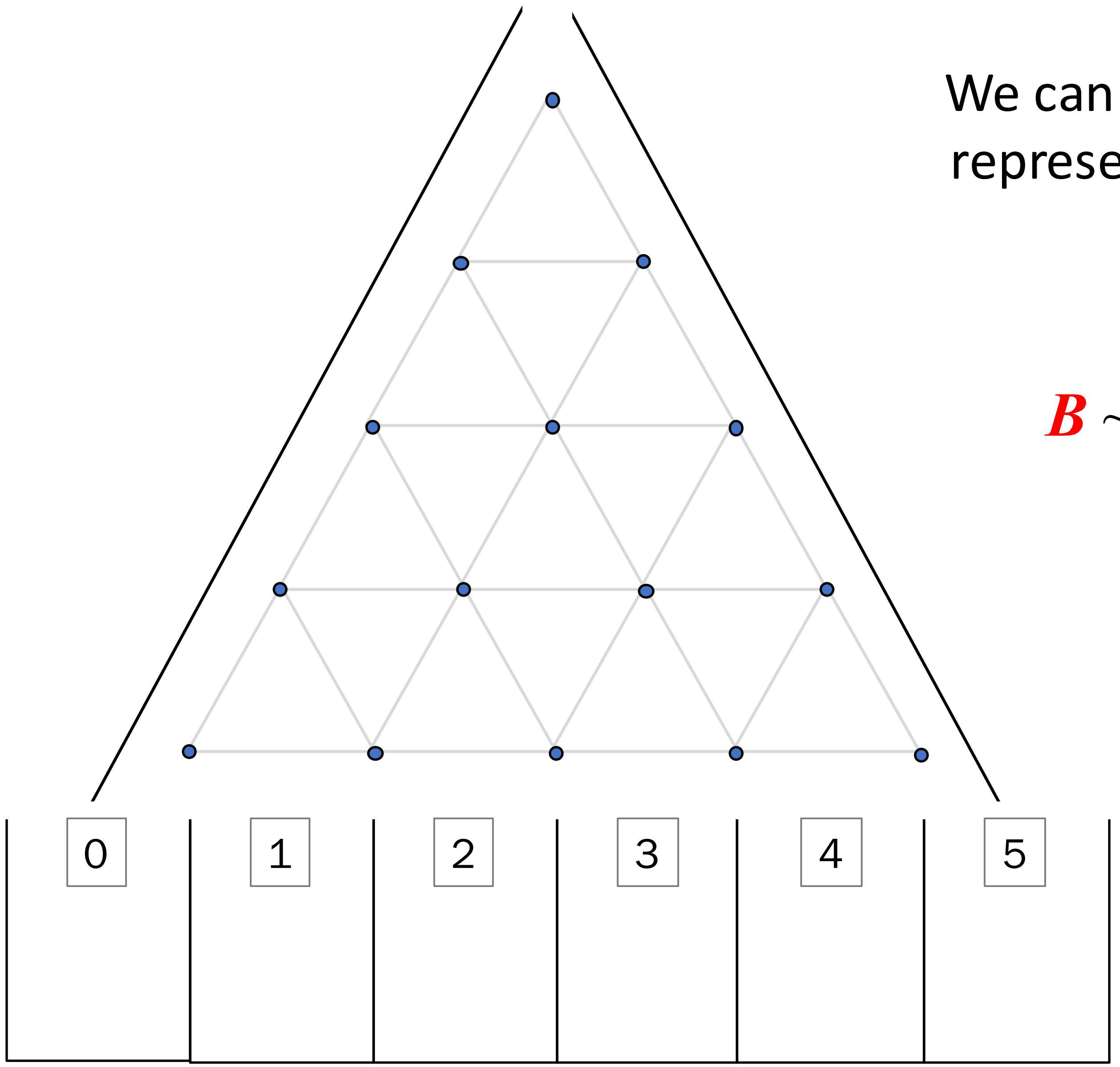


When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Which bucket a marble lands in corresponds to the number of times the marble went right.

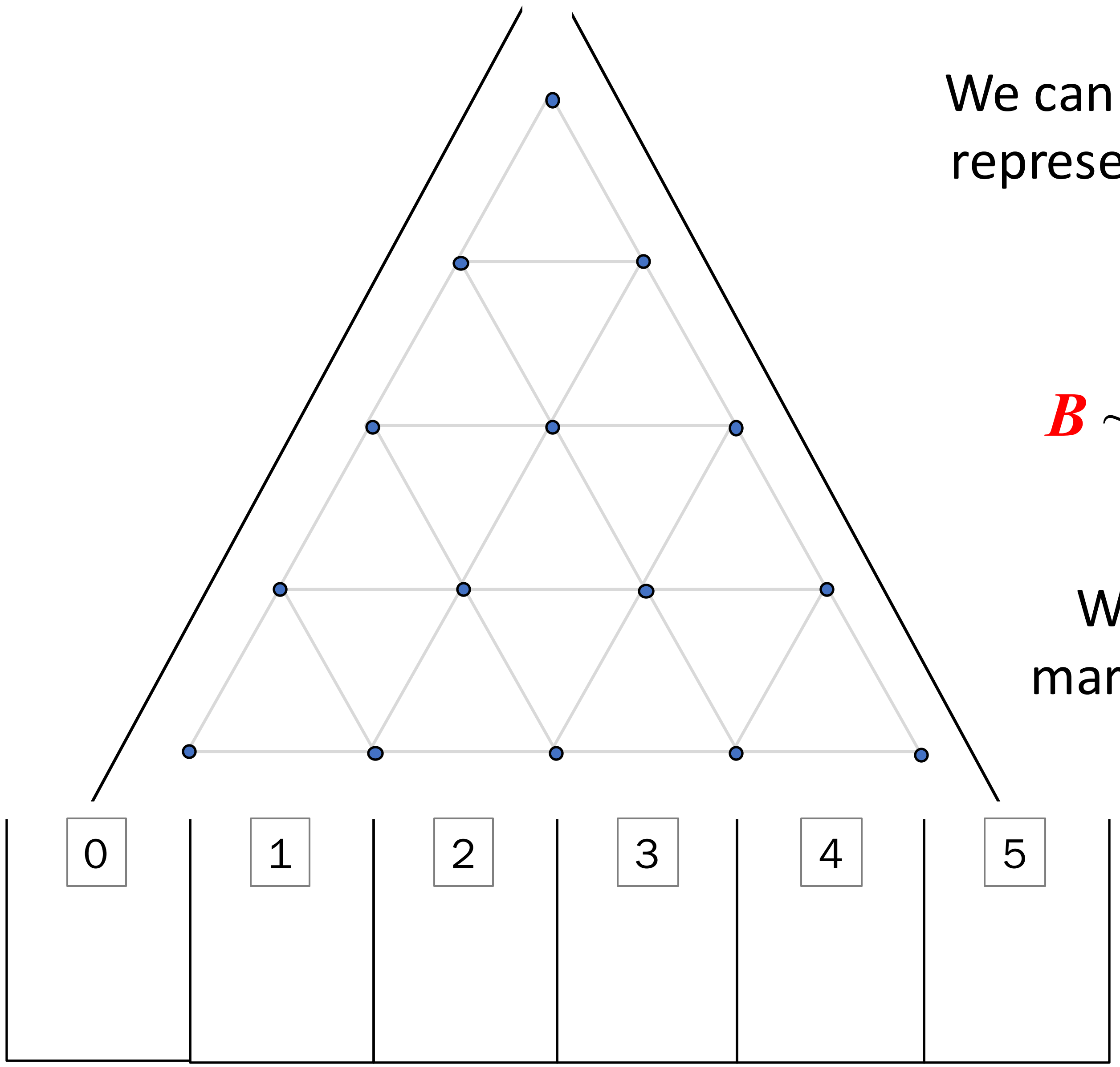
Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

Galton Board Fun



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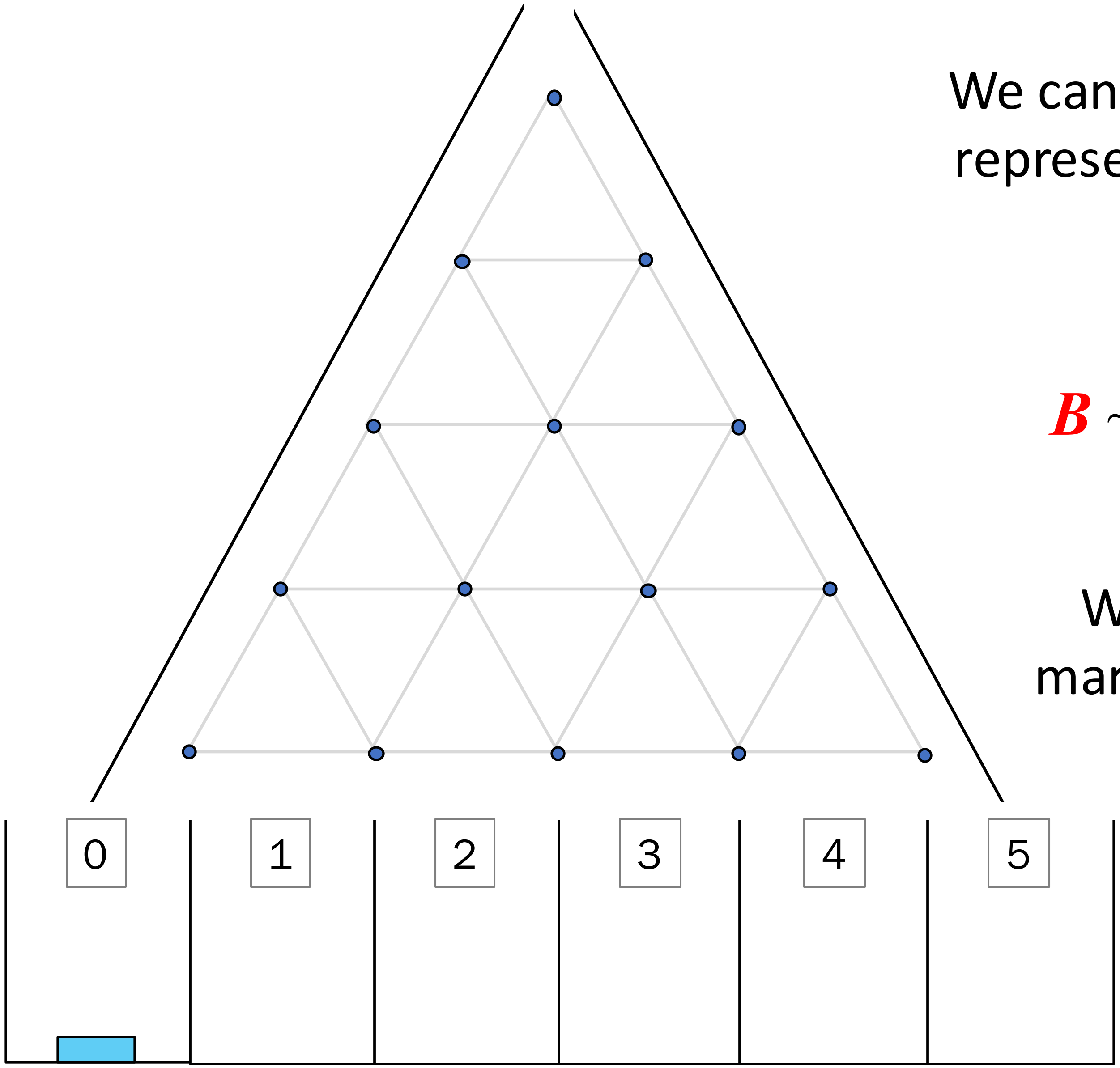
What is the probability of a marble landing in each bucket?

Galton Board Fun



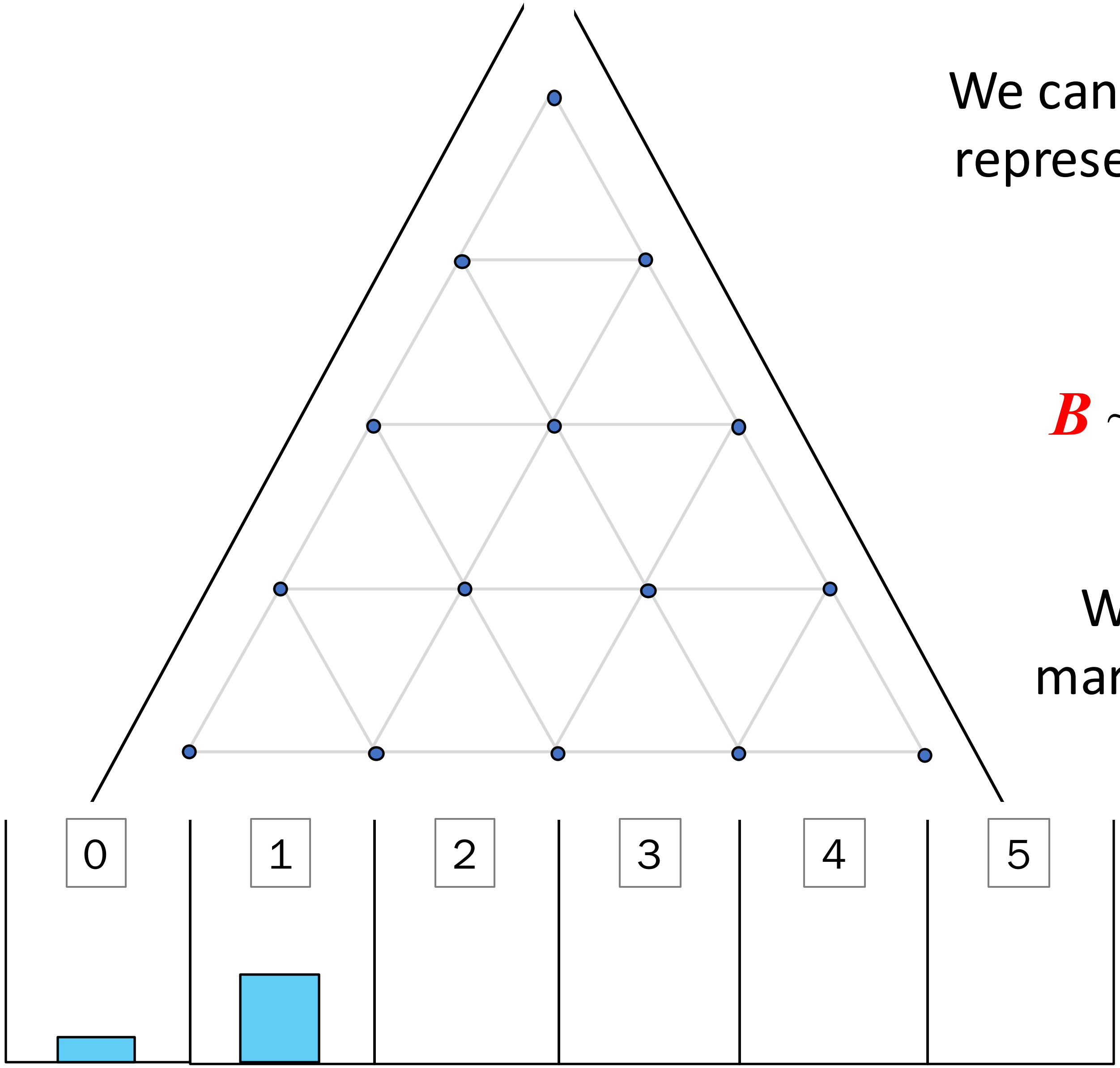
We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

$$P(B = 0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

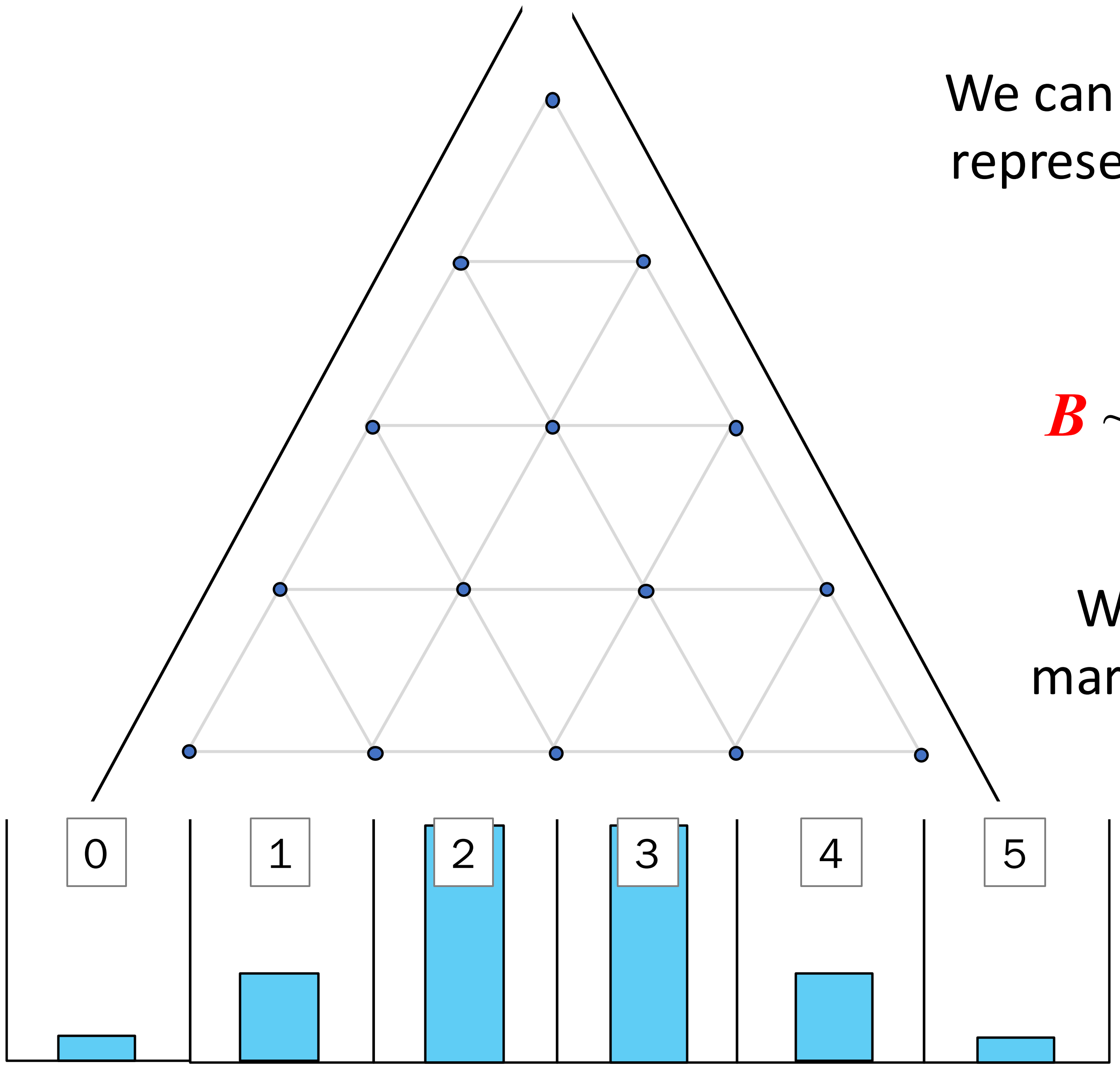
$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

$$P(B = 0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

$$P(B = 1) = \binom{5}{1} \frac{1}{2}^5 \approx 0.16$$

Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

This is the PMF of the binomial!

From Chaos to Order


6
Rakuten
30

What is the probability of winning a 7 game series?

Warriors are going to play the Celtics in a best of 7 series during the 2050 NBA finals. What is the probability that the warriors win the series? Each game is **independent**. Each game, the warriors have a 0.55 probability of winning. Win series if you win at least 4 games.

Let X be the number of games won. $X \sim \text{Bin}(n = 7, p = 0.55)$. $P(X > 3)$?

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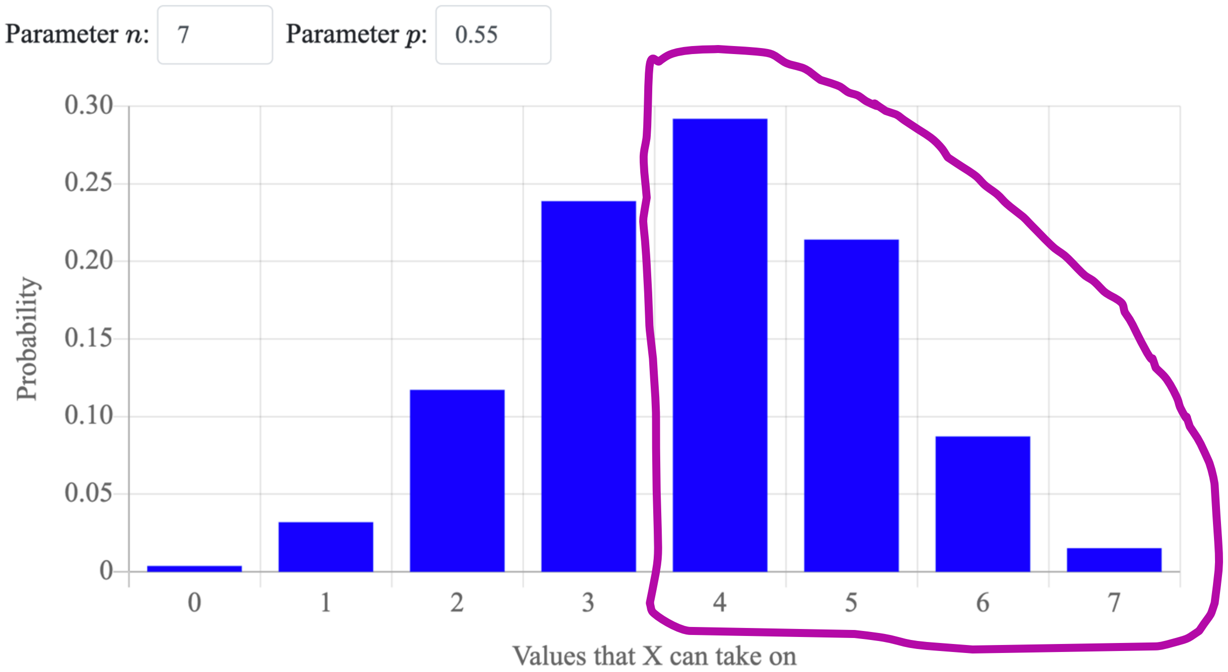

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Let X be the number of games won. $X \sim \text{Bin}(n = 7, p = 0.55)$. $\text{P}(X > 3)$?

$$P(X \geq 4) = \sum_{i=4}^{7} P(X = i)$$

$$= \sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i}$$

$$= \sum_{i=4}^{7} \binom{7}{i} 0.55^i (0.45)^{7-i} \qquad \approx 0.61$$

Debugging Probability

Debugging Probability

How to calculate the probability of at least k successes in n independent trials?

- X is number of successes in n trials each with probability p
- $P(X \geq k) =$

Chose slots for success, don't care about rest

$$\binom{n}{k} p^k$$

ways to choose slots for success

Probability that each is success

Debugging Probability

How to calculate the probability of at least k successes in n independent trials?

- X is number of successes in n trials each with probability p
- $P(X \geq k) =$

Chose slots for success, don't care about rest

$$\binom{n}{k} p^k$$

ways to choose slots for success

Probability that each is success

First clue that something is wrong. Think about p = 1

Not mutually exclusive…

Correct: $P(X \geq k) = \sum_{i=k}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$

Where are We in CS109?

You are here

↓

Counting Theory

Core Probability

Random Variables

Probabilistic Models

Uncertainty Theory

Machine Learning

Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

Successes in one trial

$X \sim \text{Geo}(p)$

Trials until one success

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$Y \sim \text{NegBin}(r, p)$

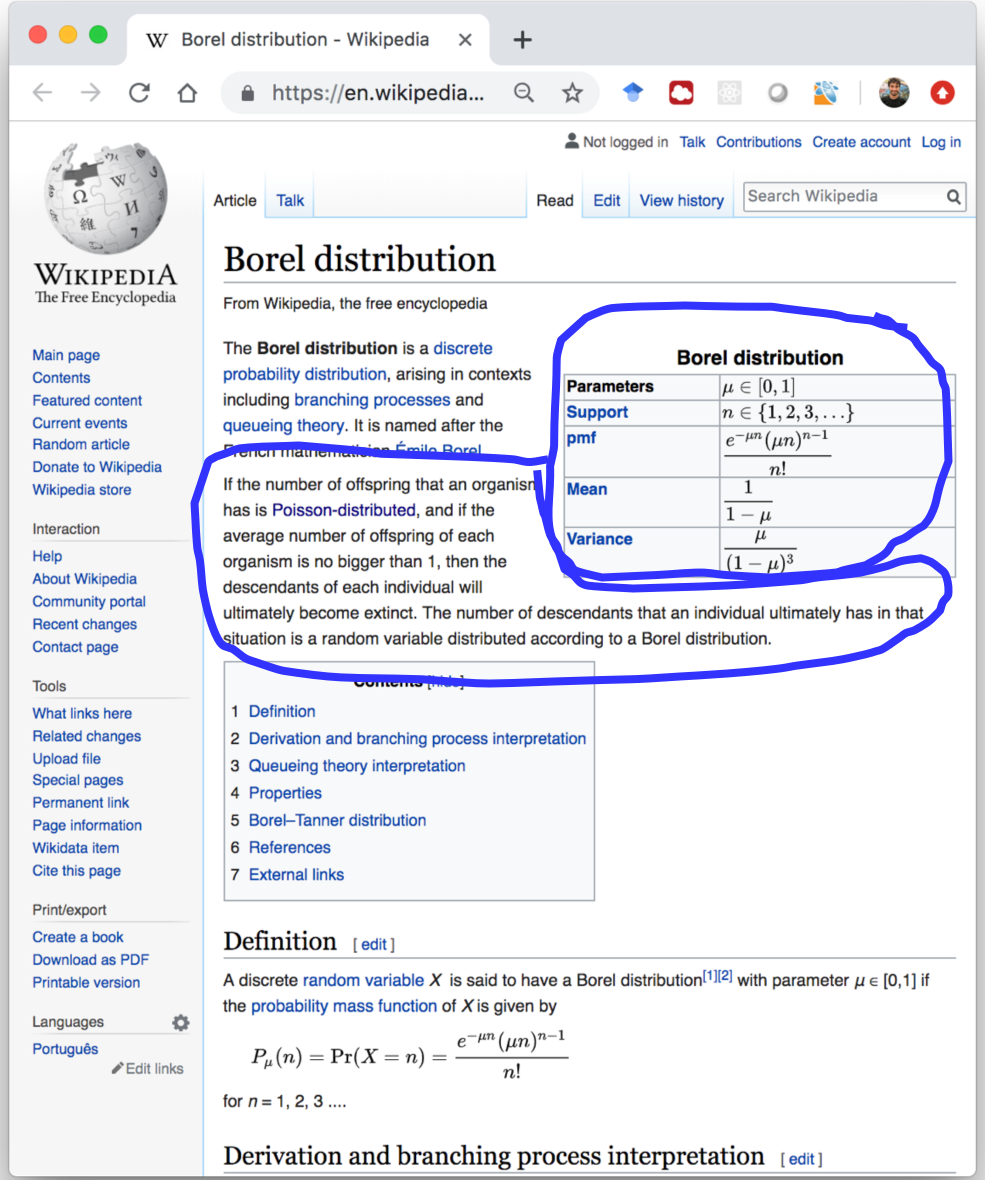
Trials until *r* success

Goal: Be Able to Use a New Random Variable

You are learning about servers…

You read about the MD1 queue…

You find a paper that says the length of a server "busy period" is distributed as a Borel with parameter μ = 0.2 …



Geometric Random Variable

X is **Geometric** Random Variable: X ~ Geo(*p*)

- X is number of independent trials until first success
- *p* is probability of success on each trial
- Assumes p does not change
- X takes on values 1, 2, 3, …, with probability:

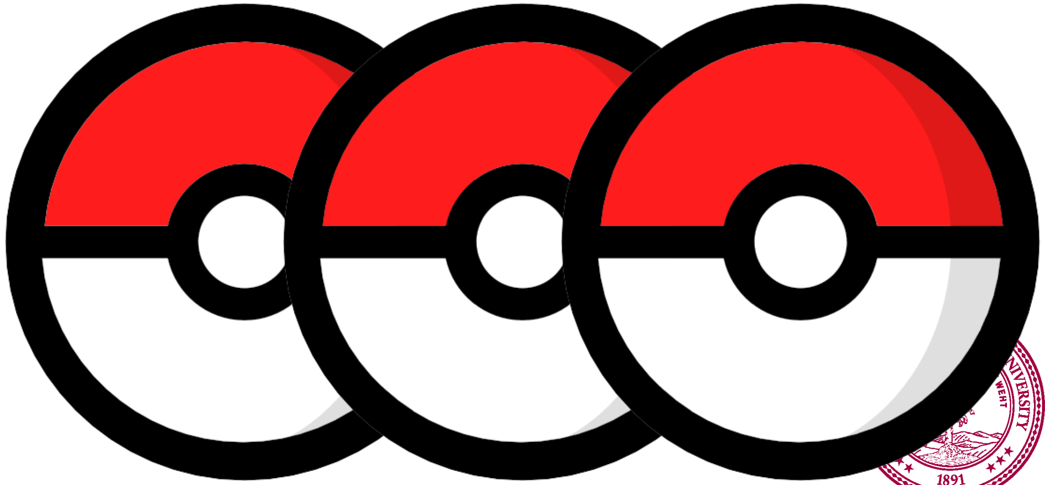
$$P(X = n) = (1 - p)^{n-1}p$$

Negative Binomial Random Variable

X is **Negative Binomial** RV: X ~ NegBin(*r*, *p*)

- X is number of independent trials until *r* successes
- *p* is probability of success on each trial
- Assumes *p* does not change.
- X takes on value *n* with probability:

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \text{ where } n = r, r+1, \ldots$$



Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

$X \sim \text{Geo}(p)$

Trials until one success

$$P(X = x) = (1 - p)^{x-1} p$$

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$Y \sim \text{NegBin}(r, p)$

Trials until *r* success

$$P(X = x) = \binom{x-1}{r-1} p^r (1 - p)^{x-r}$$

Recipe For Solving Problems:

1. Recognize a classic random variable type

2. Define a random variable to be that type, with parameters

$$X \sim \text{Bin}(n, p)$$

3. Profit off the PMF

Dating at Stanford

Each person you date has a 0.2 probability of being someone you spend your life with. What is the probability you need to date more than 5 people? **Your meta goal: what steps would you take to answer this question?**

Equity in the Courts

Berghuis v. Smith

If a group is underrepresented in a jury pool, how do you tell?

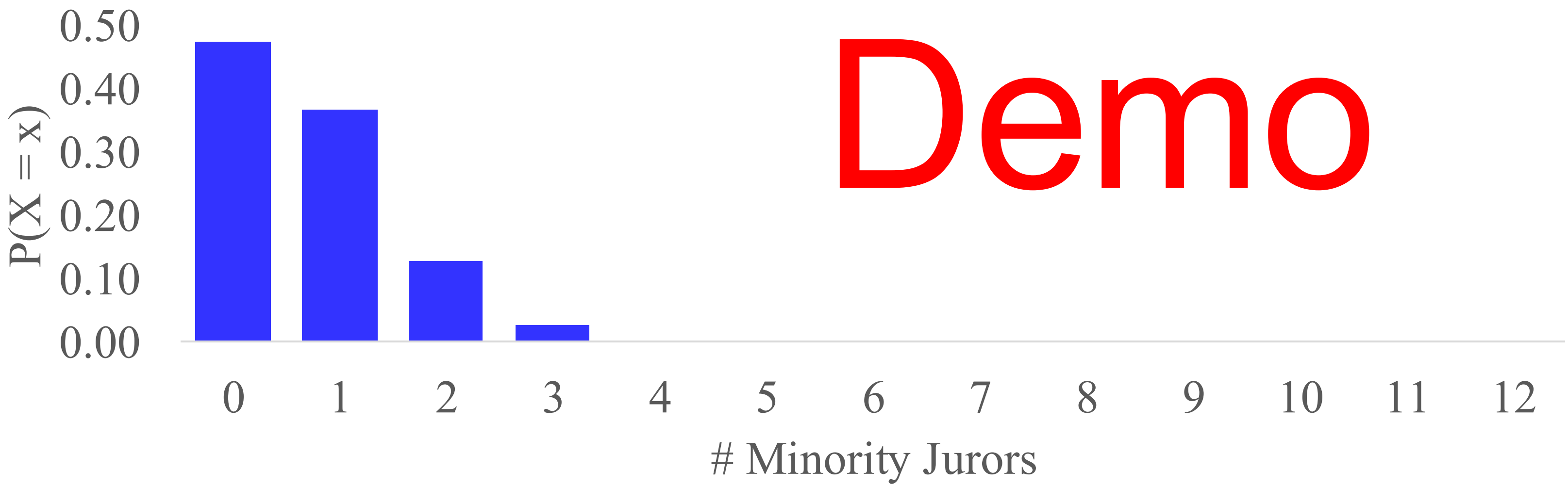
Justice Breyer [Stanford Alum] opened the questioning by invoking the binomial theorem. He hypothesized a scenario involving **“an urn with a thousand balls, and sixty are yellow, and nine hundred forty are navy-blue, and then you select them at random… twelve at a time.”** According to Justice Breyer and the binomial theorem, if the purple balls were under represented jurors then **“you would expect… something like a third to a half of juries would have at least one yellow ball”** on them.

Equity in the Courts

Approximation using Binomial distribution

- Assume P(blue ball) constant for every draw = 60/1000
- X = # blue balls drawn. $X \sim Bin(12, 60/1000 = 0.06)$
- $P(X \geq 1) = 1 - P(X = 0) \approx 1 - 0.4759 = 0.5240$

In Breyer's description, should actually expect just <u>over half</u> of juries to have at least one non-white person on them



Demo

Bitcoin Mining

SHA-256 Hash(,)

Data
Fixed

Salt
Choice

Number that looks like random bits

You "mine a bitcoin" if, for given data *D*, you find a salt number *N* such that Hash(*D*, *N*) produces a string that starts with *g* zeroes.

You "mine a bitcoin" if, for given data *D*, you find a number *N* such that Hash(*D*, *N*) produces a string that starts with *g* zeroes.

(a) What is the probability that Hash outputs a bit string which starts with *g* zeroes (in other words you mine a bitcoin)?

Let X be the number of zeros in the first *g* bits. $X \sim \text{Bin}(n = g, p = 0.5)$

$$P(X = g) = \binom{g}{g} \frac{1}{2}^g = \frac{1}{2}^g$$

Call this answer p_a

(b) What is the probability that you will need under 100 attempts to mine 2 bit coins?

Let Y be the number of tries until you mine 2 bitcoins. $Y \sim \text{NegBin}(r = 2, p = p_a)$

$$P(Y < 100) = \sum_{x=2}^{99} P(Y = x)$$

$$= \sum_{x=2}^{99} \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

$X \sim \text{Geo}(p)$

Trials until one success

$$P(X = x) = (1 - p)^{x-1} p$$

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$Y \sim \text{NegBin}(r, p)$

Trials until *r* success

$$P(X = x) = \binom{x-1}{r-1} p^r (1 - p)^{x-r}$$

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

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Yes - the Bernoulli random variable: $X \sim \text{Bern}(p)$

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

Yes - the Bernoulli random variable: $X \sim \text{Bern}(p)$

- The Bernoulli is an **indicator** random variable (value is either 0 or 1).
- $P(X = 1) = p$
- $P(X = 0) = 1 - p$

(this is the whole PMF)

- Examples: a single coin flip, one ad click, any binary event

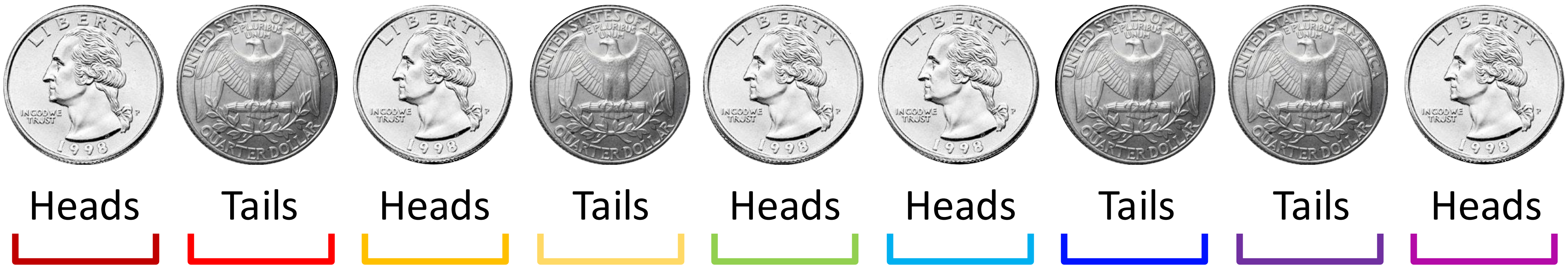
Random Variable Sums

The Binomial

Heads Tails Heads Tails Heads Heads Tails Tails Heads

Random Variable Sums

The Binomial ...is a sum of Bernoulli random variables



Random Variable Sums

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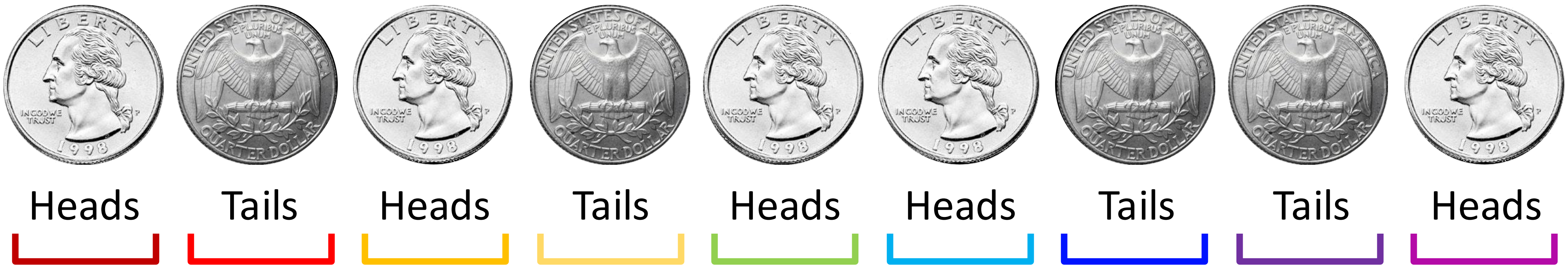


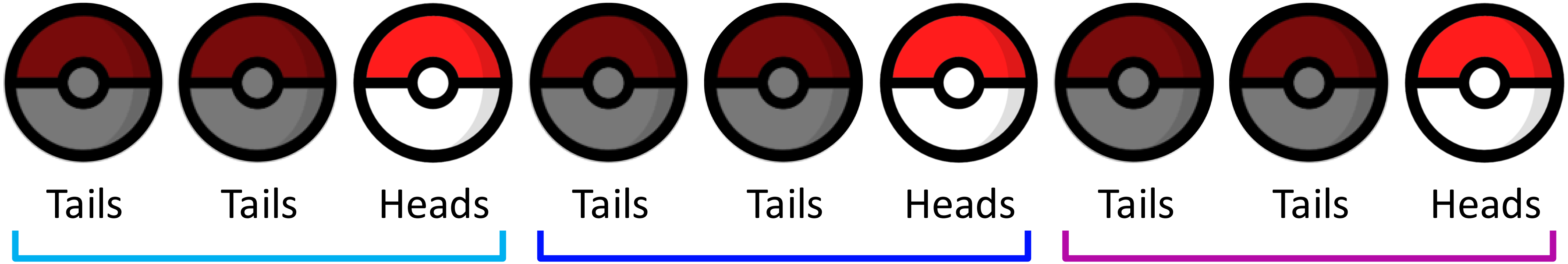
Let $X_1 \sim \text{Bern}(p = 1/2)$ and $X_2 \sim \text{Bern}(p = 1/2)$.

$$Y \sim \text{Bin}(n = 2, p = 1/2)$$

$$Y = X_1 + X_2$$

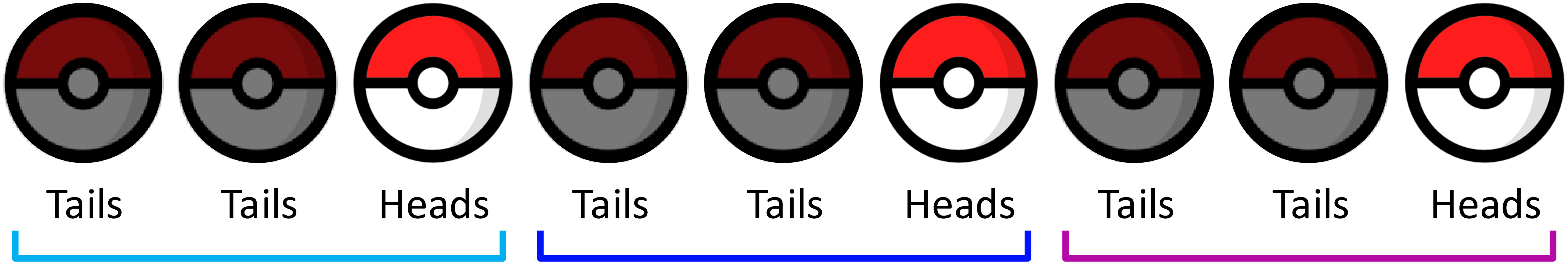
Random Variable Sums

The Negative Binomial



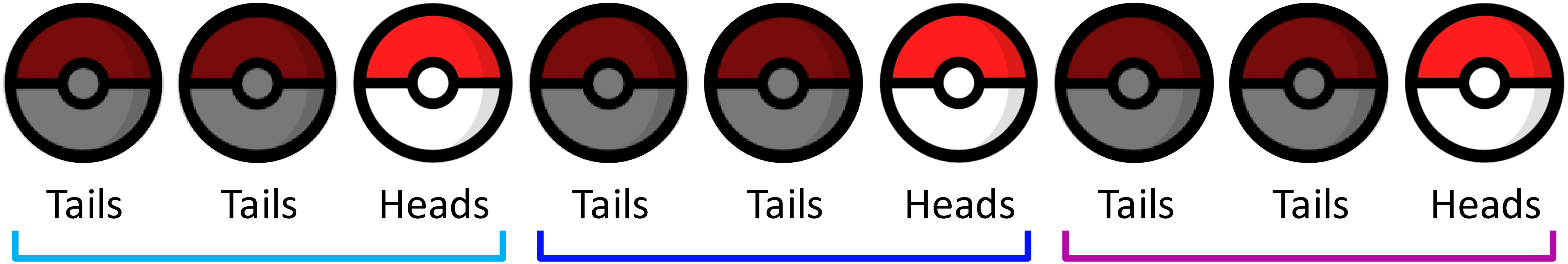
Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables



Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables



Let $X_1 \sim \text{Geo}(p = 1/3)$, $X_2 \sim \text{Geo}(p = 1/3)$, and $X_3 \sim \text{Geo}(p = 1/3)$.

$$Y \sim \text{NegBin}(r = 3, p = 1/3)$$

$$Y = X_1 + X_2 + X_3$$

Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

Successes in one trial

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

$X \sim \text{Geo}(p)$

Trials until one success

$$P(X = x) = (1 - p)^{x-1} p$$

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$Y \sim \text{NegBin}(r, p)$

Trials until *r* success

$$P(X = x) = \binom{x - 1}{r - 1} p^r (1 - p)^{x-r}$$

[Short Pedagogical Pause]

Time for some tender moments*

*Moments: numbers that summarize different aspects of a random variable

Expectation

Expected Value

The probability of that value

The value

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Loop over all values x that X can take on

Expected Value

Expected value answers the question:

What is the average value we could expect some random variable to be?

Also called: ***Mean***, *Expectation*, ***Weighted Average***, ***Center of Mass***, *1st Moment*

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$

Example: Expected Value of Dice Roll

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$\begin{aligned} E[X] &= \sum_{x=1}^{6} x \cdot P(X = x) \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \\ &= 3.5 \end{aligned}$$

Example: Expected Value of Dice Roll

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

E[X] is not always an actual possible outcome for X

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **class** with equal probability.

Let X be the chosen class's size. What is $\mathrm{E}[X]$?

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **class** with equal probability.

Let X be the chosen class's size. What is $\mathrm{E}[X]$?

$$P(X = 5) = 1/3$$
$$P(X = 10) = 1/3$$
$$P(X = 150) = 1/3$$

$$E[X] = \sum_{x \in \{5,10,150\}} x \cdot P(X = x)$$
$$= 5 \cdot \frac{1}{3} + 10 \cdot \frac{1}{3} + 150 \cdot \frac{1}{3}$$
$$= 55$$

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **student** with equal probability.

Let X be the chosen student's class size. What is $\mathrm{E}[X]$?

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **student** with equal probability.

Let X be the chosen student's class size. What is $\mathrm{E}[X]$?

$$P(X = 5) = 5/165$$
$$P(X = 10) = 10/165$$
$$P(X = 150) = 150/165$$

$$E[X] = \sum_{x \in \{5,10,150\}} x \cdot P(X = x)$$
$$= 5 \cdot \frac{5}{165} + 10 \cdot \frac{10}{165} + 150 \cdot \frac{150}{165}$$
$$= 137$$

Expectation from Data

List called data

X
3
2
6
10
1
1
5
4
...

$$E[X] = \sum_x x \cdot P(X = x)$$

$$\approx \sum_x x \cdot \frac{\text{count}(X = x)}{N}$$

Length of data

$$\approx \frac{1}{N} \sum_x x \cdot \text{count}(X = x)$$

$$\approx \frac{1}{N} \sum_{v \in \text{data}} v$$

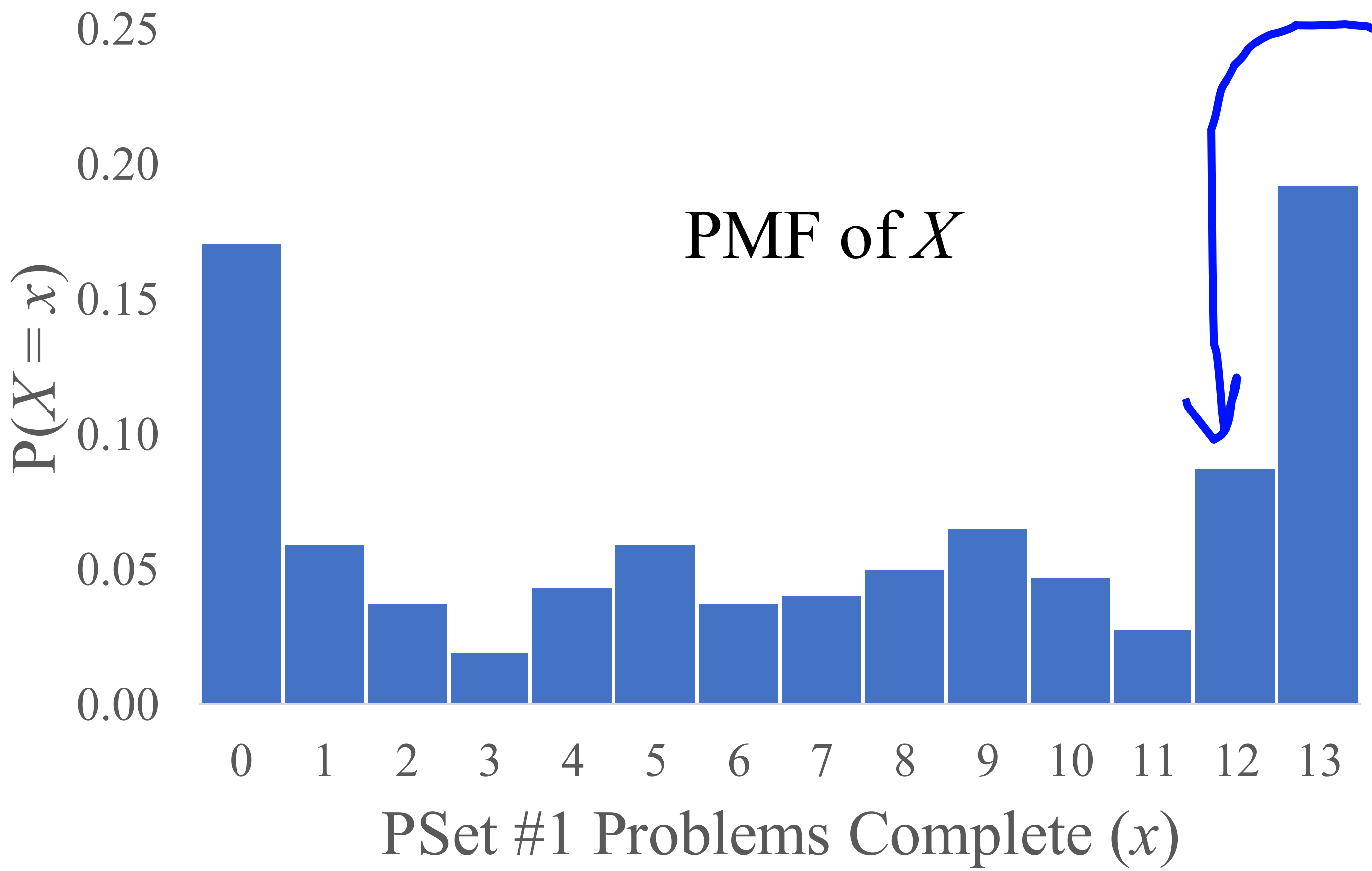
Expectation is a single number summary…

Expectation leaves much to be desired…

Expectation vs PMF

Let X be the number of problems that a randomly selected student has completed, as of 11a today.

X takes on values, with uncertainty. X is a random variable.



$$P(X = 12) \approx \frac{\text{Count}(X = 12)}{N}$$

$$E[X] = 7.4$$

Why People Care?

Properties of Expectation (proof later)

Linearity:

$$E[aX + b] = aE[X] + b$$

- Consider X = 6-sided die roll, Winnings = 2X – 1.
- E[X] = 3.5 E[2X-1] = 6

Expectation of a sum is the sum of expectations

$$E[X + Y] = E[X] + E[Y]$$

Unconscious statistician:

$$E[g(x)] = \sum_{x \in X} g(x)P(X = x)$$

Law of the Unconscious Statistician (LOTUS)

$$E[g(X)] = \sum_x g(x) P(X = x)$$

This lets you get the expectation of **any** function of a random variable.

Examples:

$$E[X^2] = \sum_x x^2 \cdot P(X = x)$$

$$E[\sin(X)] = \sum_x \sin(x) \cdot P(X = x)$$

$$E[\sqrt{(X)}] = \sum_x \sqrt{x} \cdot P(X = x)$$

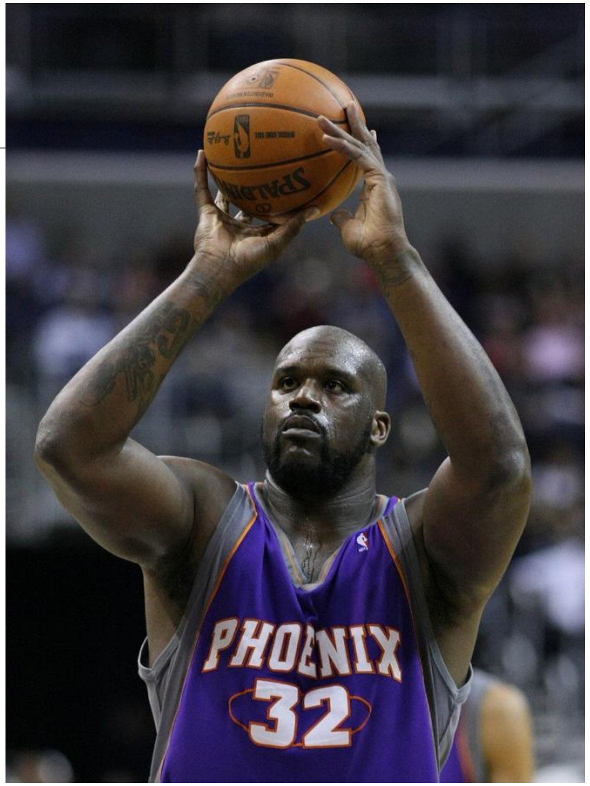
Expectation of Classic Random Variables

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

Let X be the points gained from Shaq attempting a free throw. What is $\mathrm{E}[X]$?



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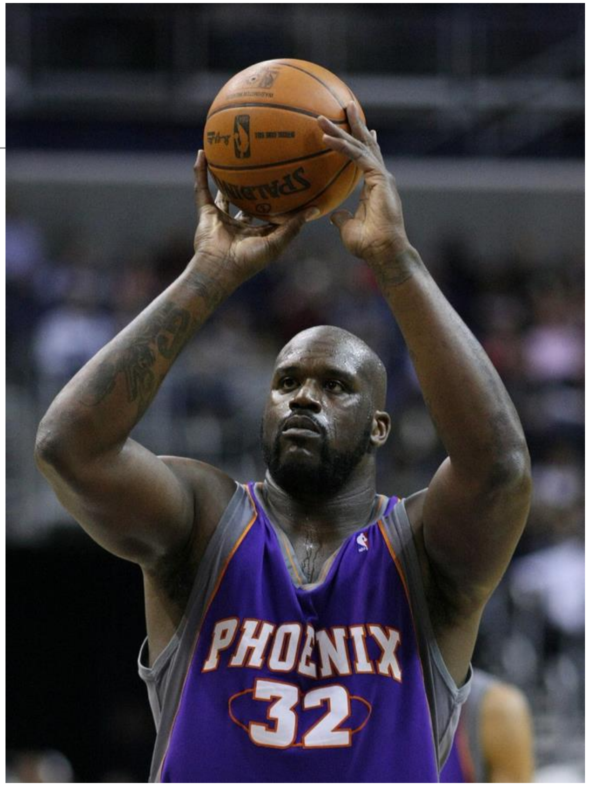

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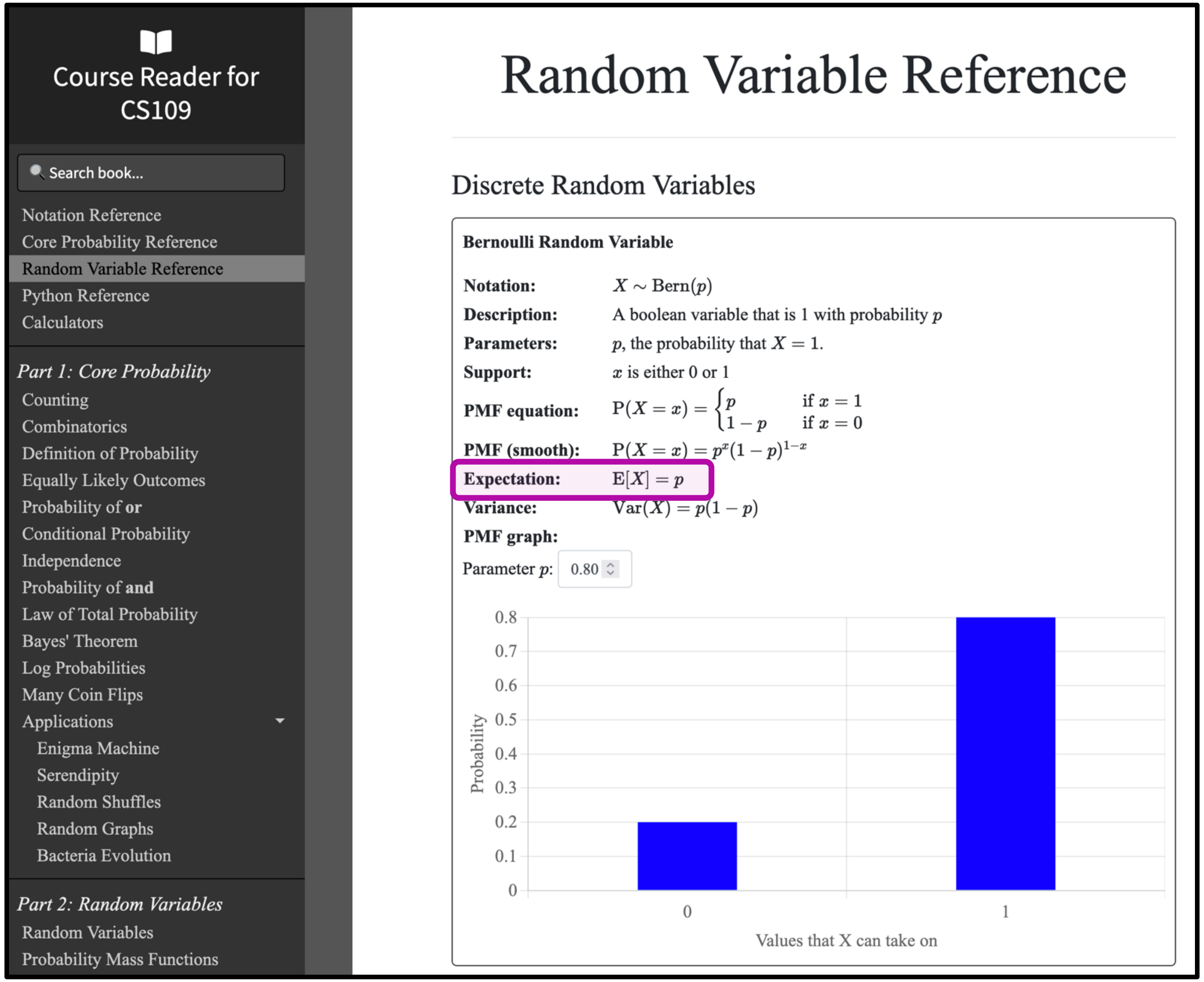
Let X be the points gained from Shaq attempting a free throw. What is $\mathrm{E}[X]$?

$$X \sim \mathrm{Bern}(p = 0.53)$$

$$E[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 1)$$
$$= 0 \cdot 0.47 + 1 \cdot 0.53 = 0.53$$

For Bernoulli random variables, $\mathrm{E}[X] = p$ (always)

With Classic RVs, You Get Expectations For Free Too!



We Can Now Calculate Expectation of Binomial

$$X \sim \text{Bin}(n, p)$$

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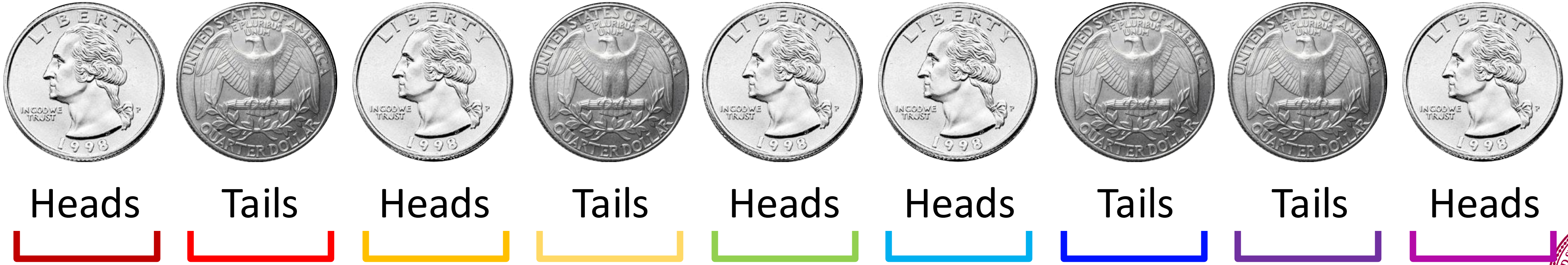
Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \text{Bern}(p)$.

The Binomial …is a sum of Bernoulli random variables



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$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} Y_i\right] \qquad \text{Since } X = \sum_{i=1}^{n} Y_i$$

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$$= \sum_{i=1}^{n} \mathrm{E}[Y_i] \qquad \text{Expectation of sum}$$

Expectation of a sum is the sum of expectations: $E[X+Y] = E[X] + E[Y]$

We Can Now Calculate Expectation of Binomial

$X \sim \text{Bin}(n, p)$

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$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n} Y_i\right] && \text{Since } X = \sum_{i=1}^{n} Y_i \\ &= \sum_{i=1}^{n} E[Y_i] && \text{Expectation of sum} \\ &= \sum_{i=1}^{n} p && \text{Expectation of Bernoulli} \\ &= n \cdot p && \text{Sum } n \text{ times} \end{aligned}$$

True for every binomial ever

You Get So Much For Free!

Binomial Random Variable

Notation: $X \sim \text{Bin}(n, p)$

Description: Number of "successes" in n identical, independent experiments each with probability of success p.

Parameters: $n \in \{0, 1, \ldots\}$, the number of experiments.
$p \in [0, 1]$, the probability that a single experiment gives a "success".

Support: $x \in \{0, 1, \ldots, n\}$

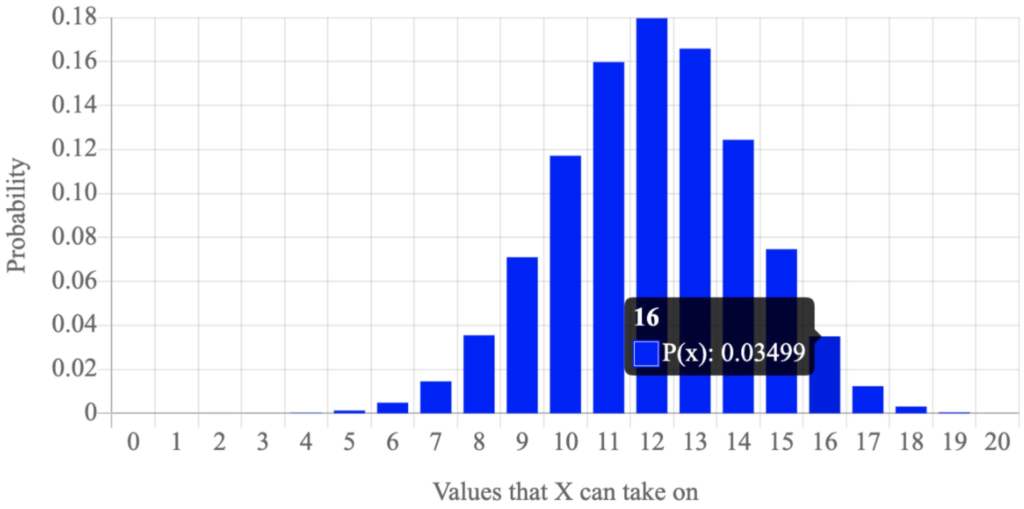
PMF equation: $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

Expectation: $\text{E}[X] = n \cdot p$

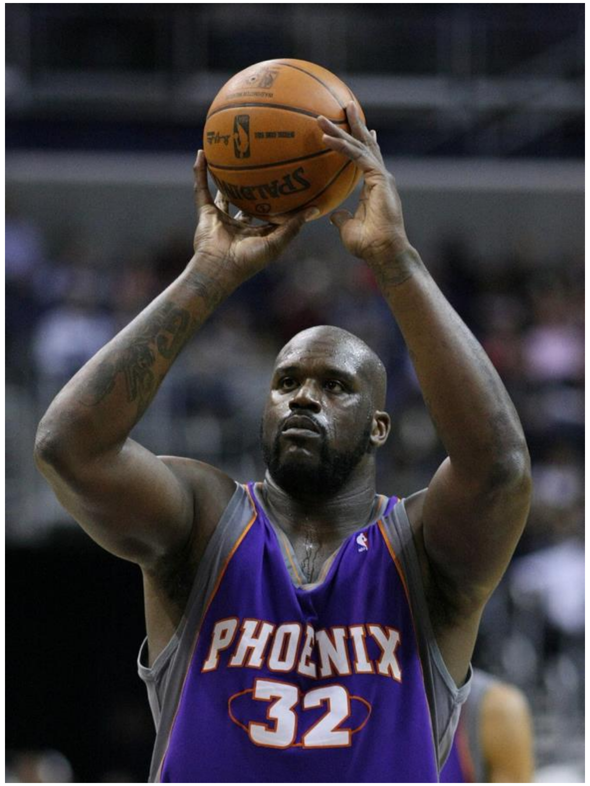
Variance: $\text{Var}(X) = n \cdot p \cdot (1-p)$

PMF graph:

Parameter n: 20 Parameter p: 0.60



Expected Value of Free Throws



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Let Y be the points gained from Shaq attempting **500** free throws. What is $\mathrm{E}[Y]$?

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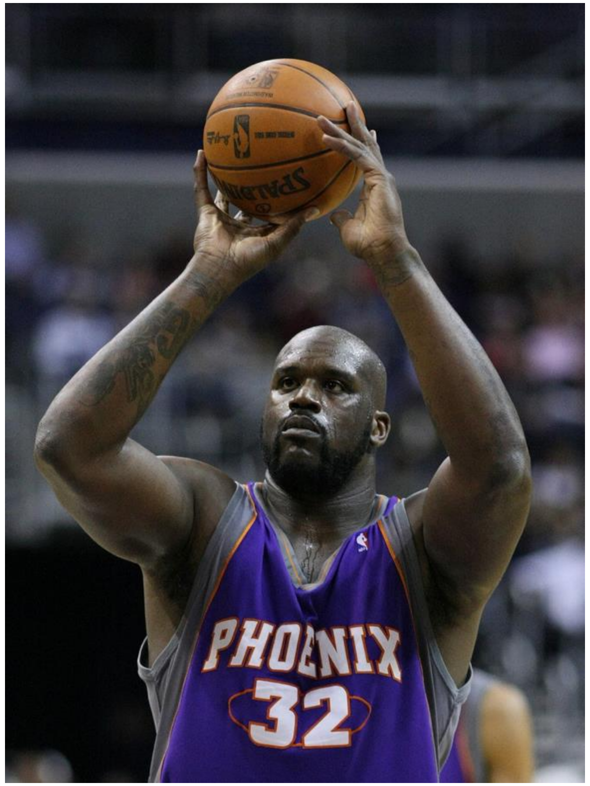

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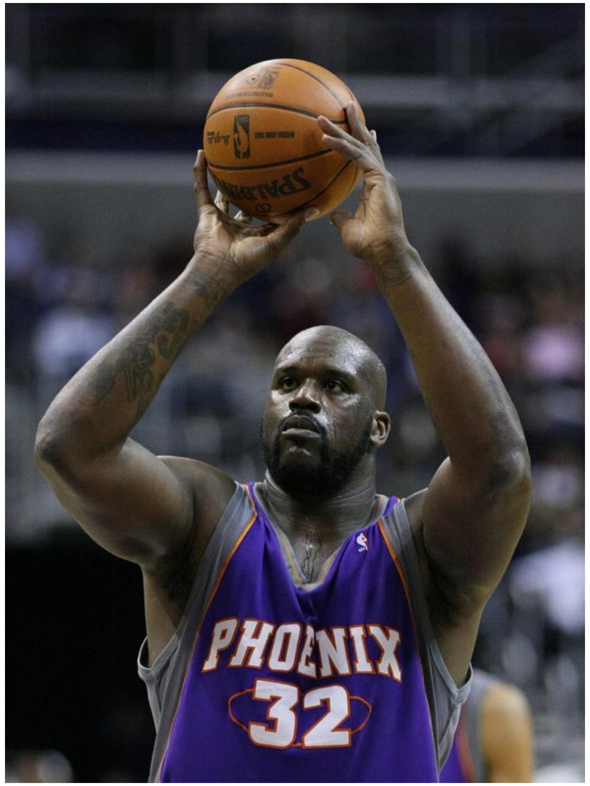

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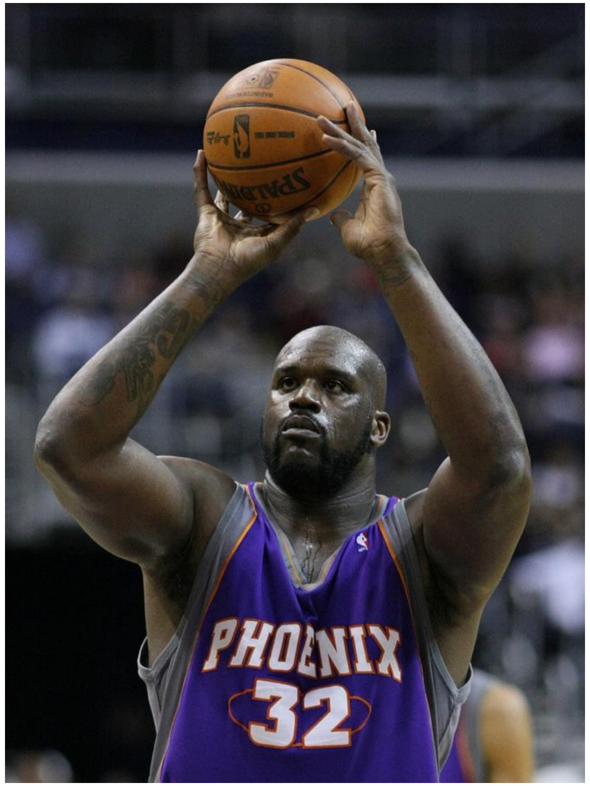
$$E[Y] = n \cdot p = 500 \cdot 0.53 = 265$$

Expected Value of Free Throws



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Challenge: If Shaq was 10% better at shooting free throws, how many *more* free throws would you expect him to make, out of 500?

Expected Value of The Geometric

If $X \sim \mathrm{Geo}(p)$, then $E[X] = \frac{1}{p}$

This definition has intuition built in:

- If Shaq makes about half his free throws, then on average, it will take him two shots to make one free throw. E[X] = $(1/2)^{-1}$ = 2.

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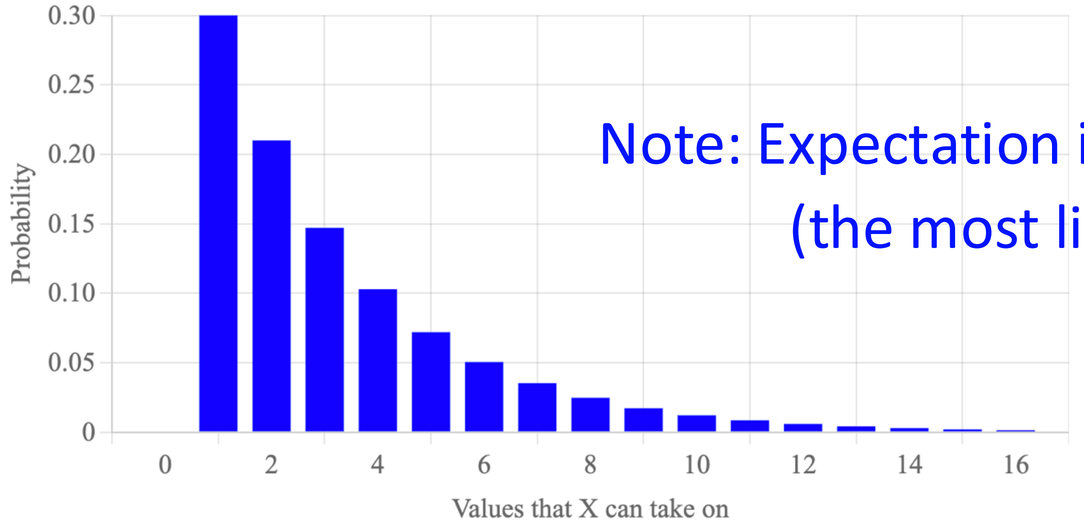
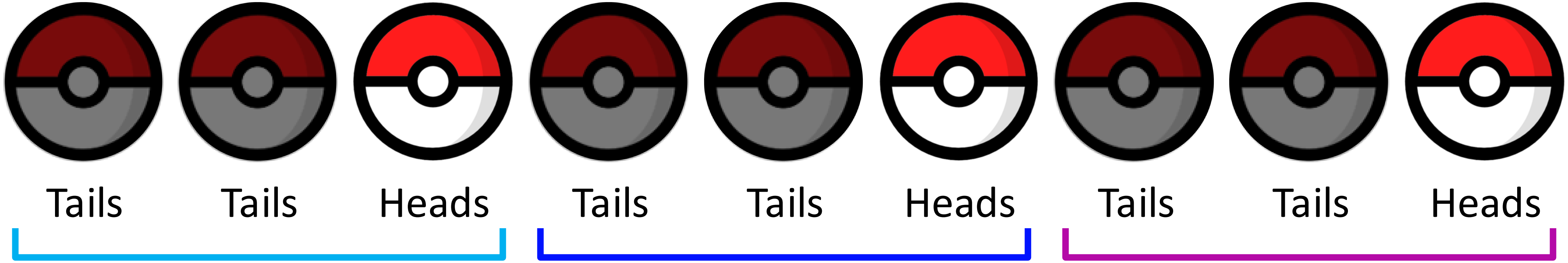


Note: Expectation is often **not** the mode (the most likely outcome)

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

The Negative Binomial …is a sum of Geometric random variables



Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

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$$= \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

Expectations of Classic Random Variables

$X \sim \text{Geo}(p)$

$$E[X] = \frac{1}{p}$$

$X \sim \text{Bern}(p)$

$$E[X] = p$$

$Y \sim \text{NegBin}(r, p)$

$$E[Y] = \frac{r}{p}$$

$Y \sim \text{Bin}(n, p)$

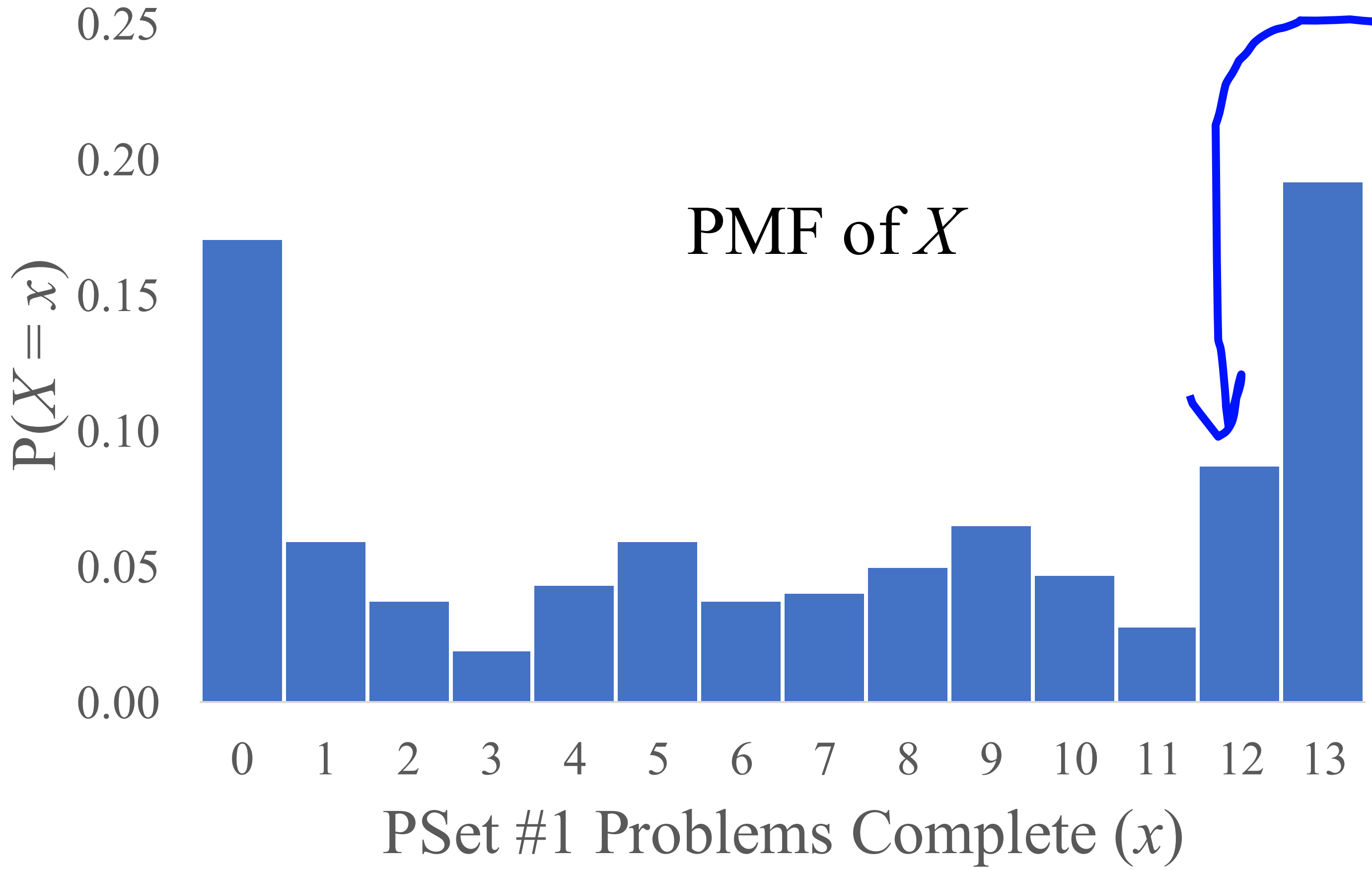
$$E[Y] = n \cdot p$$

Expectation is easy to work with, but still leaves much to be desired

Expectation vs PMF

Let X be the number of problems that a randomly selected student has completed, as of 11a today.

X takes on values, with uncertainty. X is a random variable.



$$P(X = 12) \approx \frac{\text{Count}(X = 12)}{N}$$

$$E[X] = 7.4$$

If extra time…

St. Petersburg Paradox

The Game:

- We have a fair coin (lands on heads with p = 0.5)
- Let n = number of coin flips (tails) to get the first heads
- You will win: \$$2^n$

How much would you pay to play?

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Let X be your winnings.

$$E[X] = \left(\frac{1}{2}\right)^1 2^1 + \left(\frac{1}{2}\right)^2 2^2 + \left(\frac{1}{2}\right)^3 2^3 + \ldots = \sum_{i=0}^{\infty} 1 = \infty$$

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What if you could play this game for only $1000…but just once?