

CS109 Lecture 14 Worksheet: ANSWER KEY

Beta

Summer 2026

Lecture 14 Worksheet: Answer Key

Instructor / TA copy. Problem-set solutions are hidden in the standard build; run `./convert-to-pdf.sh 14 solutions` for the full instructor/TA edition.

Problem 1 (Review): Five Quick Rolls

$$\binom{5}{2, 2, 1} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^1 = \frac{5!}{2! 2! 1!} \left(\frac{1}{6}\right)^5 = \frac{30}{7776} \approx 0.00386.$$

(The three faces we never roll have counts of 0, contributing factors of $0! = 1$ and $(1/6)^0 = 1$.)

Problem 2: The Robot Gripper

- (a) A uniform prior is $\text{Beta}(1, 1)$; add successes and failures: $X \sim \text{Beta}(1 + 6, 1 + 2) = \text{Beta}(7, 3)$.
- (b) $E[X] = \frac{a}{a+b} = \frac{7}{10} = 0.7$. Mode = $\frac{a-1}{a+b-2} = \frac{6}{8} = 0.75$. The posterior is skewed (bounded above by 1), so the mean is pulled below the peak. The mode equals the frequentist estimate $6/8$; the mean 0.7 is between the frequentist estimate and the prior mean 0.5 (the prior acts like one imagined success and one imagined failure).
- (c) A single hump on $[0, 1]$, peaked at 0.75, not symmetric: more probability mass trails off to the left than to the right (it must fit between the peak and 1).

Problem 3: Updating a Beta Prior

- (a) Conjugacy: $\text{Beta}(4 + 3, 2 + 5) = \text{Beta}(7, 7)$. (Symmetric, centered at $E[X] = 1/2$.)
- (b) $\text{Beta}(4, 2)$ acts like having already seen $a - 1 = 3$ imagined successes and $b - 1 = 1$ imagined failure on top of a uniform prior.
- (c) Posterior is $\text{Beta}(2 + s, 2 + (n - s))$, with mode

$$\frac{(2 + s) - 1}{(2 + s) + (2 + n - s) - 2} = \frac{s + 1}{n + 2}.$$

This is the classic Laplace-smoothed estimate: never exactly 0 or 1, even when $s = 0$ or $s = n$.

Problem 4: Puppy Training (Fall 2025 Midterm)

Uniform prior means $P \sim \text{Beta}(1, 1)$; after 3 successes and 1 failure, $P \sim \text{Beta}(4, 2)$. Then

$$\text{Var}(P) = \frac{ab}{(a+b)^2(a+b+1)} = \frac{4 \cdot 2}{6^2 \cdot 7} = \frac{8}{252} = \frac{2}{63} \approx 0.0317.$$

Problem 5: Street Parking (Fall 2023 Final)

A uniform prior between 0 and 1 is $\text{Beta}(a=1, b=1)$. We observe 1 success (open spot) and 9 failures (full spots):

$$X \sim \text{Beta}(2, 10), \quad E[X] = \frac{2}{12} = \frac{1}{6}.$$

Problem 6: Medicine with a Uniform Prior (pset4: warmup_beta)

(Problem set problem)

Problem 7: Thompson Sampling (pset4: learning_while_helping)

(Problem set problem)

Challenge Problem: Two Beta Facts

- (a) With $a = b = 1$: $x^{a-1}(1-x)^{b-1} = x^0(1-x)^0 = 1$, and $B(1, 1) = \int_0^1 1 \, dx = 1$. So $f(x) = 1$ on $[0, 1]$, exactly the $\text{Uni}(0, 1)$ PDF.
- (b) By the law of total probability, conditioning on $X = x$:

$$P(\text{heads}) = \int_0^1 P(\text{heads} \mid X = x) f(x) \, dx = \int_0^1 x f(x) \, dx = E[X] = \frac{a}{a+b}.$$

So the single best point prediction for the next flip is the mean of your belief.