CS109A Week 2

CS109A CA: Gili Rusak

Welcome everyone!

- Introductions
- Office hours start tomorrow from 10am - 12pm PT
  - Calendly link was sent out via email

If there is anything else I can help with, please email me

Agenda

- Permutations*
- Combinations*
- Divider method*
- Inclusion Exclusion Principle*
- Applications

* Relevant for HW 1

Permutations

Permutations

Key Idea: Order distinct objects

- Order *n* distinct objects $k! = k * (k-1) * \ldots * 2 * 1$

Note: notation is read **factorial**

- Order *n* semi-distinct objects: $n! / (n_1! \, n_2! \ldots n_r!)$

Permutations

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- Order *n* distinct objects $k! = k * (k-1) * \ldots * 2 * 1$

Note: notation is read **factorial**

- Order *n* semi-distinct objects: $n! / (n\_1!\ n\_2! \ldots n\_r!)$
- Examples
  - How many ways are there to order the letters in the word APPLE?
  - Suppose there are ten unique exams given to ten students. Each student receives one exam. How many different ways are there to assign the exams to the students?

Example 1

In how many ways can we order 5 distinct candy bars such that:

the chocolate bar and the granola bar are not next to each other?

Example 1

Technique: Complementary Counting

In how many ways can we order 5 distinct candy bars such that:

the chocolate bar and the granola bar are not next to each other?

Key idea: First, consider how many ways 5 distinct candy bars can be ordered.

Then, remove the ways that the chocolate and granola bar are next to each other

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Technique: Try writing out small examples if you get stuck

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Technique: Complementary Counting

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5 options for candy bar 1	4 options remain for candy bar 2	3 options remain for candy bar 3	2 options remain for candy bar 4	1 option remains for candy bar 5

Then, remove the ways that the chocolate and granola bar are next to each other

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Technique: Complementary Counting

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5 * 4 * 3 * 2* 1 = 5! ways

Then, remove the ways that the chocolate and granola bar are next to each other

Example 1

Technique: Casework

How many orderings are such that the **chocolate and granola bar are next to each other?**

Chocolate	Granola	Candy 3	Candy 4	Candy 5

Example 1

Technique: Casework

How many orderings are such that the **chocolate and granola bar are next to each other?**

Case 1: Chocolate-Granola orderings

Chocolate	Granola	Candy 3	Candy 4	Candy 5

Case 2: Granola-Chocolate orderings

Granola	Chocolate	Candy 3	Candy 4	Candy 5

Example 1

Technique: Complementary Counting

How many orderings are such that the **chocolate and granola bar are next to each other?**

Case 1: Chocolate-Granola orderings

Chocolate Granola	Candy 3	Candy 4	Candy 5	= 4! orders

Case 2: Granola-Chocolate orderings

Granola Chocolate	Candy 3	Candy 4	Candy 5	= 4! orders

Example 1

Technique: Complementary Counting

In how many ways can we order 5 distinct candy bars such that:

the chocolate bar and the granola bar are not next to each other?

Key idea: First, consider how many ways 5 distinct candy bars can be ordered

= 5! ways

Then, remove the ways that the chocolate and granola bar are next to each other

= - 2 * 4! ways

Solution: 5! - 2 * 4!

Combinations

Combinations

Key Idea: Choose k objects where order doesn't matter

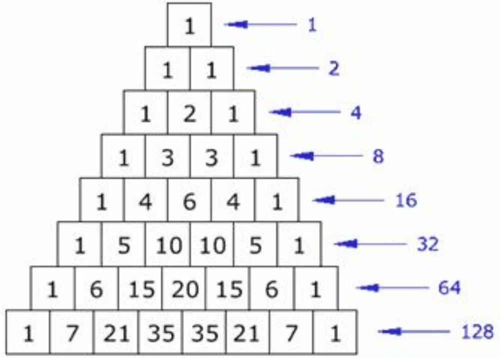
- Notation: *nCk = n! / k! / (n-k)!*

Note: also known as the *binomial coefficient* and has many interesting properties.
Notation is read: **choose**

- Beyond CS109: Combinatorial Identities
  - Pascal's triangle: provable algebraic properties hold, e.g.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

  - Binomial theorem.



Pascals Triangle

* https://artofproblemsolving.com/wiki/index.php/Pascal%27s_Identity

* Image from: https://www.mathsisfun.com/pascals-triangle.html

Combinations

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- Examples
  - Given a class of 200 students. In how many ways can we select a group of 20 students to receive a candy bar?

Example 2

A typical starting line-up for a soccer team has 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards. The 2019 Men's soccer team roster has 4 goalkeepers, 8 defenders, 18 midfielders and 3 forwards. In how many ways can the head coach choose the starting line-up?

Hint: Product rule of counting

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Choose 1 goalkeeper from 4 options	Choose 4 defenders from 8 options	Choose 4 midfielders from 18 options	Choose 2 forwards from 3 options

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Goalkeeper: 4 C 1	*	Defenders: 8 C 4	*	Midfielders: 18 C 4	*	Forwards: 3 C 2	= 4 C 1 * 8 C 4 * 18 C 4 * 3 C 2

Example 2

Key Idea: Visualizing Product rule of counting

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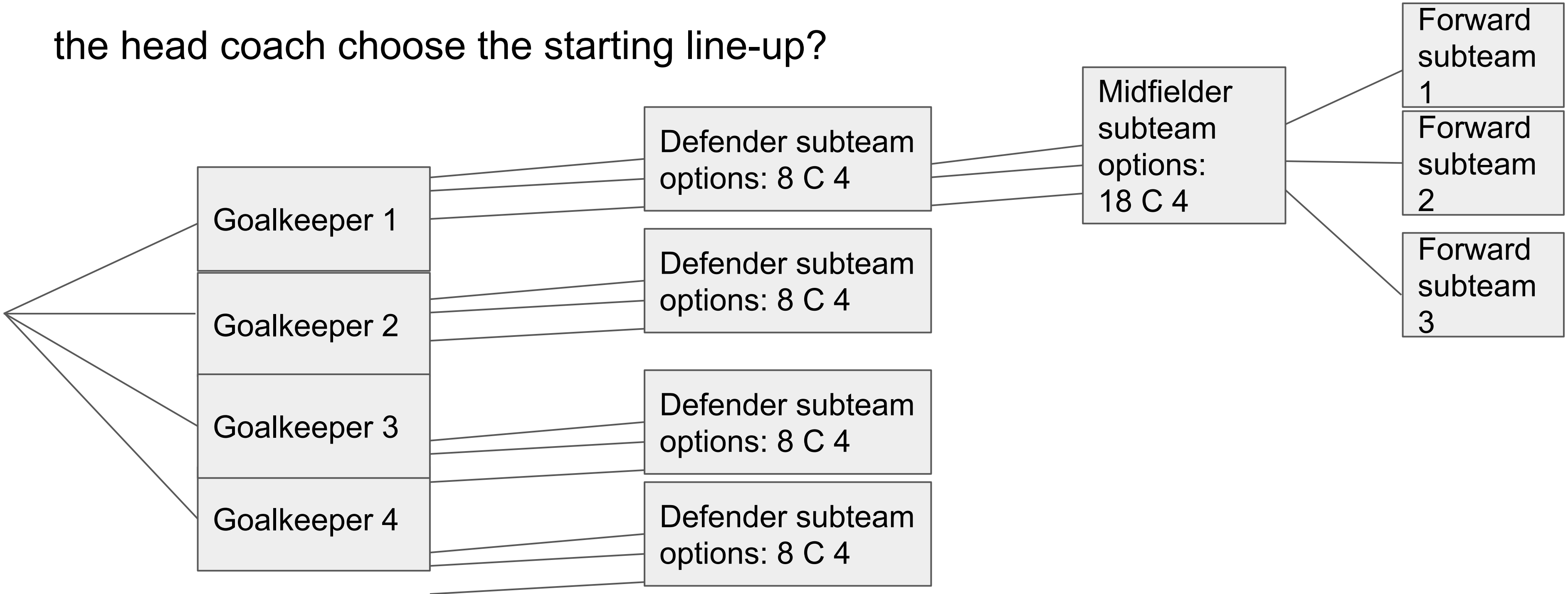

Divider Method

Divider method

Divider Method: Suppose you want to place n indistinguishable items into r containers. The divider method works by imagining that you are going to solve this problem by sorting two types of objects, your n original elements and $(r-1)$ dividers. Thus, you are permuting $n+r-1$ objects, n of which are same (your elements) and $r-1$ of which are same (the dividers). Thus the total number of outcomes is:

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{n} = \binom{n+r-1}{r-1}.$$

Example 3

Suppose x, y, z are three non-negative whole numbers. How many ways are there to assign values to x, y, z such that $\mathbf{x + y + z = 7}$?

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Key idea: Write 7 out as $1 + 1 + 1 + 1 + 1 + 1 + 1 = x + y + z$. Since the problem tells us that x, y, z are whole numbers, we can imagine that we are trying to place these 7 indistinguishable 1's into 3 distinguishable buckets (divider method!).

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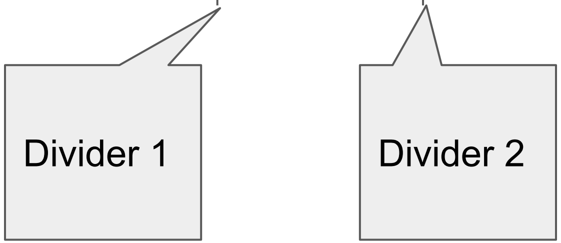
For example: 1 + 1 + | 1 + 1 + 1 | + 1 + 1



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= x   = y   = z

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Solution: By applying the divider method, we have (7 + 3 - 1) C (3 - 1) ways to permute 7 indistinguishable 1's and 2 indistinguishable dividers. (remember we only need 2 = 3 - 1 dividers to establish 3 buckets).

Example 3.5

Technique:
"Pre-allocate" values to meet a certain condition

Suppose x, y, z are three non-negative whole numbers.

Suppose **$x >= 1$, $y >= 1$, $z >= 1$**.

How many ways are there to assign values to x, y, z such that **$x + y + z = 7$**?

Divider Method with Pre-allocation

Technique:
"Pre-allocate" values to meet a certain condition

Suppose x, y, z are three non-negative whole numbers.

Suppose **x >= 1, y >= 1, z >= 1**.

How many ways are there to assign values to x, y, z such that **x + y + z = 7**?

Key Idea: Pre-allocate one 1 to each bucket to ensure constraint is satisfied.

Divider Method with Pre-allocation

For example: 1 + ___ | + 1 + ___ | + 1 + ___

= x   = y   = z

Remaining Problem: How do we allocate 4 indistinguishable 1's to 3 distinguishable buckets?

Divider Method with Pre-allocation

For example: 1 + ___ | + 1 + ___ | + 1 + ___

= x   = y   = z

Remaining Problem: How do we allocate 4 indistinguishable 1's to 3 buckets?

Solution: Divider method

(4 + 3 - 1) C (3 - 1)

Practice Problem 2 from last week [Midterm 1 2020]

In how many distinct ways can the 100 (indistinct) socials be allocated to the 3 quarters?

Practice Problem 2 from last week [Midterm 1 2020]

In how many distinct ways can the 100 (indistinct) socials be allocated to the 3 quarters?

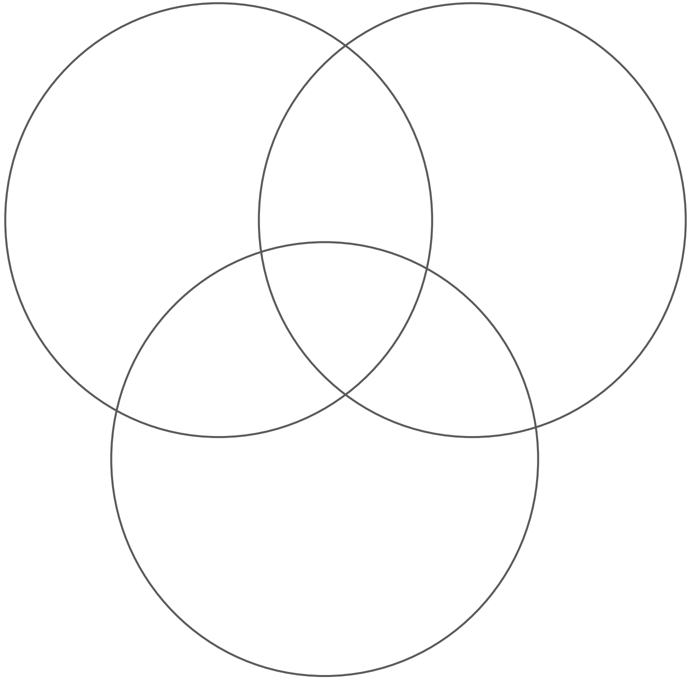
Solution:

By divider method, (100 + 3 - 1) C (3 - 1) = 102 C 2 ways.

Inclusion Exclusion Principle

Inclusion Exclusion Principle

Technique: draw out Venn diagrams



Applications

Applications

- Distributed computing and networking [CS144]
  - Routing different requests to different servers
  - Load balancing
- Cryptography and network security [CS155, CS255]
  - Password
  - Encryption techniques: e.g. one-time pad
- Card games

See you next Tuesday!