Welcome to CS109A

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Agenda

● Conditional probability*
  ○ Bayes' Theorem*
  ○ Independence*

● Random variables (more next week)
  ○ Probability mass function
  ○ Cumulative distribution function
  ○ Expectation*

● Applications

* Relevant for HW2
Conditional Probability
Definitions

**Definition of Conditional Probability**
The probability of $E$ given that (aka conditioned on) event $F$ already happened:

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{P(E \cap F)}{P(F)}$$

**The Law of Total Probability**
For events $E$ and $F$,

$$P(F) = P(F \mid E)P(E) + P(F \mid E^C)P(E^C)$$
Bayes' Theorem

Bayes’ Theorem
The most common form of Bayes’ Theorem is:

\[ P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)} \]
Independence

**Independence**

Two events, $E$ and $F$, are independent if and only if:

$$P(EF) = P(E)P(F)$$

Otherwise, they are called dependent events.

Two events $E$ and $F$ are called conditionally independent given a third event $G$, if

$$P EF | G = P E | G P F | G$$

Or, equivalently:

$$P E | FG = P E | G$$
Example Problem 1.1

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

**Problem:** What is the probability that you choose Coin A, i.e. $P(A)$?

**Hint:** You can read off the answer from the problem statement! No calculation is needed.

* Problem by Alex Tsun
Example Problem 1.1

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

What is the probability that you choose Coin A, i.e. P(A)?

Hint: You can read off the answer from the problem statement! No calculation is needed.

Solution: 0.3333
Example Problem 1.2

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

**Problem:** Given that you choose Coin A, what is the probability you get heads, i.e. $P(H|A)$?

**Hint:** You can read off the answer from the problem statement! No calculation is needed.
Example Problem 1.2

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

**Problem:** Given that you choose Coin A, what is the probability you get heads, i.e. $P(H|A)$?

Hint: You can read off the answer from the problem statement! No calculation is needed.

**Solution:** 0.5
Example Problem 1.3

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

**Problem:** Use one of the definitions from the slides to calculate the probability you get heads, i.e. $P(H)$. Hint: Multiple cases.
Example Problem 1.3

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

Problem: Use one of the definitions from the slides to calculate the probability you get heads, i.e. $P(H)$. Hint: Multiple cases.

Solution: By Law of total probability

$$P(H) = P(H \mid A) \cdot P(A) + P(H \mid B) \cdot P(B) + P(H \mid C) \cdot P(C)$$

$$= 0.5 \cdot \frac{1}{3} + 0.2 \cdot \frac{1}{3} + 0.9 \cdot \frac{1}{3} = 0.5333$$
Example Problem 1.4

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

**Problem:** Use one of the definitions from the slides to calculate the probability that you chose the Coin A, given that you got heads, i.e. $P(A|H)$. 
Example Problem 1.4 Solution

Consider a scenario in which you have three coins. Coin A comes up heads with probability 0.5, Coin B with probability 0.2, and Coin C with probability 0.9. You choose a coin randomly with equal probability for all three.

Problem: Use one of the definitions from the slides to calculate the probability that you chose the Coin A, given that you got heads, i.e. $P(A|H)$.

Solution: By Bayes' Theorem,

$$P(A|H) = \frac{P(H \mid A) \cdot P(A)}{P(H)}$$

$$= \frac{0.5 \cdot \frac{1}{3}}{0.5 \cdot \frac{1}{3} + 0.2 \cdot \frac{1}{3} + 0.9 \cdot \frac{1}{3}}$$

$$= \frac{0.5 \cdot \frac{1}{3}}{0.533} = (0.3125)$$
A robot can be in exactly one of two locations: L1 or L2. The probability that the robot is in location L1 is $P(L1)$ and the probability it is in location L2 is $P(L2)$. Based on all past observations, the robot thinks that there is a 0.8 probability it is in L1 and a 0.2 probability that it is in L2.

The robot’s vision algorithm detects a window, and although there is only a window in L2, it can’t conclude that it is in fact in L2 because its image recognition algorithm is not perfect.

The probability of observing a window given there is no window at its location is 0.2 and the probability of observing a window given there is a window is 0.9. After incorporating the observation of a window, what is the robot’s new values for $P(L1)$ and $P(L2)$?
Example Problem 2 Solution

Let O be the event that we observe a window.

Problem Goal: "After incorporating the observation of a window, what is the robot’s new values for P(L1) and P(L2)?"
Example Problem 2 Solution

Let O be the event that we observe a window.

**Problem Goal:** "After incorporating the observation of a window, what is the robot’s new values for P(L1) and P(L2)?"

i.e. calculate P(L1 | O) and P(L2 | O).
Example Problem 2 Solution

Let $O$ be the event that we observe a window.

Problem Goal: Calculate $P(L1 \mid O)$ and $P(L2 \mid O)$.

Key information in the problem:

"A robot can be in exactly **one of two locations: L1 or L2.**"
Example Problem 2 Solution

Let $O$ be the event that we observe a window.

Problem Goal: Calculate $P(L1 \mid O)$ and $P(L2 \mid O)$.

Key information in the problem:

"A robot can be in exactly one of two locations: $L1$ or $L2$."

1. Since $L1$ and $L2$ are the only locations, $L1 = L2^C$ and vice versa.
Example Problem 2 Solution

Let $O$ be the event that we observe a window.

Problem Goal: Calculate $P(L1 \mid O)$ and $P(L2 \mid O)$.

Key information in the problem:

"Based on all past observations, the robot thinks that there is a 0.8 probability it is in $L1$ and a 0.2 probability that it is in $L2$."

Key idea: translate paragraph to mathematical notation
Let O be the event that we observe a window.

Problem Goal: Calculate P(L1 | O) and P(L2 | O).

Key information in the problem:

"Based on all past observations, the robot thinks that there is a 0.8 probability it is in L1 and a 0.2 probability that it is in L2."

2. P(L1) = 0.8 and P(L2) = 0.2.
Example Problem 2 Solution

Let $O$ be the event that we observe a window.

Problem Goal: Calculate $P(L1 \mid O)$ and $P(L2 \mid O)$.

Key information in the problem:

"The probability of observing a window given there is no window at its location is 0.2 and the probability of observing a window given there is a window is 0.9."
Example Problem 2 Solution

Let $O$ be the event that we observe a window.

Problem Goal: Calculate $P(L1 \mid O)$ and $P(L2 \mid O)$.

**Key information in the problem:**

"The probability of observing a window given there is no window at its location is 0.2 and the probability of observing a window given there is a window is 0.9."

3. Therefore, $P(O\mid L1) = 0.2$ and $P(O\mid L2) = 0.9$. 
Example Problem 2 Solution

Let $O$ be the event that we observe a window.

Problem Goal: Calculate $P(L_1 | O)$ and $P(L_2 | O)$.

**Key information in the problem:**

1. Since $L_1$ and $L_2$ are the only locations, $L_1 = L_2^C$ and vice versa.

2. $P(L_1) = 0.8$ and $P(L_2) = 0.2$.

3. Therefore, $P(O|L_1) = 0.2$ and $P(O|L_2) = 0.9$. 
Key idea: Bayes' Theorem

Example Problem 2 Solution

Use Bayes’ theorem.

\[
P(L_1 \mid O) = \frac{P(O \mid L_1)P(L_1)}{P(O \mid L_1)P(L_1) + P(O \mid L_1^C)P(L_1^C)}
\]

\[
= \frac{P(O \mid L_1)^{(0.8)}}{P(O \mid L_1)^{(0.8)} + P(O \mid L_1^C)^{(0.2)}}
\]

\[
= (0.2) \times (0.8) / (0.2 \times 0.8 + 0.9 \times 0.2)
\]
Random Variables
Random Variables

- A random variable (RV) is a variable that probabilistically takes on different values.
- The probability mass function (PMF) of a random variable is a function that maps possible outcomes of a random variable to the corresponding probabilities.
- The cumulative distribution function (CDF) of a random variable $X$ is a function $F$ specified as $F(a) = P(X \leq a)$, the probability that $X$ takes on a value less than or equal to some value $a$. 

![Graph showing PMF of a single 6-sided die roll and the sum of two dice rolls.]
The **expectation** of a discrete random variable $X$ is defined as:

$$E[X] = \sum_{x: P(x) > 0} xP(x)$$
Preview for week 3 and 4 in class

- Common random variables
  - Bernoulli RV
  - Binomial RV
  - Geometric RV
  - Poisson RV
  - Normal RV
  - Others!

- Properties of RVs
  - PDF, CDF, E[X], applications
Applications
Applications

- Naive bayes' model: a machine learning model (end of course)
- Classification machine learning problems
- Healthcare and biology
  - Punnett squares (see HW2)
See you next Tuesday!
Example Problem 2 [Midterm Spring 2020]

A home robot has two different sensors for motion detection. If there is a moving object, sensor $V$ (video camera) will detect motion with probability 0.95, and sensor $L$ (laser) will detect motion with probability 0.8. If there is no moving object, there is a 0.1 probability that sensor $V$ will detect motion (even though there is no object), and a 0.05 probability that sensor $L$ will detect motion.

Based on empirical evidence, the probability that there is a moving object is 0.7. Note that these sensors use independent detection algorithms to identify motion, so that conditioned on there being a moving object (or not), the events of detecting motion (or not) for each Sensor is independent.

**Problem:** Given that there is a moving object and that sensor $V$ does not detect motion, what is the probability that sensor $L$ detects motion? Give a numerical answer.