Welcome to CS109A

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Agenda

- Resources
  - Study groups
  - Review session
  - Practice exams
- Expectation
  - Definition of Expectation
  - Linearity of Expectation
- Practice midterm problems

Expectation

Random Variables

- A **random variable** (RV) is a variable that probabilistically takes on different values.
- The **probability mass function** (PMF) of a random variable is a function that maps possible outcomes of a random variable to the corresponding probabilities.

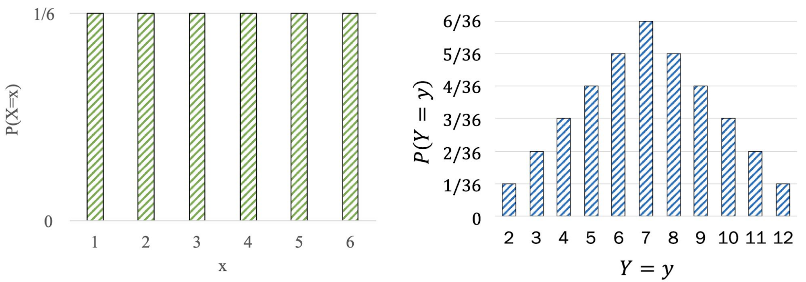

Figure 1: On the left, the PMF of a single 6 sided die roll. On the right, the PMF of the sum of two dice rolls.

- The **cumulative distribution function** (CDF) of a random variable X is a function F specified as $F(a) = P(X \leq a)$, the probability that X takes on a value less than or equal to some value a.

Expectation

1. Definition of Expectation for a discrete random variable

$$E[X] = \sum_{x:P(x)>0} xP(x)$$

2. Linearity of expectation

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

3. Expectation of a function

$$E[g(X)] = \sum_{x} g(x) \cdot p_X(x)$$

Variance

The variance of a discrete random variable X with expected value μ is:

$$\text{Var}(X) = E[(X - \mu)^2]$$

When computing the variance, we often use a different form of the same equation:

$$\text{Var}(X) = E[X^2] - E[X]^2$$

Example Problem

a. Let X be a random variable with PMF:

$$p_X(k) = \begin{cases} 4/7 & k = 7 \\ 2/7 & k = 14 \\ 1/7 & k = 28 \end{cases}$$

b. Let Y be a random variable with PMF:

$$p_Y(k) = \begin{cases} 2/6 & k = 6 \\ 1/6 & k = 12 \\ 3/6 & k = 18 \end{cases}$$

What is E[X]? What is E[Y]?

Solutions

a. E[X] = $4 \cdot \frac{4}{7} + 14 \cdot \frac{2}{7} + 28 \cdot \frac{1}{7} = 12$

b. E[Y] = $6 \cdot \frac{2}{6} + 12 \cdot \frac{1}{6} + 18 \cdot \frac{3}{6} = 13$

Example Problem Continued

c. Let $Z = X + Y$. What is $E[Z]$?

Hint: use linearity of expectation. Imagine how difficult it would be to calculate $p_z(k)$, and the range of z (i.e. the values that Z can take on).

d. Suppose X represents the number of televisions I sell in one day. Let Y represent the number of televisions you sell in one day. Suppose we sell each television for exactly $100 at a store, but we need to pay $1000 per day in operating charges in order to sell at that store. All together, what is our expected profit in a day?

Solutions

c. $$E[Z] = E[X + Y] = E[X] + E[Y] = 12 + 13 = 25$$

d. E[100Z - 1000] = 100 E[Z] - 1000 = 2500 - 1000 = $1500

Proof of Linearity of Expectation

For a single random variable X, $E[aX + b] = aE[X] + b$.

Proof: Let X be a random variable with PMF $p(x) = P(X = x)$.

$$
\begin{aligned}
E[aX + b] &= \sum_x (ax + b)p(x) = \sum_x (axp(x) + bp(x)) && \text{(LOTUS:} g(X) = aX + b\text{)} \\
&= a \sum_x xp(x) + b \sum_x p(x) && \text{(linearity of summation)} \\
&= aE[X] + b \sum_x p(x) && \text{(definition of expectation)} \\
&= aE[X] + b \cdot 1 && \text{(definition of PMF)}
\end{aligned}
$$

Practice Problems: Counting

Counting [Midterm 2017]

Shazam is an app that listens to a 10 second sample of background music playing (for example in a restaurant) and guesses the song. A user has just sent in a sample which we will use to explore how Shazam works. Within the 10 second sample there are: 50 sounds heard from the background ***music*** and 2000 sounds heard from background ***noise.***

Problem 1: There are 2050 total sounds in the sample. How many distinct ***pairs*** of two sounds are there? Sounds can not be paired with themselves. Sound-pairs with the same two sounds are ***not*** distinct.

Problem 2: How many distinct sound pairs exist such that ***both*** sounds in the sound-pair are from the music (as opposed to being from background noise)?

Solution

Problem 1:

2050 C 2

Common mistakes: 2050 · 2049 (order shouldn't matter)

Problem 2:

50 C 2

Common mistakes: 50 · 49 (order shouldn't matter)

Counting continued [Midterm 2017]

Problem 3: Every distinct pair of sounds from the 10 second sample casts a vote as to what song the pair thinks is playing. If both sounds in a sound-pair are from the music, the pair always casts a vote for the correct song. Otherwise, since the pair contains background noise, the pair casts a vote uniformly at random from a set of 5 songs (always the same five songs, including the correct song).

We want **more than 1/5 of the total number of votes** to go to the correct song. How many of the pairs containing background noise must vote for the correct song in order for the correct song to get 1/5 of the votes?

Solution

Problem 3:

(2050 C 2) / 5 - 50 C 2

Practice Problems: Conditional Probability

Conditional Probability [Extra Practice Problems]

A student is taking the computer-based GRE test. As the student answers questions, the test adapts its level of difficulty to their abilities. Let's say that the student is in the math section, and the test is trying to estimate whether or not they know geometry based on whether they answer the first geometry question correctly. The test's prior belief that the student knows geometry is q. The test also knows the following: Given that a student knows geometry, the probability that they solve the first question correctly is p1. Additionally, given that a student does not know geometry, the probability that the solve the first questions correctly is p2. Given that the student solves the first question correctly, what is the probability that they know geometry?

Solutions

Let C be the event that the student knows geometry. Let S be the event that the student solves the first geometry question correctly. Then P(C) = q, P(C^c) = 1-q, P(S|C) = p1, and P(S|C^c) = p2. The value we need to solve for is P(C|S). We use Bayes Rule:

$$P(C|S) = \frac{P(S|C)P(C)}{P(S)}$$

$$P(C|S) = \frac{P(S|C)P(C)}{P(S|C)P(C) + P(S|C^c)P(C^c)}$$

$$P(C|S) = \frac{p_1 q}{p_1 q + p_2(1-q)}$$

Conditional Probability [Extra Practice Problems]

There are 4 levels for a certain exam: Beginner, Intermediate, Expert, and Euclid, and each student is in exactly 1 level. The test's prior belief that a student is in a certain level is as follows: Beginner is b1, Intermediate is b2, Expert is b3, and Euclid is b4. The conditional probability that a student solves the first question correctly given their level is as follows: Beginner is d1, Intermediate is d2, Expert is d3, and Euclid is d4. Find the conditional probability that a student is in a certain level given that they solve the first question correctly. Do this for each of the 4 levels.

Solution

Let S be the event the student solves the first question correctly.

A = the event the student is in the Beginner level

B = the event the student is in the Intermediate level

C = the event the student is in the Expert level

D = the event the student is in the Euclid level

Then $P(A) = b1$, $P(B) = b2$, $P(C) = b3$, $P(D) = b4$. And $P(S|A) = d1$, $P(S|B) = d2$, $P(S|C) = d3$, $P(S|D) = d4$. We want to solve for $P(A|S)$, $P(B|S)$, $P(C|S)$, $P(D|S)$. Here we show the example for $P(A|S)$.

Solution Continued

We use Bayes Rule:

$$P(A|S) = \frac{P(S|A)P(A)}{P(S)}$$

$$P(A|S) = \frac{P(S|A)P(A)}{P(SA) + P(SB) + P(SC) + P(SD)}$$

$$P(A|S) = \frac{P(S|A)P(A)}{P(S|A)P(A) + P(S|B)P(B) + P(S|C)P(C) + P(S|D)P(D)}$$

$$P(A|S) = \frac{d_1 b_1}{d_1 b_1 + d_2 b_2 + d_3 b_3 + d_4 b_4}$$

Good luck on the midterm!

More problems

Conditional Probability [Midterm Spring 2020]

A home robot has two different sensors for motion detection. If there is a moving object, sensor *V* (video camera) will detect motion with probability 0.95, and sensor *L* (laser) will detect motion with probability 0.8. If there is no moving object, there is a 0.1 probability that sensor *V* will detect motion (even though there is no object), and a 0.05 probability that sensor *L* will detect motion.

Based on empirical evidence, the probability that there is a moving object is 0.7. Note that these sensors use independent detection algorithms to identify motion, so that **conditioned** on there being a moving object (or not), the events of detecting motion (or not) for each Sensor is **independent**.

Problem: Given that there is a moving object and that sensor *V* does not detect motion, what is the probability that sensor *L* detects motion? Give a numerical answer.