Welcome to CS109A

Gili Rusak

Agenda

- Discrete Joint Distributions*
- Multinomial Distribution*
- Independent Discrete RVs*
- Applications*

* Relevant for HW4

Discrete Joint Distributions

Discrete joint distributions

- Given random variables X,Y, a joint probability mass function tells us the probability of any combination of events $X = a$ and $Y = b$

$$p_{X,Y}(a, b) = P(X = a, Y = b)$$

- Marginal probabilities from the joint probability mass function

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

Example Problem 1.1*

Suppose we toss a fair coin **three times** and record the sequence of heads (h) and tails (t). Let random variable X denote the number of heads obtained. Let random variable Y denote the winnings earned in a single play of a game with the following rules:

- player wins $1 if first h occurs on the first toss
- player wins $2 if first h occurs on the second toss
- player wins $3 if first h occurs on the third toss
- player loses $1 if no h occur

What are the possible values X can take on? What are the possible values Y can take on?

*https://stats.libretexts.org/Courses/Saint_Mary%27s_College_Notre_Dame/MATH_345__-_Probability_(Kuter)/5%3A_Probability_Distributions_for_Combinations_of_Random_Variables/5.1%3A_Joint_Distributions_of_Discrete_Random_Variables

Example Problem 1.1*

X, the number of heads, can take on values x= 0,1,2,3.

Y, the possible winnings in a single game, can take on values y= -1,1,2,3.

* Problem from:
https://stats.libretexts.org/Courses/Saint_Mary%27s_College_Notre_Dame/MATH_345__-_Probability_(Kuter)/5%3A_Probability_Distributions_for_Combinations_of_Random_Variables/5.1%3A_Joint_Distributions_of_Discrete_Random_Variables

Example Problem 1.2*

Suppose we toss a fair coin **three times** and record the sequence of heads (h) and tails (t). Let random variable X denote the number of heads obtained. Let random variable Y denote the winnings earned in a single play of a game with the following rules:

- player wins $1 if first h occurs on the first toss
- player wins $2 if first h occurs on the second toss
- player wins $3 if first h occurs on the third toss
- player loses $1 if no h occur

What is the joint probability mass function for p(x,y)? Represent this as a table.

*https://stats.libretexts.org/Courses/Saint_Mary%27s_College_Notre_Dame/MATH_345__-_Probability_(Kuter)/5%3A_Probability_Distributions_for_Combinations_of_Random_Variables/5.1%3A_Joint_Distributions_of_Discrete_Random_Variables

Example Problem 1.2*

X, the number of heads, can take on values x= 0,1,2,3.

Y, the possible winnings in a single game, can take on values y= -1,1,2,3.

$p(x, y)$	X			
Y	0	1	2	3
-1	1/8	0	0	0
1	0	1/8	2/8	1/8
2	0	1/8	1/8	0
3	0	1/8	0	0

* Problem from:
https://stats.libretexts.org/Courses/Saint_Mary%27s_College_Notre_Dame/MATH_345__-_Probability_(Kuter)/5%3A_Probability_Distributions_for_Combinations_of_Random_Variables/5.1%3A_Joint_Distributions_of_Discrete_Random_Variables

Example Problem 1.3*

Suppose we toss a fair coin **three times** and record the sequence of heads (h) and tails (t). Let random variable X denote the number of heads obtained. Let random variable Y denote the winnings earned in a single play of a game with the following rules:

- player wins $1 if first h occurs on the first toss
- player wins $2 if first h occurs on the second toss
- player wins $3 if first h occurs on the third toss
- player loses $1 if no h occur

Calculate the marginal probability mass function p(y). Represent this as a table

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Example Problem 1.3*

X, the number of heads, can take on values x= 0,1,2,3.

Y, the possible winnings in a single game, can take on values y= -1,1,2,3.

Joint PMF:

$p(x, y)$	X			
Y	0	1	2	3
-1	1/8	0	0	0
1	0	1/8	2/8	1/8
2	0	1/8	1/8	0
3	0	1/8	0	0

Marginal PMF:

x	$p_X(x)$	y	$p_Y(y)$
0	1/8	-1	1/8
1	3/8	1	1/2
2	3/8	2	1/4
3	1/8	3	1/8

*https://stats.libretexts.org/Courses/Saint_Mary%27s_College_Notre_Dame/MATH_345__-_Probability_(Kuter)/5%3A_Probability_Distributions_for_Combinations_of_Random_Variables/5.1%3A_Joint_Distributions_of_Discrete_Random_Variables

Multinomial Distribution

Multinomial Distribution

- Suppose you perform *n* independent trials of an experiment where each trial results in one of *m* outcomes with probabilities p1, p2, …, pm. Define Xi to be the number of trails of outcome *i*. What is the probability that there are *ci* trials with outcome *i*?

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \ldots p_m^{c_m}$$

Example Problem 2

A 6-sided die is rolled 7 times. What is the probability that you roll: 1 one, 1 two, 0 threes, 2 fours, 0 fives, 3 sixes (disregarding order).

Example Problem 2 Solution

A 6-sided die is rolled 7 times. What is the probability that you roll: 1 one, 1 two, 0 threes, 2 fours, 0 fives, 3 sixes (disregarding order).

Remember: multinomial distribution

Example Problem 2 Solution

A 6-sided die is rolled 7 times. What is the probability that you roll: 1 one, 1 two, 0 threes, 2 fours, 0 fives, 3 sixes (disregarding order).

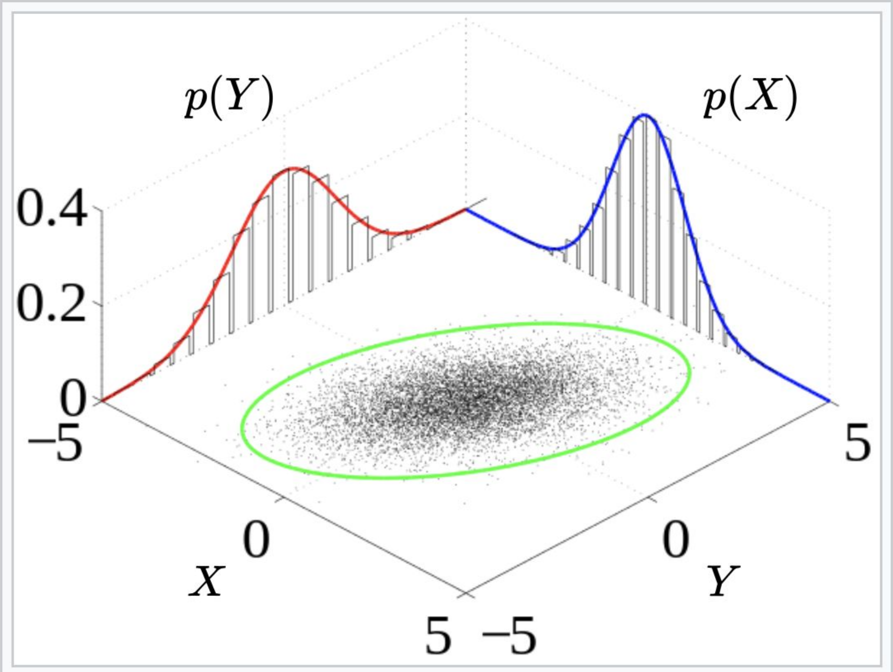
Remember: multinomial distribution

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3) = \frac{7!}{2!3!}\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^3$$

$$= 420\left(\frac{1}{6}\right)^7$$

Visualizing Joint Distributions

$p(x, y)$	X			
Y	0	1	2	3
-1	1/8	0	0	0
1	0	1/8	2/8	1/8
2	0	1/8	1/8	0
3	0	1/8	0	0



*https://en.wikipedia.org/wiki/Joint_probability_distribution and https://stats.libretexts.org/Courses/Saint_Mary%27s_College_Notre_Dame/MATH_345__-_Probability_(Kuter)/5%3A_Probability_Distributions_for_Combinations_of_Random_Variables/5.1%3A_Joint_Distributions_of_Discrete_Random_Variables

Independent Discrete Joint RVs

Independent Discrete Joint RVs

Two discrete random variables X and Y are called **independent** if:

$$P(X = x, Y = y) = P(X = x)P(Y = y) \text{ for all } x, y$$

Independent Binomials with equal p

For any two Binomial random variables with the same “success" probability: $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$ the sum of those two random variables is another binomial: $X+Y \sim \text{Bin}(n_1+n_2, p)$. This does not hold when the two distributions have different parameters p.

Independent Poissons

For any two Poisson random variables: $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ the sum of those two random variables is another Poisson: $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$. This holds even if λ_1 is not the same as λ_2.

Example Problem 3

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Calculate the P(X+Y = 3).

Example Problem 3 Solution

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Let Z = X+Y. Calculate the P(Z = 3).

Step 1: What distribution does Z follow?

Example Problem 3 Solution

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Let Z = X+Y. Calculate the P(Z = 3).

Step 1: What distribution does Z follow?

Z ~ Poi(14)

Example Problem 3 Solution

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Let Z = X+Y. Calculate the P(Z = 3).

Step 1: What distribution does Z follow?

Z ~ Poi(14)

Step 2: Calculate P(Z = 3).

Example Problem 3 Solution

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Let Z = X+Y. Calculate the P(Z = 3).

Step 1: What distribution does Z follow?

Z ~ Poi(14)

Step 2: Calculate P(Z = 3). By applying the PMF of the Poisson RV:

P(Z=3) = e^(-14) 14^3 / 3!

Example Problem 3 Solution

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Let Z = X+Y. Calculate the P(Z = 3).

Step 1: What distribution does Z follow?

Z ~ Poi(14)

Step 2: Calculate P(Z = 3). By applying the PMF of the Poisson RV:

P(Z=3) = e^(-14) 14^3 / 3!

Extension: What is P(Z > 3)?

Example Problem 3 Solution

Suppose we have X ~ Poi(6) and Y ~ Poi(8). Suppose X and Y are independent random variables. Let Z = X+Y. Calculate the P(Z = 3).

Step 1: What distribution does Z follow?

Z ~ Poi(14)

Step 2: Calculate P(Z = 3). By applying the PMF of the Poisson RV:

P(Z=3) = e^(-14) 14^3 / 3!

Extension: What is P(Z > 3)? Remember from last week, complementary counting: 1 - P(Z = 0) - P(Z=1) - P(Z=2).

Applications

Applications

Federalist Papers Example

- In HW4, you will apply your understandings of conditional probability and random variables to build a model that determines authorship probabilistically, given the contents of the document and historical evidence of the two authors' penmanship.

See you next Tuesday!