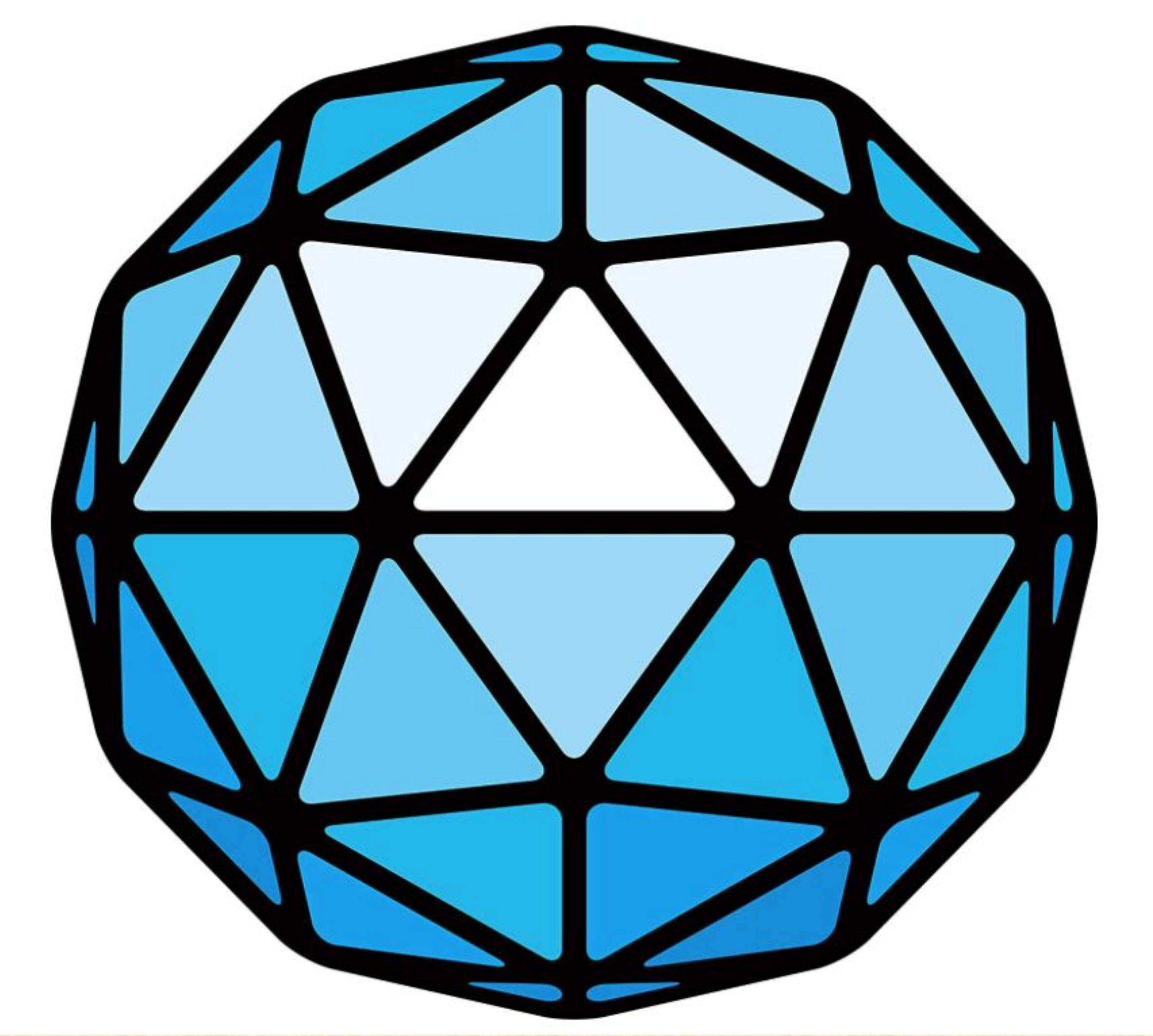
Triangles



Lots of Triangles

Stanford Bunny
69,451 triangles

David (Digital Michelangelo Project)
56,230,343 triangles

Why Triangles?

- Can focus on specializing/optimizing the geometry pipeline for only one geometric primitive
- Software and algorithms can be optimized for one geometric primitive
- Hardware (e.g. GPUs) can be specialized to treat one geometric primitive

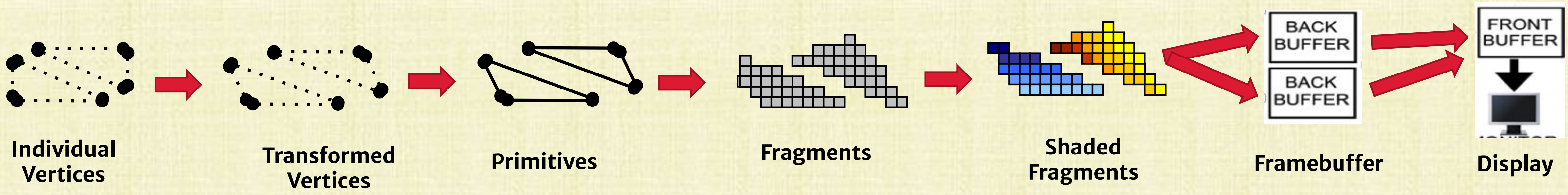
- Triangles have many inherent benefits:
  - Complex objects are well approximated (piecewise linear convergence) using enough triangles
  - Easy to break other polygons into triangles
  - Triangles are guaranteed to be planar (unlike quadrilaterals)
  - Transformations (from last lecture) only need be applied to the triangle vertices
  - Barycentric interpolation can be used to robustly interpolate information from the triangle's vertices to the triangle's interior
  - Etc.

OpenGL

- Blender uses OpenGL for it's real-time scanline renderer

- OpenGL was started by SGI in 1991 (went into the public domain in 2006)
- It's a drawing API for 2D/3D graphics
- Designed to be implemented mostly on hardware
- Many books and other documentation
- Main competitor is DirectX

- OpenGL is highly optimized for triangles:

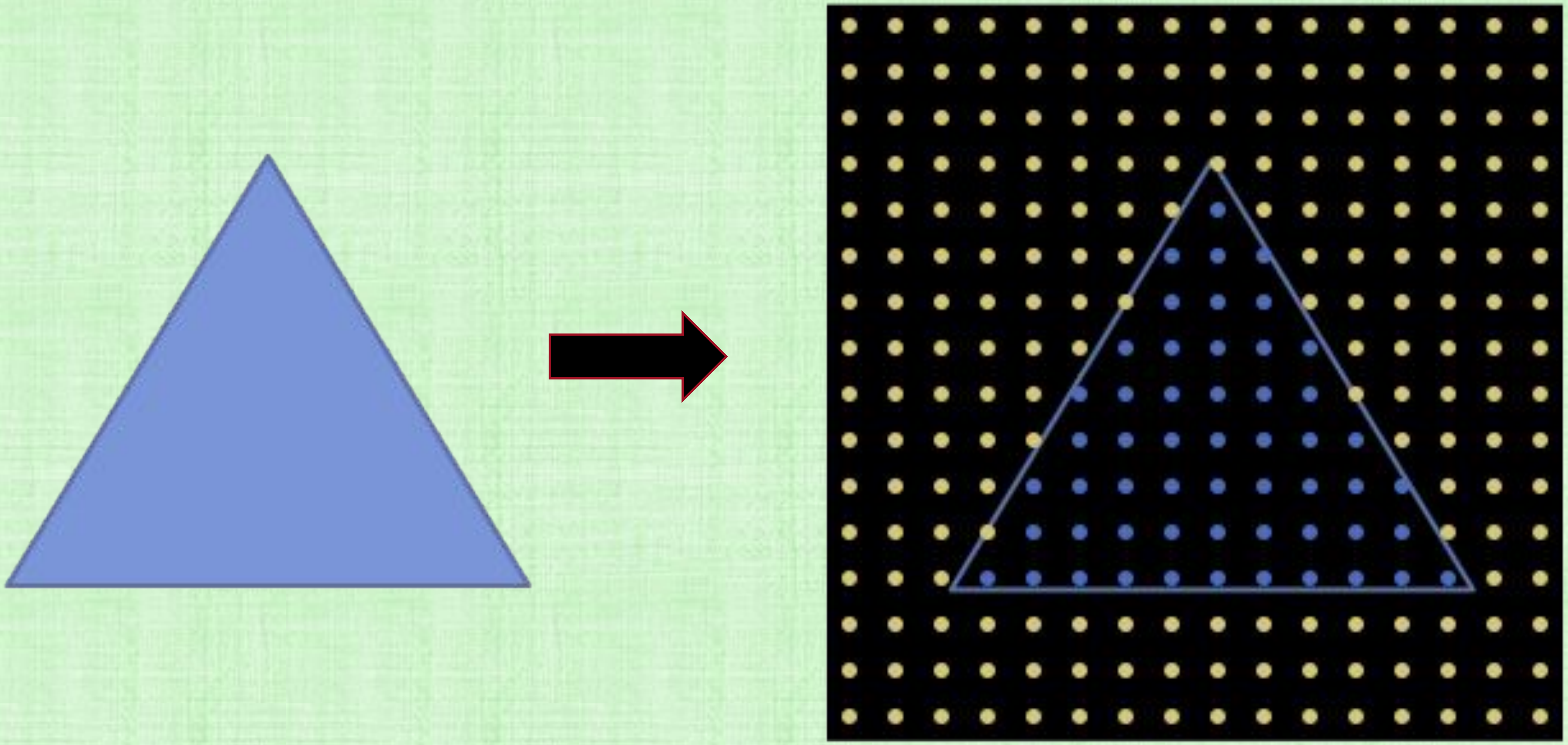


GPUs and Gaming Consoles

- GPUs and Consoles are highly optimized for the graphics geometry pipeline
  - They now support ray tracing, as does Blender

Rasterization

- Screen Space Projection transforms triangle vertices from 3D to screen space
- Find all the pixels inside the projected 2D triangle
- Color the pixels inside the triangle with the RGB-color of the triangle

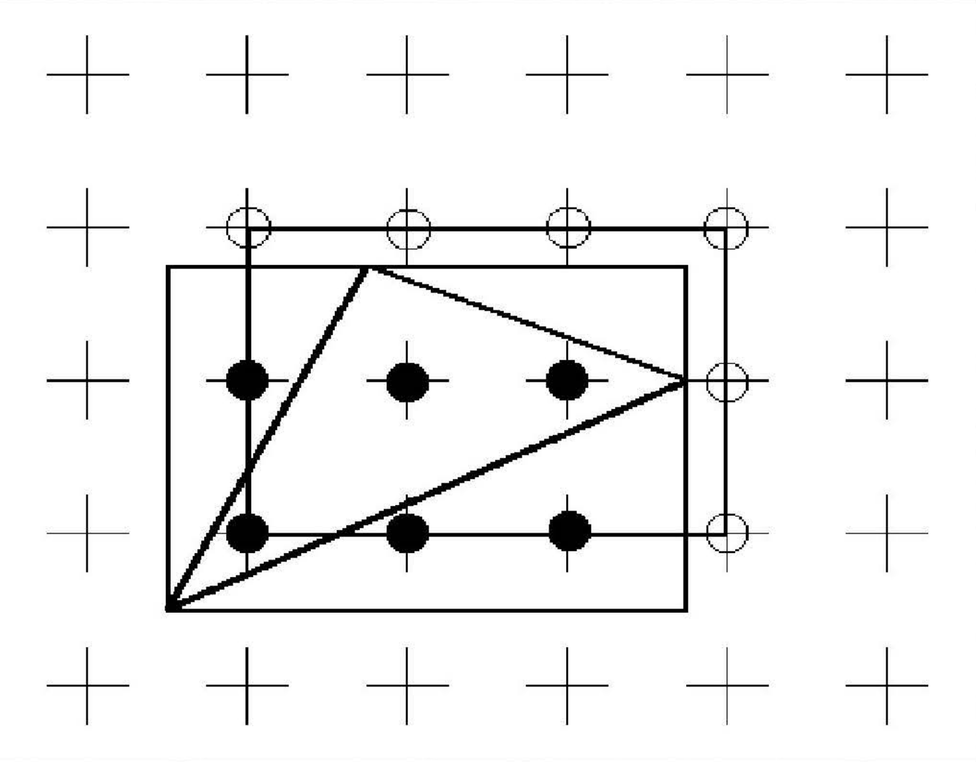


Aside: Bounding Box Acceleration

- Checking every pixel against every triangle is computationally expensive
- Calculate a bounding box around the triangle, with diagonal corners:

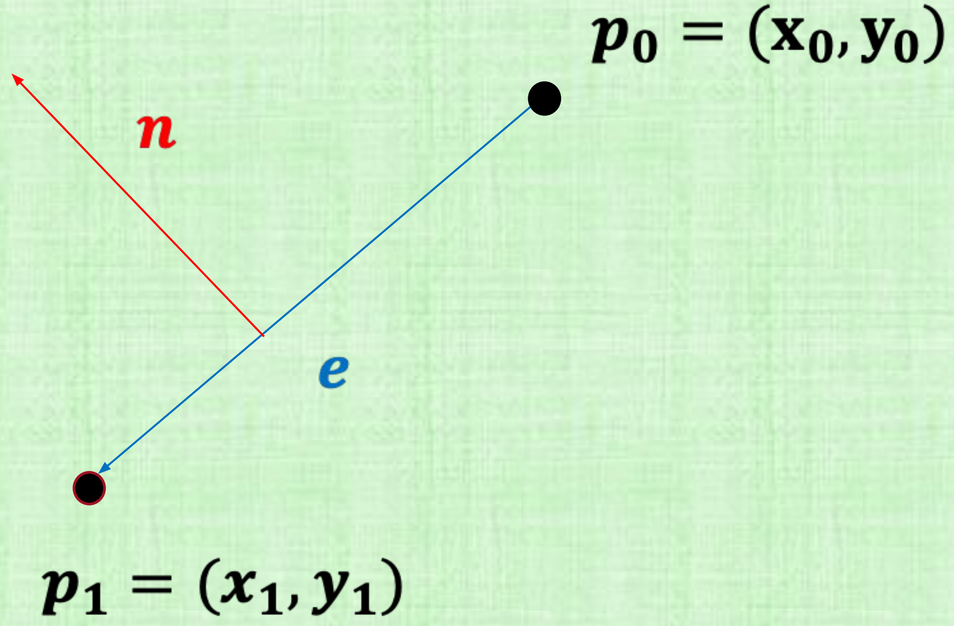
$$(\min(x_o, x_1, x_2), \min(y_0, y_1, y_2)) \text{ and } (\max(x_o, x_1, x_2), \max(y_0, y_1, y_2))$$

- Then, round coordinates upward to the nearest integer to find all relative pixels



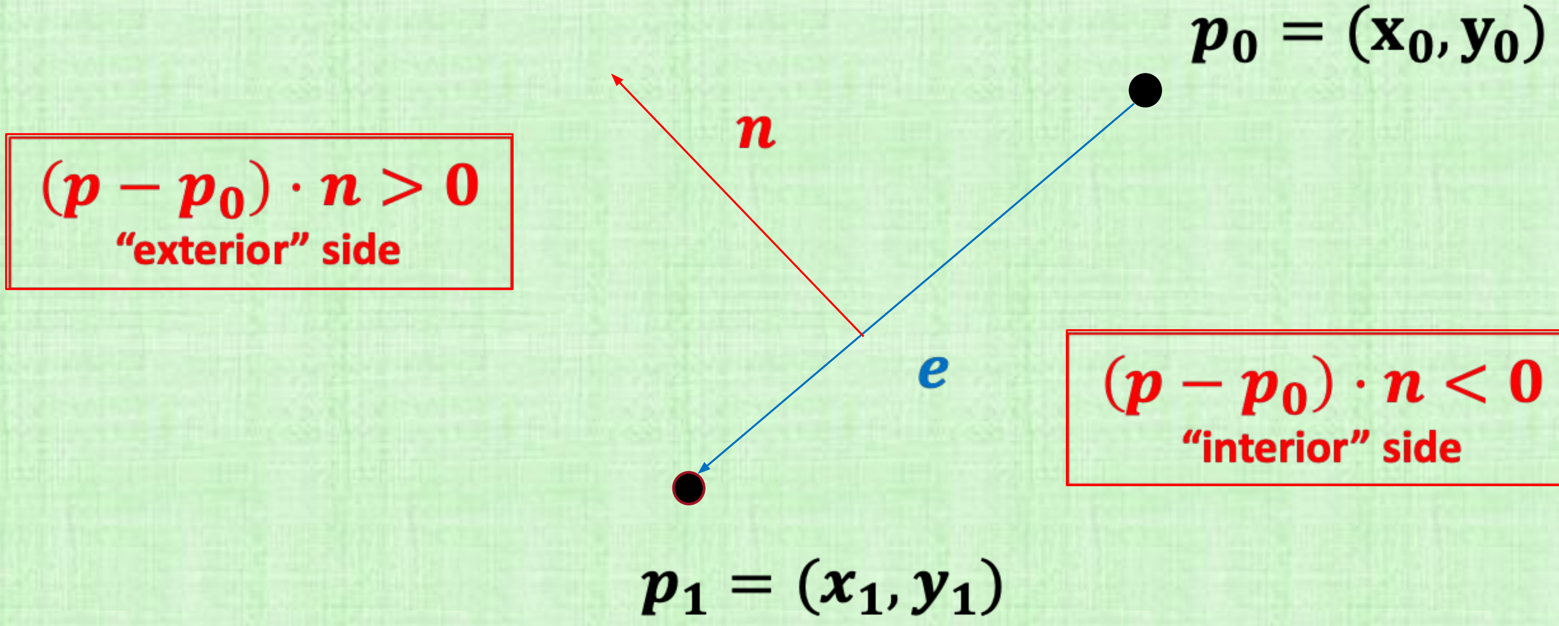
Implicit Equation for a 2D line

- Compute a directed edge vector $e = p_1 - p_0 = (x_1 - x_0, y_1 - y_0)$
- Compute the 2D normal $n = (y_1 - y_0, -(x_1 - x_0))$, which doesn't need be unit length
- This 2D normal is "rightward" with respect to the 2D ray direction ("leftward" normal is $-n$)
- Points p lying exactly on the 2D line have: $(p - p_0) \cdot n = 0$
  - This is the same equation used for planes in 3D

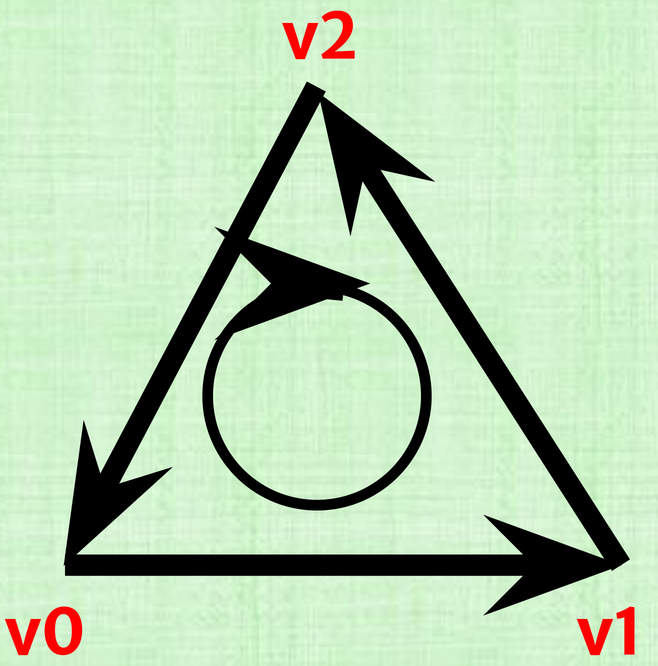


("Leftward") Interior Side of a 2D Ray

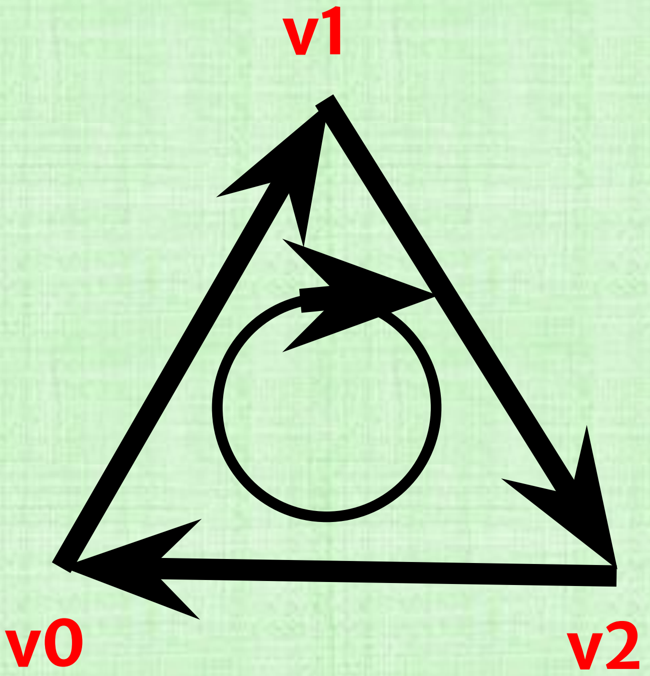
- Points p on the interior side of the 2D ray have: $(p - p_0) \cdot n < 0$
- Points p exactly on the 2D line have: $(p - p_0) \cdot n = 0$
- Points p on the exterior side of the 2D ray have: $(p - p_0) \cdot n > 0$
- This same concept can be used for planes in 3D



2D Point Inside a 2D Triangle



Counter-Clockwise Vertex Ordering
(Facing Camera)



Clockwise Vertex Ordering
(Facing Away from Camera)

- A 2D point is considered inside a 2D triangle, when it is interior to (to the left of) all 3 rays
- <u>Vertex ordering matters</u>: backward facing triangles are not rendered, since no points are to the left of all three rays

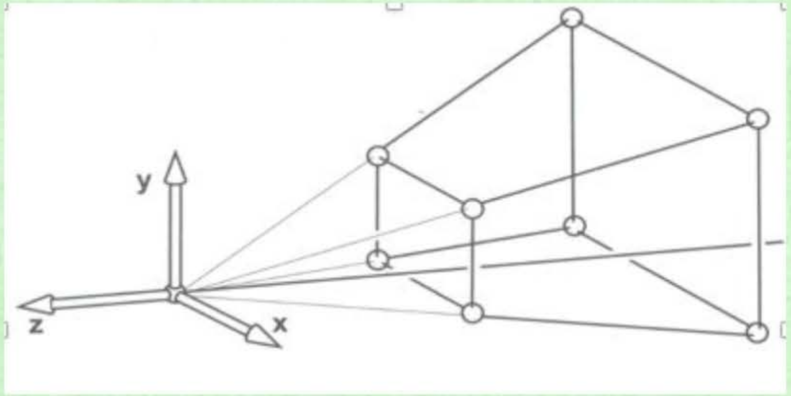
Boundary Cases

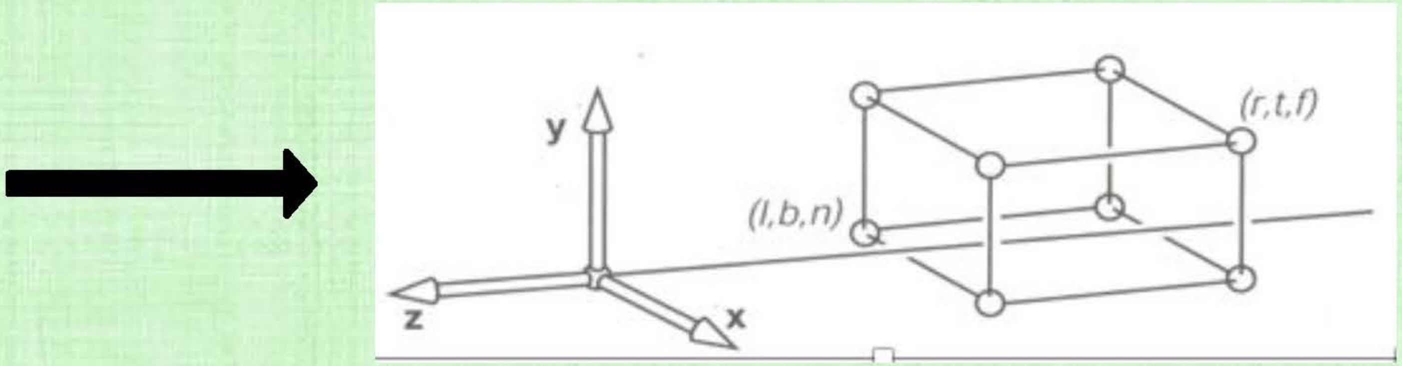
- Pixels lying exactly on a triangle boundary with $(p - p_0) \cdot n = 0$ for one of the edges won't be rendered
  - Causes gaps between adjacent (sharing an edge) triangles, when that shared edge overlaps a pixel
- Changing the inside test to $(p - p_0) \cdot n \leq 0$ instead of $(p - p_0) \cdot n < 0$ rectifies the problem, but both triangles attempt to color the same pixel
  - Inefficient, and can cause disagreements that lead to artifacts
- Instead, points on the shared edge can be consistently rendered with one triangle or the other:
  - The edge normals point in opposite directions for the two adjacent triangles
  - When $n_x > 0$ or ($n_x = 0$ and $n_y > 0$), rasterize pixels on that edge
  - When $n_x < 0$ or ($n_x = 0$ and $n_y < 0$), do not rasterize pixels on that edge
  - Note: n_x and n_y are never both zero for non-degenerate 2D triangles

Overlapping Triangles

- If one object is in front of another, two triangles may both try to color the same pixel

- Recall (last lecture): screen space projection computes $z' = n + f - \frac{fn}{z}$ that can be used for occlusion/transparency (via the alpha channel)



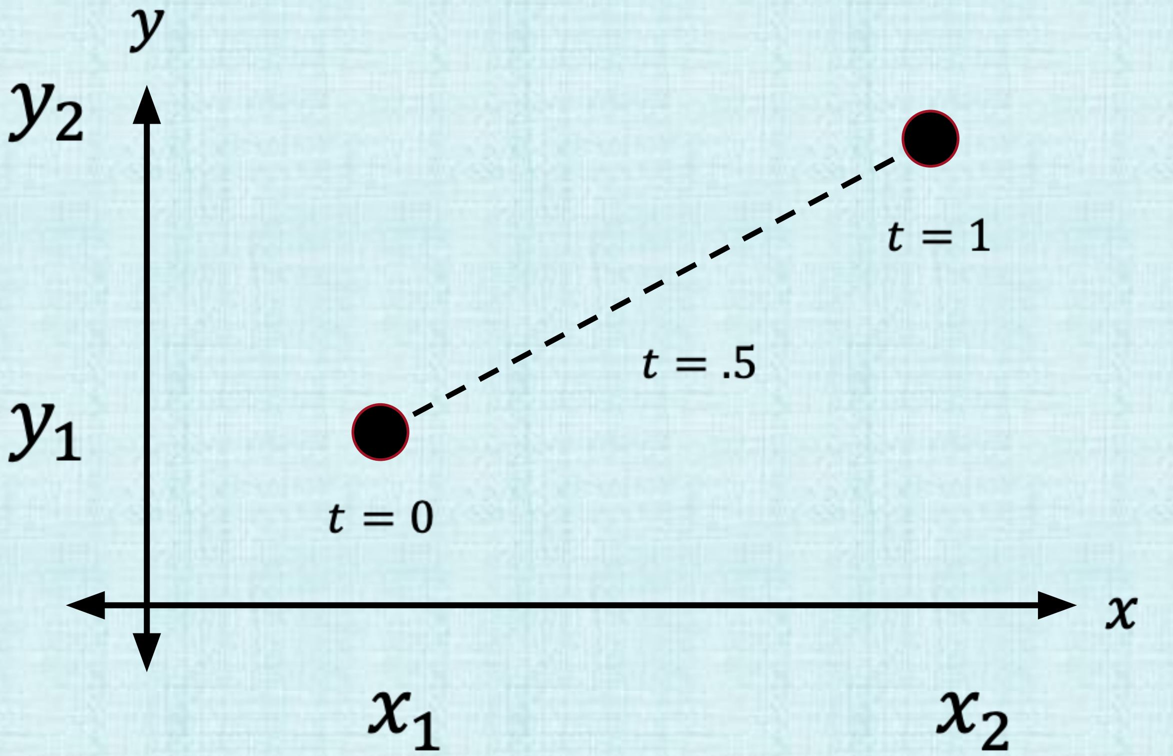


- Color the pixel based on which triangle gives the smallest z' value (closest to the camera)

- This requires interpolating z' values from the vertices of the triangle to the pixel locations
- In order to do this, we use *proper* screen space barycentric weight interpolation

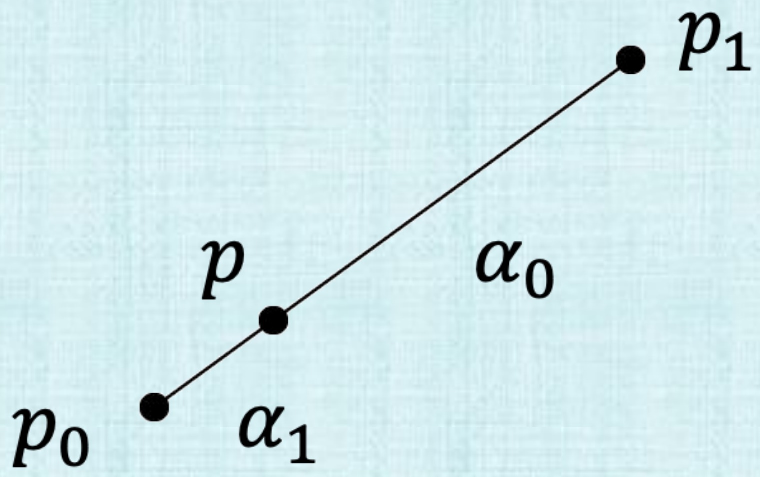
1D Linear Interpolation

- Given two points (x_1, y_1) and (x_2, y_2) in 1D, linearly interpolate between them via:
$$y(x) = \frac{y_2 - y_1}{x_2 - x_1}x - \frac{y_2 - y_1}{x_2 - x_1}x_1 + y_1 \quad \text{or} \quad y(x) = \left(1 - \frac{x - x_1}{x_2 - x_1}\right)y_1 + \frac{x - x_1}{x_2 - x_1}y_2$$
- Alternatively, $y(t) = (1 - t)y_1 + ty_2$ where $t = \frac{x - x_1}{x_2 - x_1}$ ranges from 0 to 1 (and can be seen as the fraction of the way from x_1 to x_2)



2D/3D Line Segments

- This can be extended to line segments in both 2D and 3D
- Given endpoints p_0 and p_1, intermediate points are defined based on the fraction of the distance that point is from p_0 to p_1 via $p(t) = (1 - t)p_0 + tp_1$
- $t = \frac{\|p - p_0\|_2}{\|p_1 - p_0\|_2}$, since p_0 and p_1 are multidimensional points
- Barycentric weights reformulate this using weights $\alpha_0, \alpha_1 \in [0,1]$ where $\alpha_0 + \alpha_1 = 1$ and $p = \alpha_0 p_0 + \alpha_1 p_1$, i.e. $\alpha_0 = \frac{\|p - p_1\|_2}{\|p_1 - p_0\|_2}$ and $\alpha_1 = \frac{\|p - p_0\|_2}{\|p_1 - p_0\|_2}$
- Barycentric weights express any point p on the segment as a linear combination of the endpoints of the segment

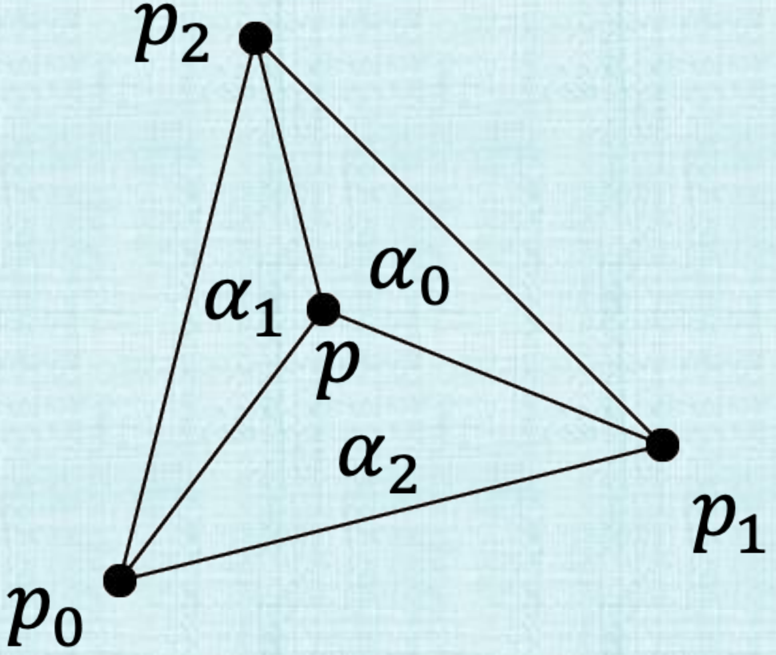


2D/3D Triangles

- Extend to triangles with 3 vertices by computing 3 barycentric weights $\alpha_0, \alpha_1, \alpha_2 \in [0,1]$ with $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and $p = \alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2$
- The weights are computed via areas:

$$\alpha_0 = \frac{Area(p,p_1,p_2)}{Area(p_0,p_1,p_2)} \quad \text{and} \quad \alpha_1 = \frac{Area(p_0,p,p_2)}{Area(p_0,p_1,p_2)} \quad \text{and} \quad \alpha_2 = \frac{Area(p_0,p_1,p)}{Area(p_0,p_1,p_2)}$$

- Note the triangle area formula: $Area(p_0, p_1, p_2) = \frac{1}{2} \| \overrightarrow{p_0 p_1} \times \overrightarrow{p_0 p_2} \|_2$

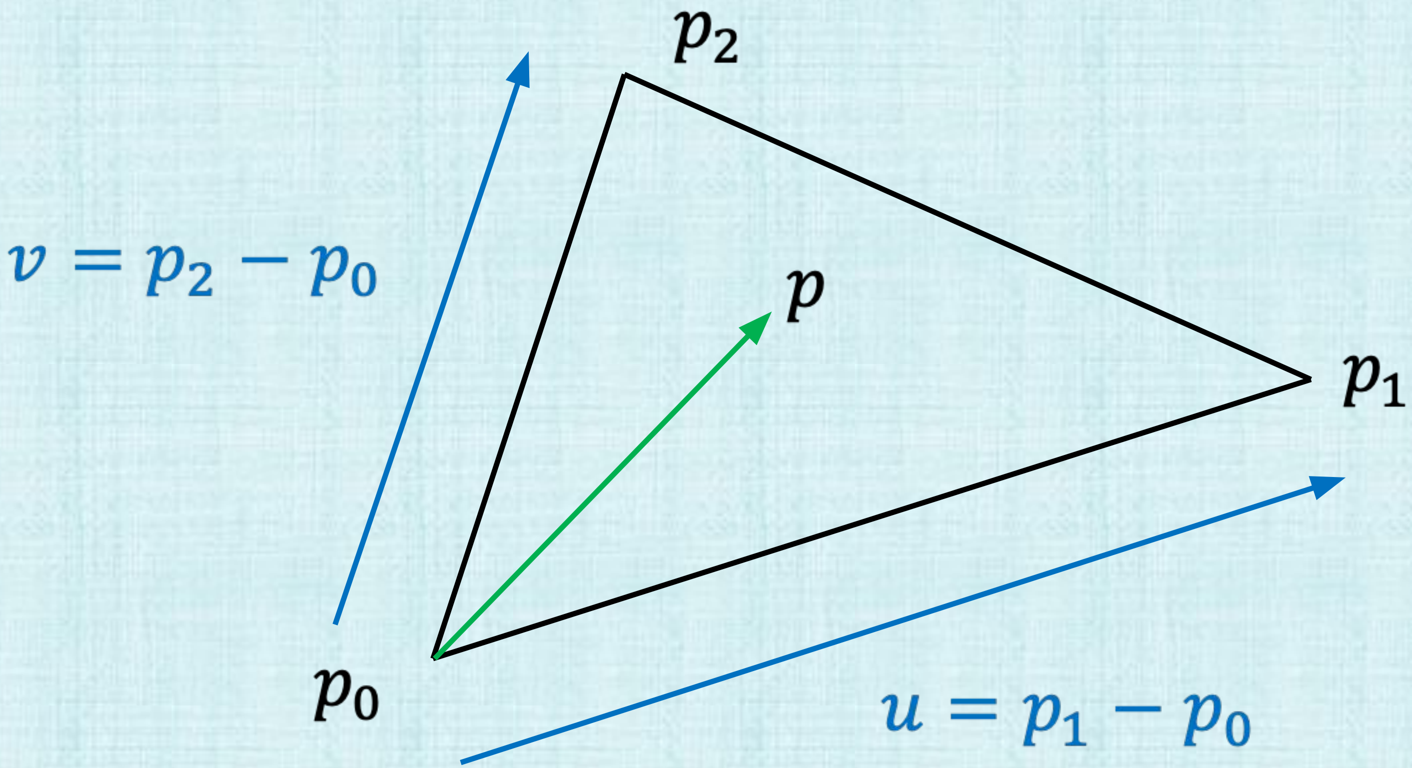


(Alternative) Algebraic Approach

- Rewrite $\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 = p$ as $\alpha_0 \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \alpha_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (1 - \alpha_0 - \alpha_1) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- Assemble into matrix form: $\begin{pmatrix} x_0 - x_2 & x_1 - x_2 \\ y_0 - y_2 & y_1 - y_2 \\ z_0 - z_2 & z_1 - z_2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} x - x_2 \\ y - y_2 \\ z - z_2 \end{pmatrix}$
- In 2D, this is a 2x2 coefficient matrix, but in 3D one has to use the normal equations to reduce to a 2x2 system, i.e. convert $A \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = b$ to $A^T A \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = A^T b$
- The coefficient matrix is rank 1 when the two vectors are colinear, implying infinite solutions for triangles with zero area (one can still embed p on an appropriate edge)
- Otherwise, invert the 2x2 coefficient matrix to solve the system of 2 equations with 2 unknowns (for α_0 and α_1, and set $\alpha_2 = 1 - \alpha_0 - \alpha_1$)

Triangle Basis Vectors

- Compute edge vectors $u = p_1 - p_0$ and $v = p_2 - p_0$
- Any point p interior to the triangle can be written as $p = p_0 + \beta_1 u + \beta_2 v$ with $\beta_1, \beta_2 \in [0,1]$ and $\beta_1 + \beta_2 \leq 1$
- Substitutions and collecting terms gives $p = (1 - \beta_1 - \beta_2)p_0 + \beta_1 p_1 + \beta_2 p_2$ implying the equivalence: $\alpha_0 = 1 - \beta_1 - \beta_2$, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$



Perspective Projection

- Project a world space triangle (vertices p_0, p_1, p_2) into screen space, vertex by vertex, to obtain p_0', p_1', p_2' via $x' = \frac{hx}{z}$ and $y' = \frac{hy}{z}$ for each vertex (x, y, z)
- A point $p = \alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2$ on the world space triangle is projected into screen space to a corresponding point p'
- Notably, $p' \neq \alpha_0 p_0' + \alpha_1 p_1' + \alpha_2 p_2'$ because the perspective projection is highly nonlinear
- The barycentric weights that describe the interior of the triangle in world space do not still hold after projecting the vertices into screen space

- Need a way of computing z' at a pixel from the z' values at the vertices of the screen space triangle
- The z' values are not linear with respect to the triangle vertices in screen space, only in world space (so can't use barycentric interpolation!)
- However, if we knew the location of the pixel on the world space triangle, we could use barycentric interpolation on the world space triangle to compute z and z' for the pixel

Screen Space Barycentric Weights

- Given a pixel at p', find valid screen space barycentric weights so that $p' = \alpha_0' p_0' + \alpha_1' p_1' + (1 - \alpha_0' - \alpha_1') p_2'$
- Define 2D triangle basis vectors (about p_2') as $u' = p_0' - p_2'$ and $v' = p_1' - p_2'$
- Then $p' = \alpha_0' u' + \alpha_1' v' + p_2' = \begin{pmatrix} u_1' & v_1' \\ u_2' & v_2' \end{pmatrix} \begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix} + \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$

- The unknown point $p = \alpha_0 p_0 + \alpha_1 p_1 + (1 - \alpha_0 - \alpha_1) p_2 = \alpha_0 (p_0 - p_2) + \alpha_1 (p_1 - p_2) + p_2$ that projects to p' has unknown barycentric weights that need to be determined (once α_0 and α_1 are known, p is then known)
- The coordinates of p obey $x = \alpha_0 (x_0 - x_2) + \alpha_1 (x_1 - x_2) + x_2$, $y = \alpha_0 (y_0 - y_2) + \alpha_1 (y_1 - y_2) + y_2$, and $z = \alpha_0 (z_0 - z_2) + \alpha_1 (z_1 - z_2) + z_2$
- Thus, $p' = \begin{pmatrix} \frac{hx}{z} \\ \frac{hy}{z} \end{pmatrix} = \begin{pmatrix} h \frac{\alpha_0 (x_0 - x_2) + \alpha_1 (x_1 - x_2) + x_2}{\alpha_0 (z_0 - z_2) + \alpha_1 (z_1 - z_2) + z_2} \\ h \frac{\alpha_0 (y_0 - y_2) + \alpha_1 (y_1 - y_2) + y_2}{\alpha_0 (z_0 - z_2) + \alpha_1 (z_1 - z_2) + z_2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_0 (z_0 x_0' - z_2 x_2') + \alpha_1 (z_1 x_1' - z_2 x_2') + z_2 x_2'}{\alpha_0 (z_0 - z_2) + \alpha_1 (z_1 - z_2) + z_2} \\ \frac{\alpha_0 (z_0 y_0' - z_2 y_2') + \alpha_1 (z_1 y_1' - z_2 y_2') + z_2 y_2'}{\alpha_0 (z_0 - z_2) + \alpha_1 (z_1 - z_2) + z_2} \end{pmatrix}$

- Or $p' = \frac{1}{\alpha_0 (z_0 - z_2) + \alpha_1 (z_1 - z_2) + z_2} \left[\begin{pmatrix} z_0 x_o' - z_2 x_2' & z_1 x_1' - z_2 x_2' \\ z_0 y_0' - z_2 y_2' & z_1 y_1' - z_2 y_2' \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} z_2 x_2' \\ z_2 y_2' \end{pmatrix} \right]$

Screen Space Barycentric Weights

- These two definitions of p' can be equated to obtain:

$$\frac{1}{\alpha_0(z_0 - z_2) + \alpha_1(z_1 - z_2) + z_2}\left[\begin{pmatrix} z_0x_0' - z_2x_2' & z_1x_1' - z_2x_2' \\ z_0y_0' - z_2y_2' & z_1y_1' - z_2y_2' \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} z_2x_2' \\ z_2y_2' \end{pmatrix}\right] = \begin{pmatrix} u_1' & v_1' \\ u_2' & v_2' \end{pmatrix}\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix} + \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$$

- Bringing $\begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$ to the left hand side, and under the brackets as $-(\alpha_0(z_0 - z_2) + \alpha_1(z_1 - z_2) + z_2)\begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$ or equivalently $\begin{pmatrix} z_2x_2' - z_0x_2' & z_2x_2' - z_1x_2' \\ z_2y_2' - z_0y_2' & z_2y_2' - z_1y_2' \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} - \begin{pmatrix} z_2x_2' \\ z_2y_2' \end{pmatrix}$ leads to:

$$\frac{1}{\alpha_0(z_0 - z_2) + \alpha_1(z_1 - z_2) + z_2}\begin{pmatrix} z_0x_0' - z_0x_2' & z_1x_1' - z_1x_2' \\ z_0y_0' - z_0y_2' & z_1y_1' - z_1y_2' \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} u_1' & v_1' \\ u_2' & v_2' \end{pmatrix}\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$$

$$\frac{1}{\alpha_0(z_0 - z_2) + \alpha_1(z_1 - z_2) + z_2}\begin{pmatrix} u_1' & v_1' \\ u_2' & v_2' \end{pmatrix}\begin{pmatrix} z_0\alpha_0 \\ z_1\alpha_1 \end{pmatrix} = \begin{pmatrix} u_1' & v_1' \\ u_2' & v_2' \end{pmatrix}\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$$

$$\frac{1}{\alpha_0(z_0 - z_2) + \alpha_1(z_1 - z_2) + z_2}\begin{pmatrix} z_0\alpha_0 \\ z_1\alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$$

- Importantly, all the terms related to x and y coordinates vanished, leaving dependence only on the z coordinates

Screen Space Barycentric Weights

- Starting from $\frac{1}{\alpha_0(z_0-z_2)+\alpha_1(z_1-z_2)+z_2}\begin{pmatrix} z_0\alpha_0 \\ z_1\alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$ or $\begin{pmatrix} z_0\alpha_0 \\ z_1\alpha_1 \end{pmatrix} = (\alpha_0(z_0 - z_2) + \alpha_1(z_1 - z_2) + z_2)\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$
- Rewrite to $\begin{pmatrix} z_0 + (z_2 - z_0)\alpha_0' & (z_2 - z_1)\alpha_0' \\ (z_2 - z_0)\alpha_1' & z_1 + (z_2 - z_1)\alpha_1' \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = z_2\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix}$
- The determinant of this 2x2 matrix is $z_0z_1 + z_1(z_2 - z_0)\alpha_0' + z_0(z_2 - z_1)\alpha_1'$
- Thus the inverse is $\frac{1}{z_0z_1+z_1(z_2-z_0)\alpha_0'+z_0(z_2-z_1)\alpha_1'}\begin{pmatrix} z_1 + (z_2 - z_1)\alpha_1' & (z_1 - z_2)\alpha_0' \\ (z_0 - z_2)\alpha_1' & z_0 + (z_2 - z_0)\alpha_0' \end{pmatrix}$
- Note that $\begin{pmatrix} z_1 + (z_2 - z_1)\alpha_1' & (z_1 - z_2)\alpha_0' \\ (z_0 - z_2)\alpha_1' & z_0 + (z_2 - z_0)\alpha_0' \end{pmatrix}\begin{pmatrix} \alpha_0' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} z_1\alpha_0' \\ z_0\alpha_1' \end{pmatrix}$
- Thus, $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \frac{z_2}{z_0z_1+z_1(z_2-z_0)\alpha_0'+z_0(z_2-z_1)\alpha_1'}\begin{pmatrix} z_1\alpha_0' \\ z_0\alpha_1' \end{pmatrix}$

- So, given barycentric coordinates of the pixel, α_0' and α_1', we can compute:

$$\alpha_0 = \frac{z_1z_2\alpha_0'}{z_0z_1+z_1(z_2-z_0)\alpha_0'+z_0(z_2-z_1)\alpha_1'} \quad \text{and} \quad \alpha_1 = \frac{z_0z_2\alpha_1'}{z_0z_1+z_1(z_2-z_0)\alpha_0'+z_0(z_2-z_1)\alpha_1'}$$

- Then α_0 and α_1 (and α_2) can be used to find the (unknown) corresponding point p on the world space triangle
- We use α_0 and α_1 to compute z (as well as $z' = n + f - \frac{fn}{z}$) for the pixel (not α_0' and α_1')

Ray Tracing

- Ray Tracing works very differently than the Scanline Rendering just discussed
- The ray tracer creates a ray going through the pixel in question, and subsequently intersects that ray with triangles in world space
- Since the ray tracer intrinsically operates in world space, as opposed to screen space, it need not worry about dealing with screen space barycentric coordinates
- Operating in world space is a huge advantage for the ray tracer when it comes to image quality, as it can thoroughly look around in world space to figure out what's going on

- A scanline renderer operates in screen space and as such has much more limited information
- On the other hand, the limited capabilities of a scanline renderer make it a fantastic candidate for real time implementation on hardware

- Only recently have hardware implementations of some aspects of ray tracing become more feasible!

Lighting and Shading

- After identifying that a pixel is inside a triangle, as discussed above, we set its color to the color of the triangle
- This ignores all the nuances of how light works (and we'll discuss that more next week)
- If you rendered a sphere based on this simplistic approach, it would look like this: