Fundamental Graph Algorithms

Part II

Outline for Today

- **Dijkstra's Algorithm**
  - An algorithm for finding shortest paths in more realistic settings
- **Depth-First Search**
  - A different graph search algorithm.
- **Directed Acyclic Graphs**
  - Graphs for representing prerequisites.
- **(ITA) Topological Sorting**
  - Algorithms for ordering dependencies.

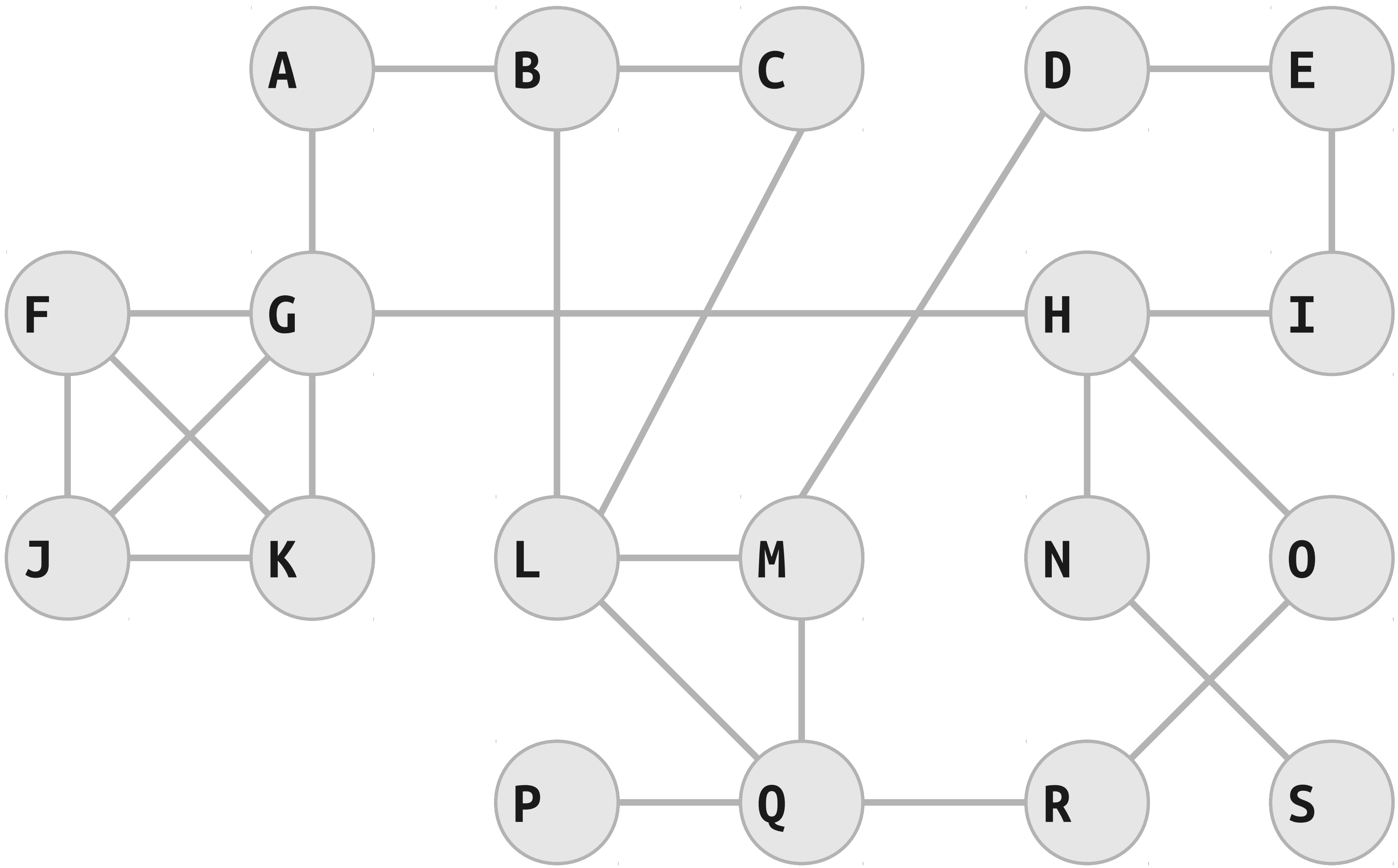
Recap from Last Time

Breadth-First Search

- Given an arbitrary graph $G = (V, E)$ and a starting node $s \in V$, **breadth-first search** finds shortest paths from s to each reachable node v.
- When implemented using an adjacency list, runs in $O(m + n)$ time, which we defined to be linear time on a graph.
- One correctness proof worked in terms of "layers:" the algorithm finds all nodes at distance 0, 1, 2, … in order.

A Second Intuition for BFS

Breadth-First Search



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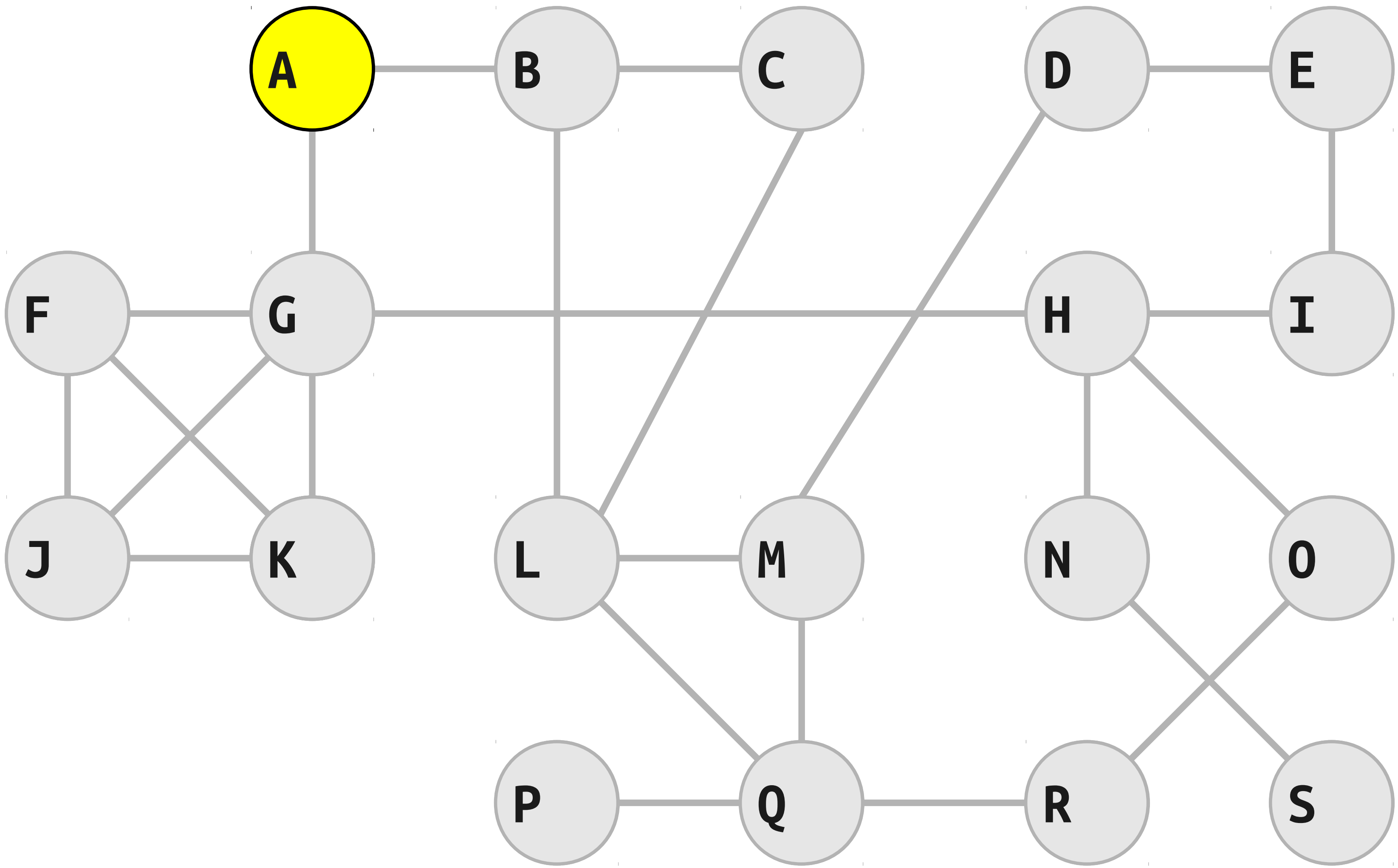

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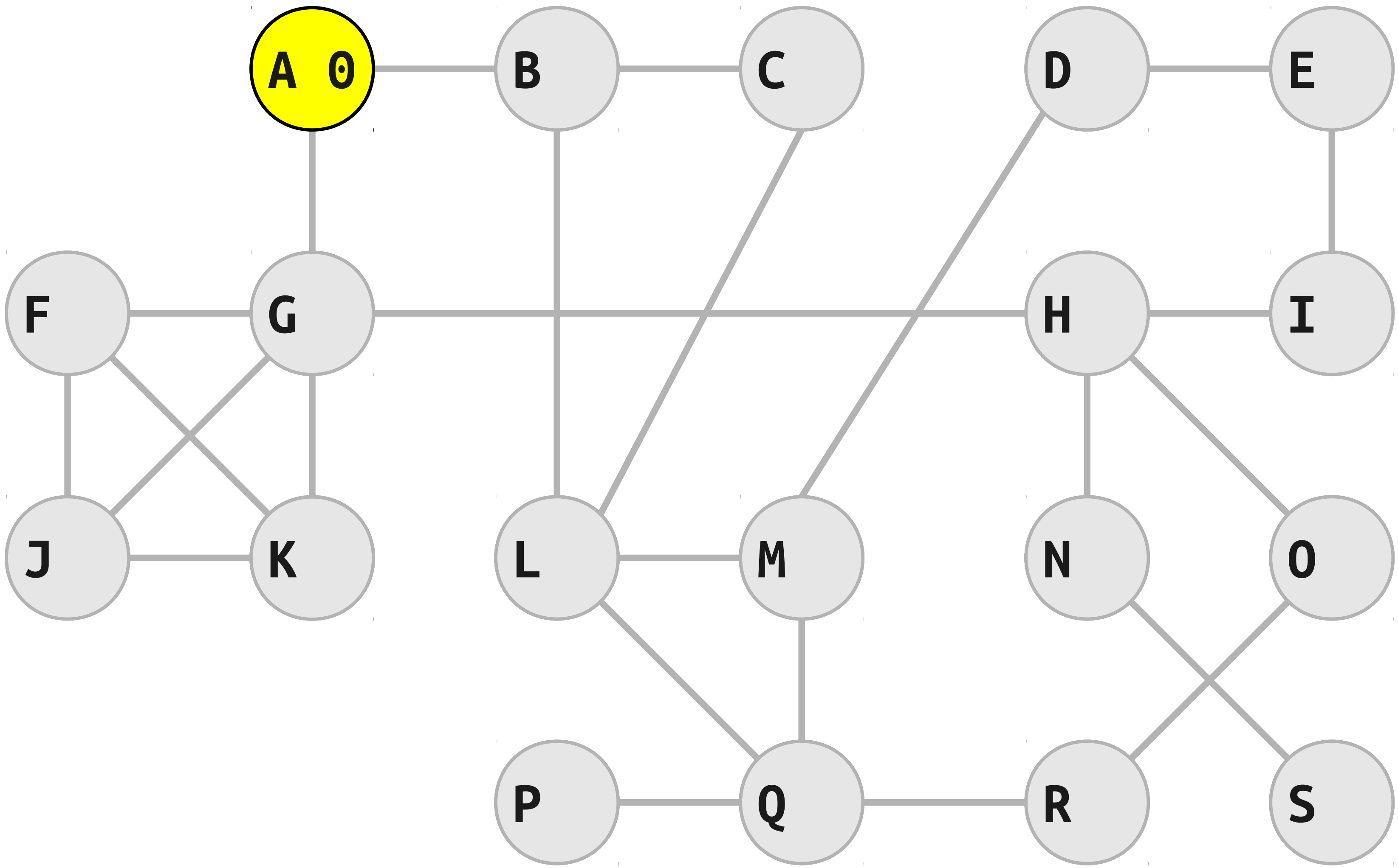

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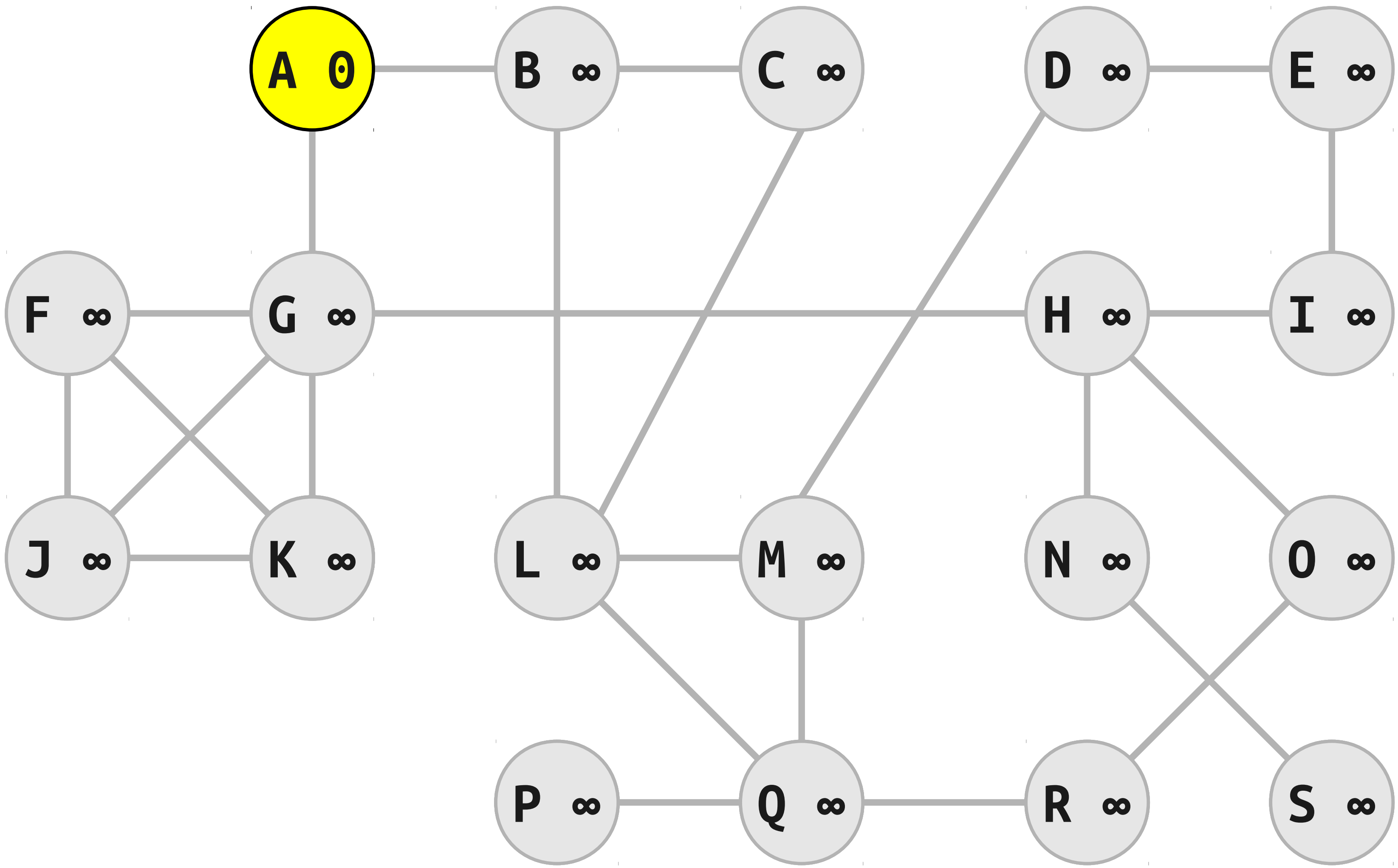

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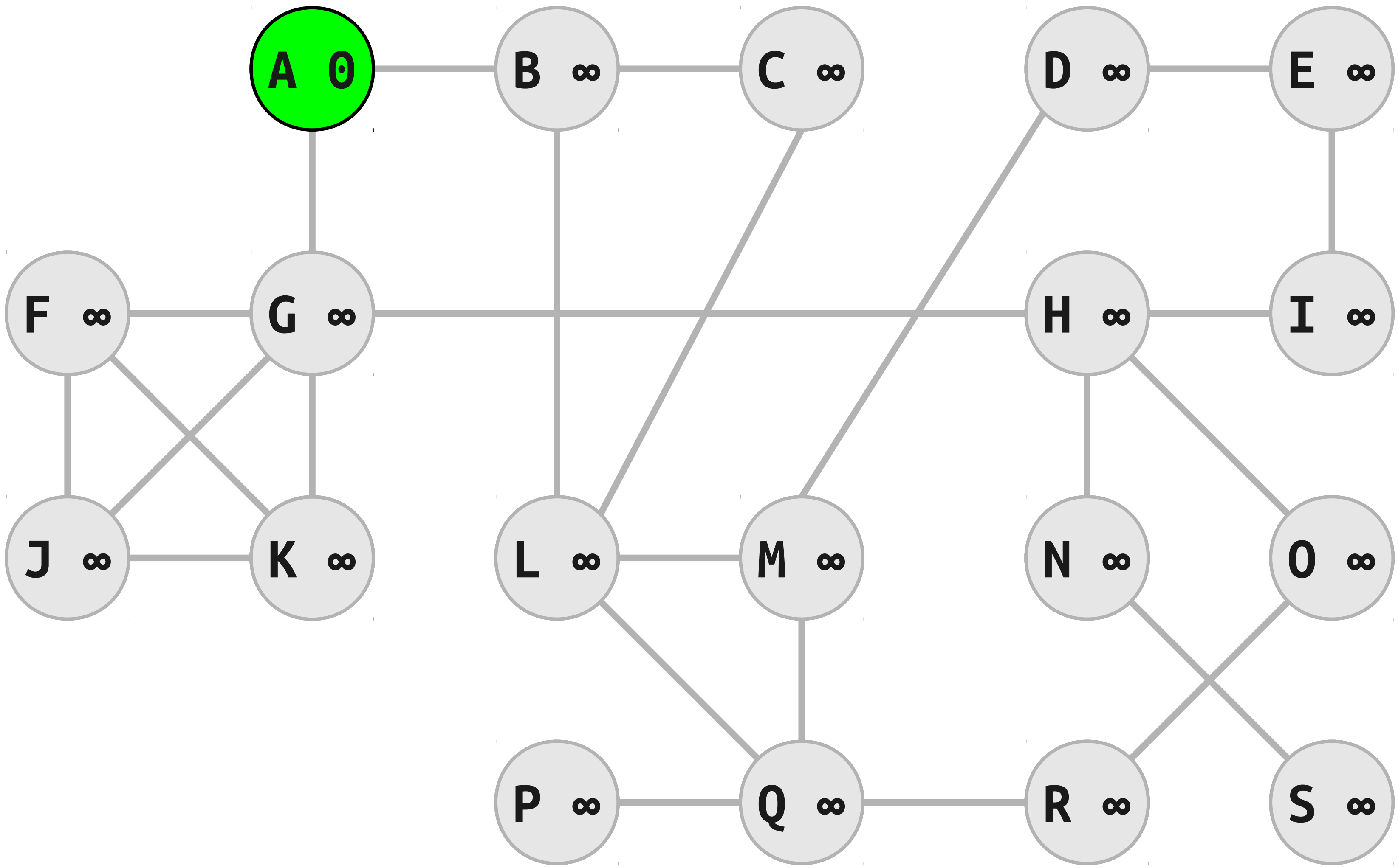

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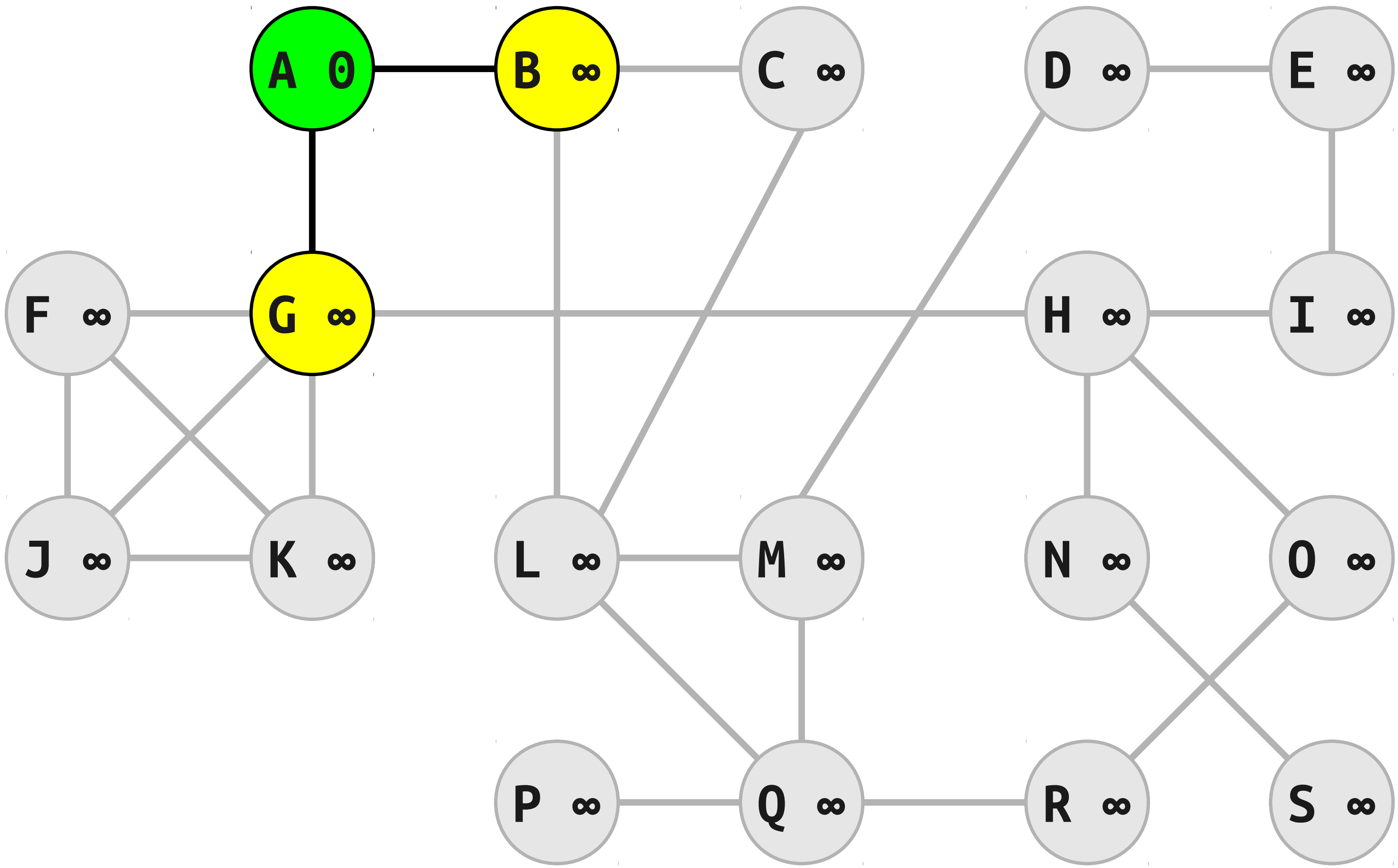

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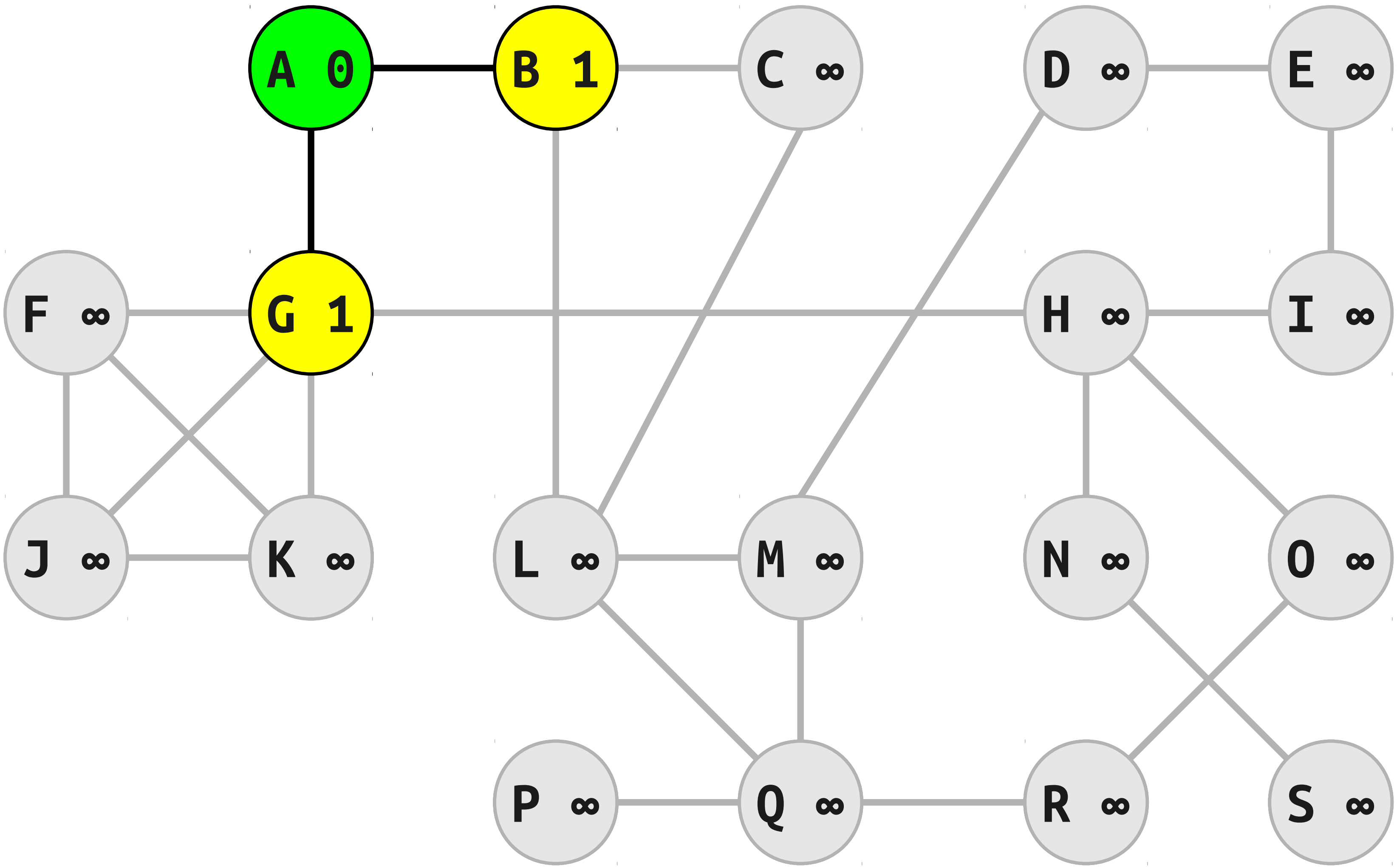

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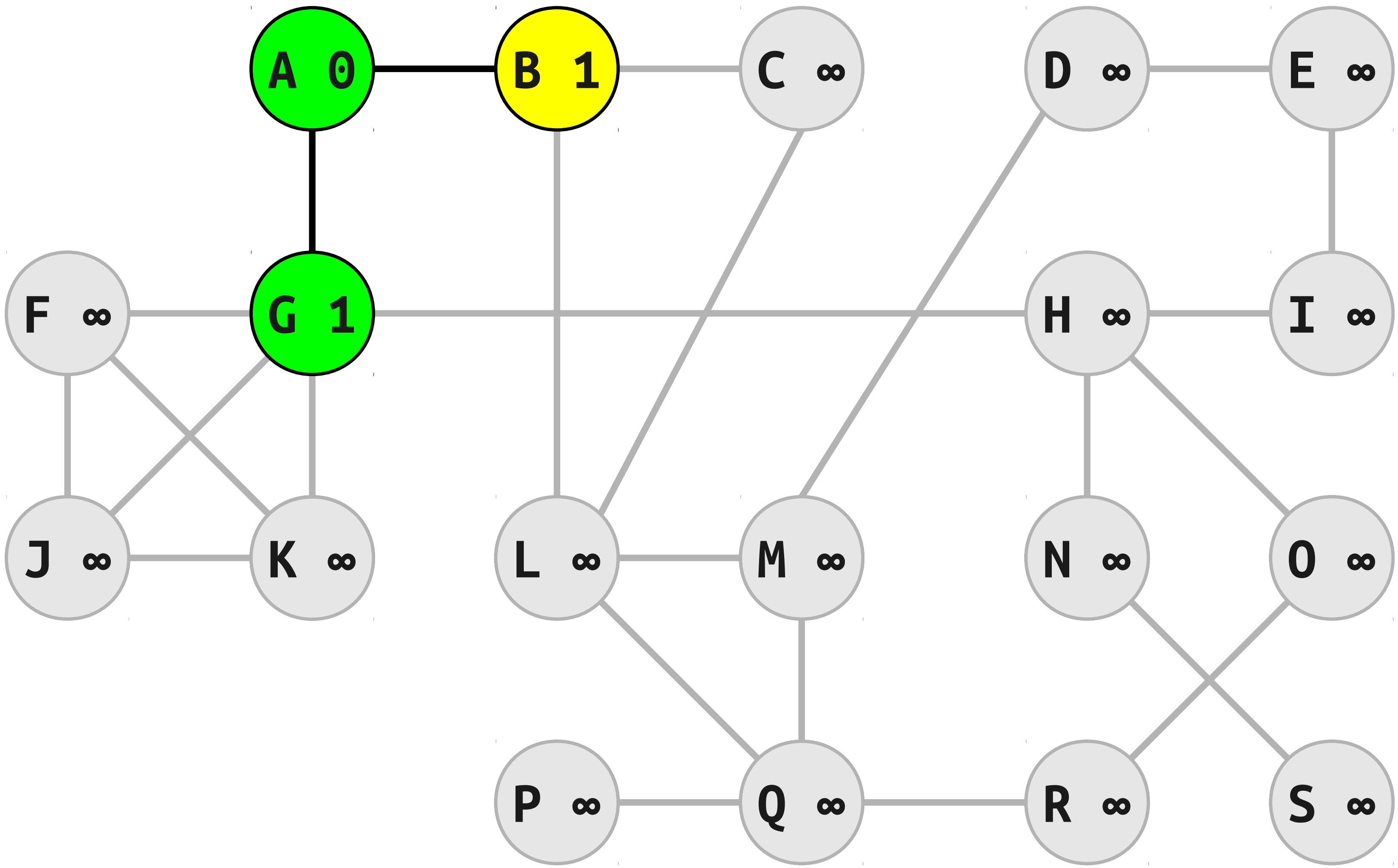

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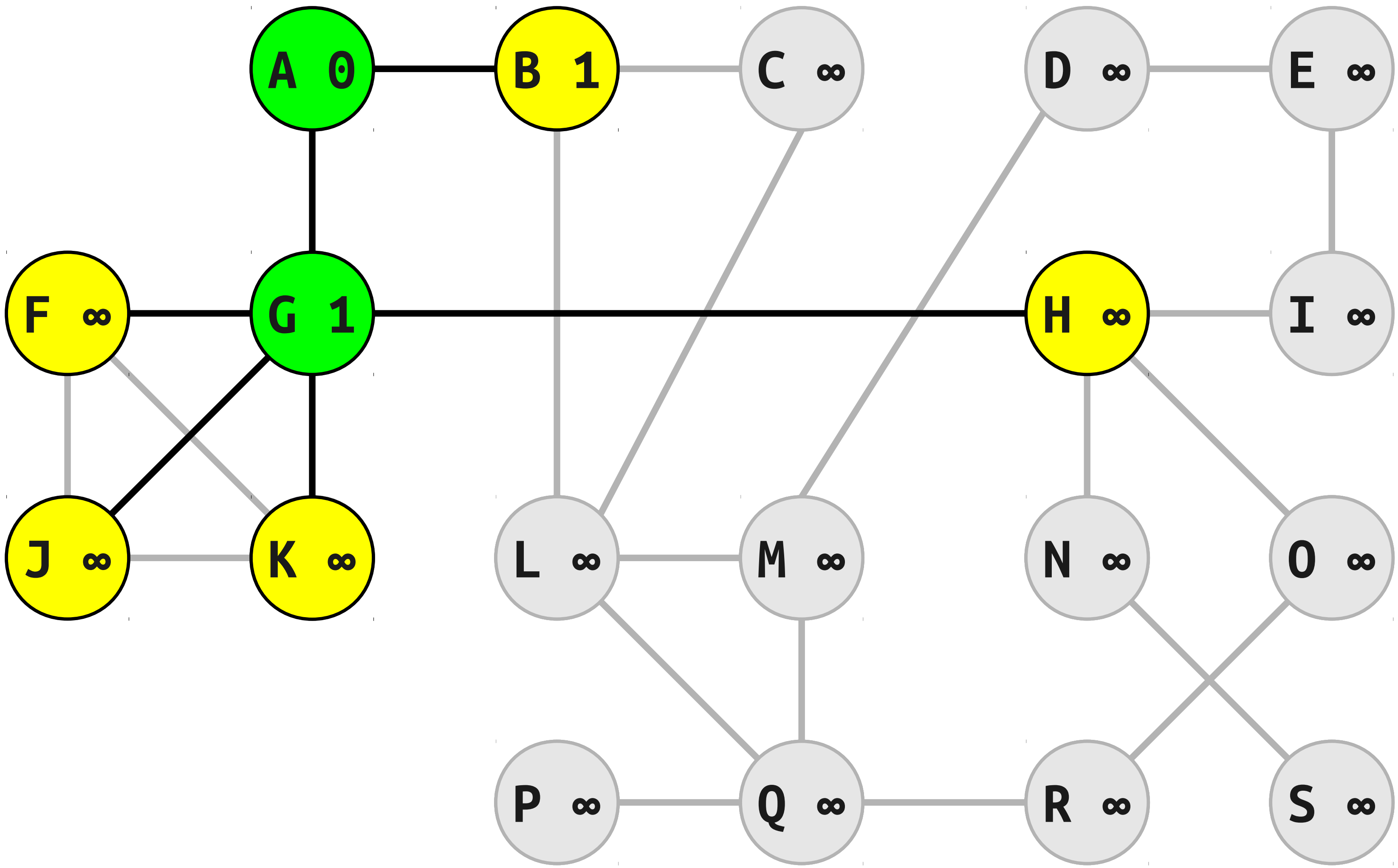

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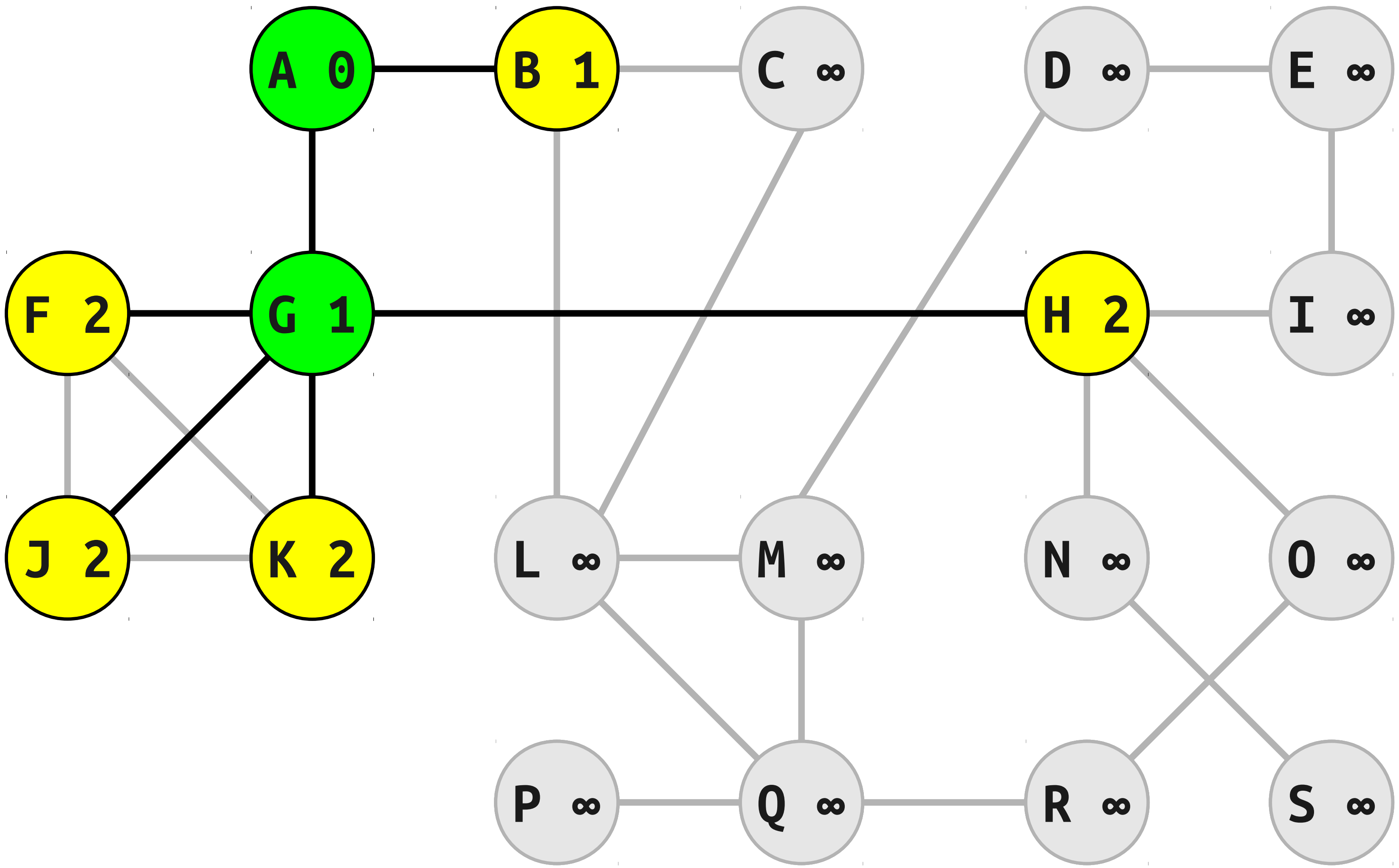

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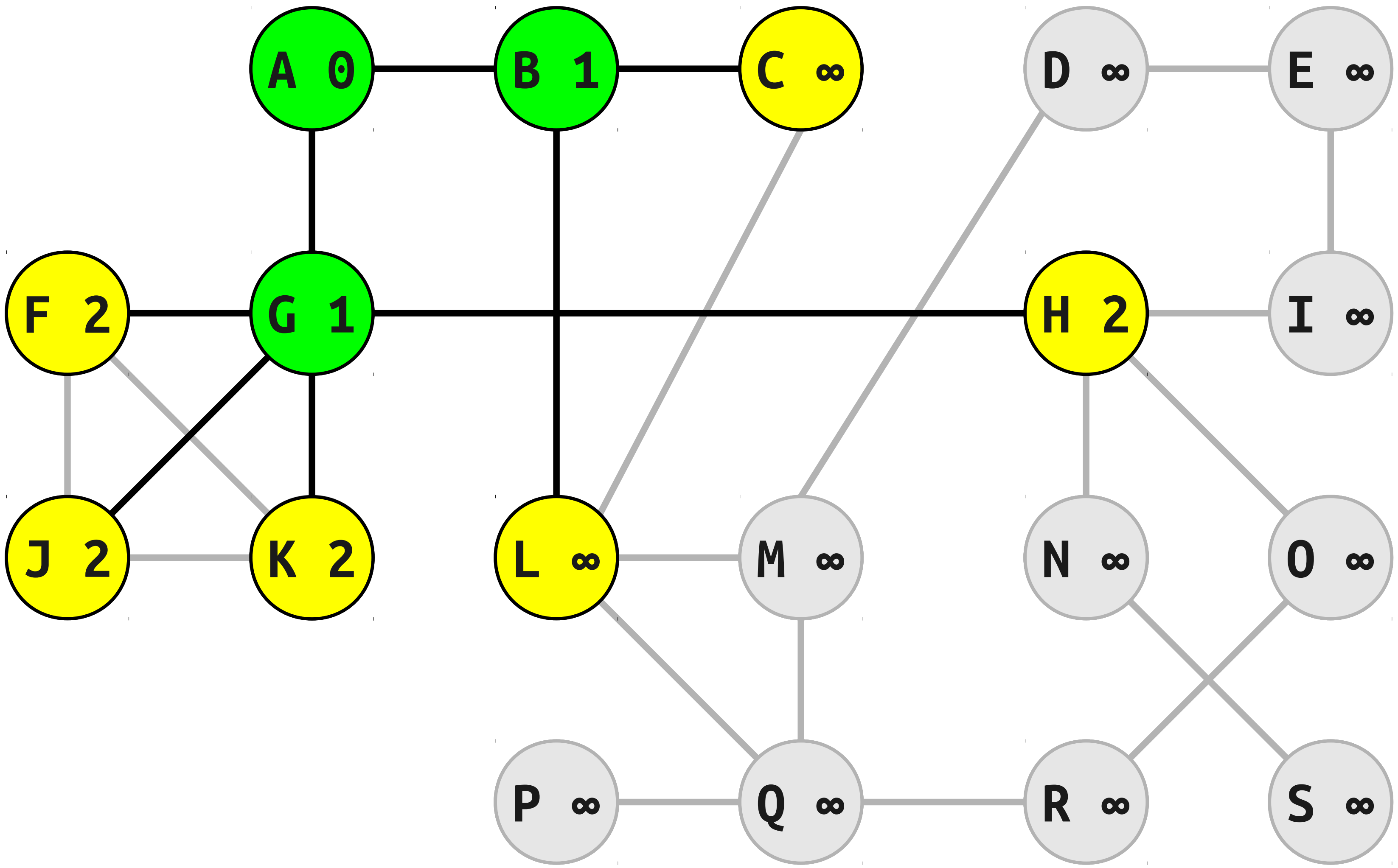

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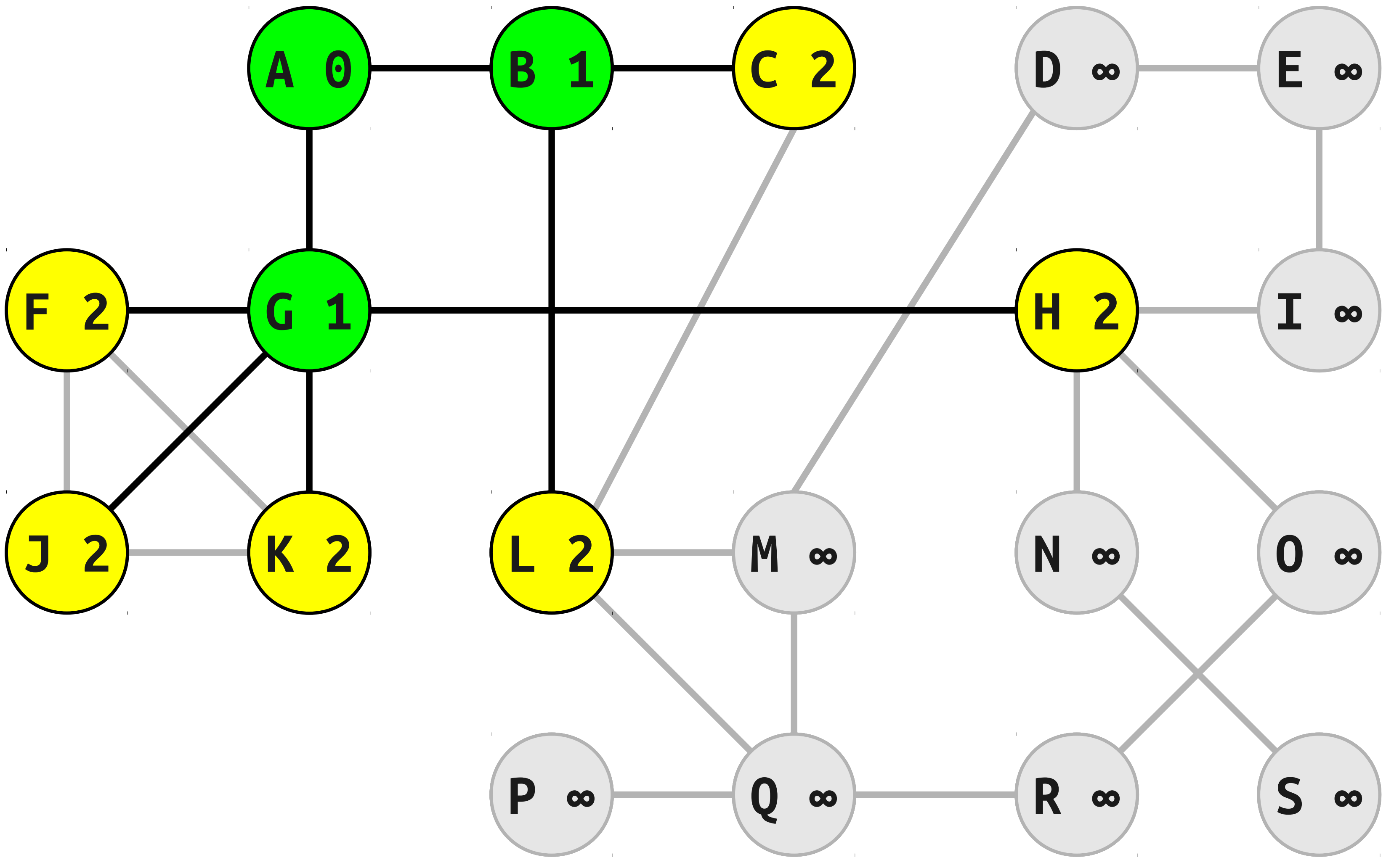

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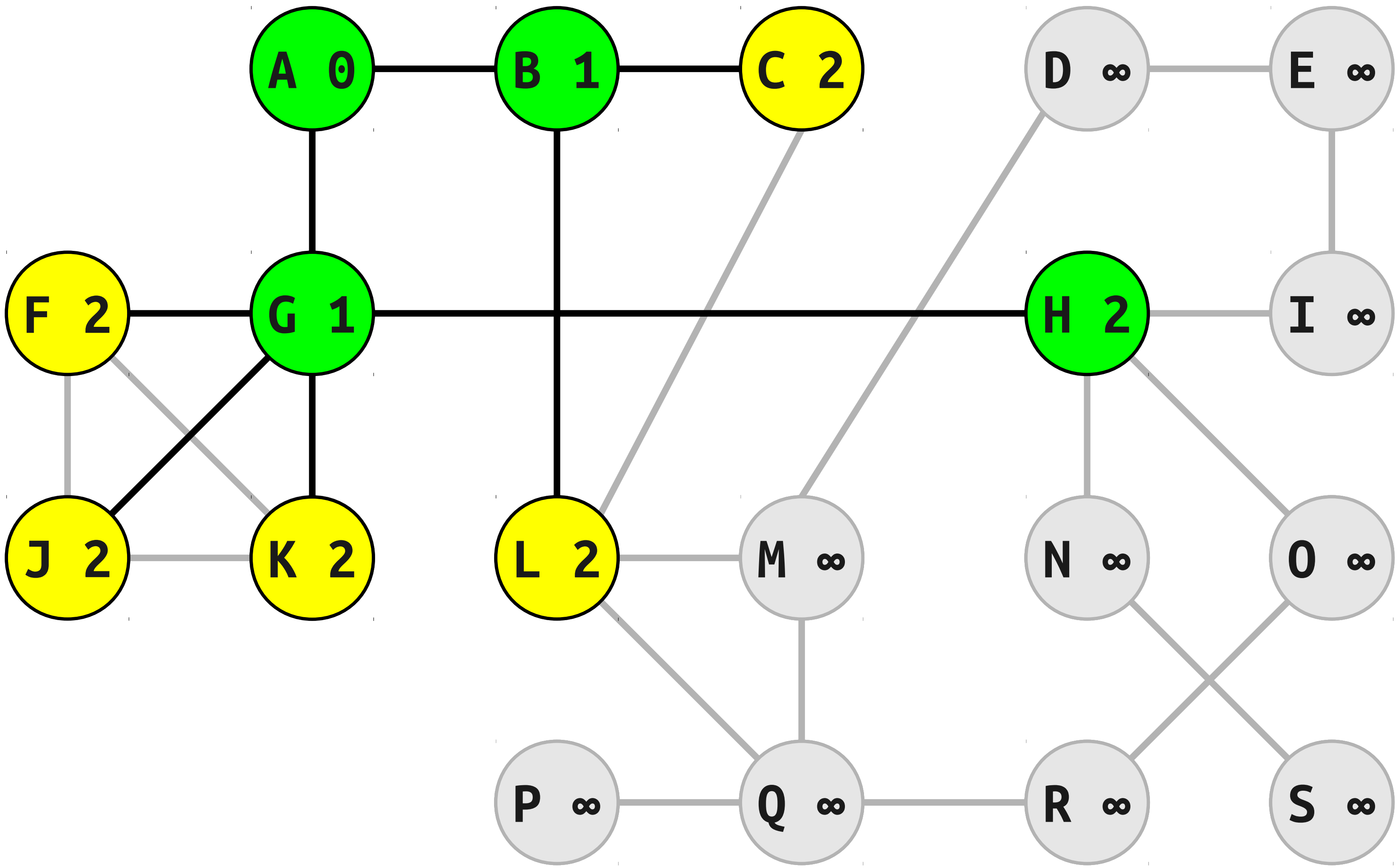

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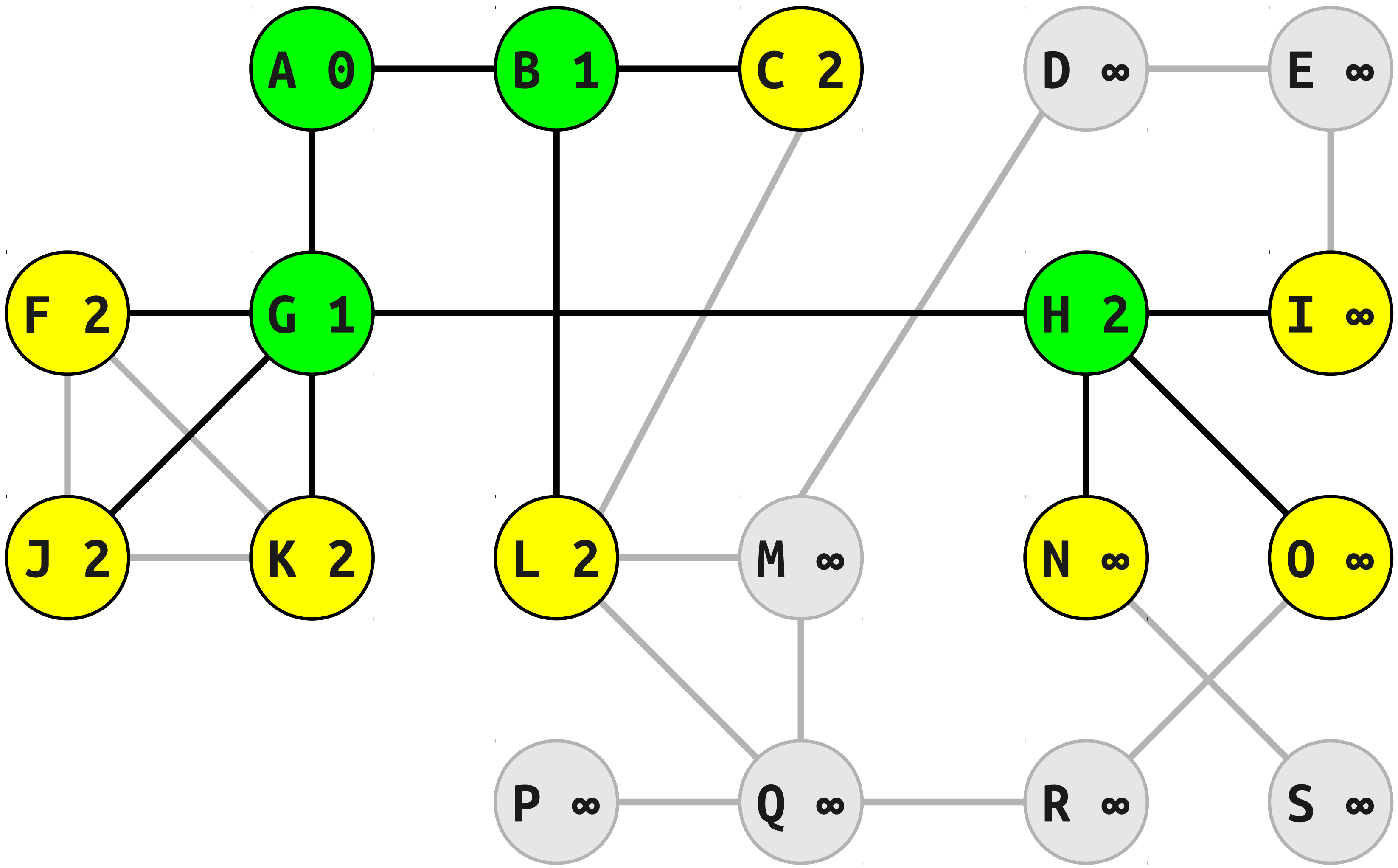

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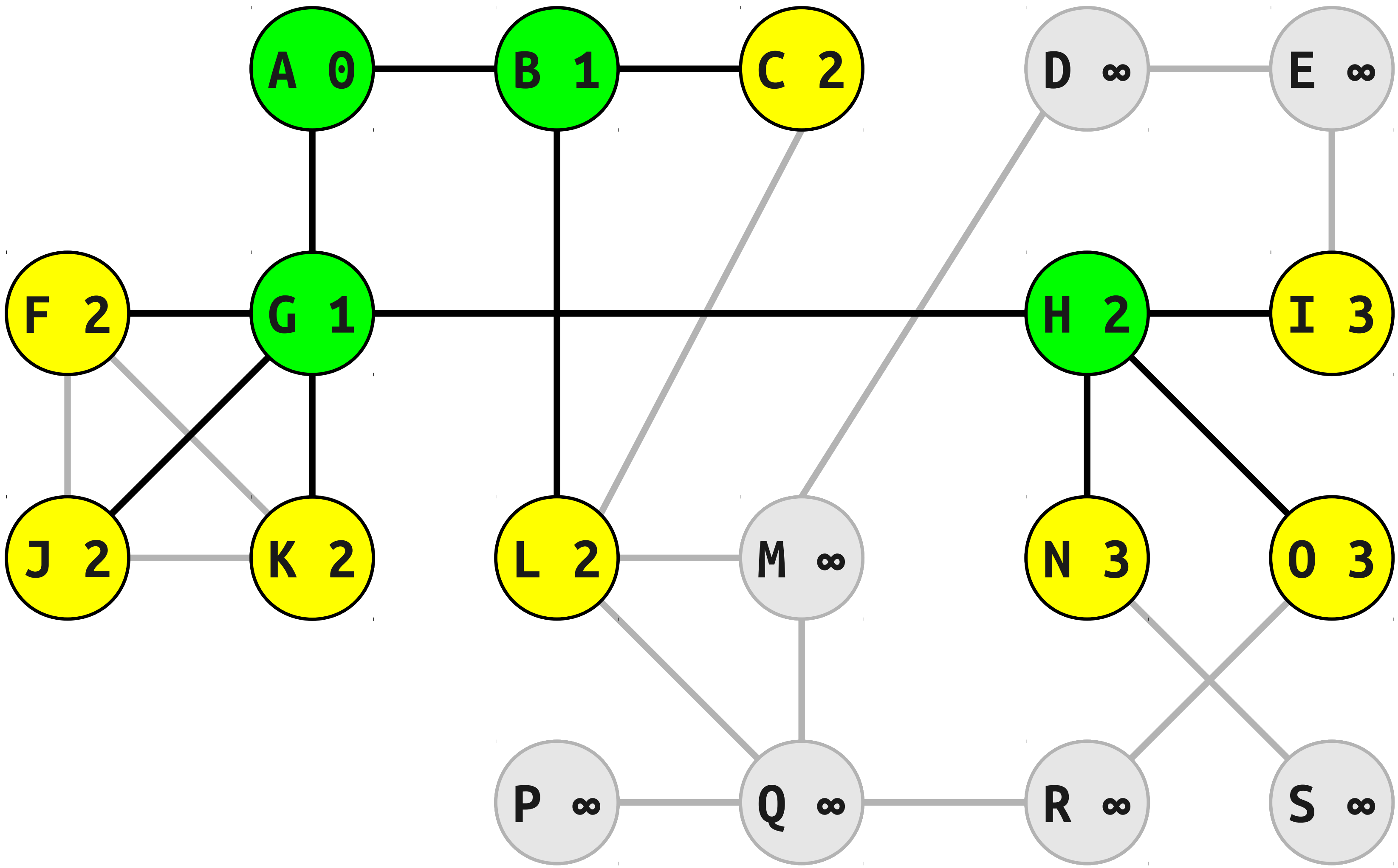


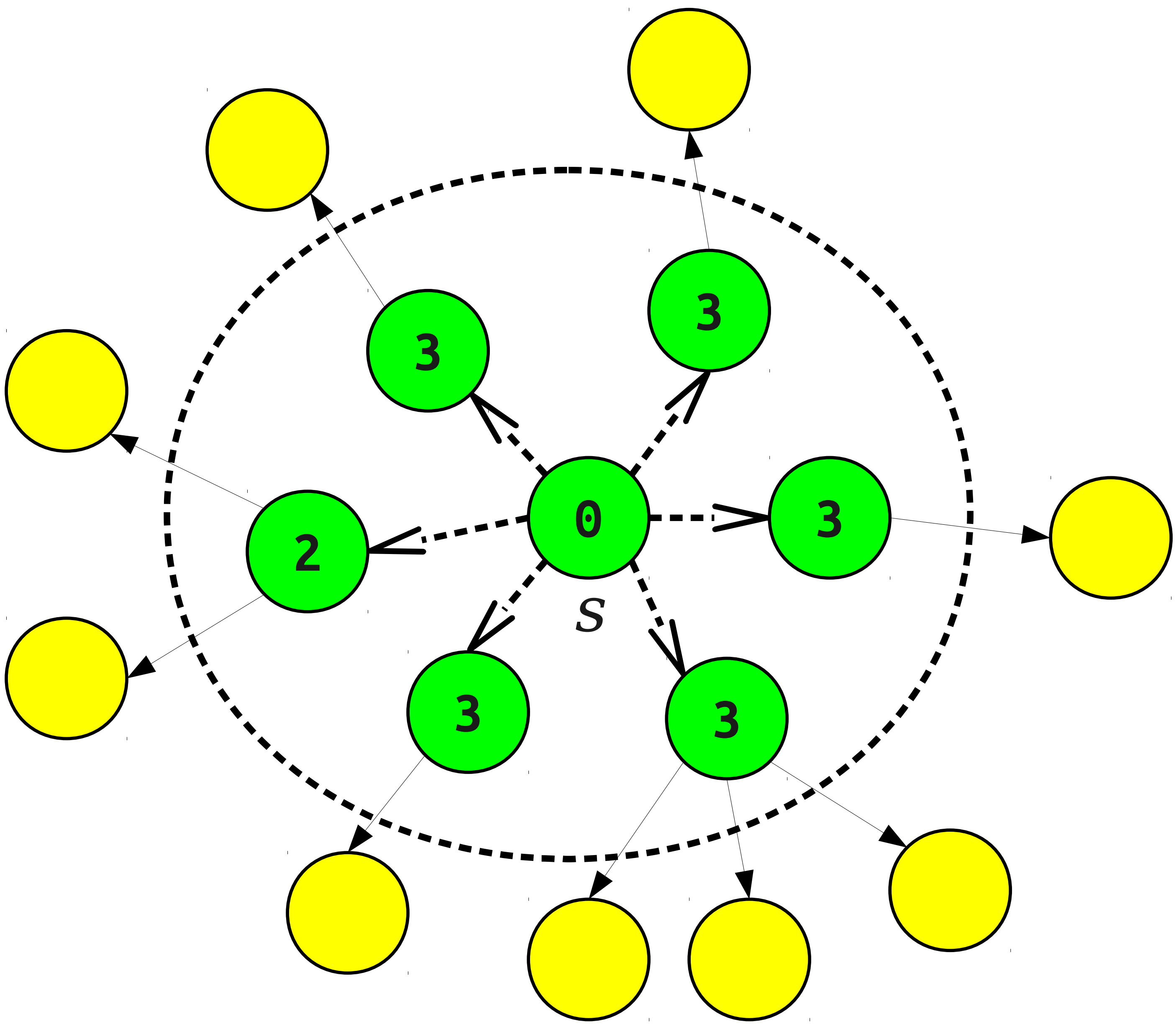
A Second Intuition

- At each point in the execution of BFS, a node v is either
  - green, and we have the shortest path to v;
  - yellow, and it is connected to some green node; or
  - gray, and v is undiscovered.
- Each iteration, we pick a yellow node with minimal distance from the start node and color it green. So what is the cost of the lowest-cost yellow node?
- If v is yellow, it is connected to a green node u by an edge.
- The cost of getting from s to v is then $\mathrm{d}(s, u) + 1$.
- BFS works by picking the yellow node v minimizing

$$\mathbf{d}(\boldsymbol{s}, \boldsymbol{u}) + \mathbf{1}$$

  where (u, v) is an edge and u is green.

Pick yellow node v minimizing d(s, u) + 1, where (u, v) is an edge and u is green.



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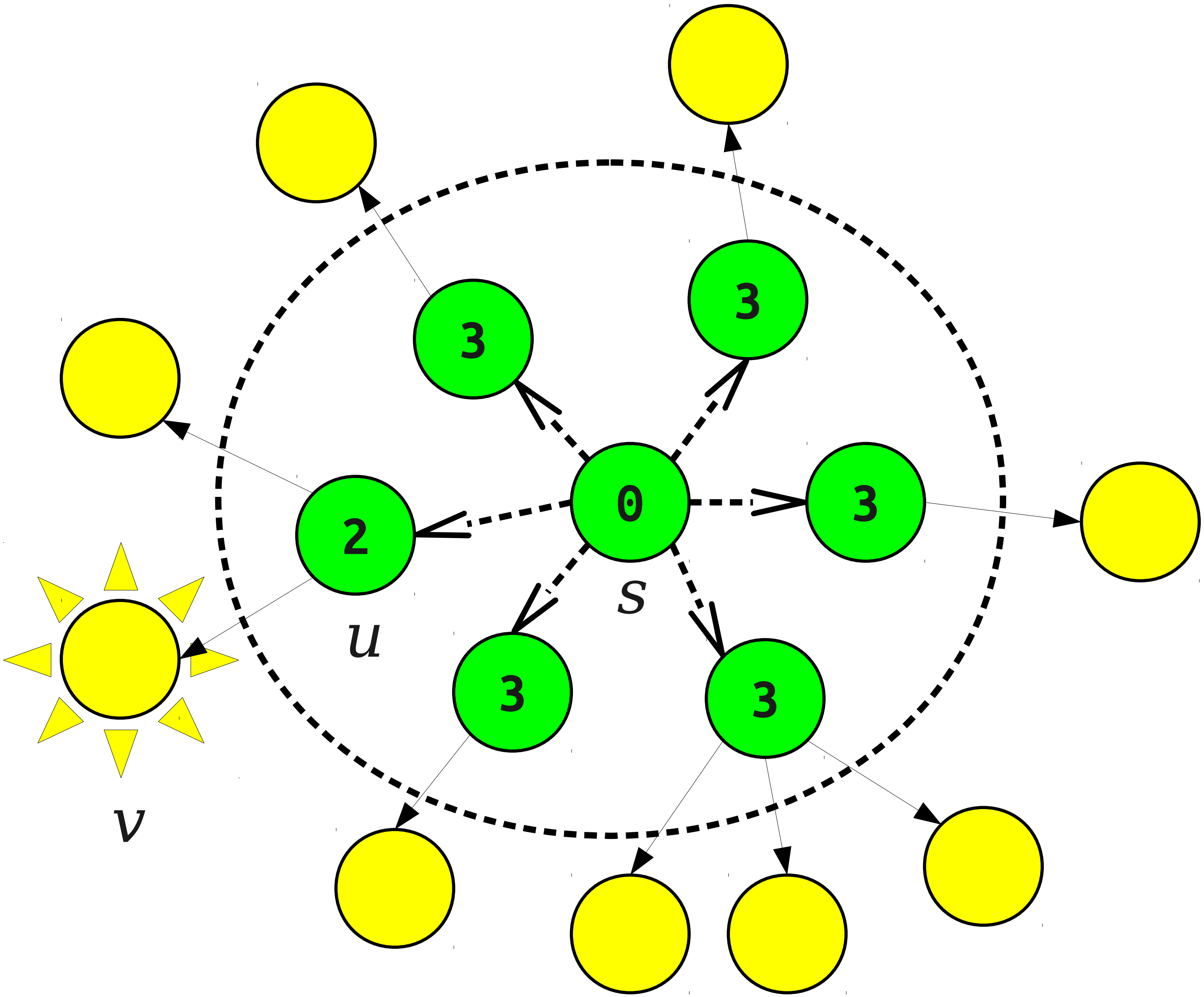

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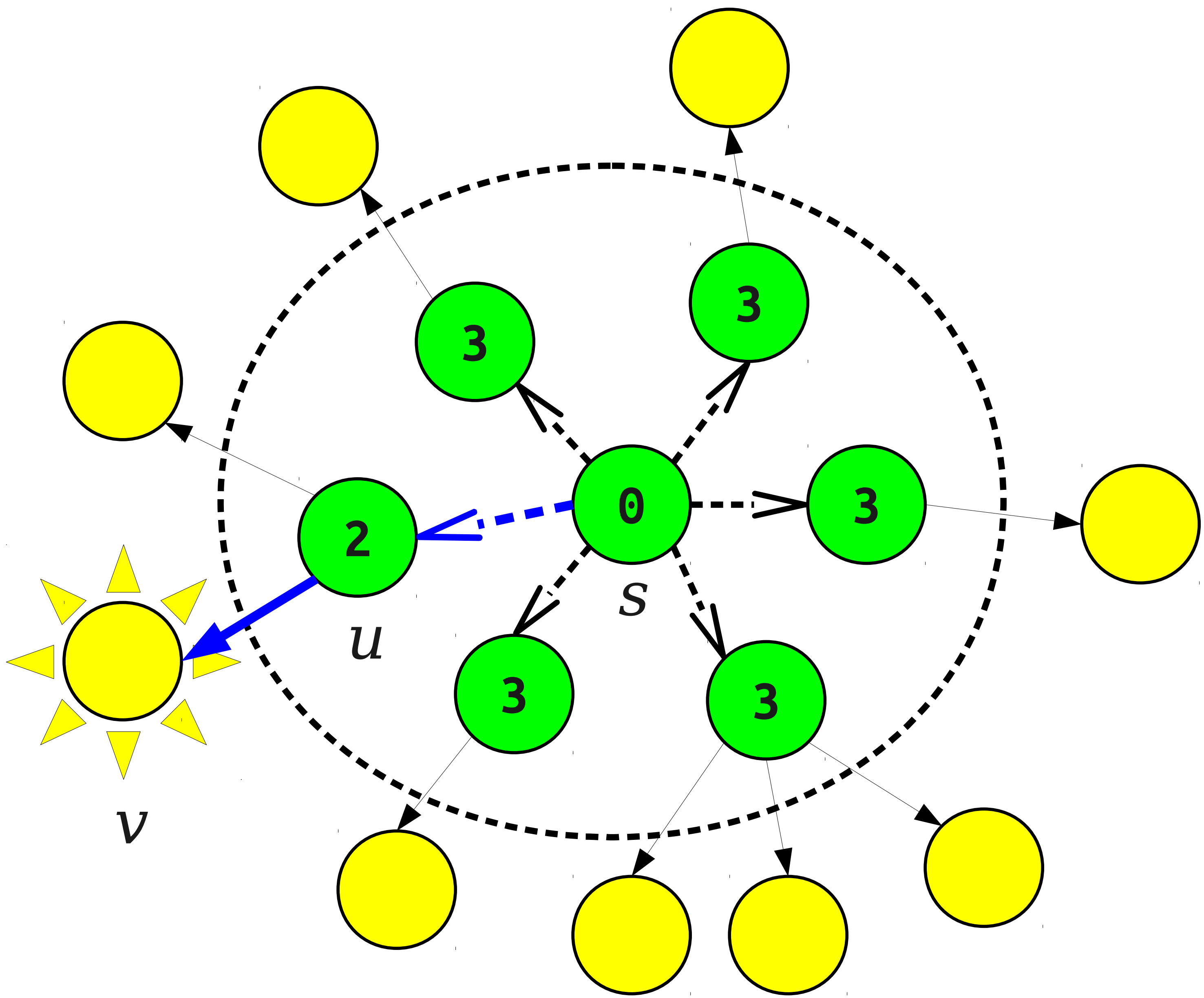

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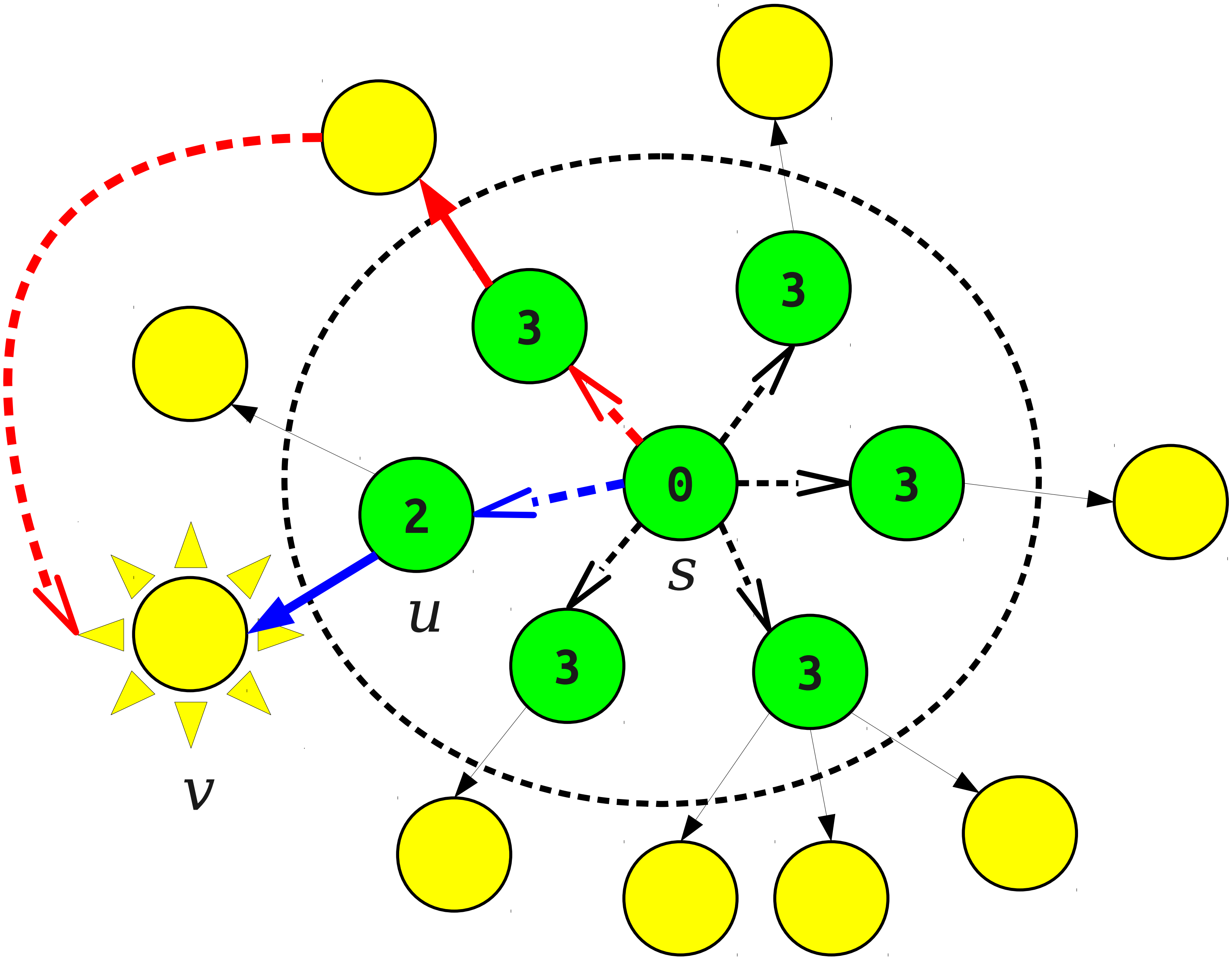

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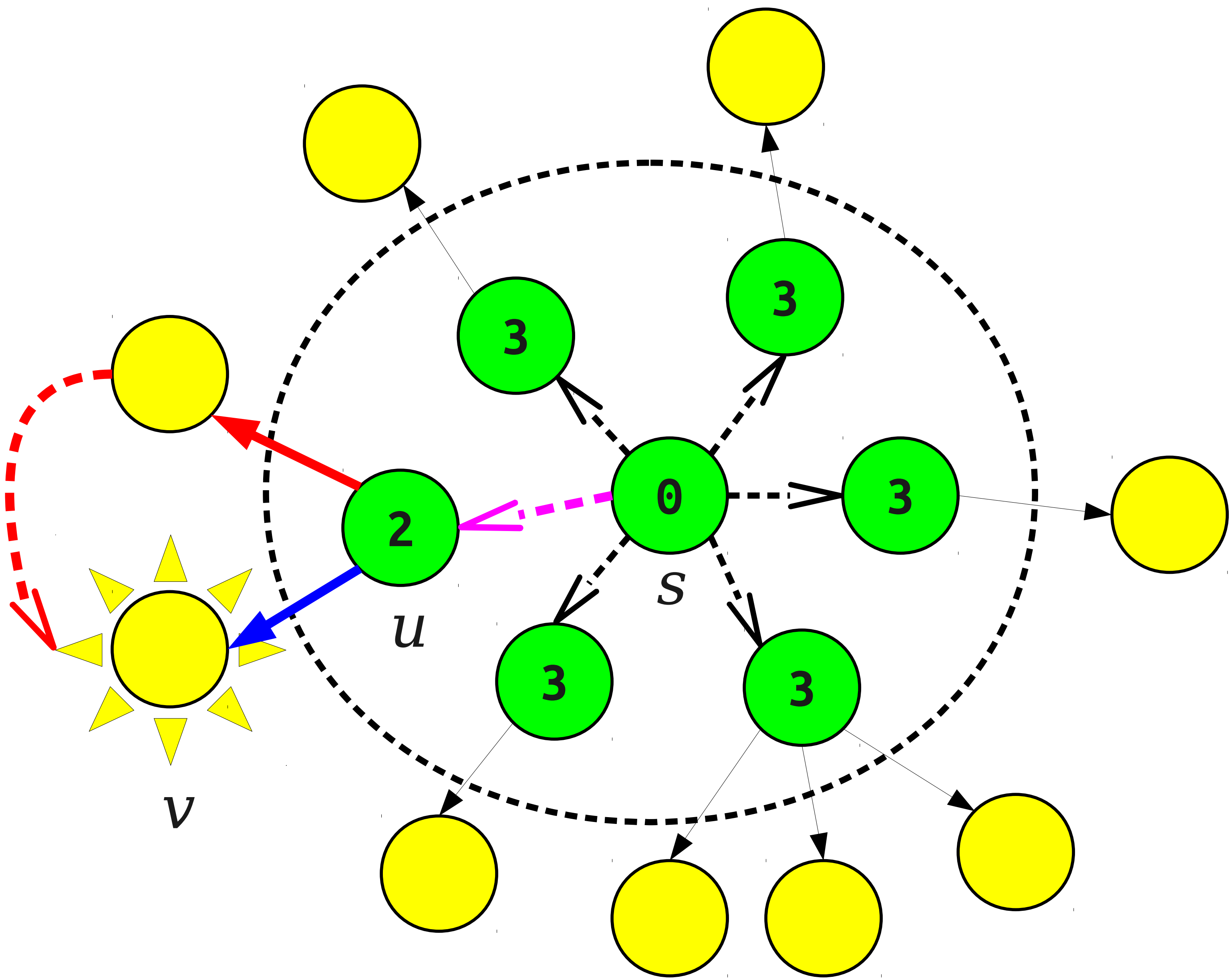

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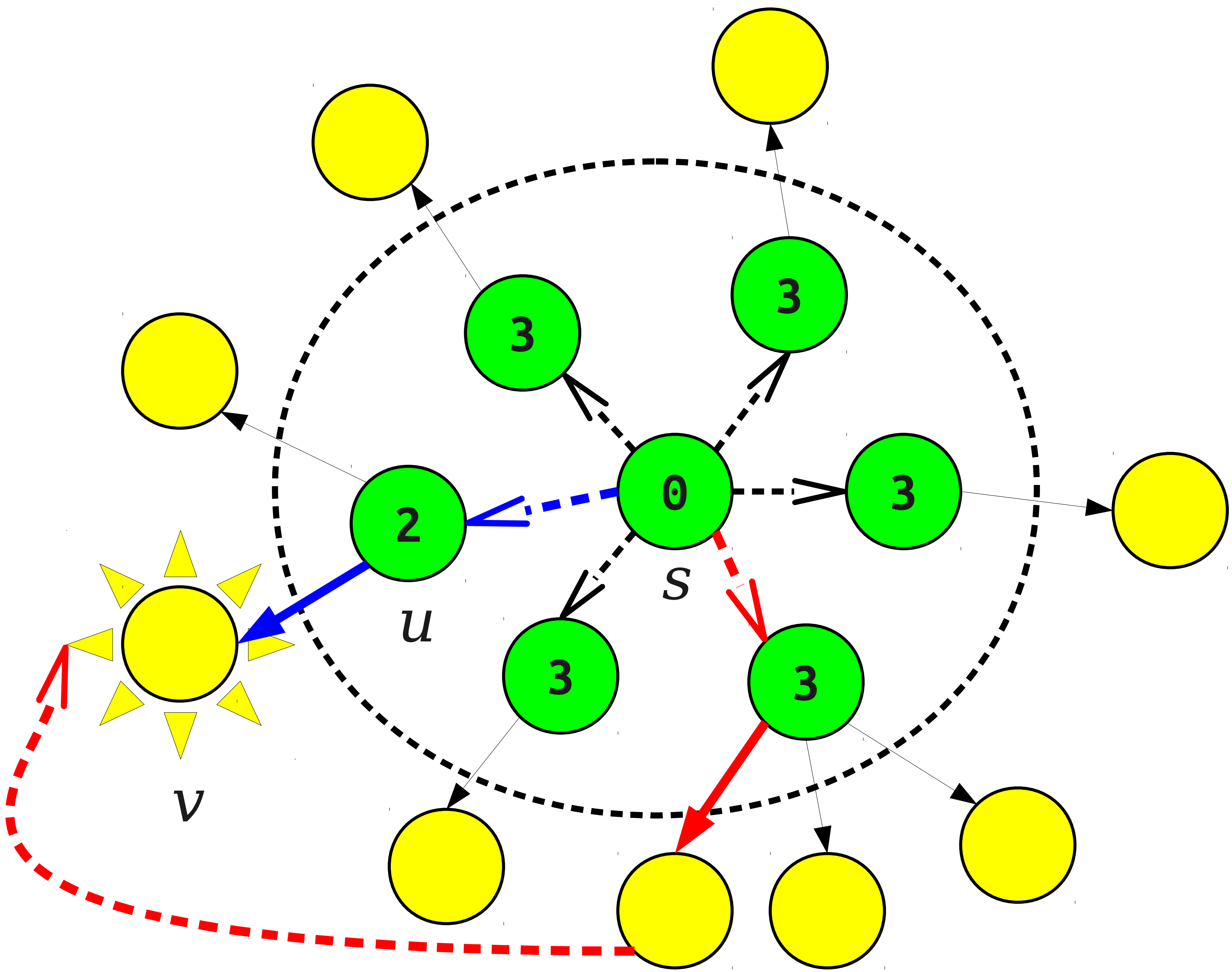
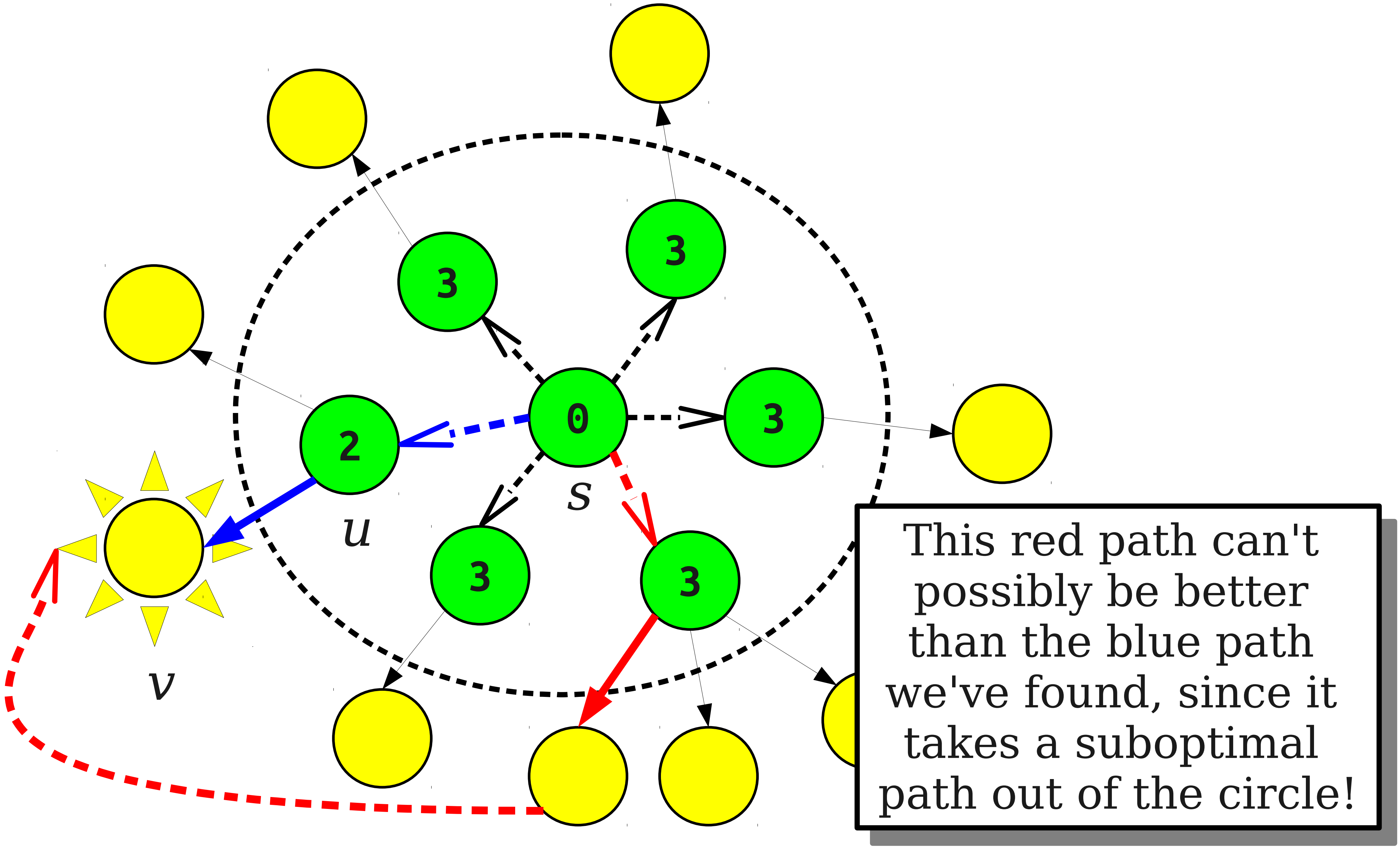


Pick yellow node *v* minimizing d(*s*, *u*) + 1, where (*u*, *v*) is an edge and *u* is green.



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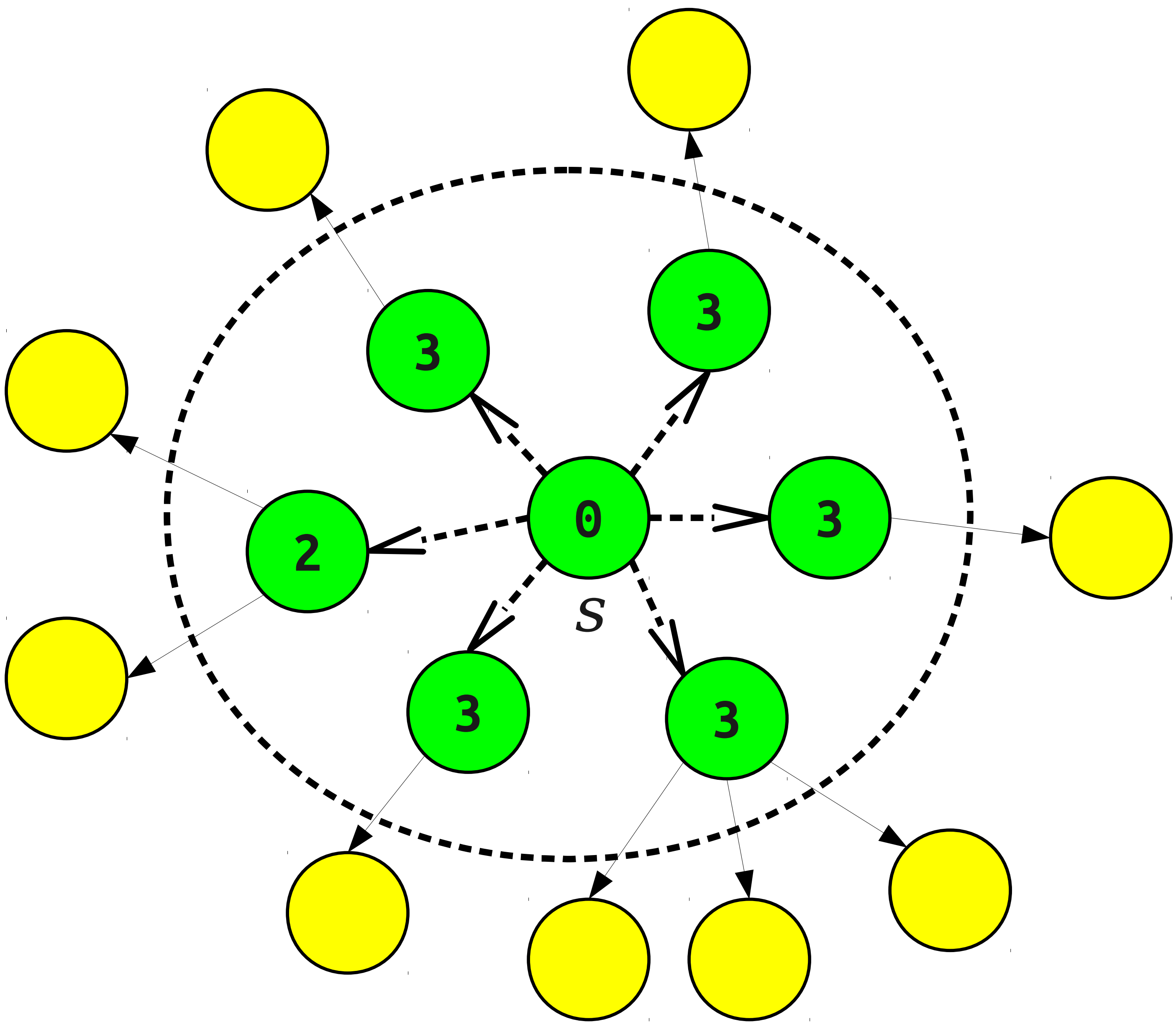

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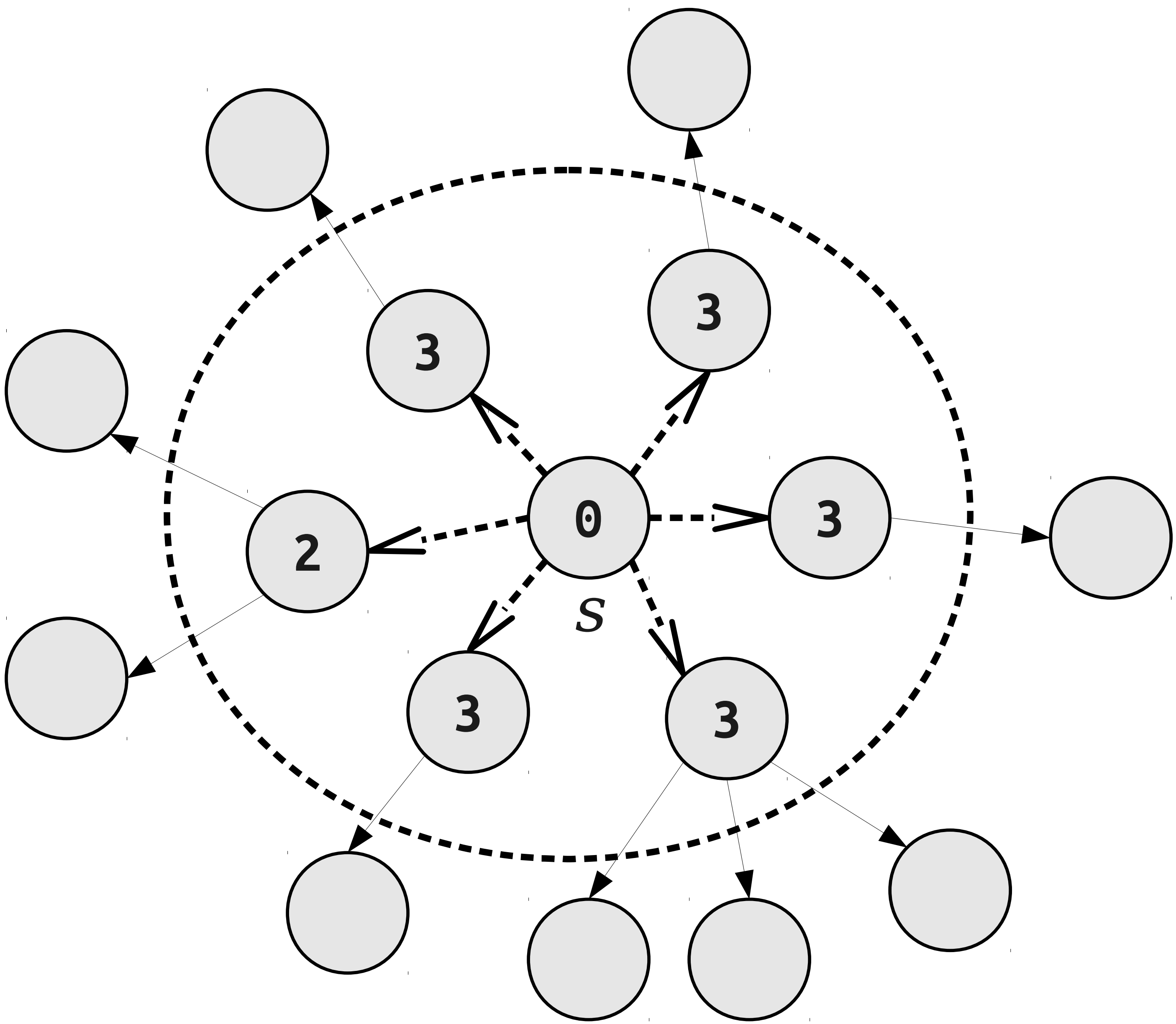
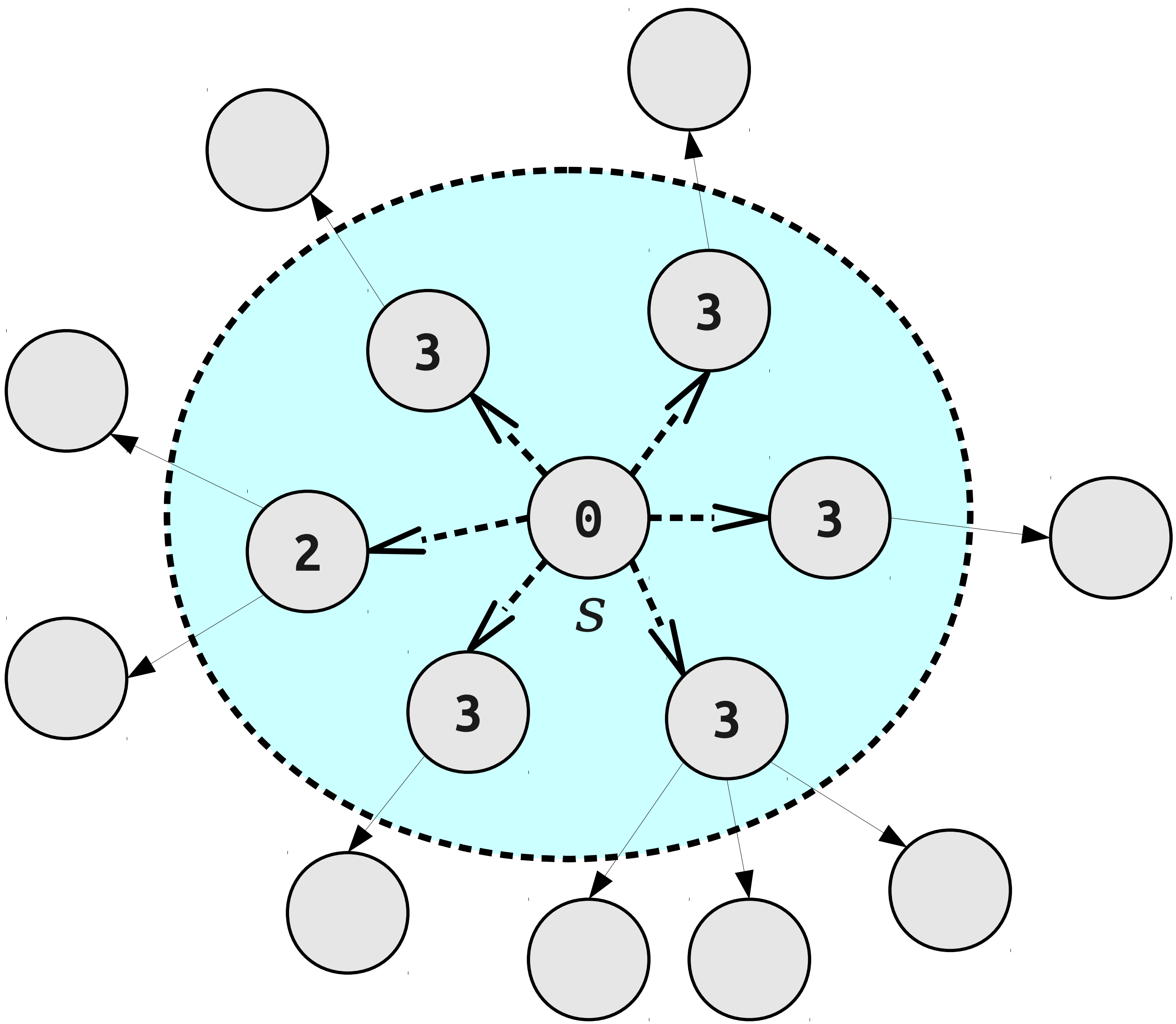


Pick yellow node v minimizing $d(s, u) + 1$, where (u, v) is an edge and u is green.



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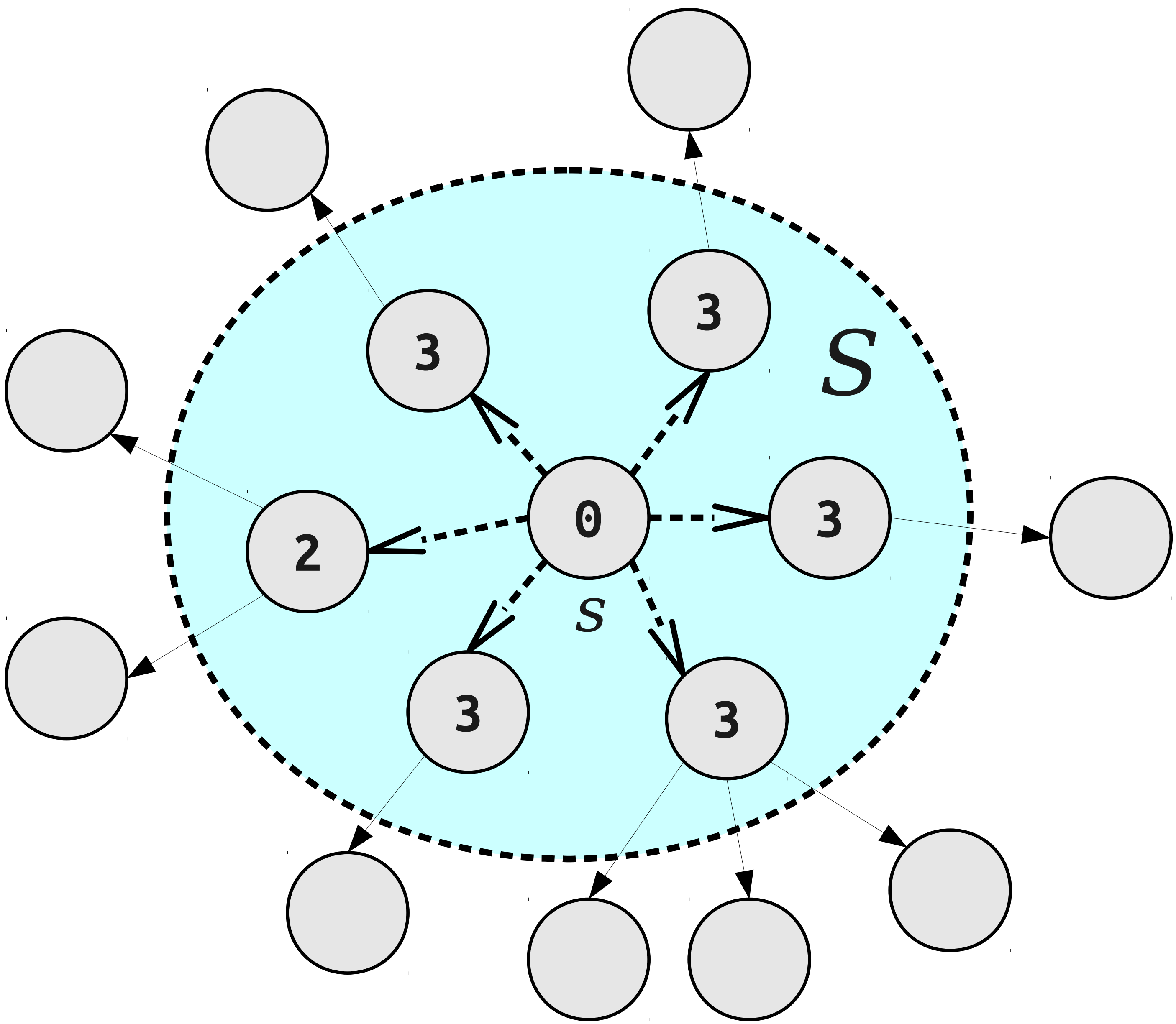
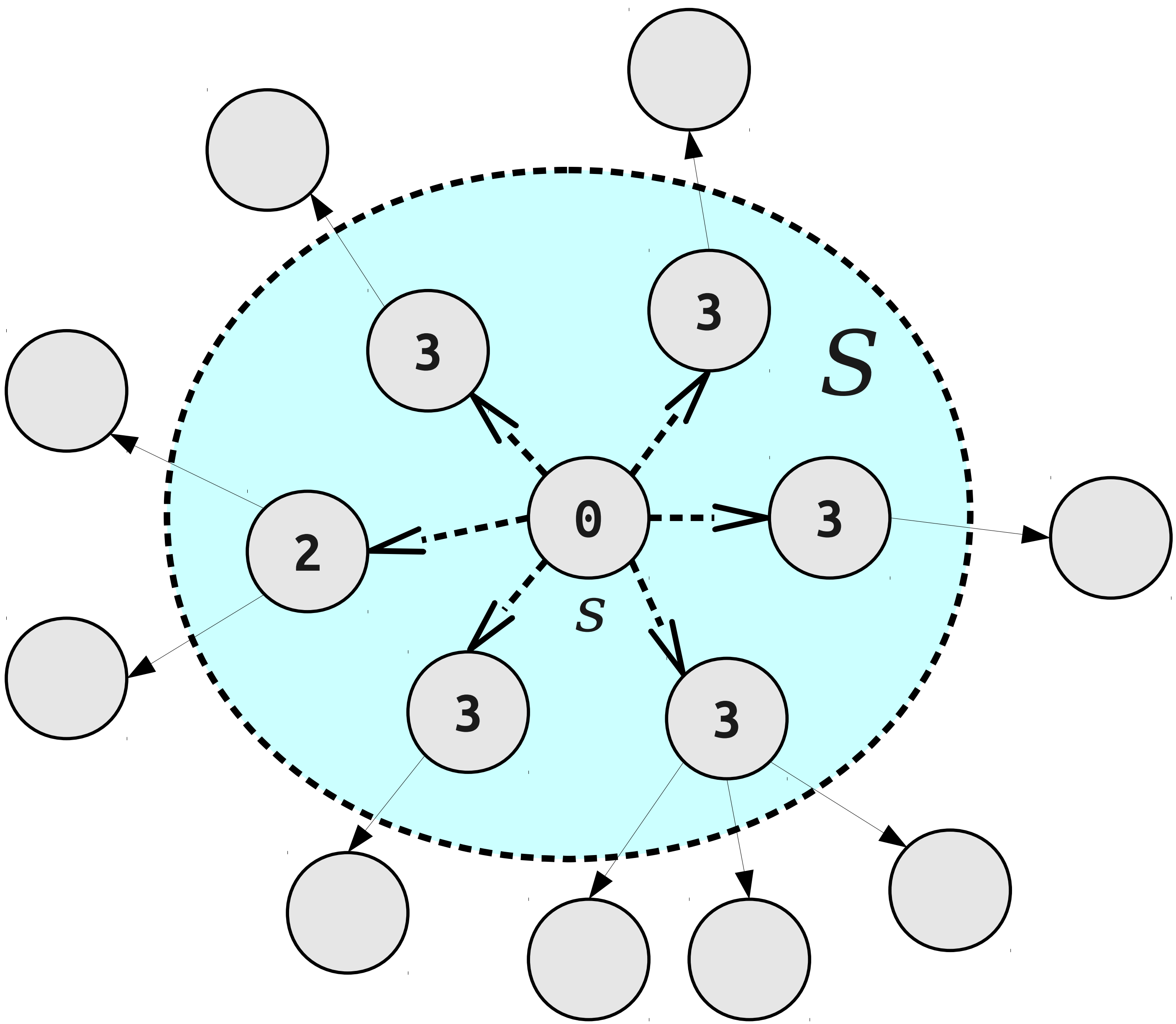


Pick node $\boldsymbol{v} \notin \boldsymbol{S}$ minimizing d(s, u) + 1,
where (u, v) is an edge and $\boldsymbol{u} \in \boldsymbol{S}$



Lemma: Suppose we have shortest paths computed for nodes $S \subseteq V$, where $s \in S$. Consider a node v where $(u, v) \in E$, $u \in S$, and the quantity $\mathrm{d}(s, u) + 1$ is minimized. Then $d(s, v) = d(s, u) + 1$.

Proof: There is a path to v of cost $\mathrm{d}(s, u) + 1$: follow the shortest path to u (which has cost $\mathrm{d}(s, u)$), then follow one more edge to v for total cost $\mathrm{d}(s, u) + 1$.

Now suppose for the sake of contradiction that there is a shorter path P to v. This path must start in S (since $s \in S$) and leave S (since $v \notin S$). So consider when P leaves S. When this happens, P must go from s to some node $x \in S$, cross an edge (x, y) to some node y, then continue from y to v. This means that $|P|$ is at least $\mathrm{d}(s, x) + 1$, since the path goes from s to x and then follows at least one more edge.

Since v was picked to minimize $\mathrm{d}(s, u) + 1$ for any choice of $u \in S$ adjacent to an edge (u, v), we know

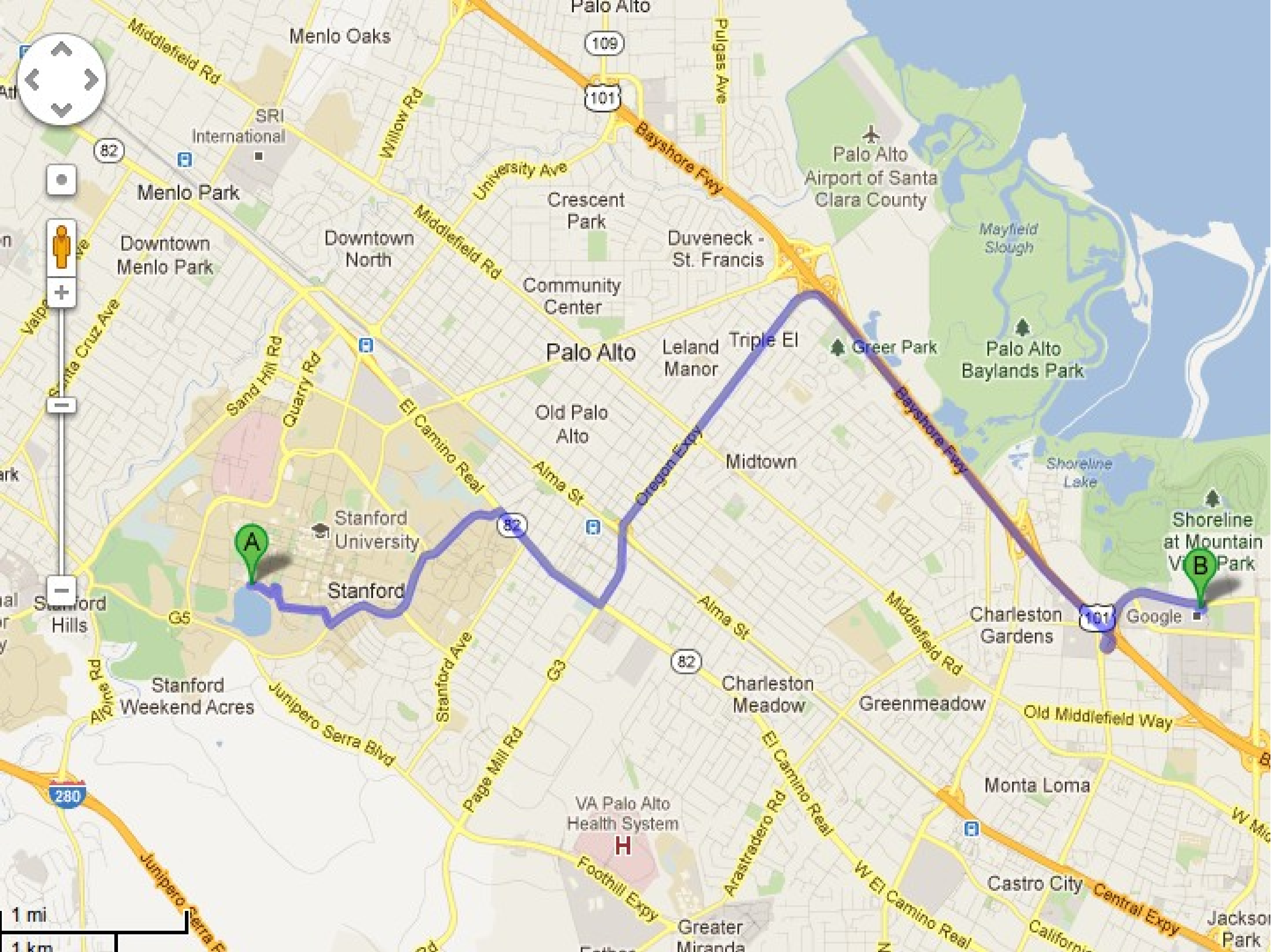
$$\mathrm{d}(s, u) + 1 \leq d(s, x) + 1 \leq |P|$$

contradicting the fact that $|P| < \mathrm{d}(s, u) + 1$. We have reached a contradiction, so our assumption was wrong and no shorter path exists.

Since there is a path of length $\mathrm{d}(s, u) + 1$ from s to v and no shorter path, this means that $\mathrm{d}(s, v) = \mathrm{d}(s, u) + 1$. ■

Why These Two Proofs Matter

- The first proof of correctness (based on layers) is based on our first observation: the nodes visited in BFS radiate outward from the start node in ascending order of distance.
- The second proof of correctness (based on picking the lowest yellow node) is based on our second observation: picking the lowest-cost yellow node correctly computes a shortest path.
- Interestingly, this second correctness proof can be generalized to a larger setting…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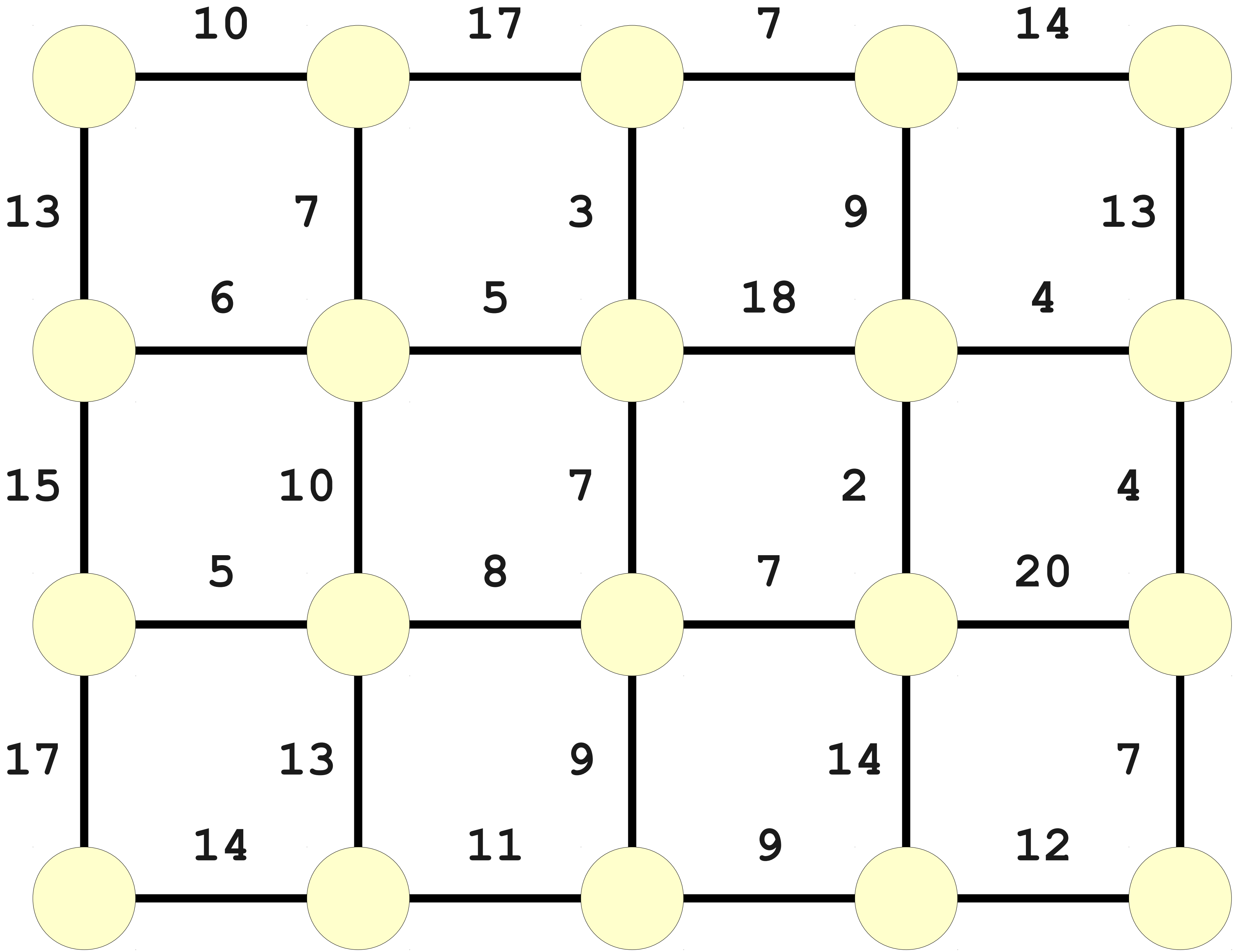
Edges with Costs

- In many applications, edges have an associated length (or cost, weight, etc.), denoted $\boldsymbol{l(u, v)}$.
- **Assumption**: Lengths are nonnegative. (We'll revisit this later in the quarter.)
- Let's say that the length of a path P (denoted $\boldsymbol{l(P)}$) is the sum of all the edge lengths in the path P.
- Goal: find the shortest path from s to every node in V, taking costs into account.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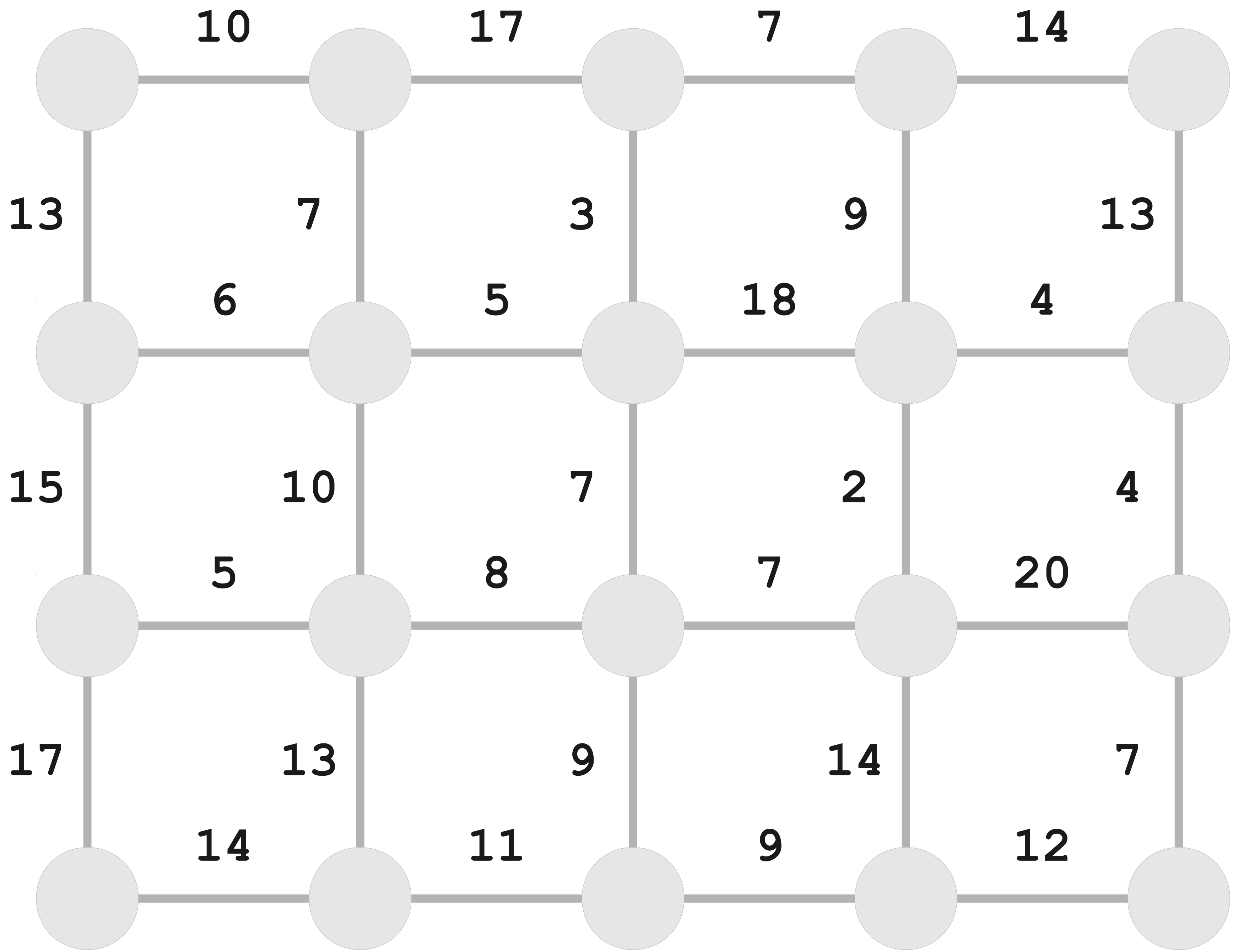
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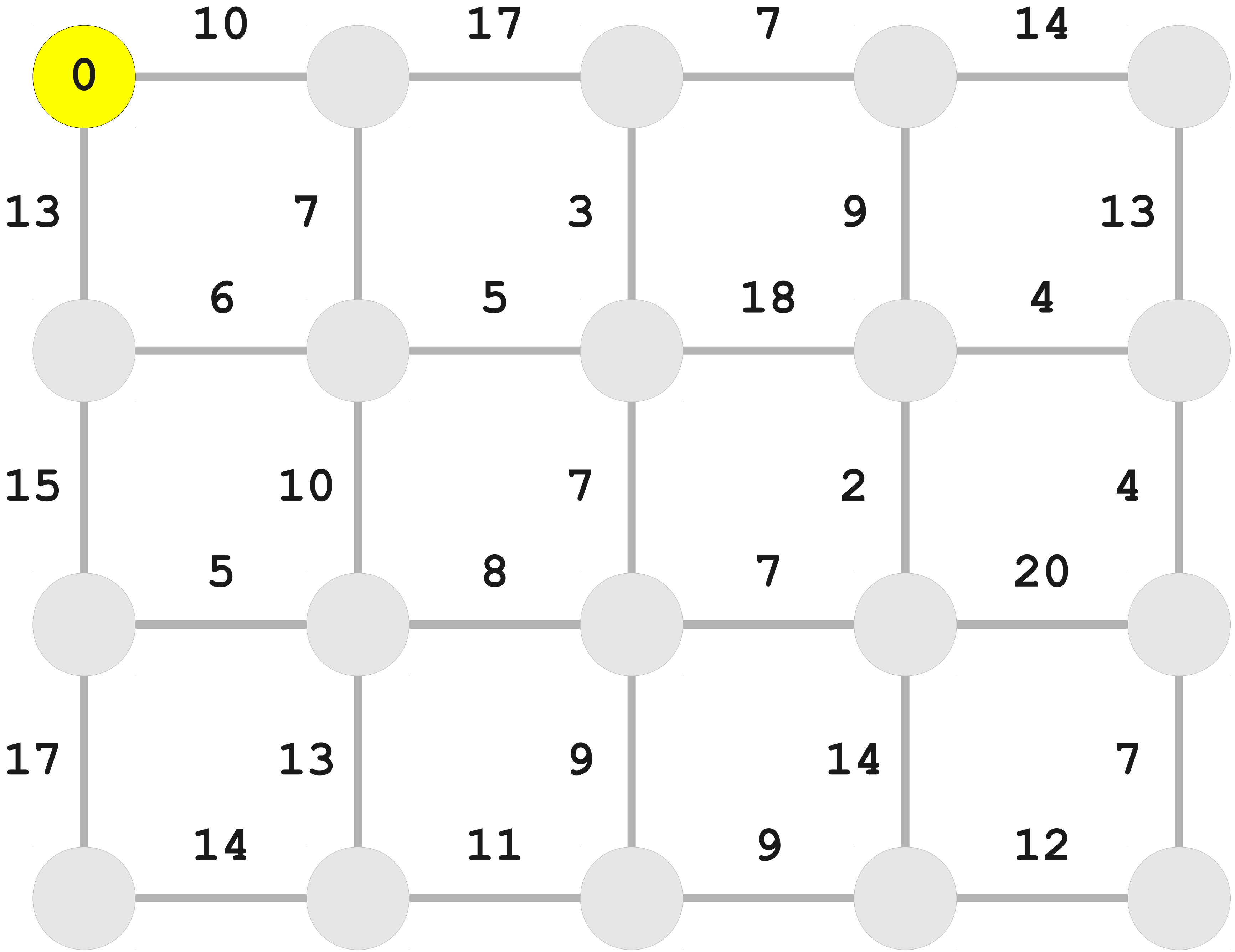
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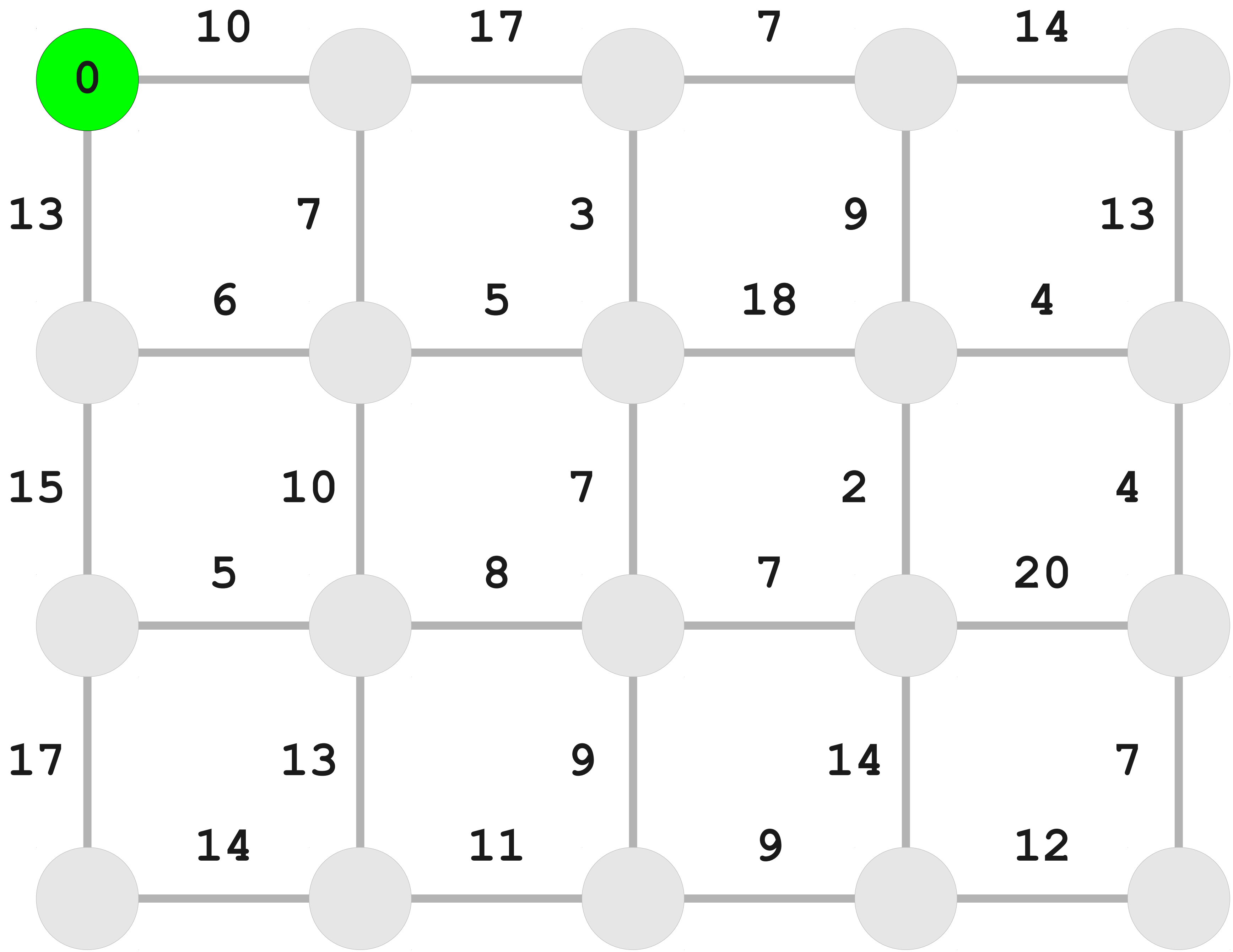
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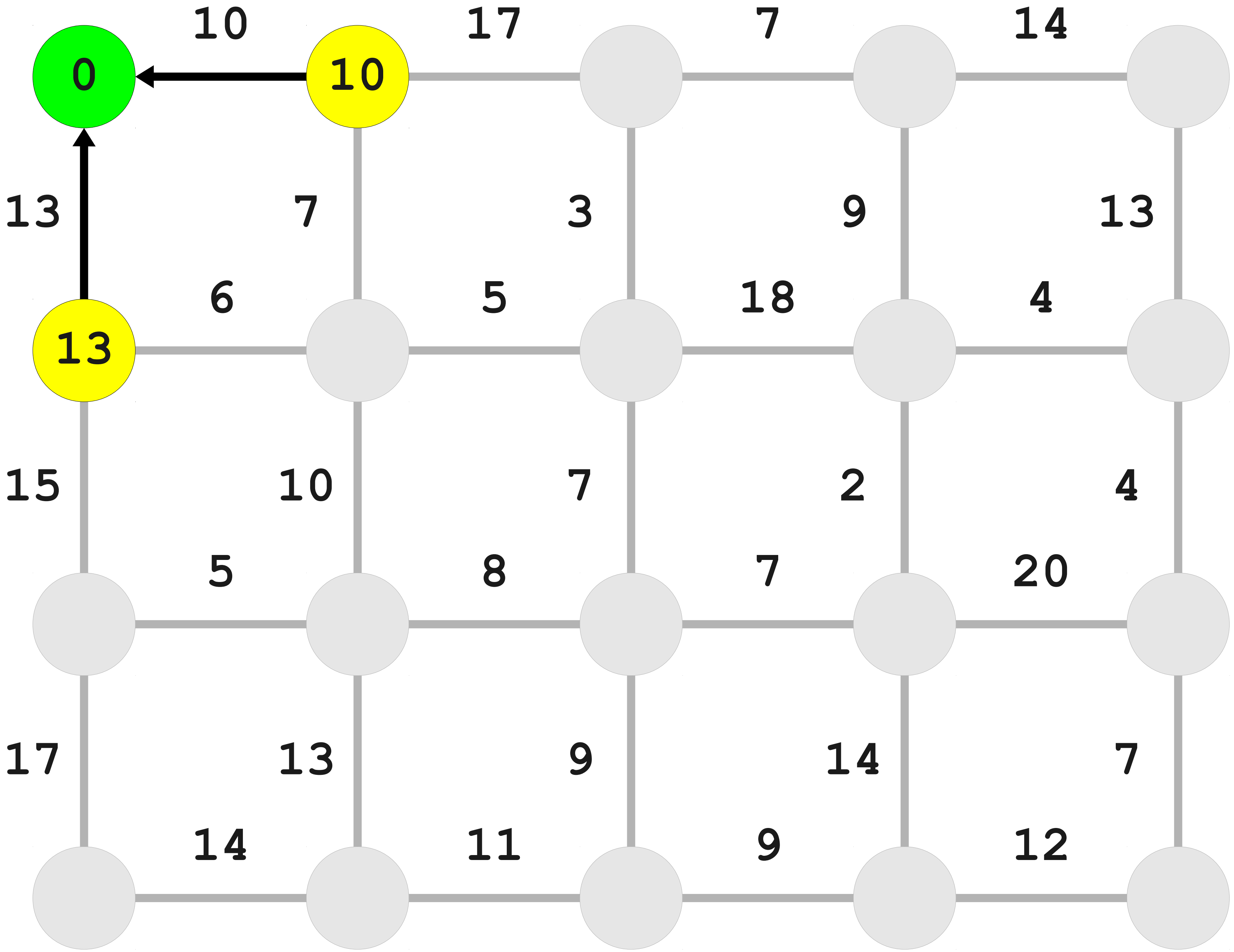
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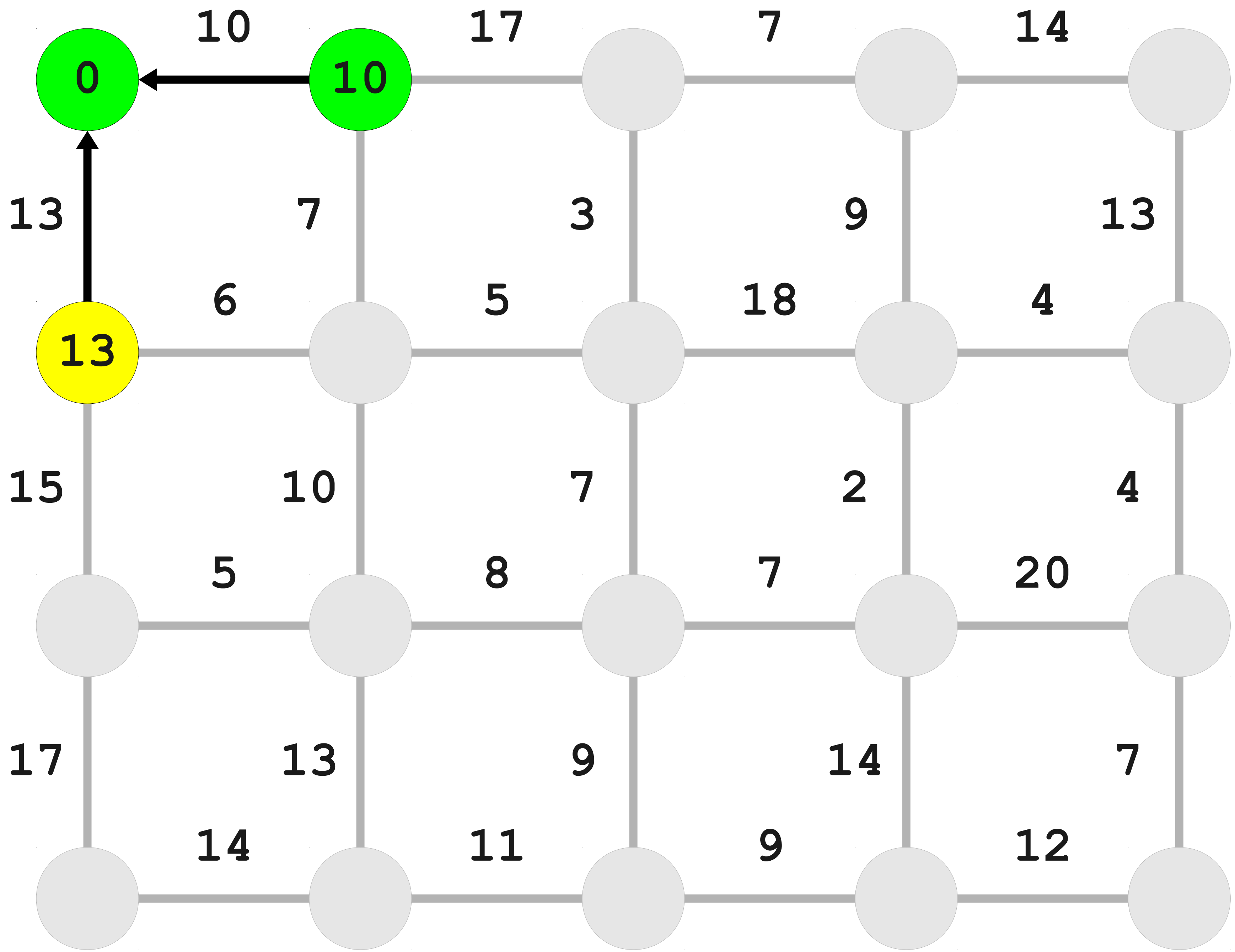
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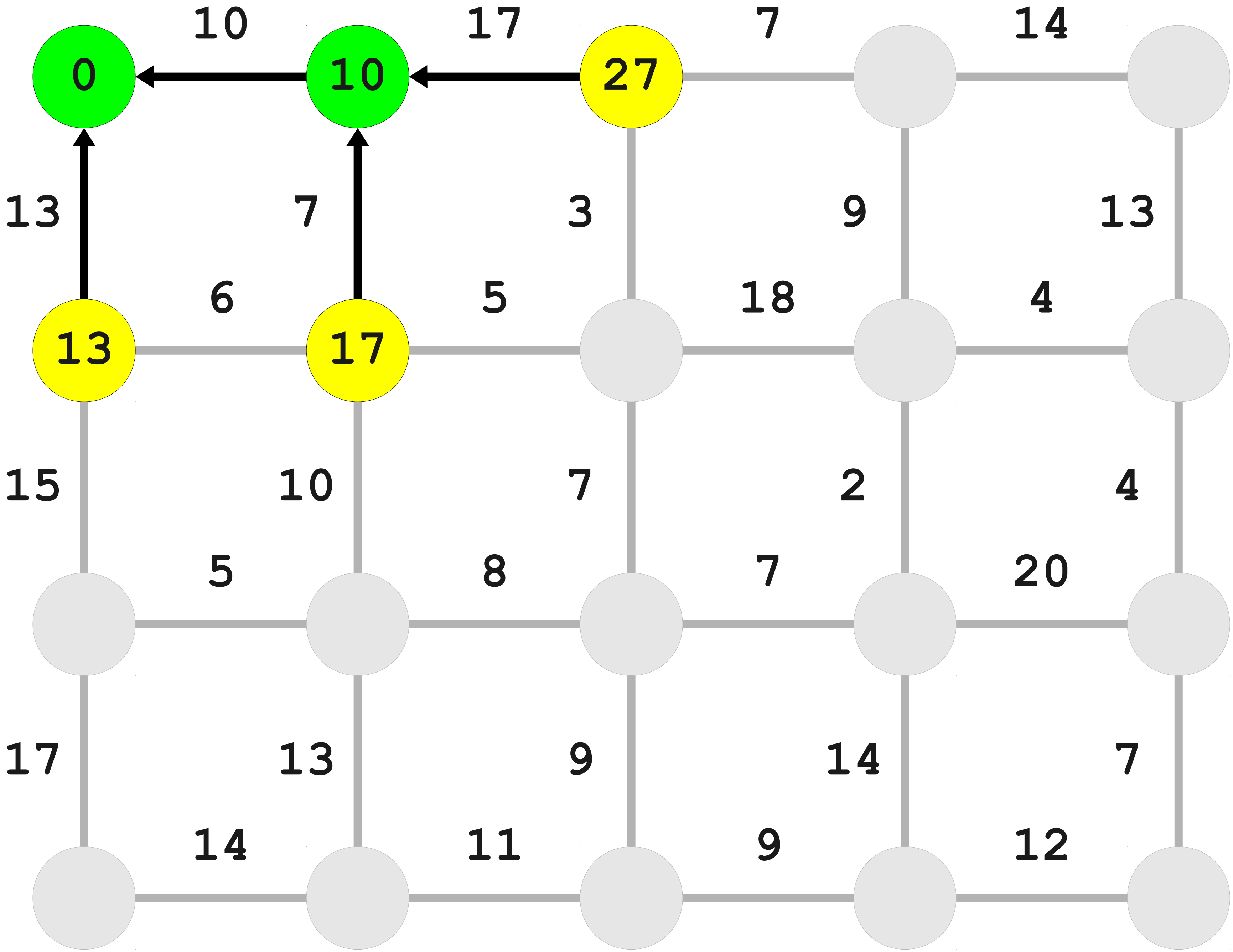
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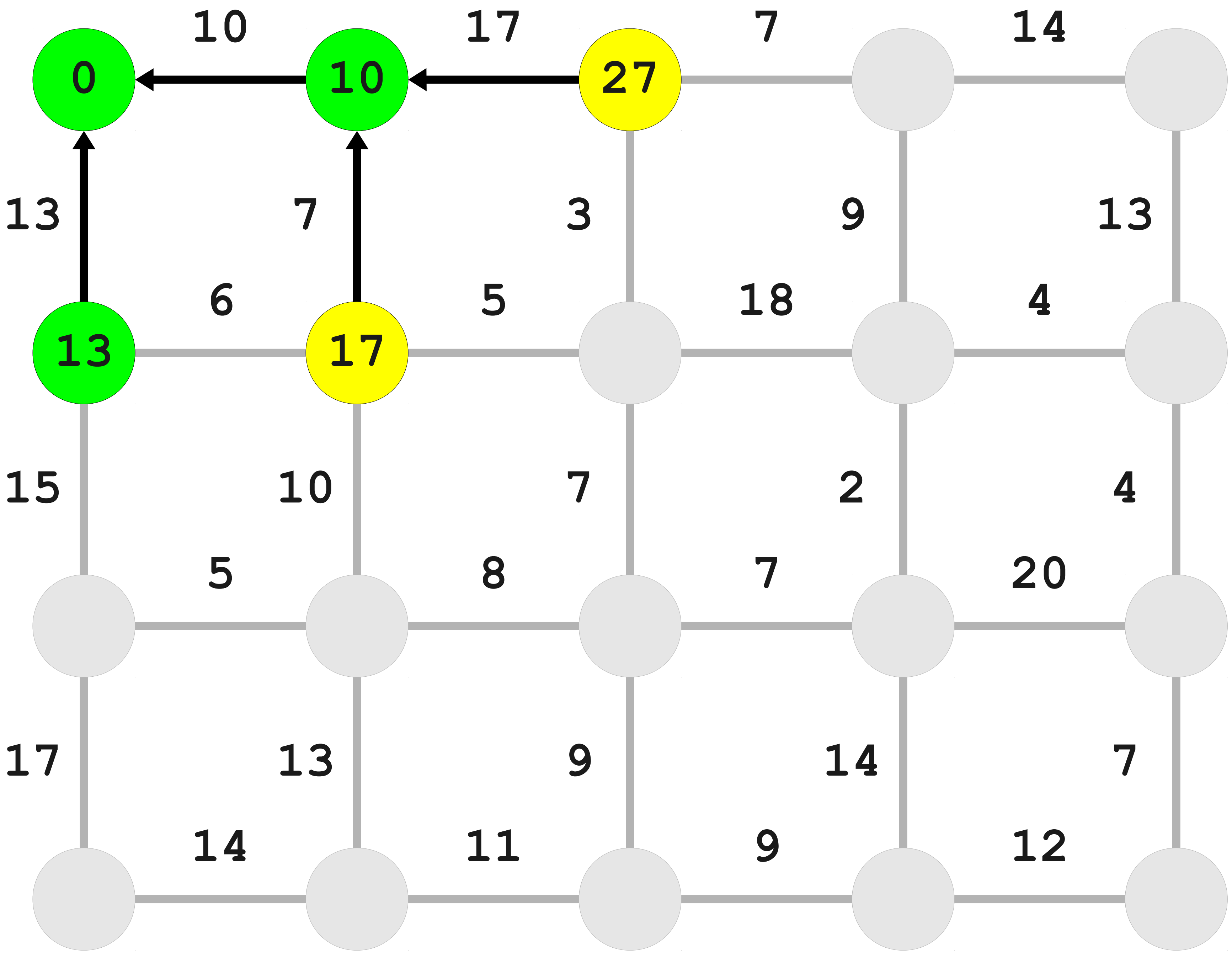
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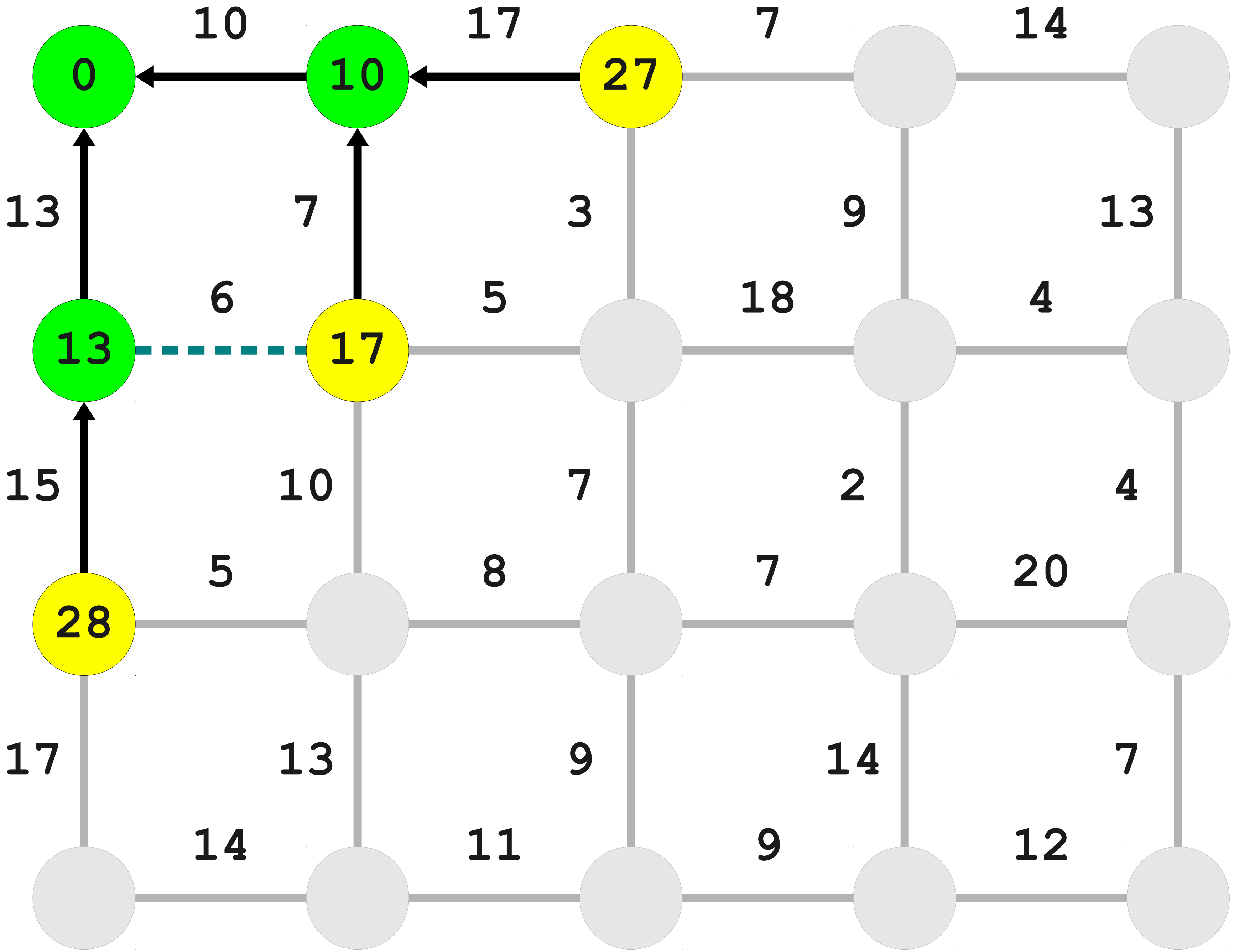

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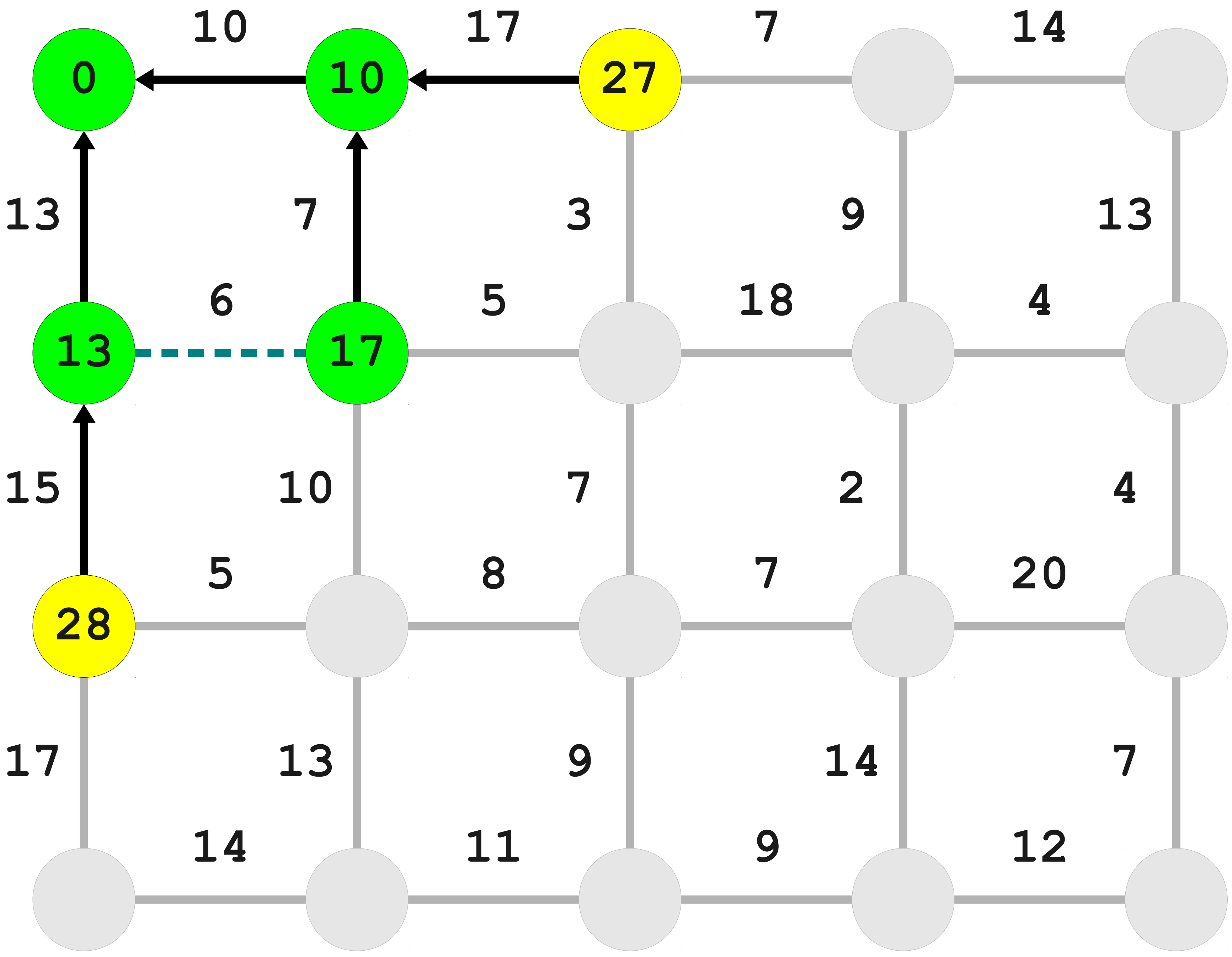

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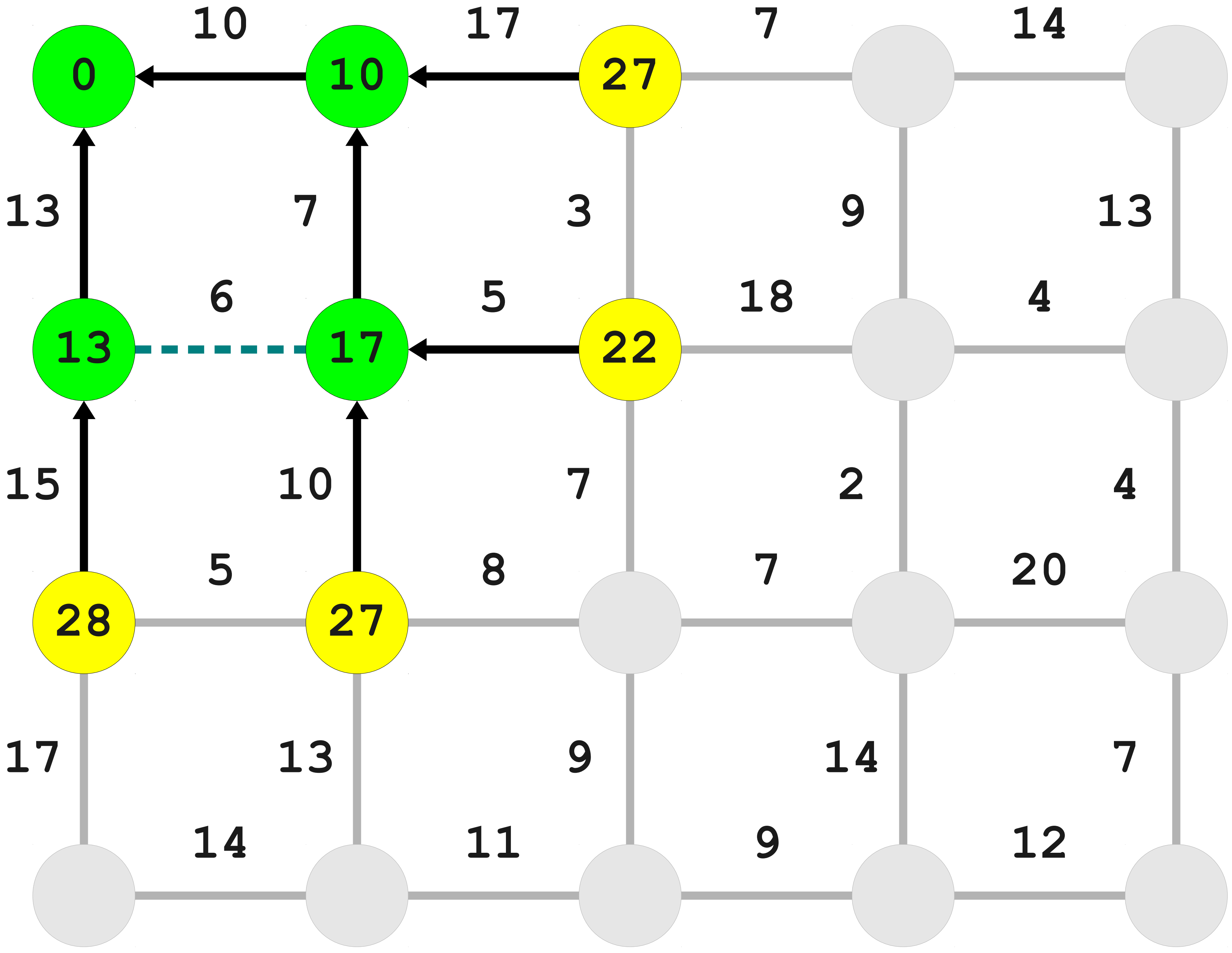

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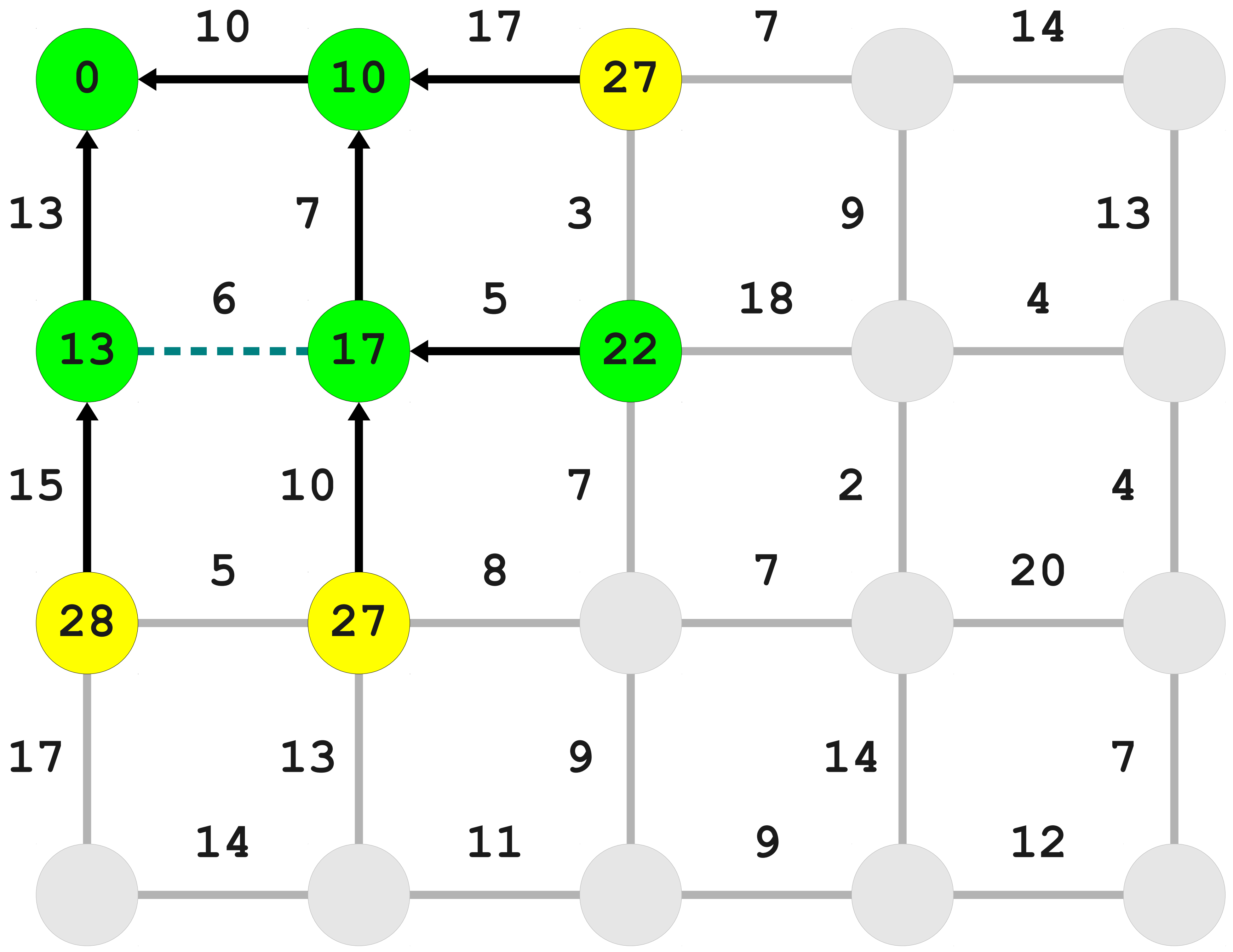
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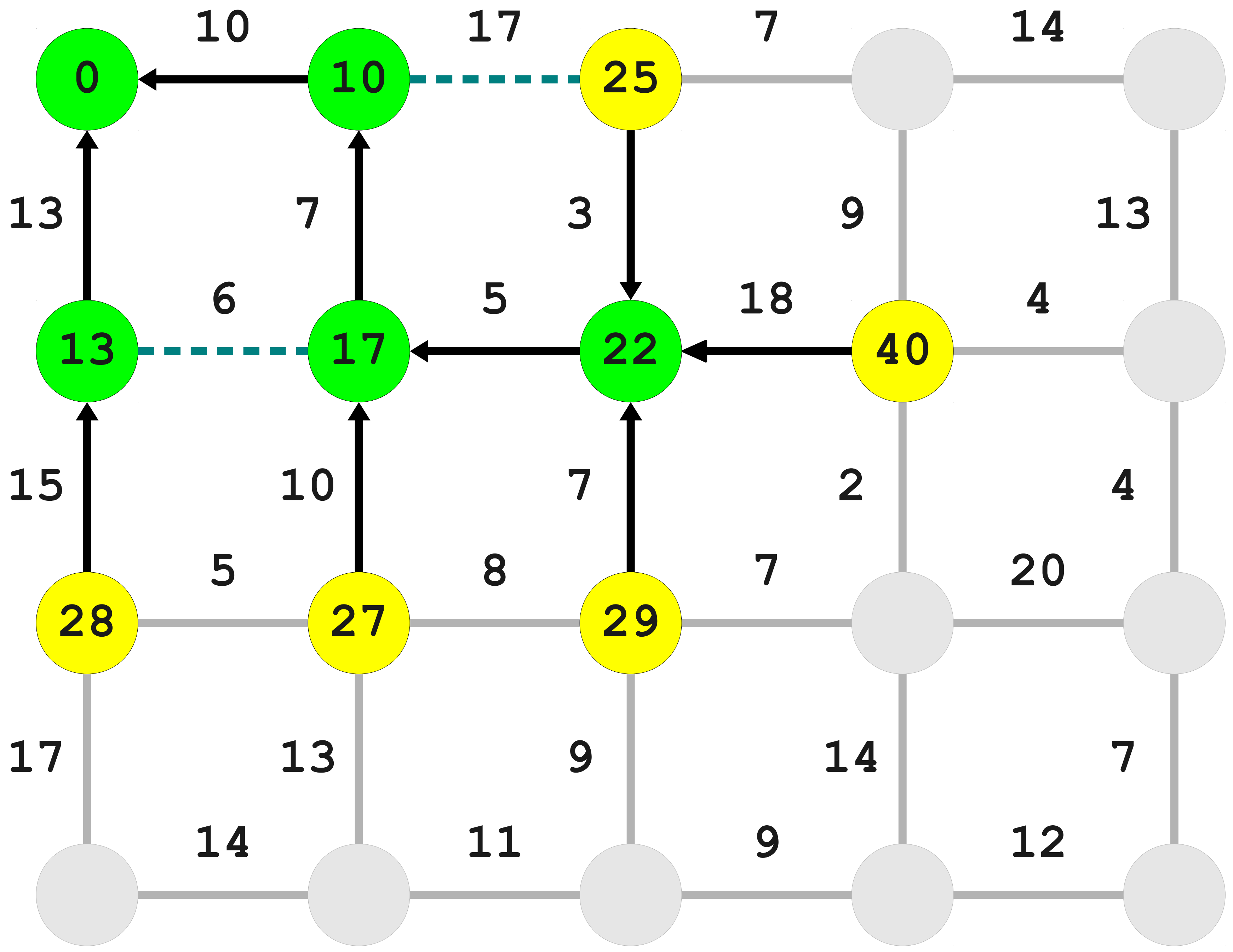
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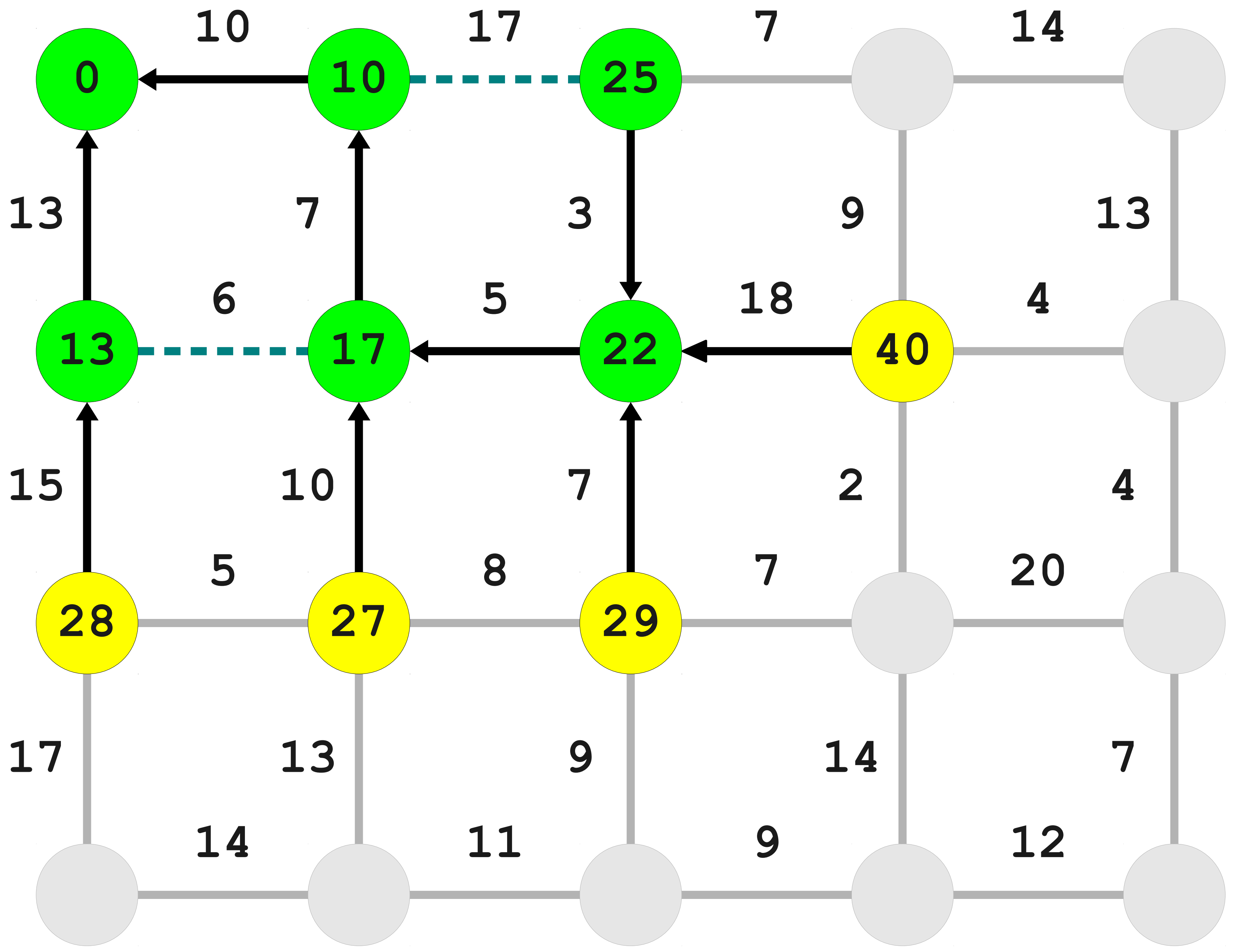
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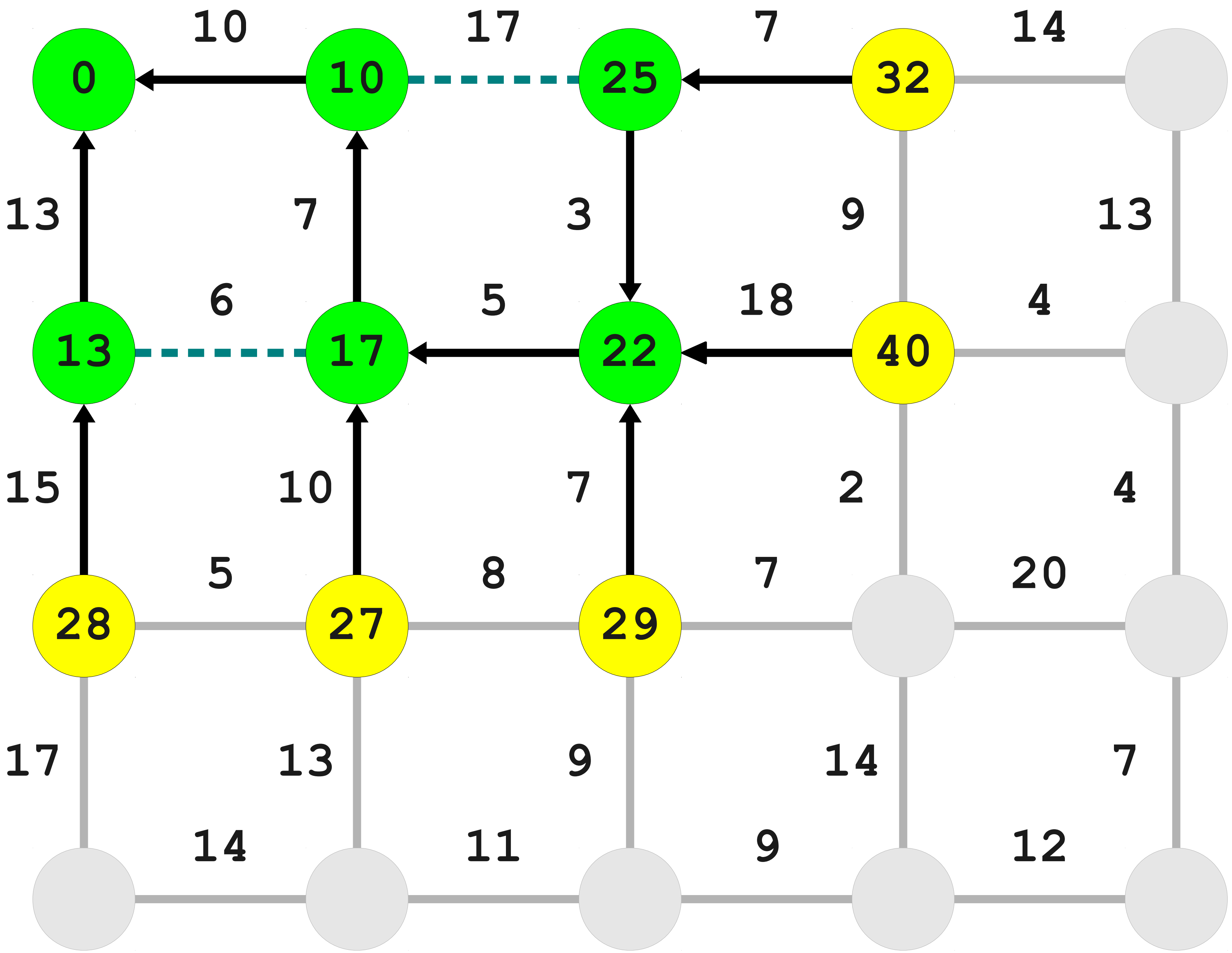
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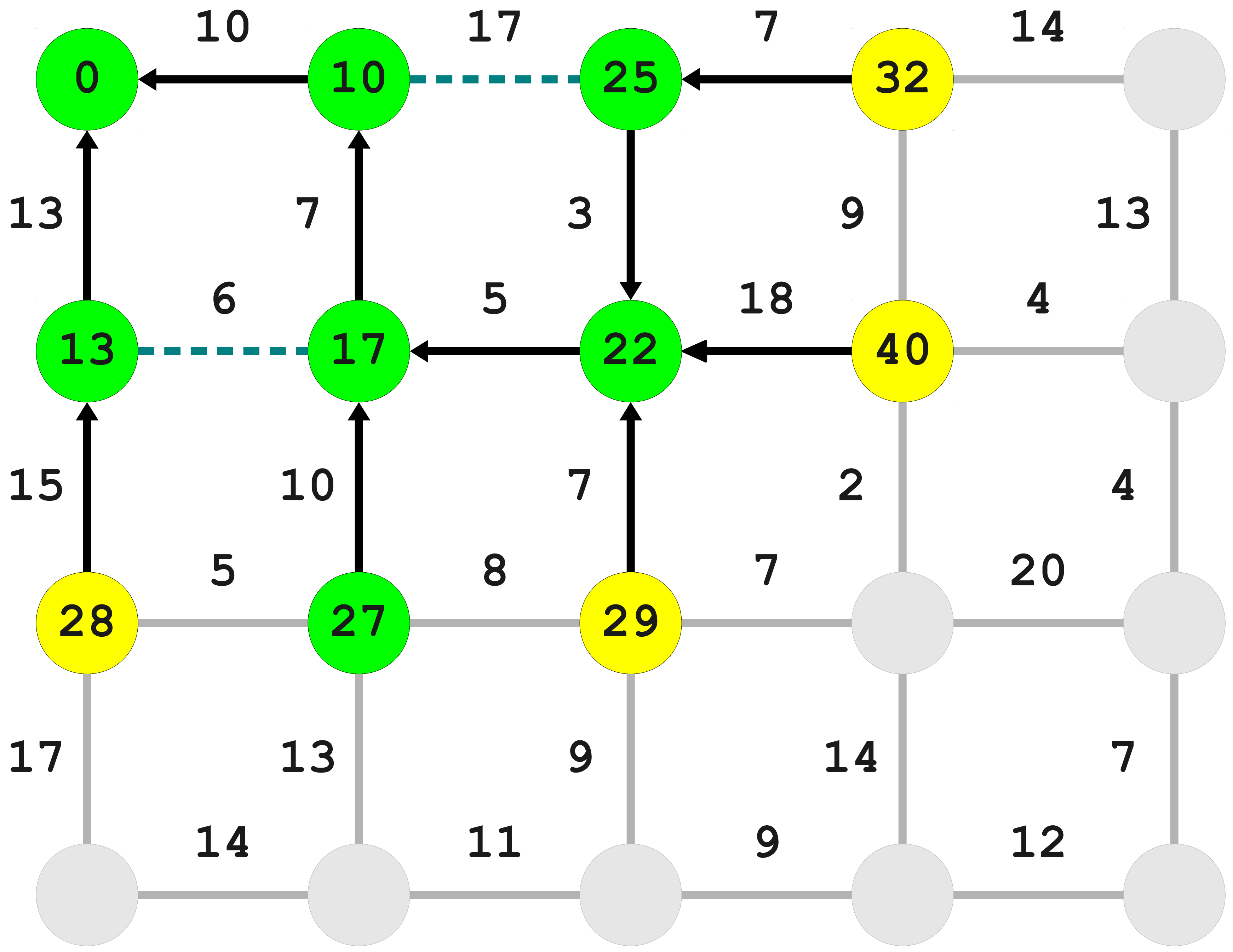
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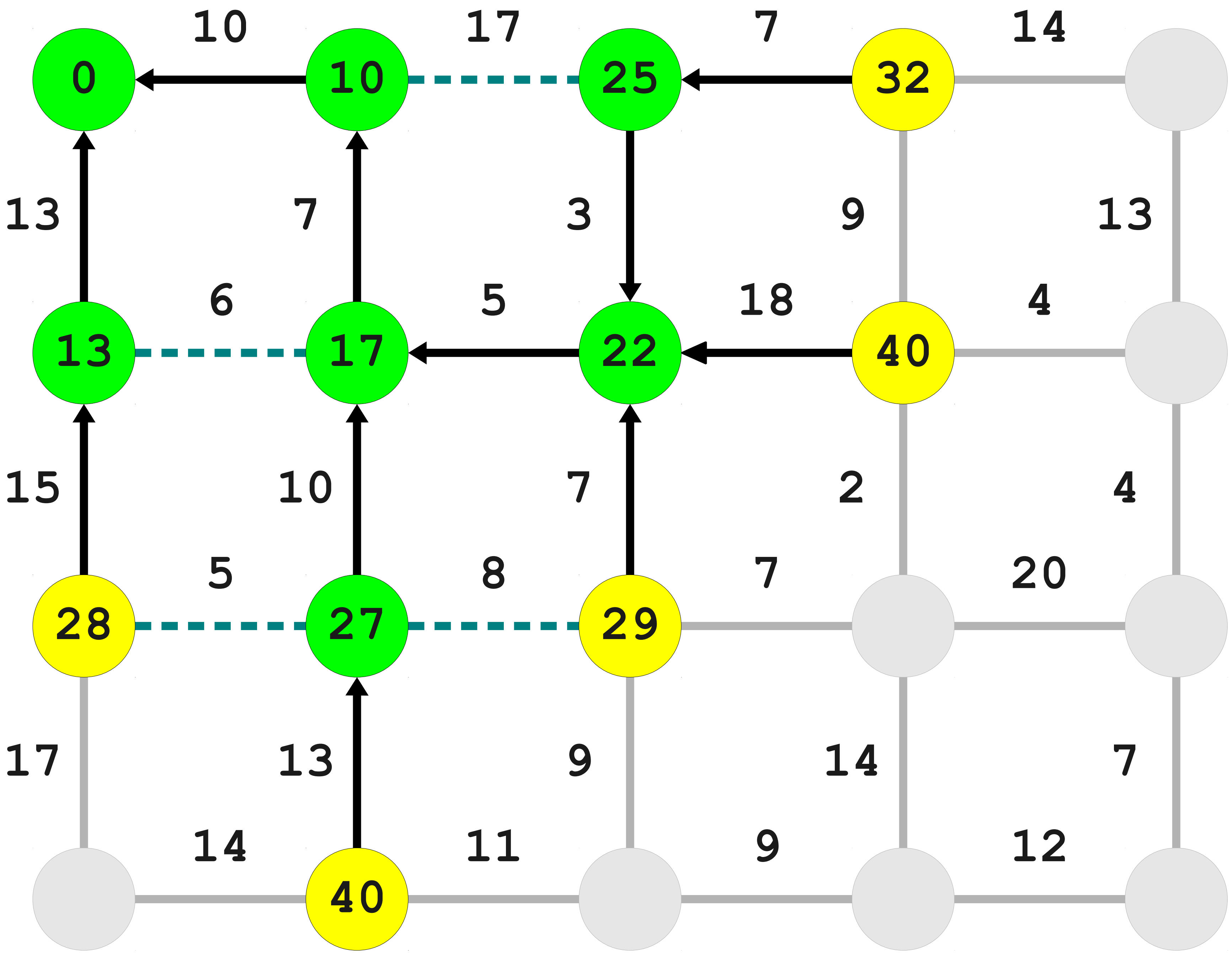

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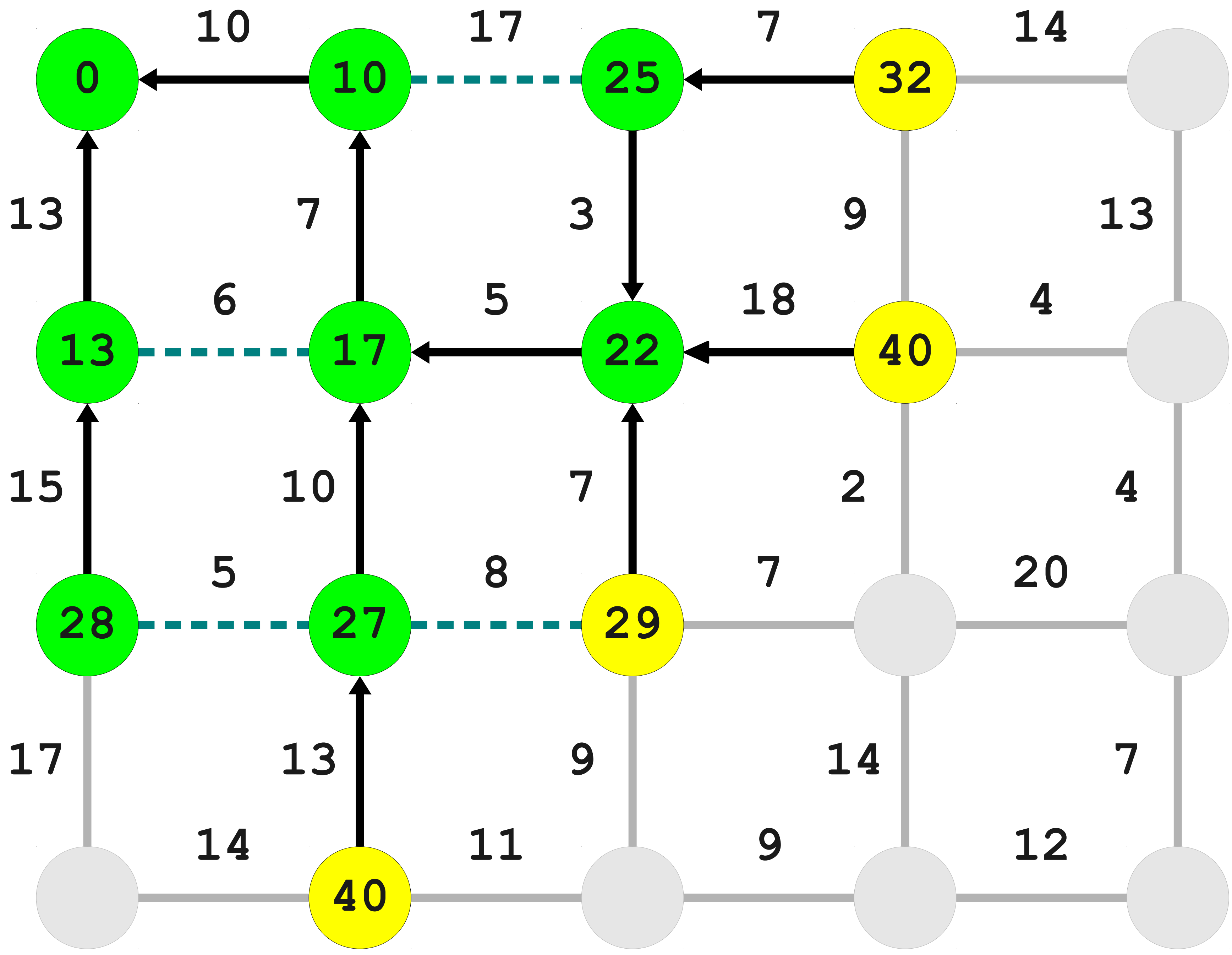
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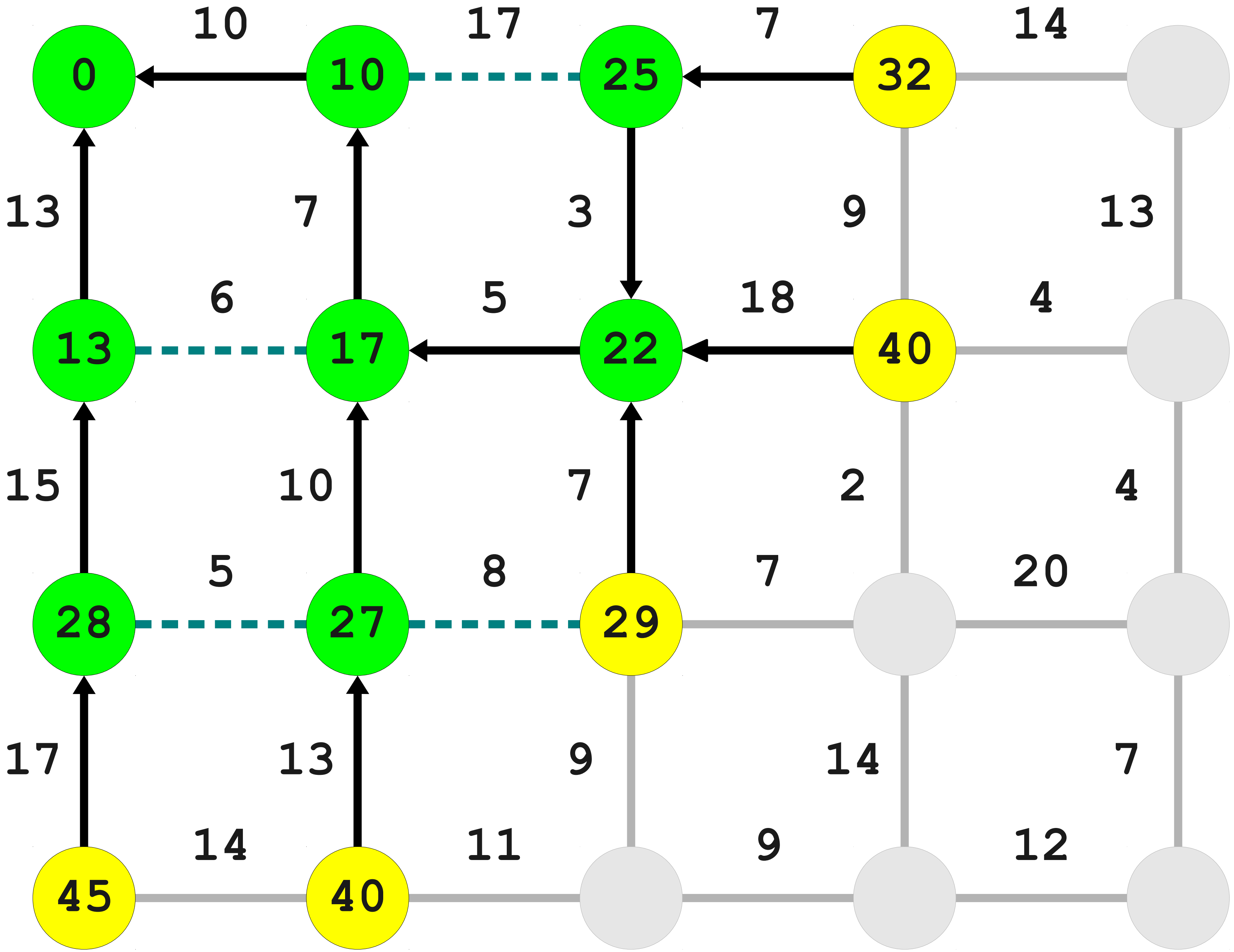
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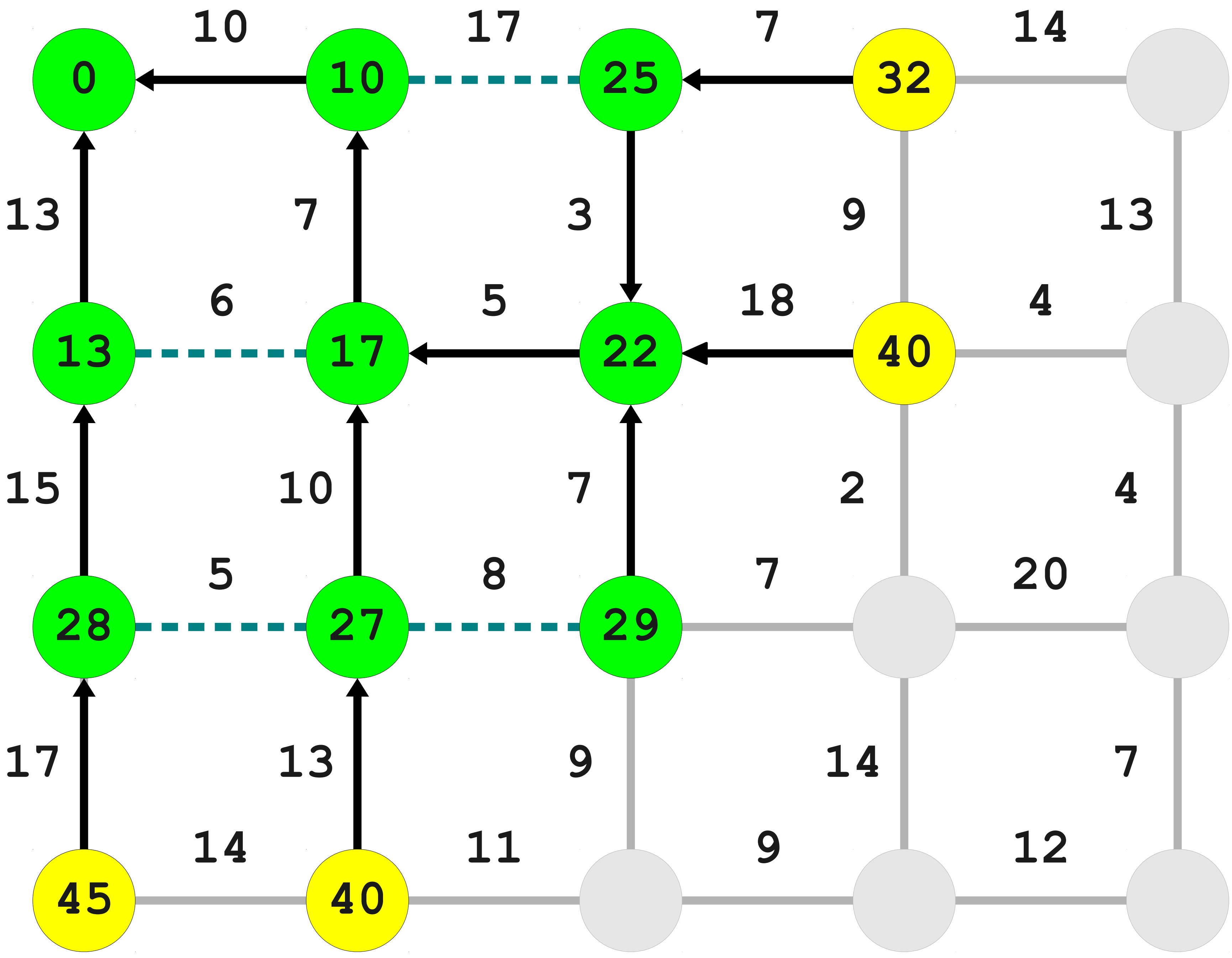

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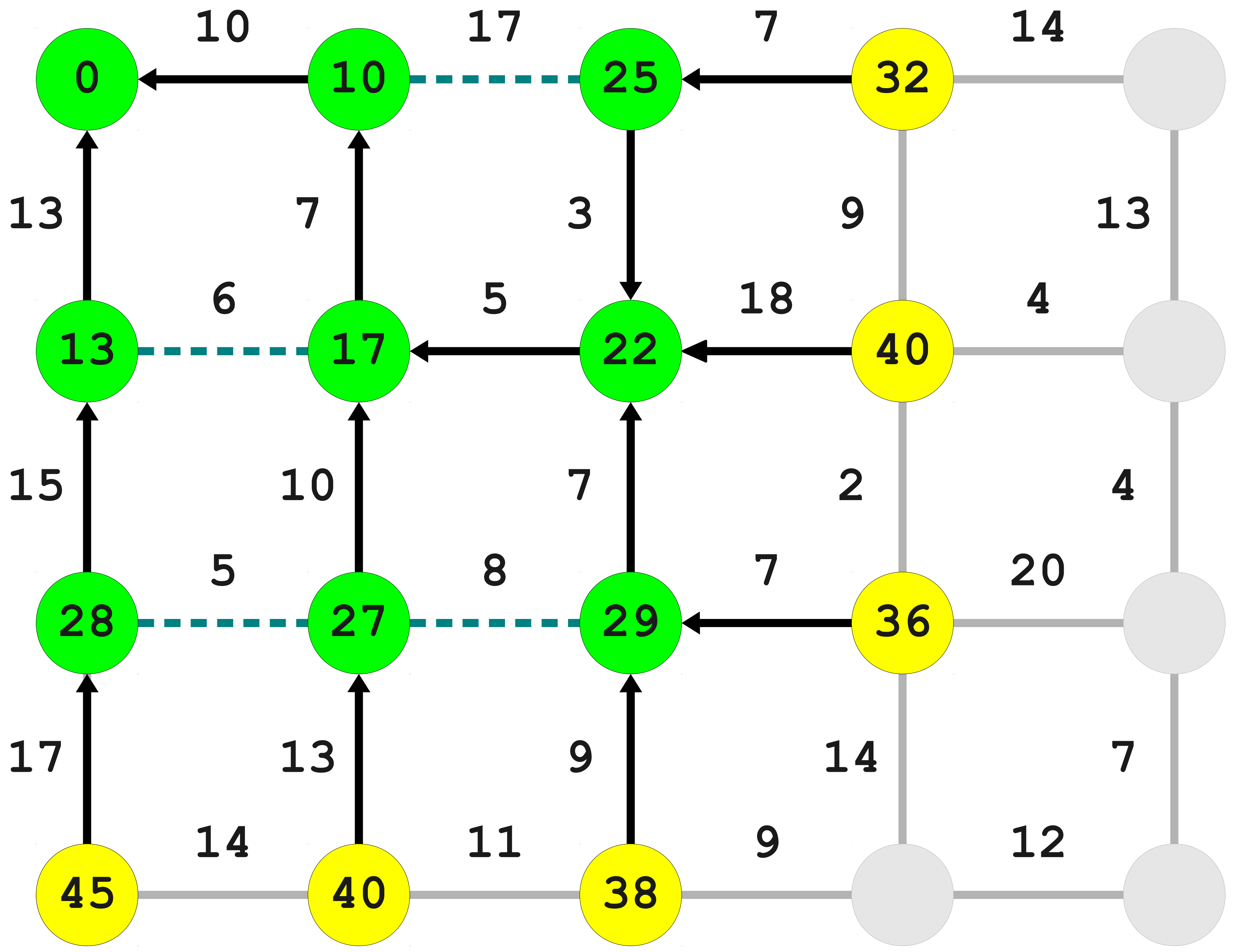
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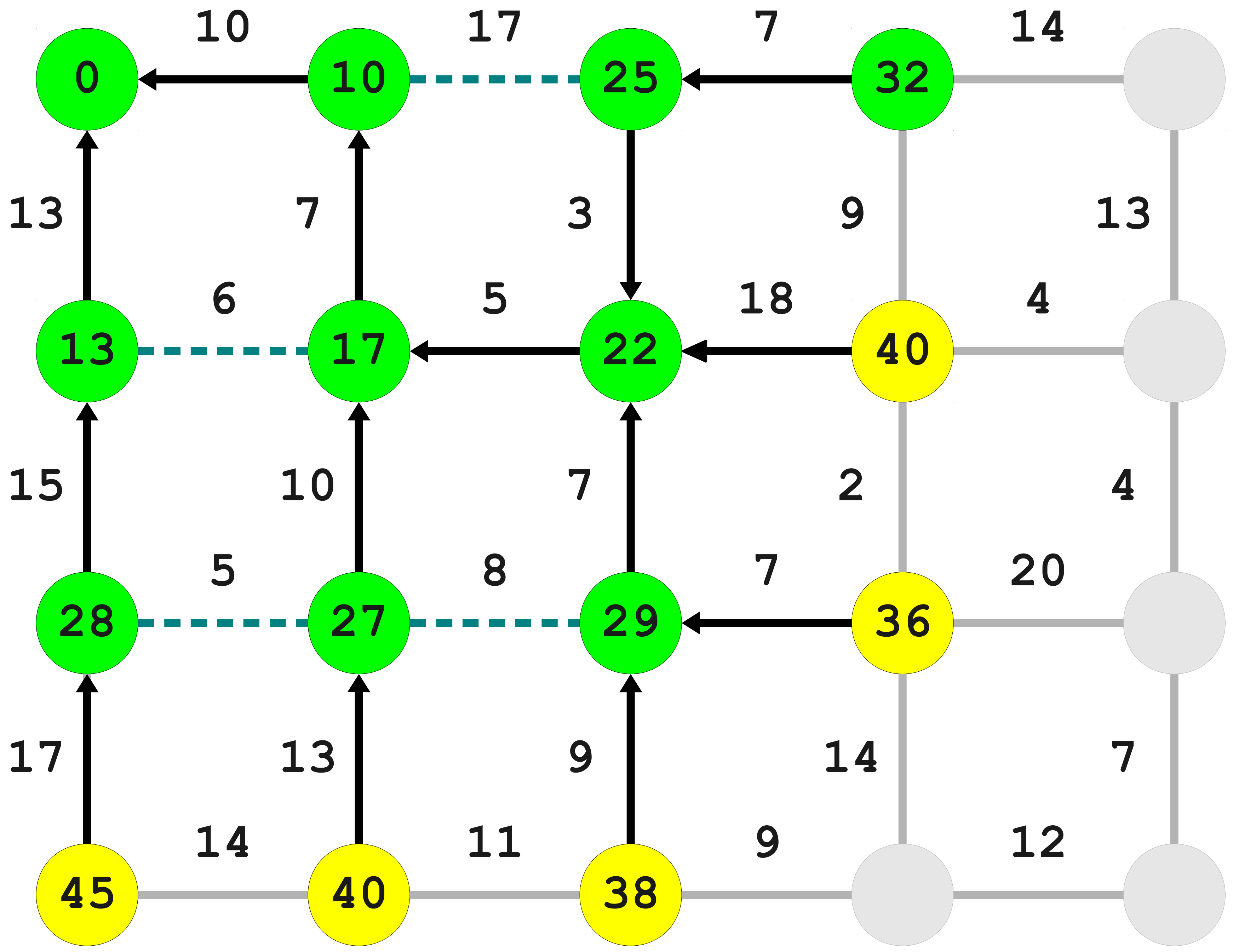
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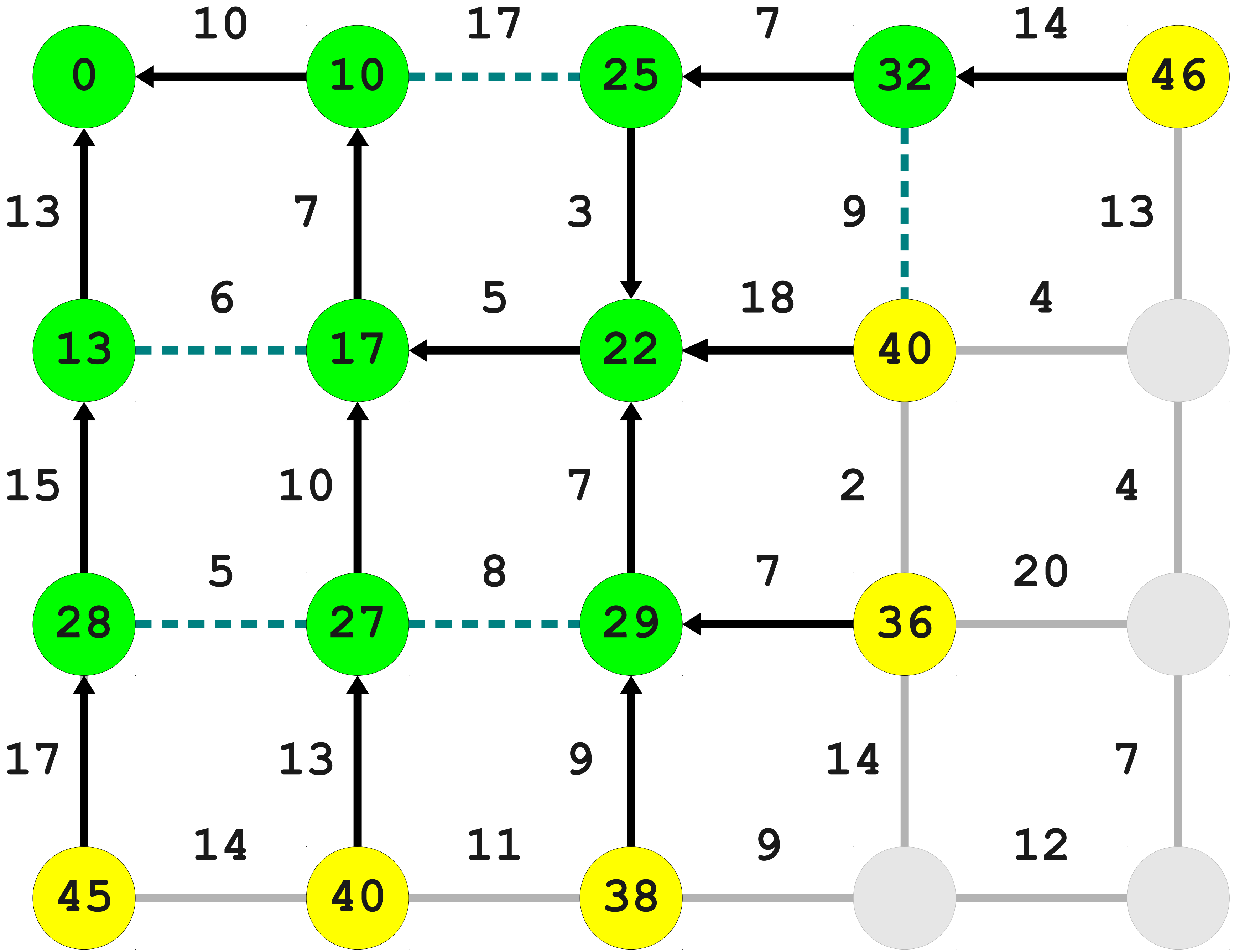
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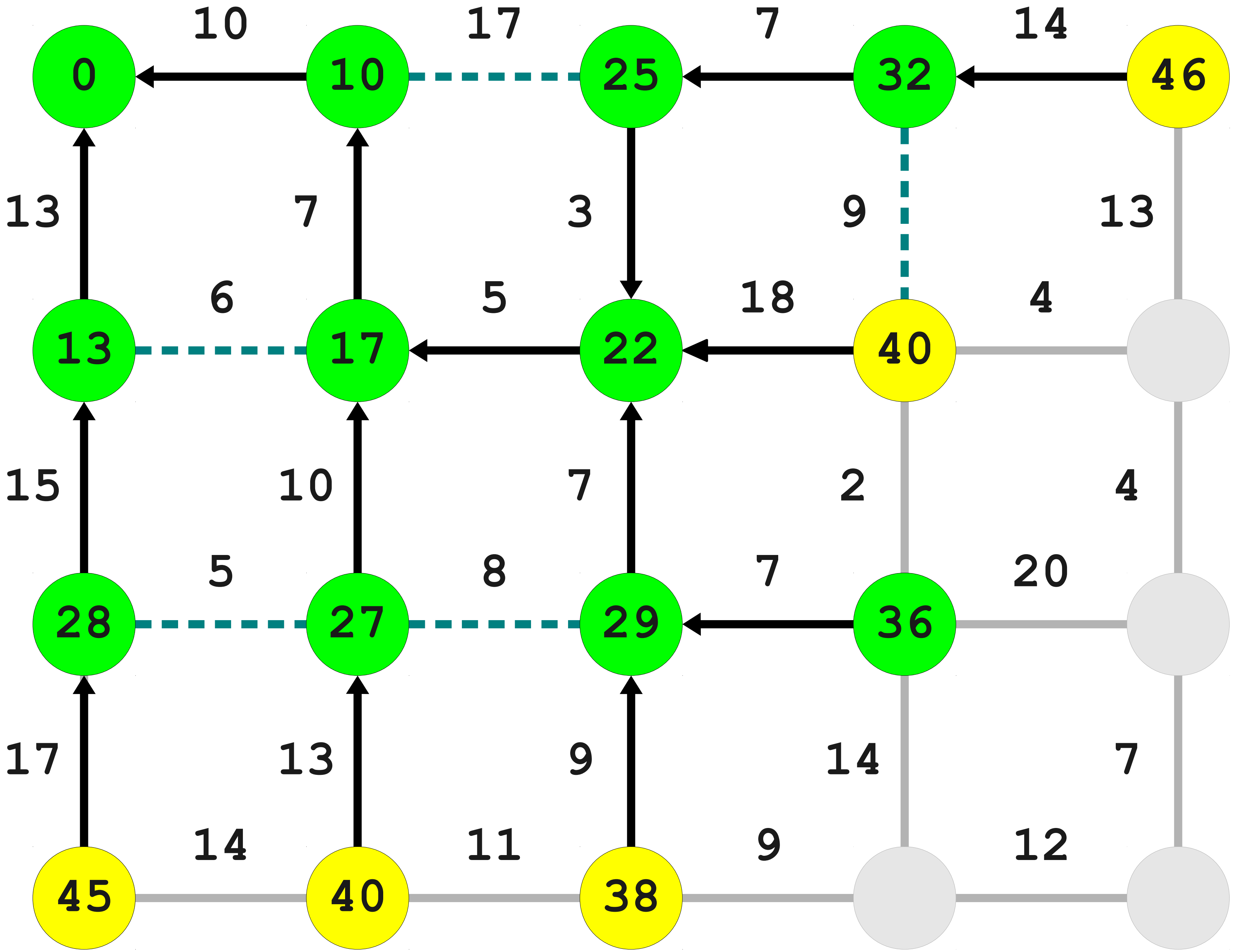

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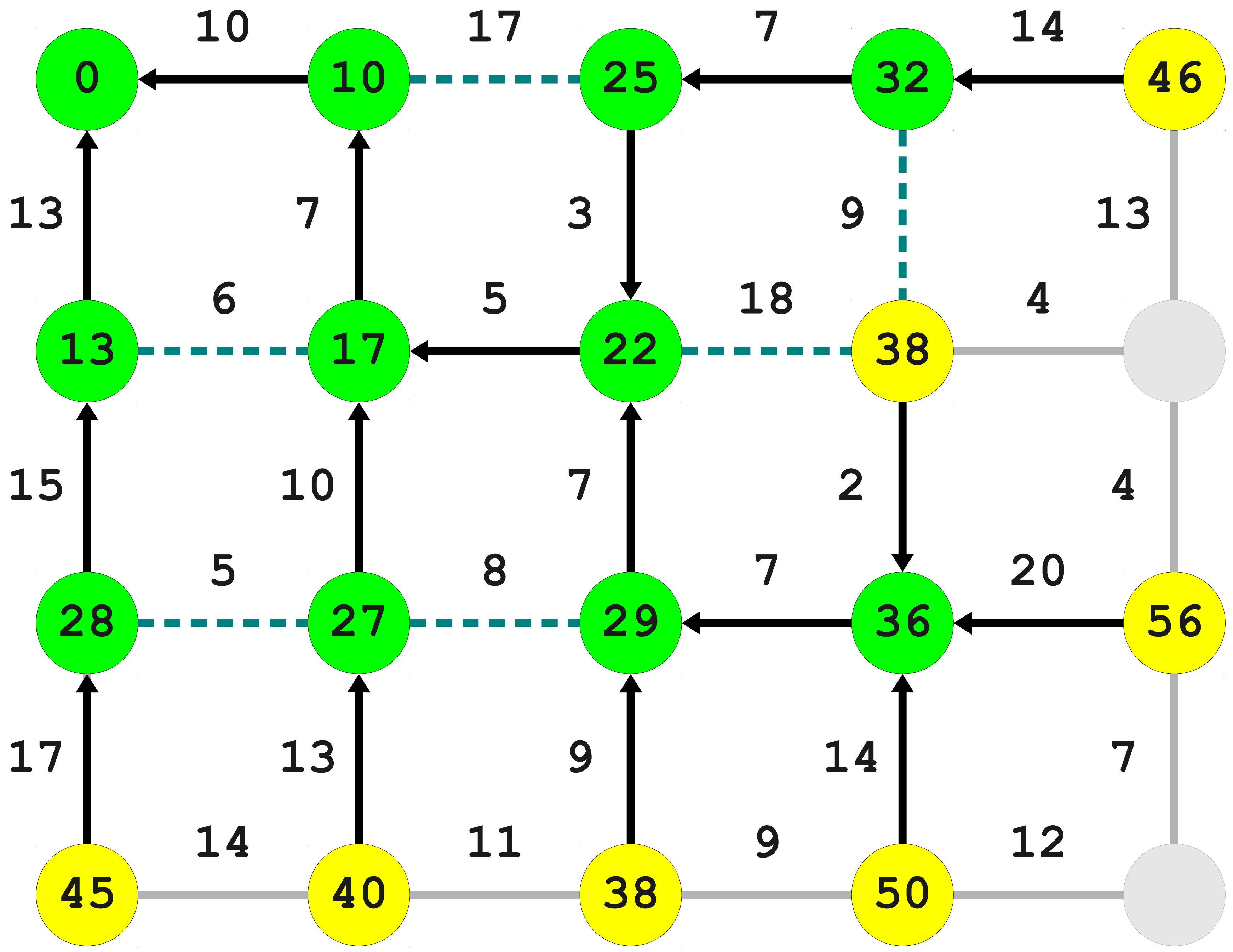

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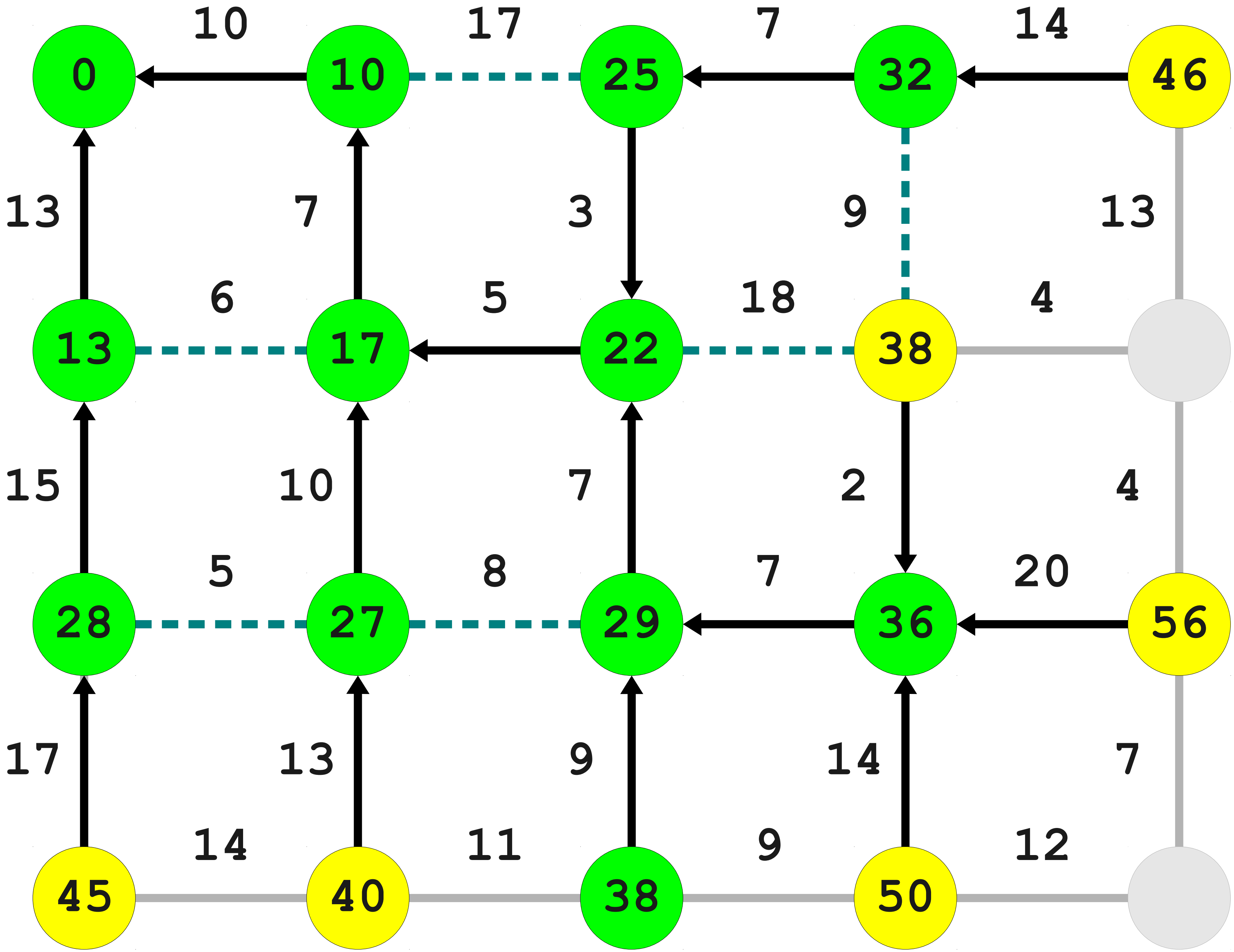

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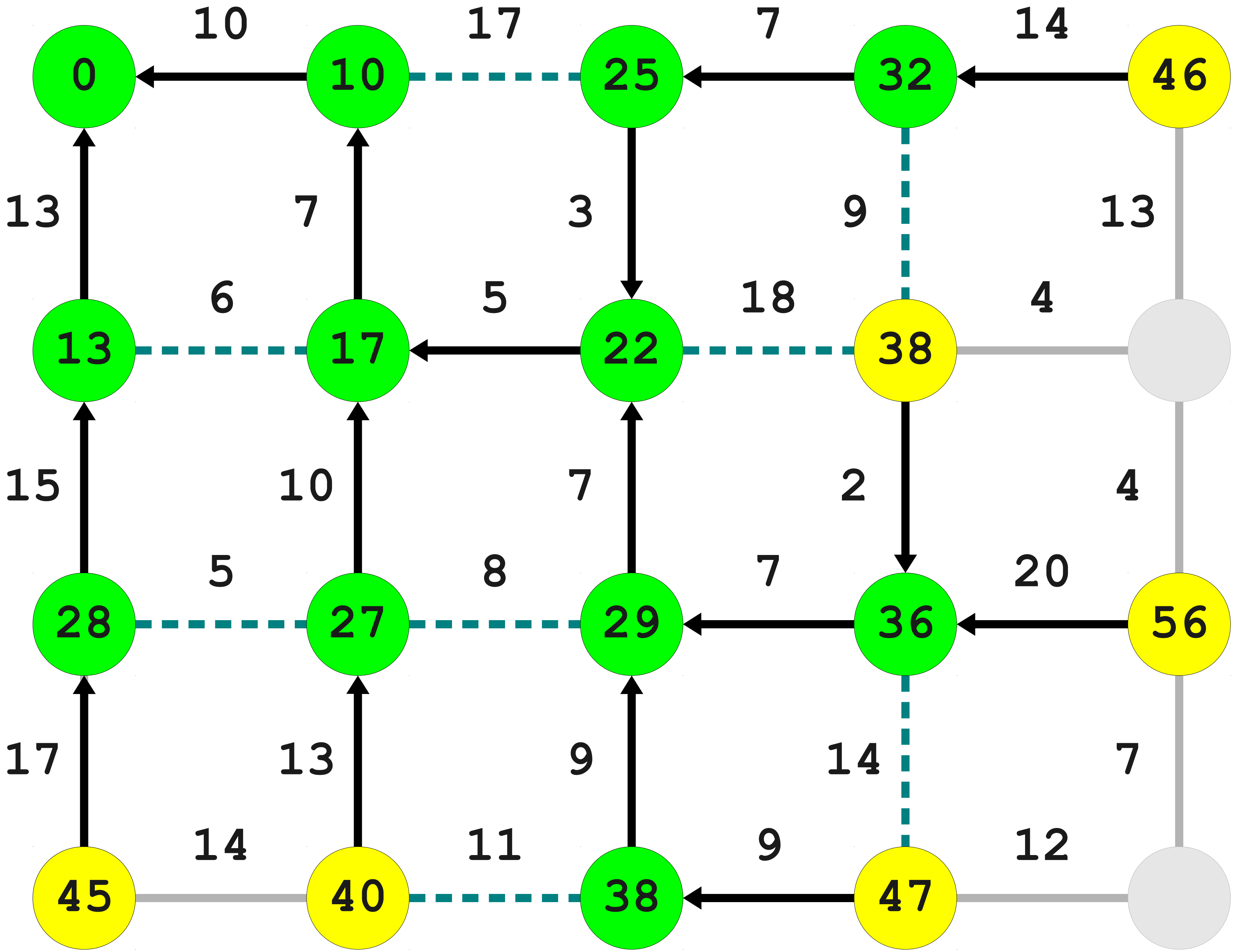

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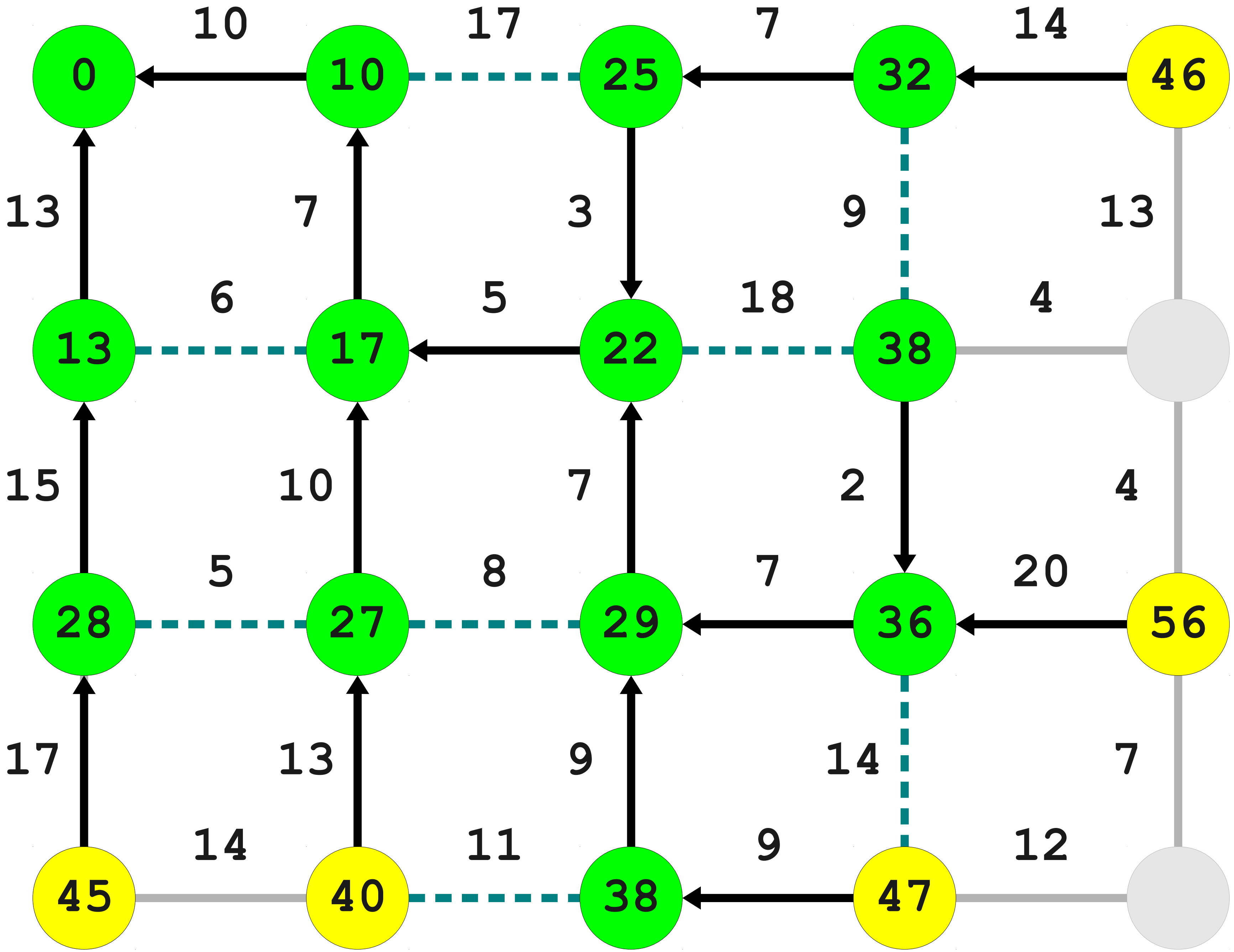
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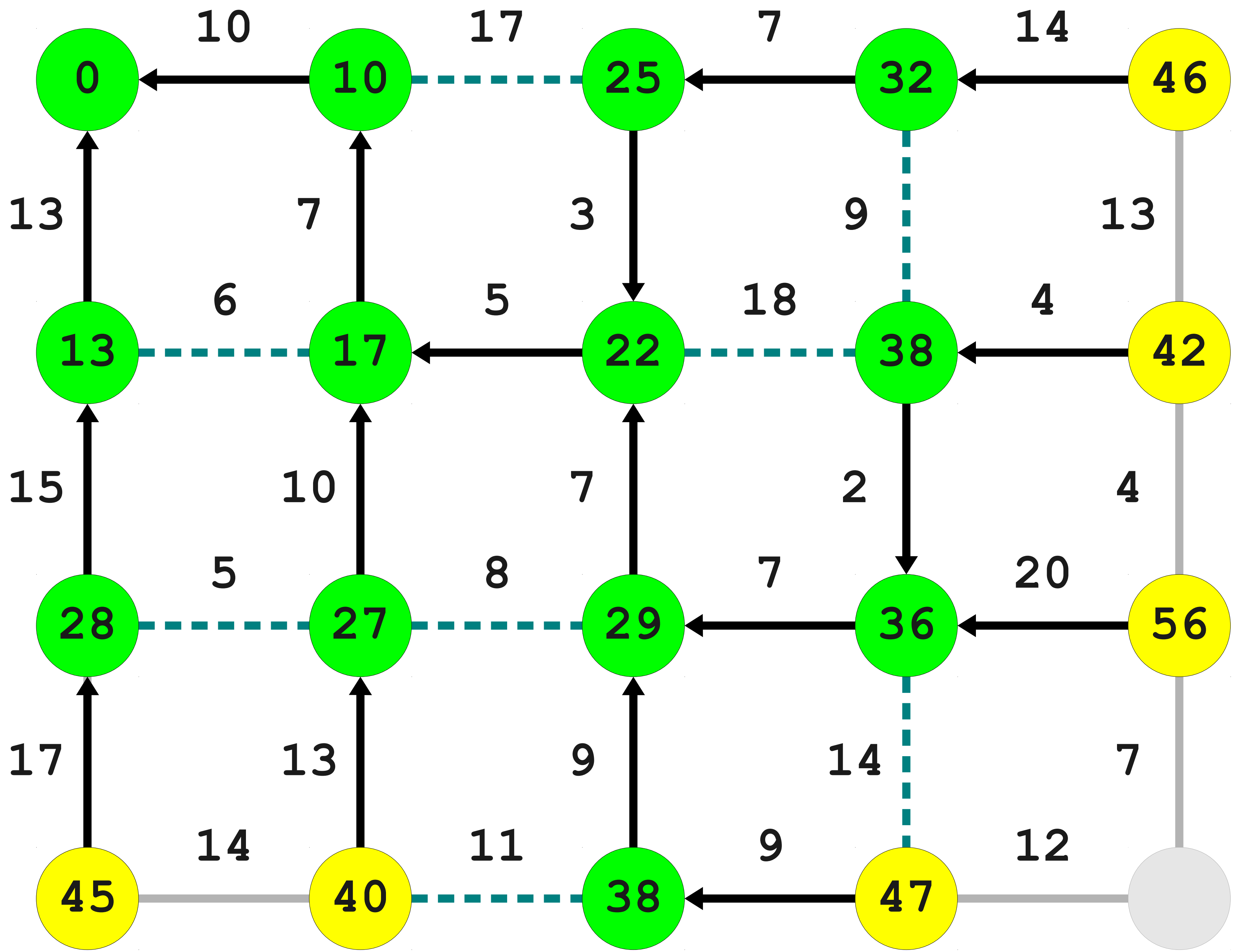
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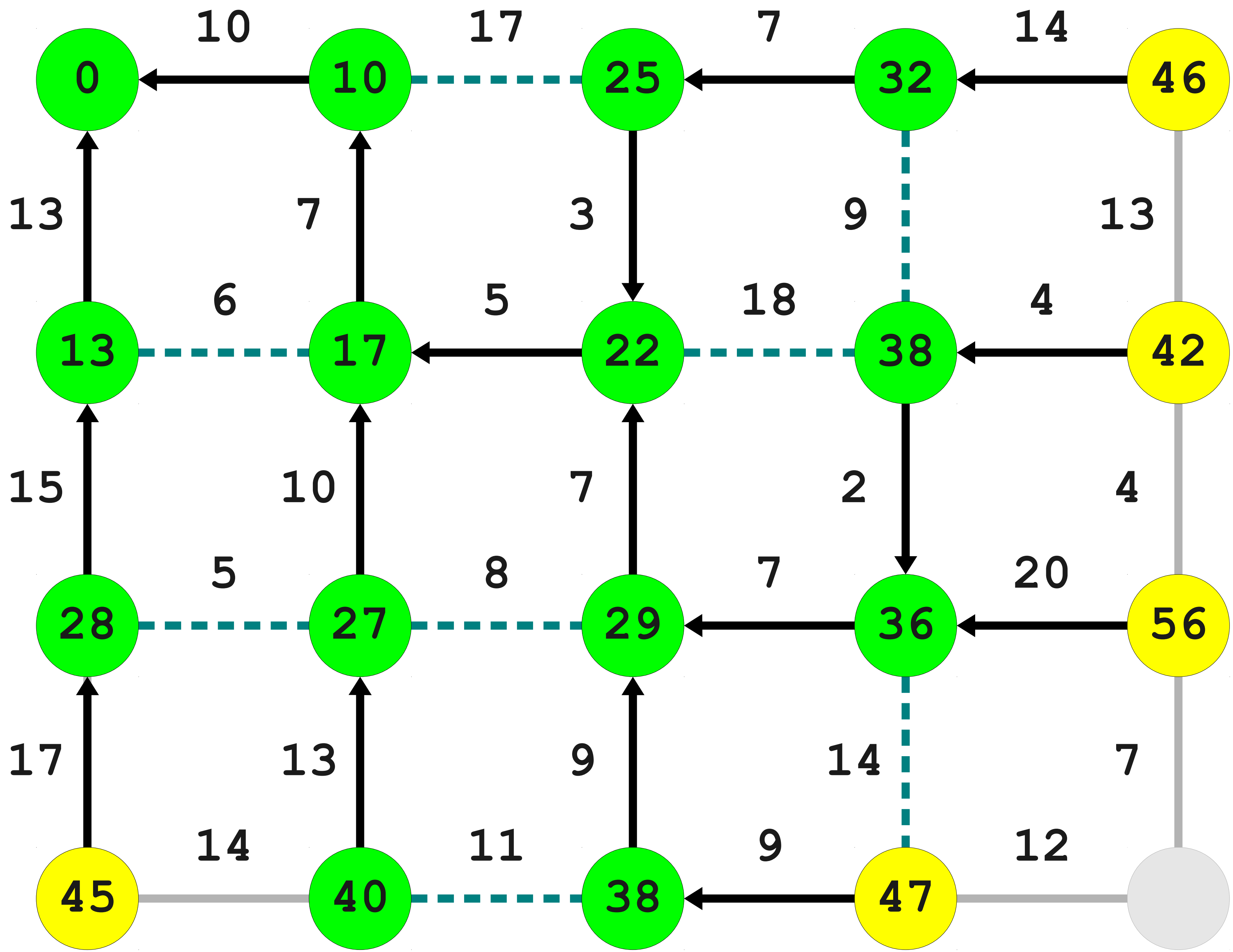

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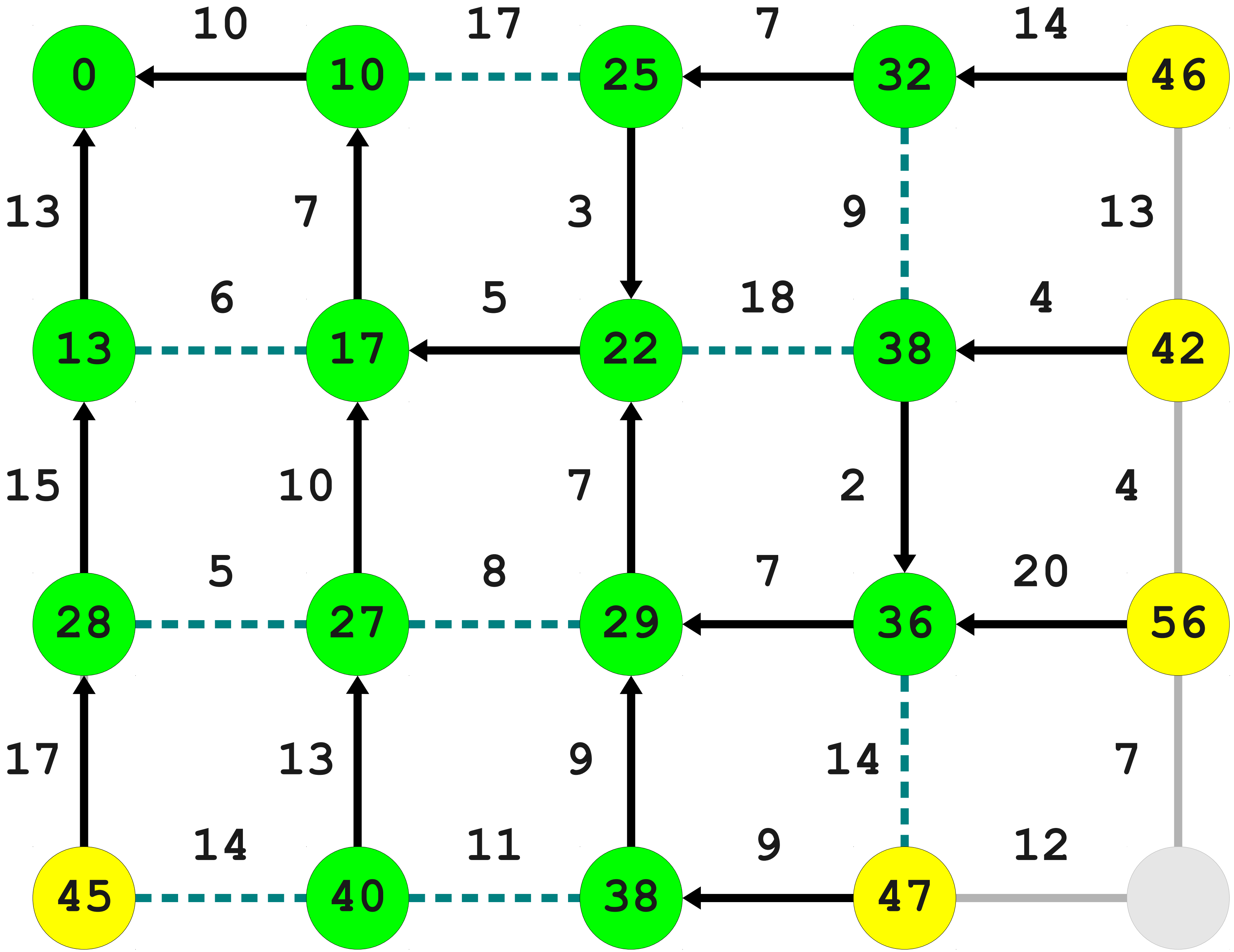
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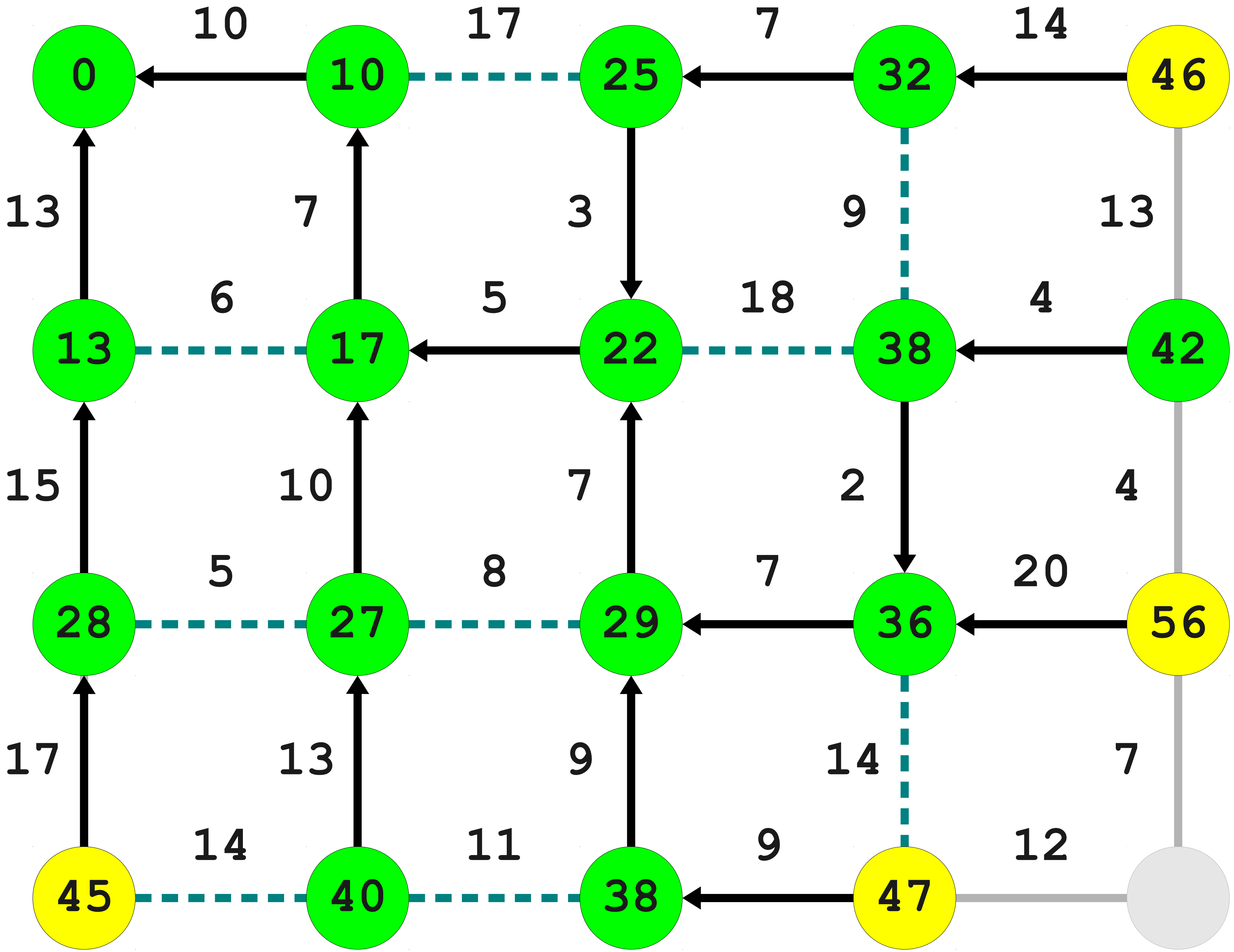
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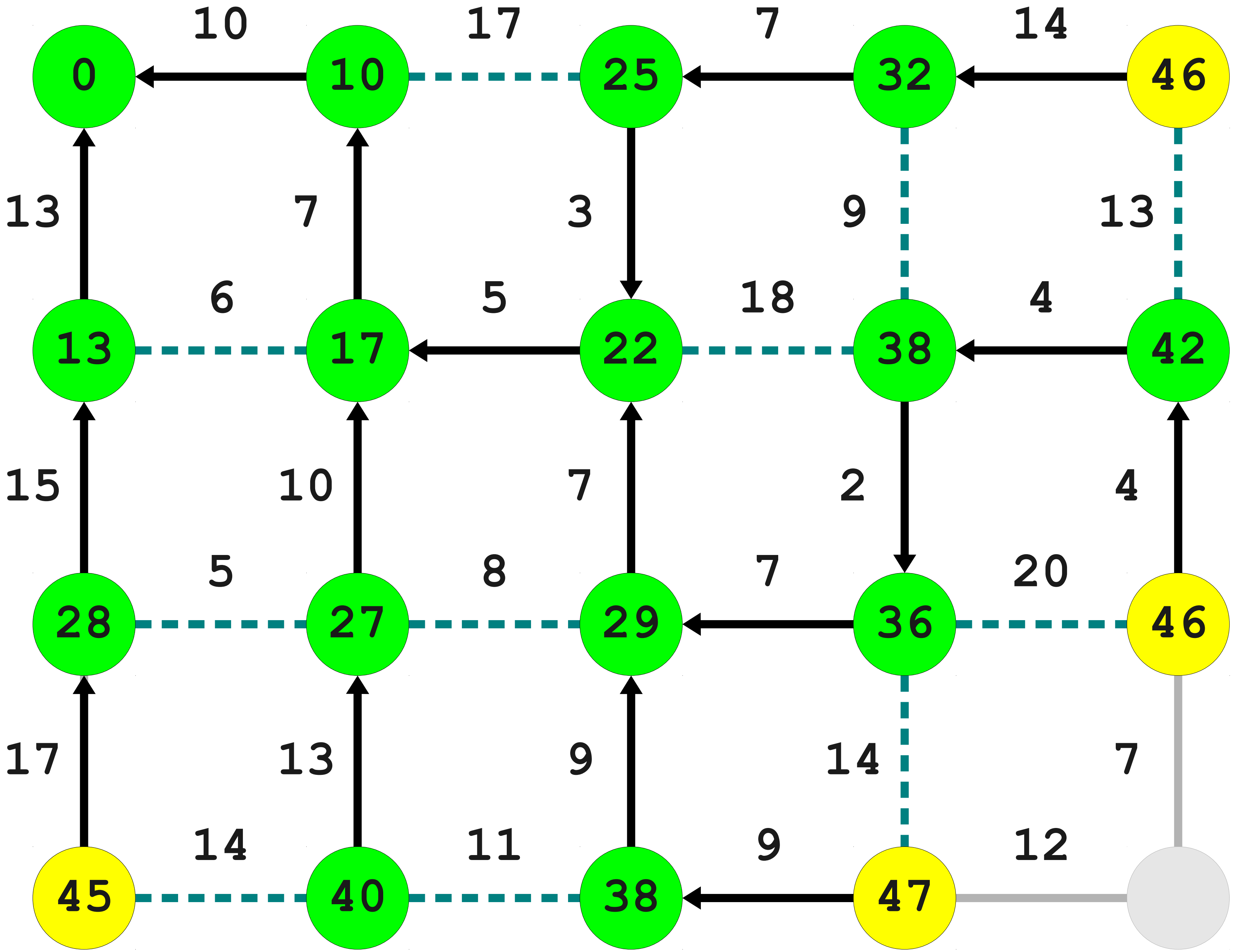
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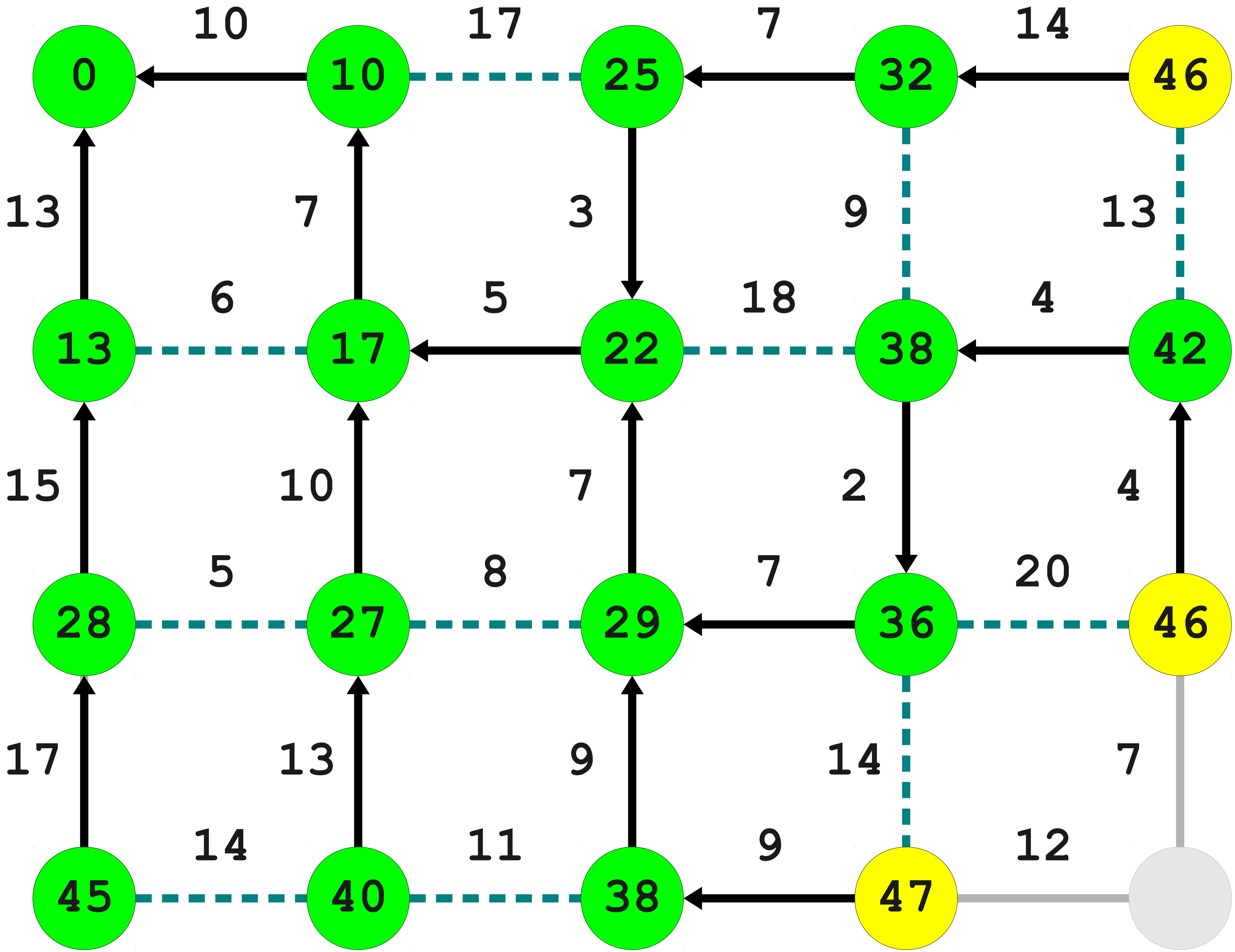
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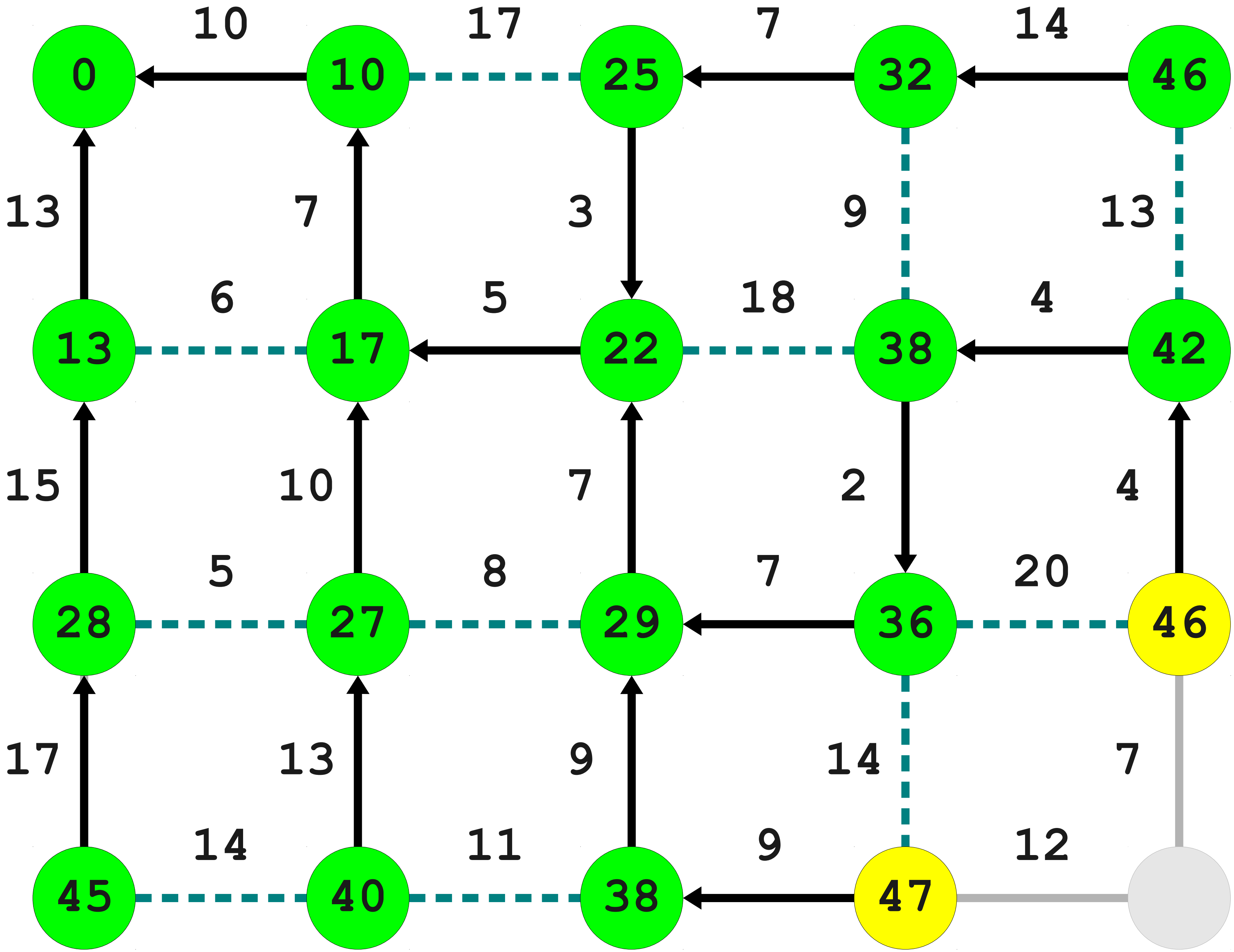
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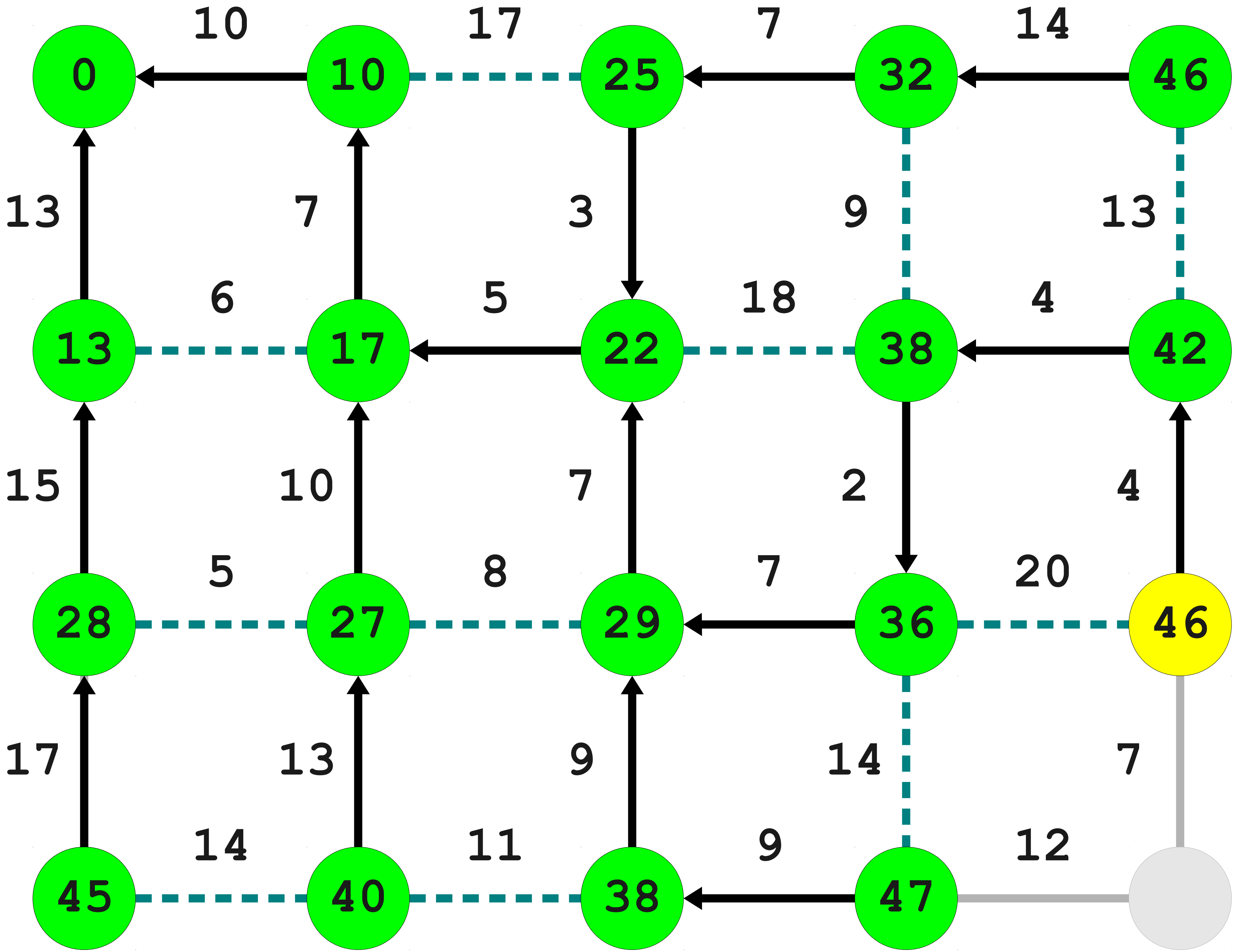
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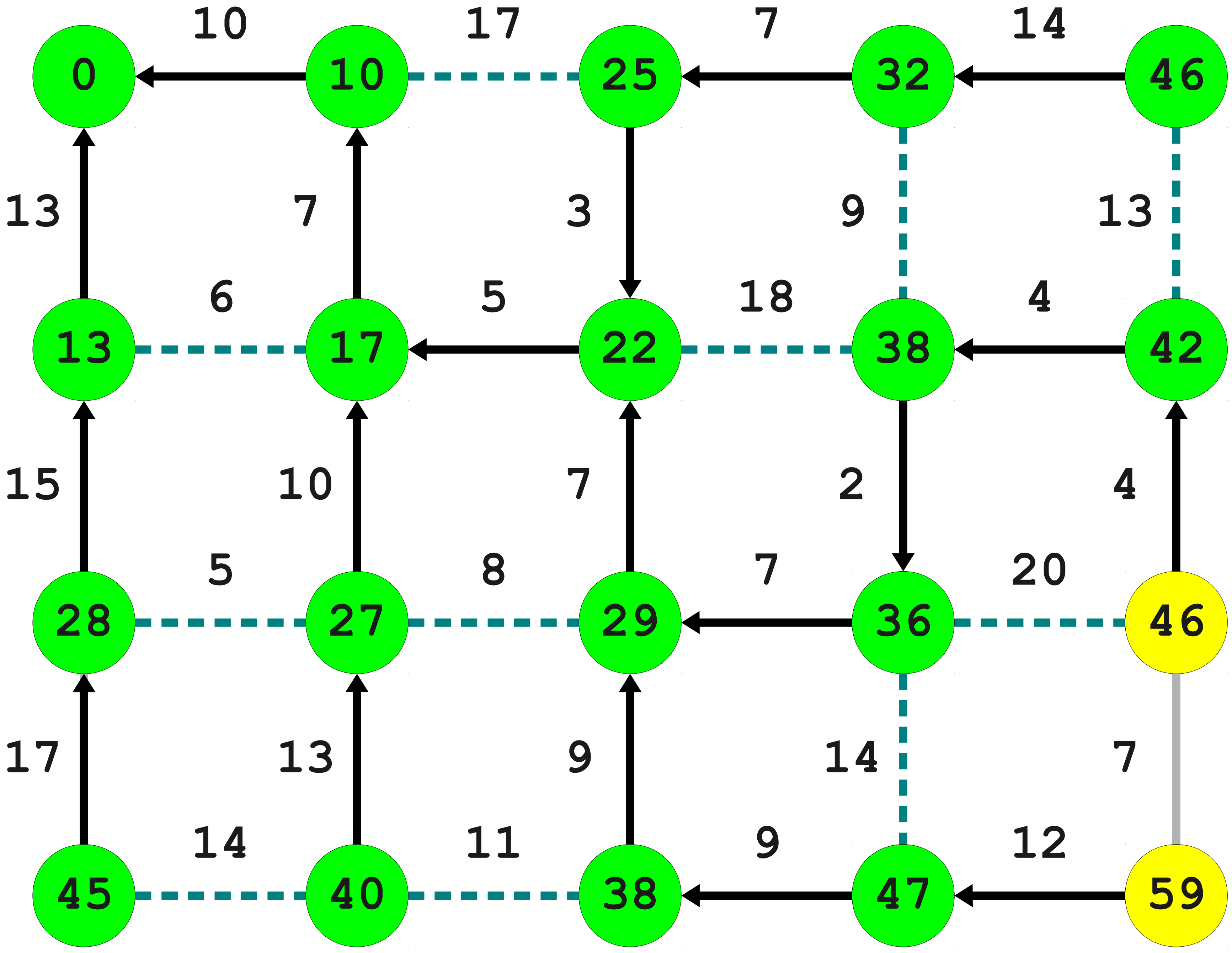

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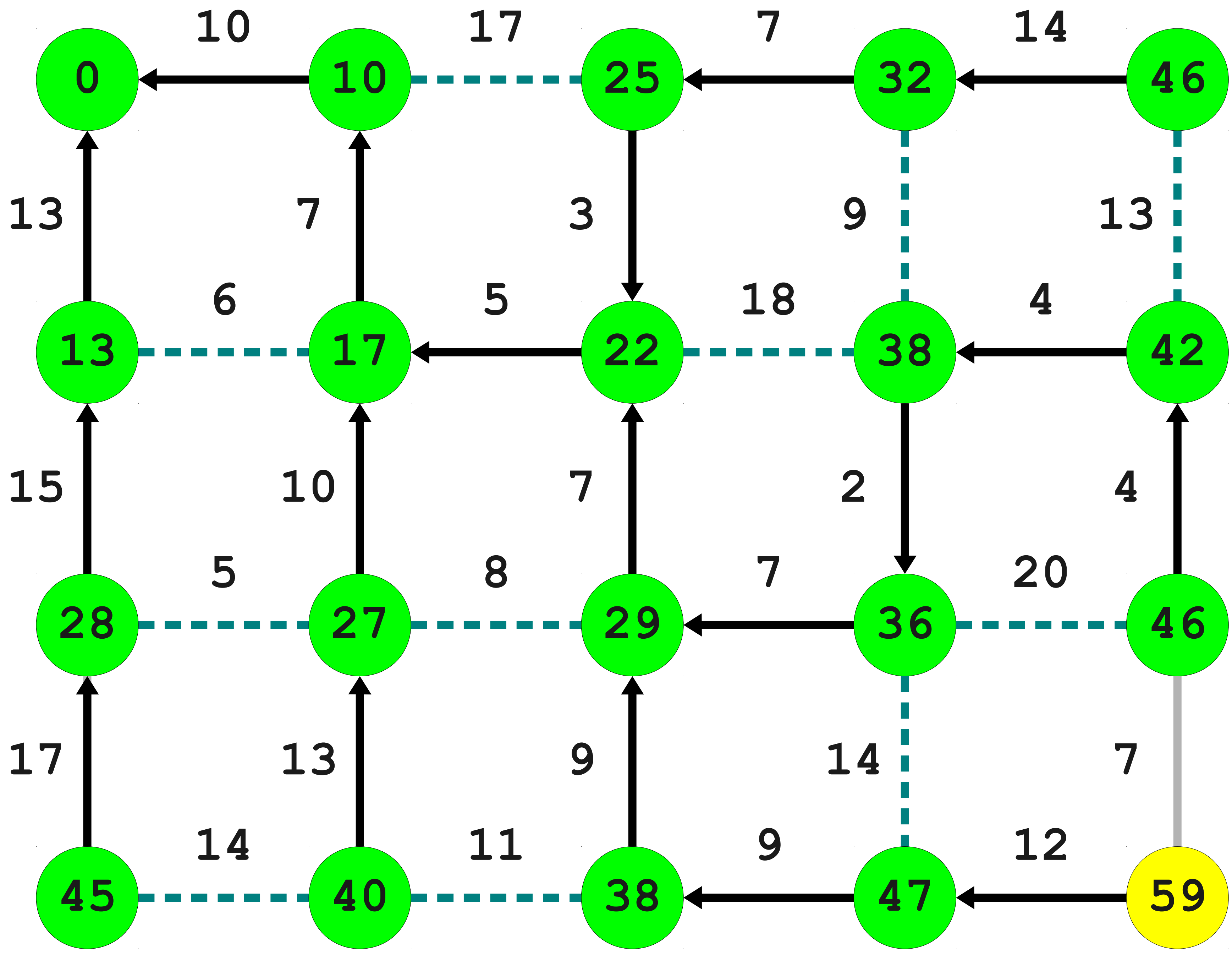
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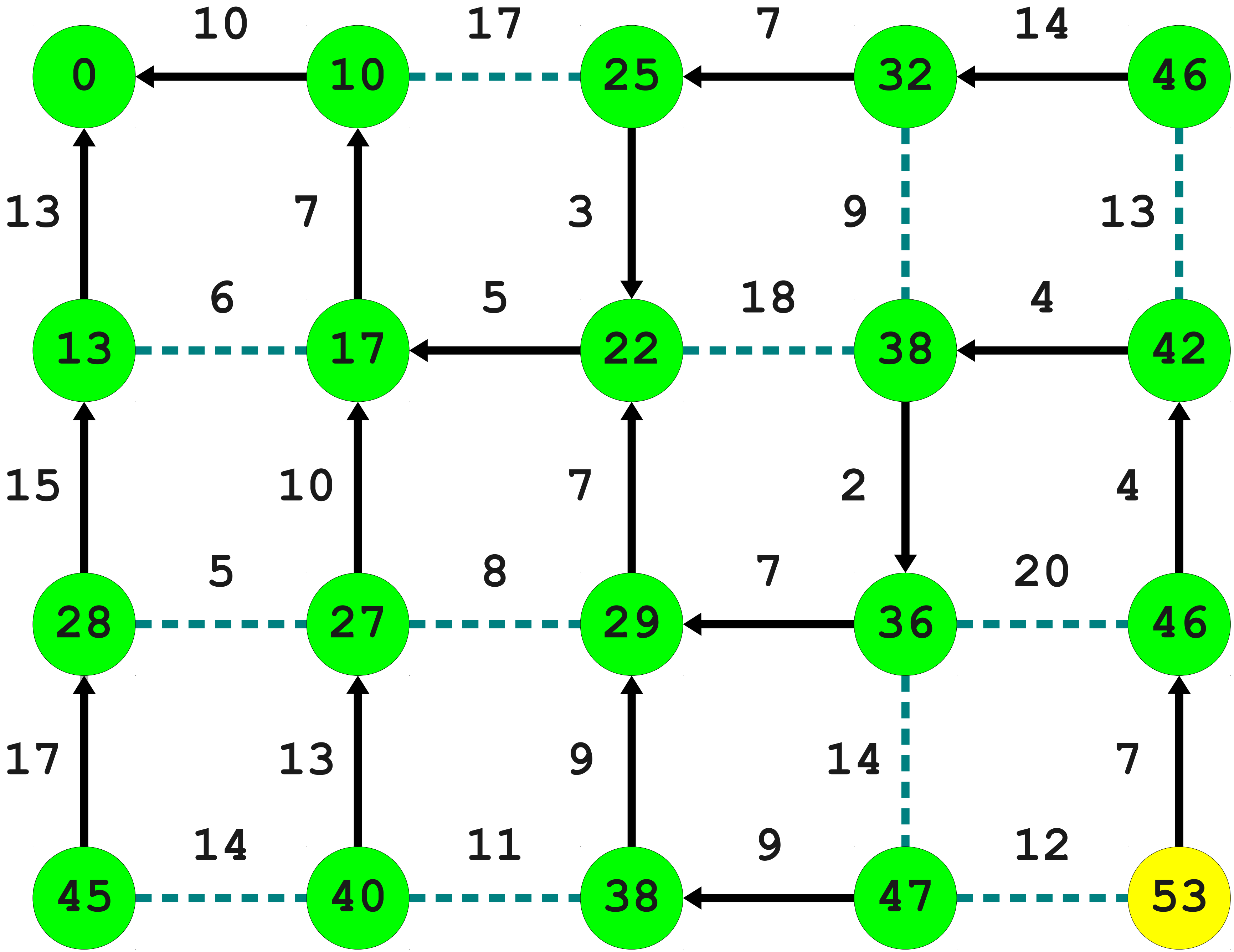

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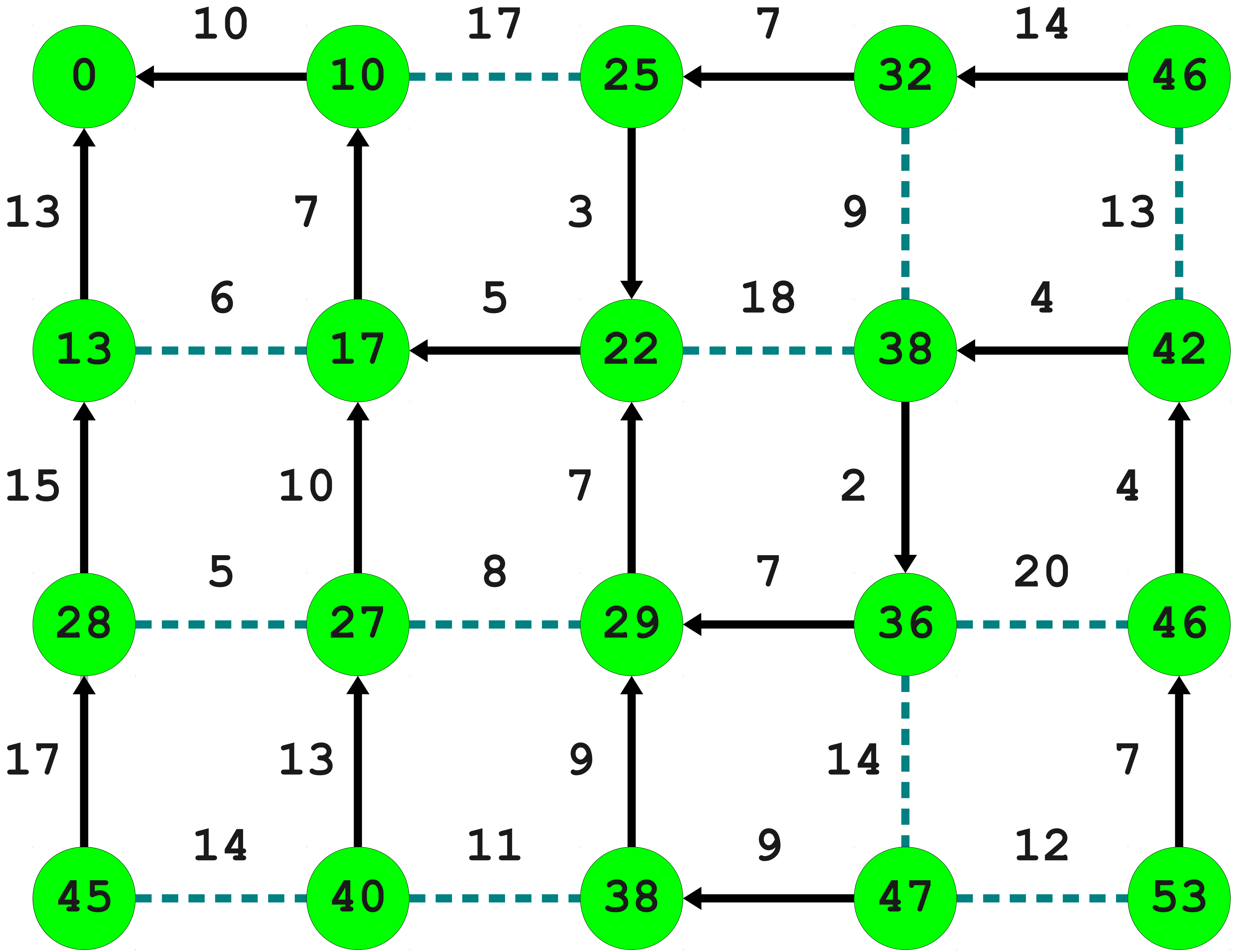

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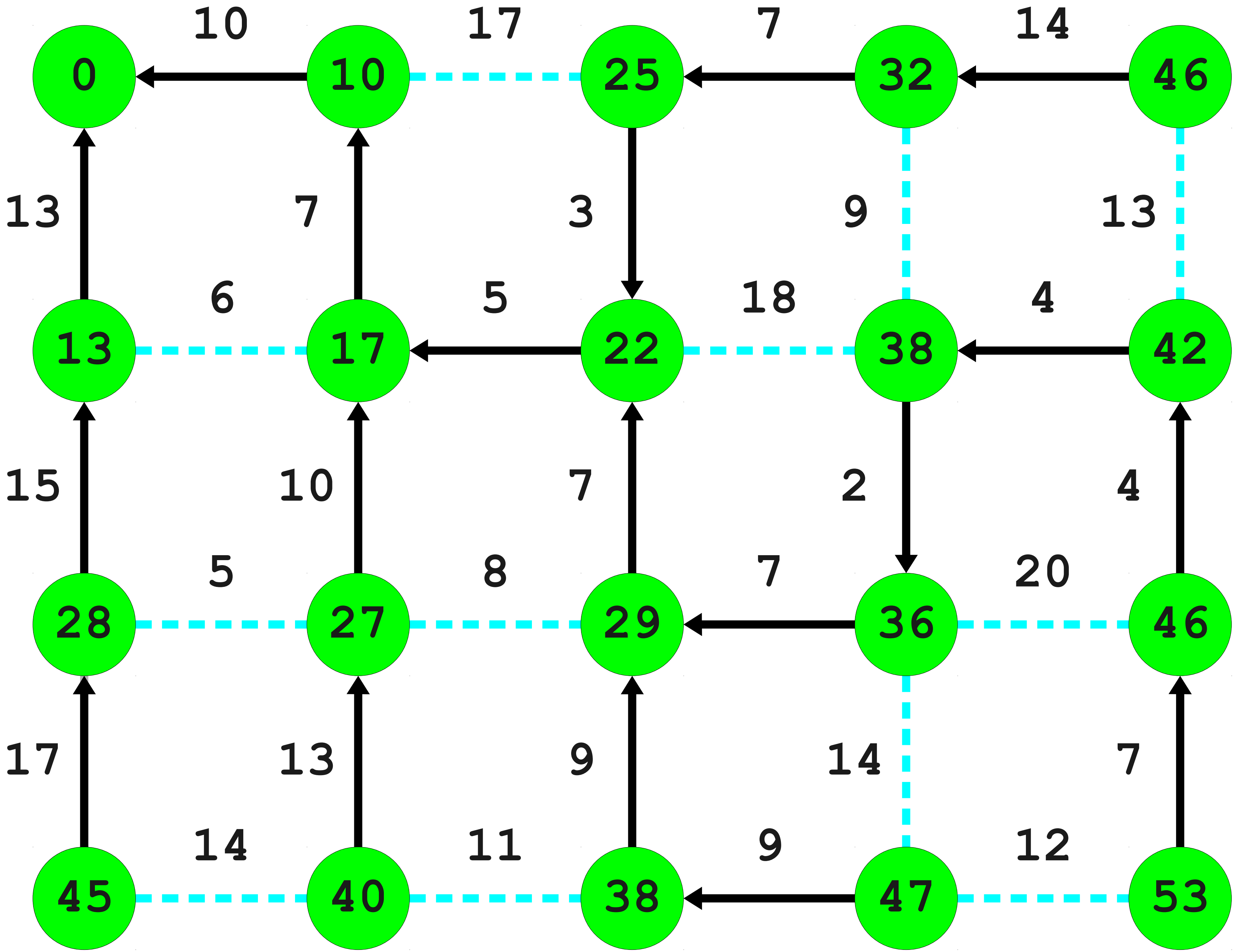
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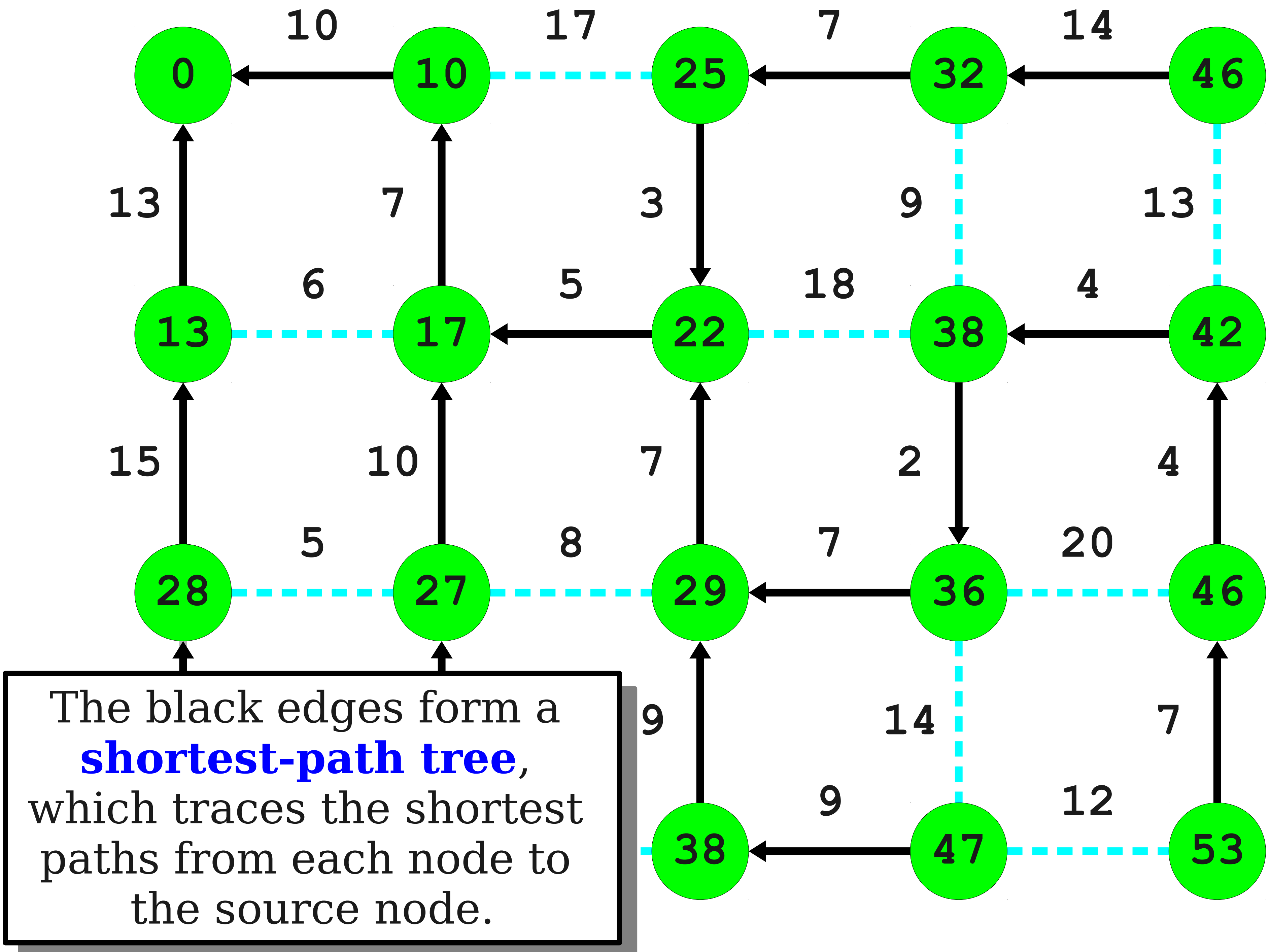
The black edges form a **shortest-path tree**, which traces the shortest paths from each node to the source node.

```
procedure dijkstrasAlgorithm(s, G):
 let q be a new queue
 for each v in V:
    dist[v] = ∞

 dist[s] = 0
 enqueue(s, q)

 while q is not empty:
    let v be a node in q minimizing dist[v]
    remove(v, q)

    for each node u connected to v:
       if dist[u] > dist[v] + l(u, v):
          dist[u] = dist[v] + l(u, v)
          if u is not enqueued into q:
             enqueue(u, q)
```

Dijkstra's Algorithm

- Assuming nonnegative edge lengths, finds the shortest path from s to each node in G.
- Correctness proof sketch is based on the second argument for breadth-first search:
  - Always picks the node v minimizing $d(s, u) + l(u, v)$ for yellow v and green u.
  - If a shorter path P exists to v, it must leave the set of green nodes through some edge (x, y).
  - But then $l(P)$ is at least $d(s, x) + l(x, y)$, which is at least $d(s, u) + l(u, v)$.
  - So the "shorter" path costs at least as much as the path we found.

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$O(m + n)$

```
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O(n^2)

Dijkstra Runtime

- Using a standard implementation of a queue, Dijkstra's algorithm runs in time $\mathbf{O(n^2)}$.
  - $O(n + m)$ time processing nodes and edges, plus $O(n^2)$ time finding the lowest-cost node.
  - Since $m = O(n^2)$, $O(n + m + n^2) = O(n^2)$.
- Using a slightly fancier data structure (a binary heap), can be made to run in time $\mathbf{O(m \log n)}$.
  - Is this necessarily more efficient?
  - More on how to do this later this quarter.
- Using a *much* fancier data structure (the *Fibonacci heap*), can be made to run in time $\mathbf{O(m + n \log n)}$.
  - Take CS166 for details!

Shortest Path Algorithms

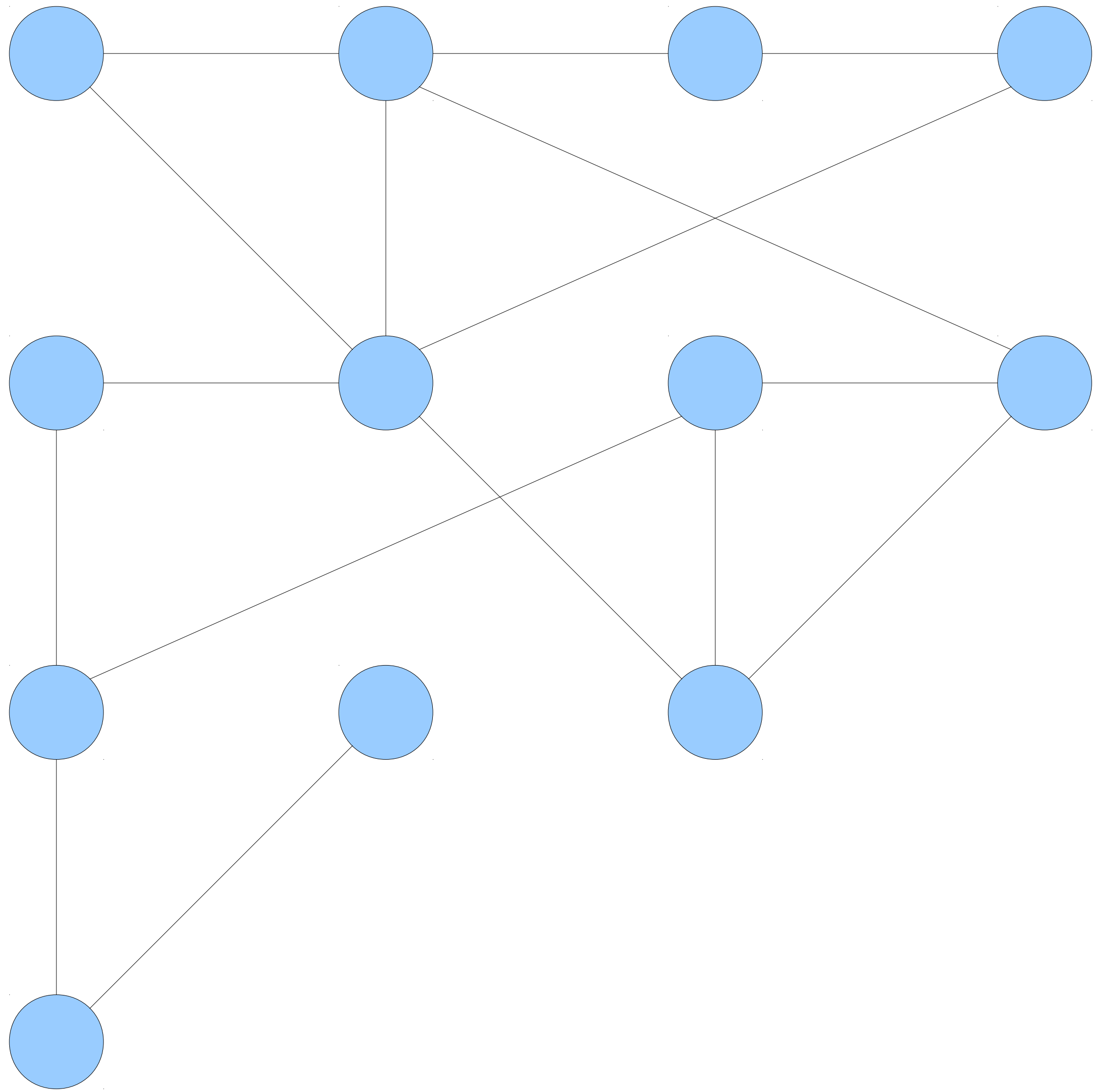
- If all edges have the same weight, can use breadth-first search to find shortest paths.
  - Takes time O(m + n).
- If edges have nonnegative weight, can use Dijkstra's algorithm.
  - Takes time O(n^2), or less using more complex data structures.
- What about the case where edges can have negative weight?
  - More on that later in the quarter...

Depth-First Search

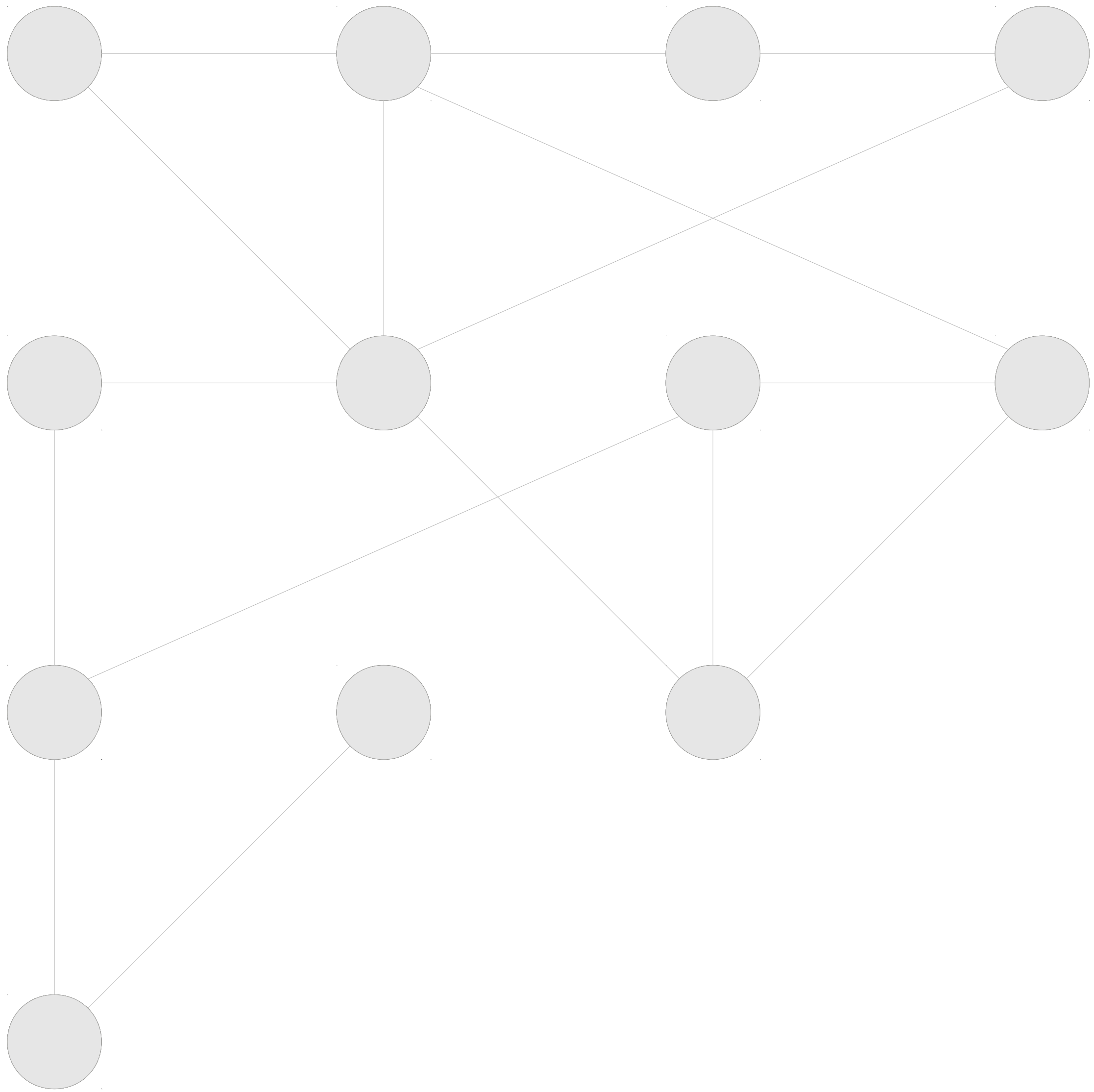
BFS and DFS

- Last time, we saw the **breadth-first search** (**BFS**) algorithm, which explored a graph and found shortest paths.
- The algorithm explored outward in all directions uniformly.
- We will now see **depth-first search** (**DFS**), an algorithm that explores out in one direction, backing up when necessary.

Depth-First Search



Depth-First Search



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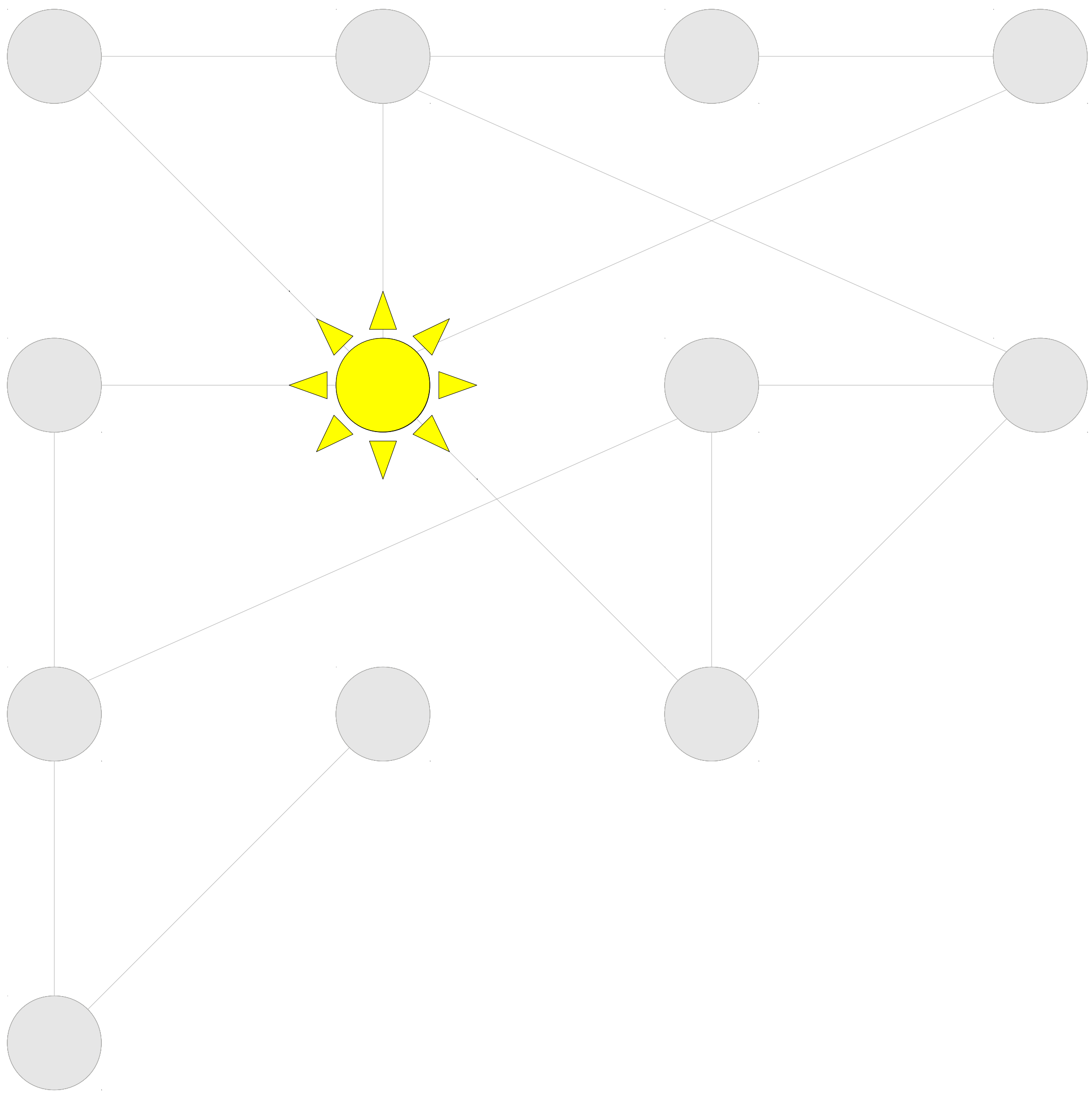

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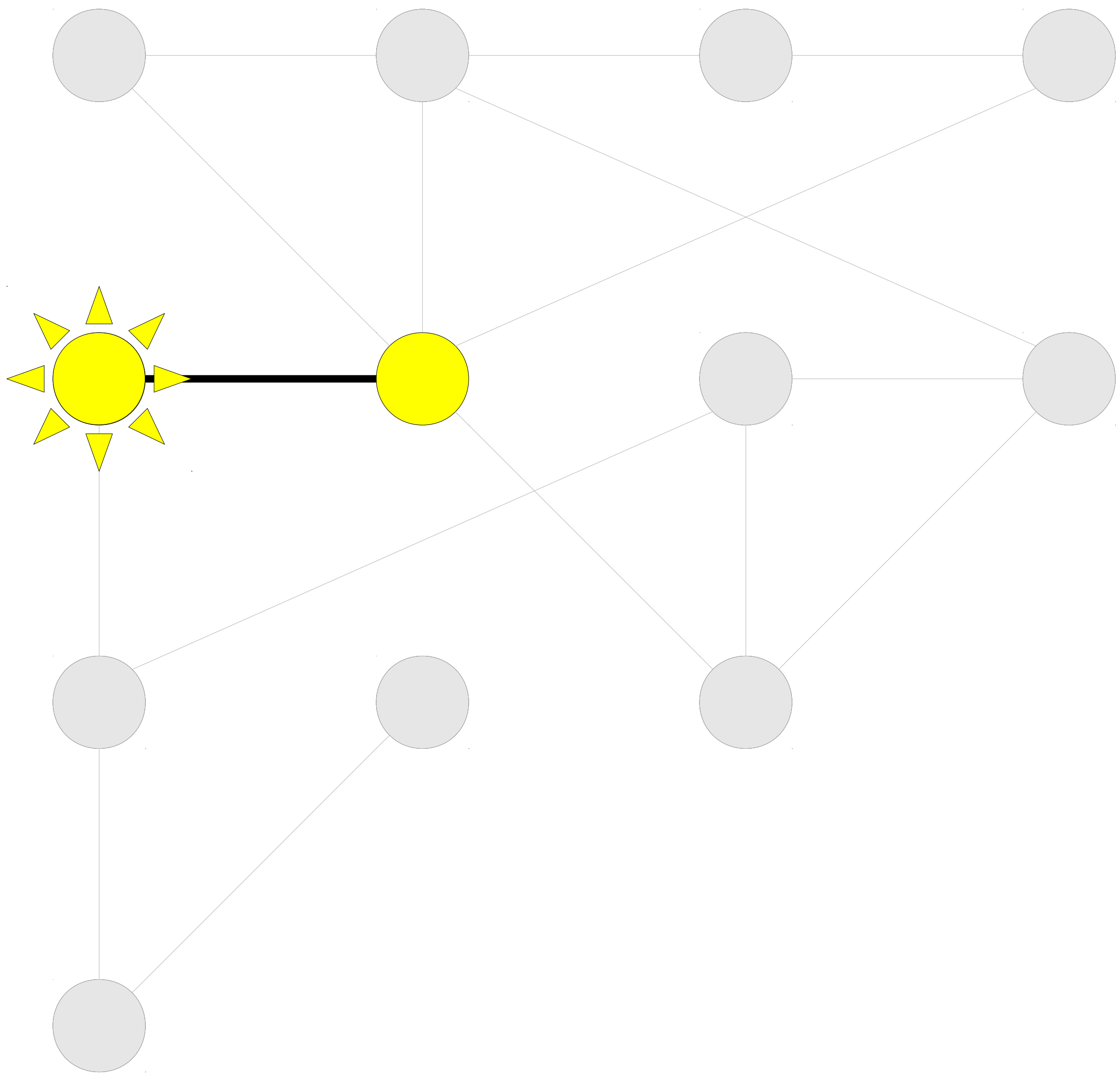

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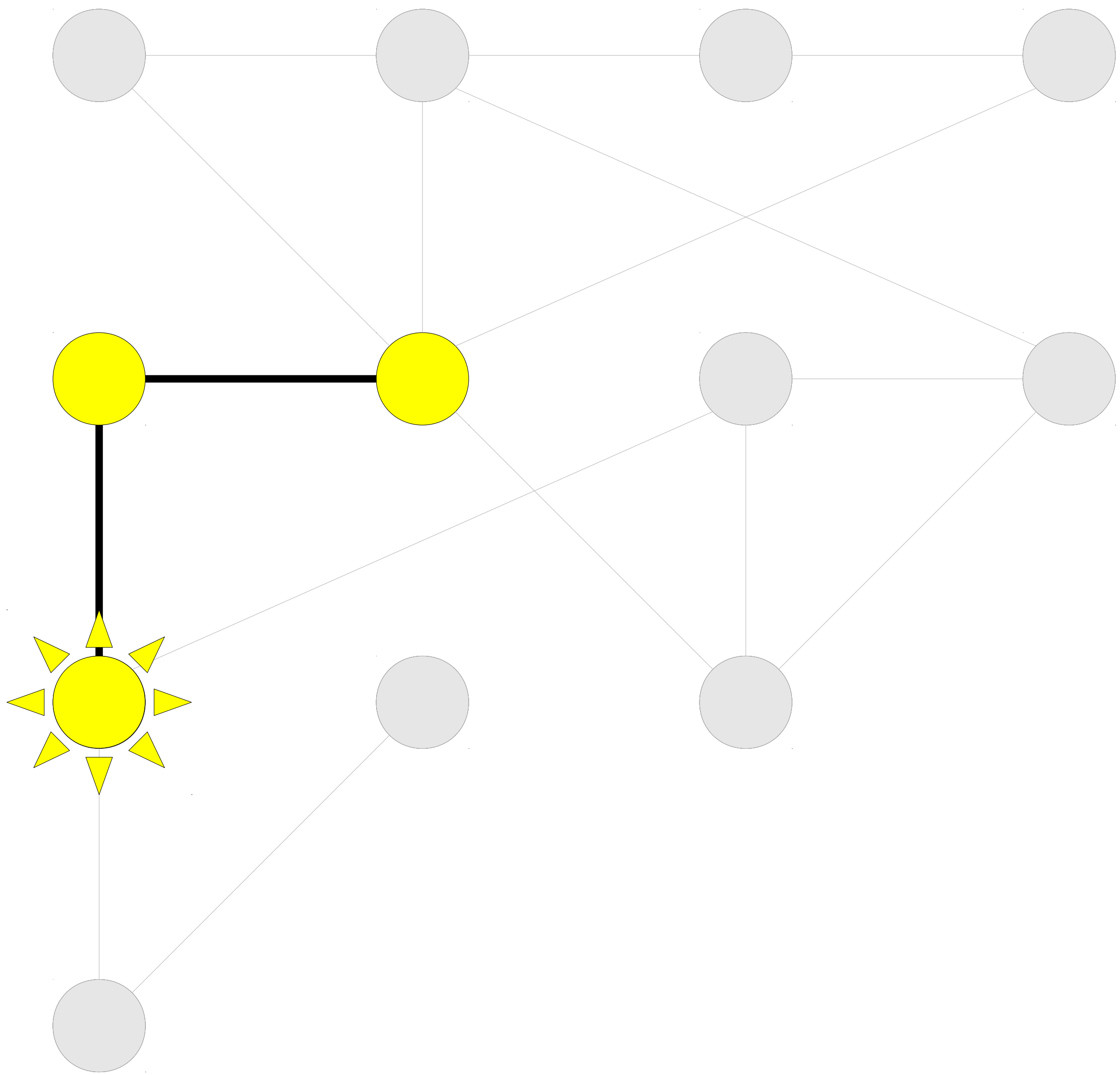

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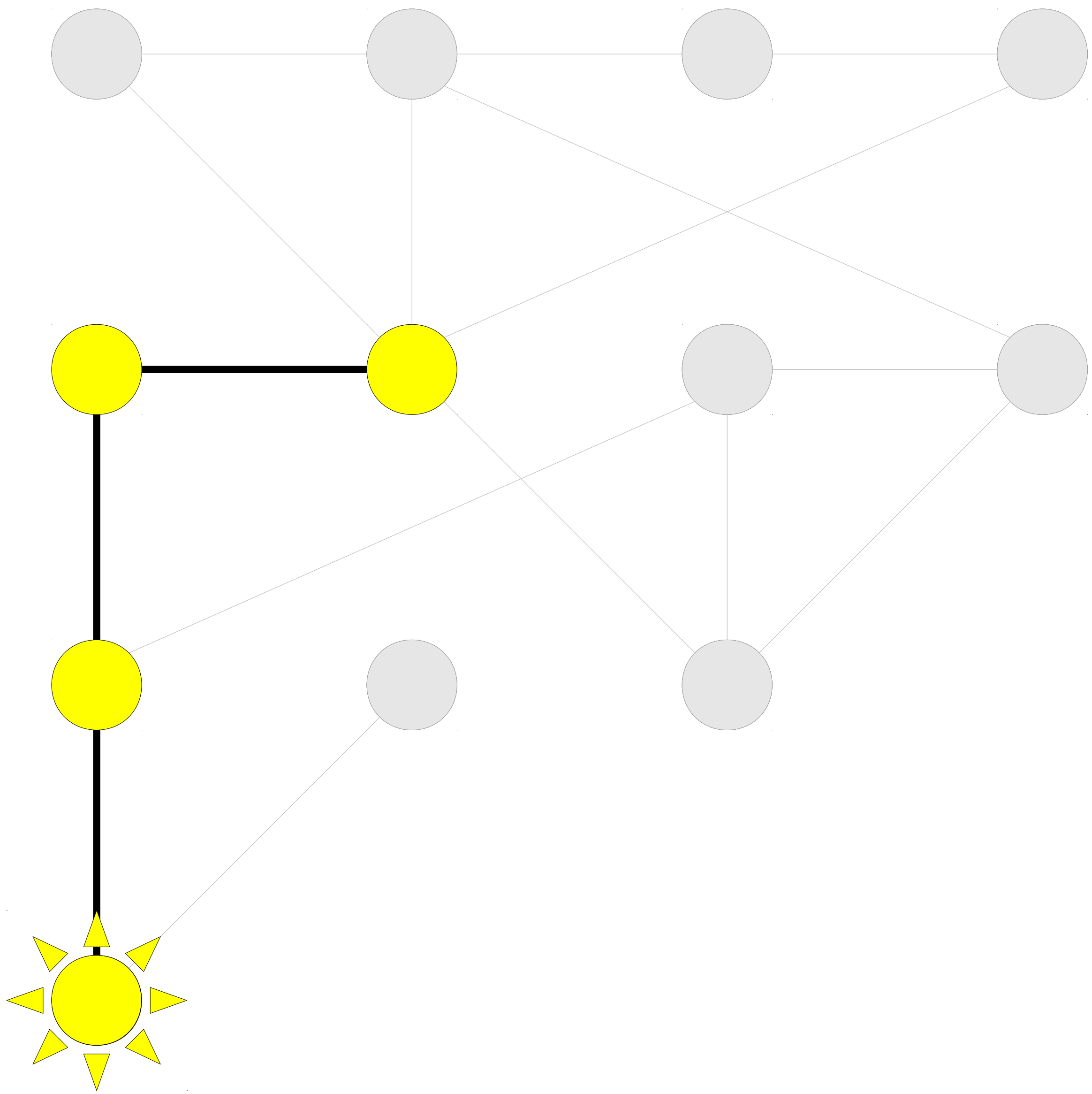

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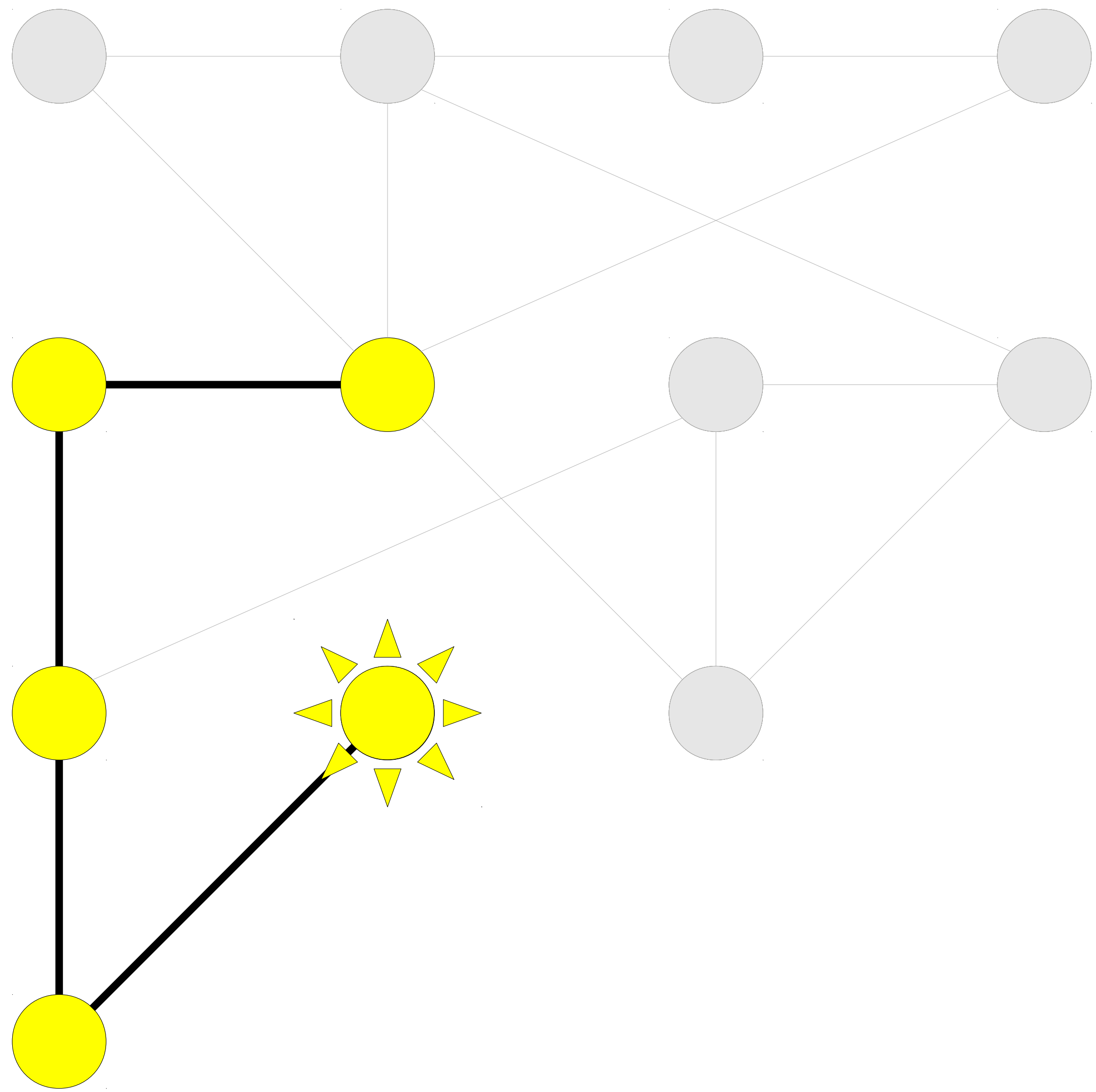

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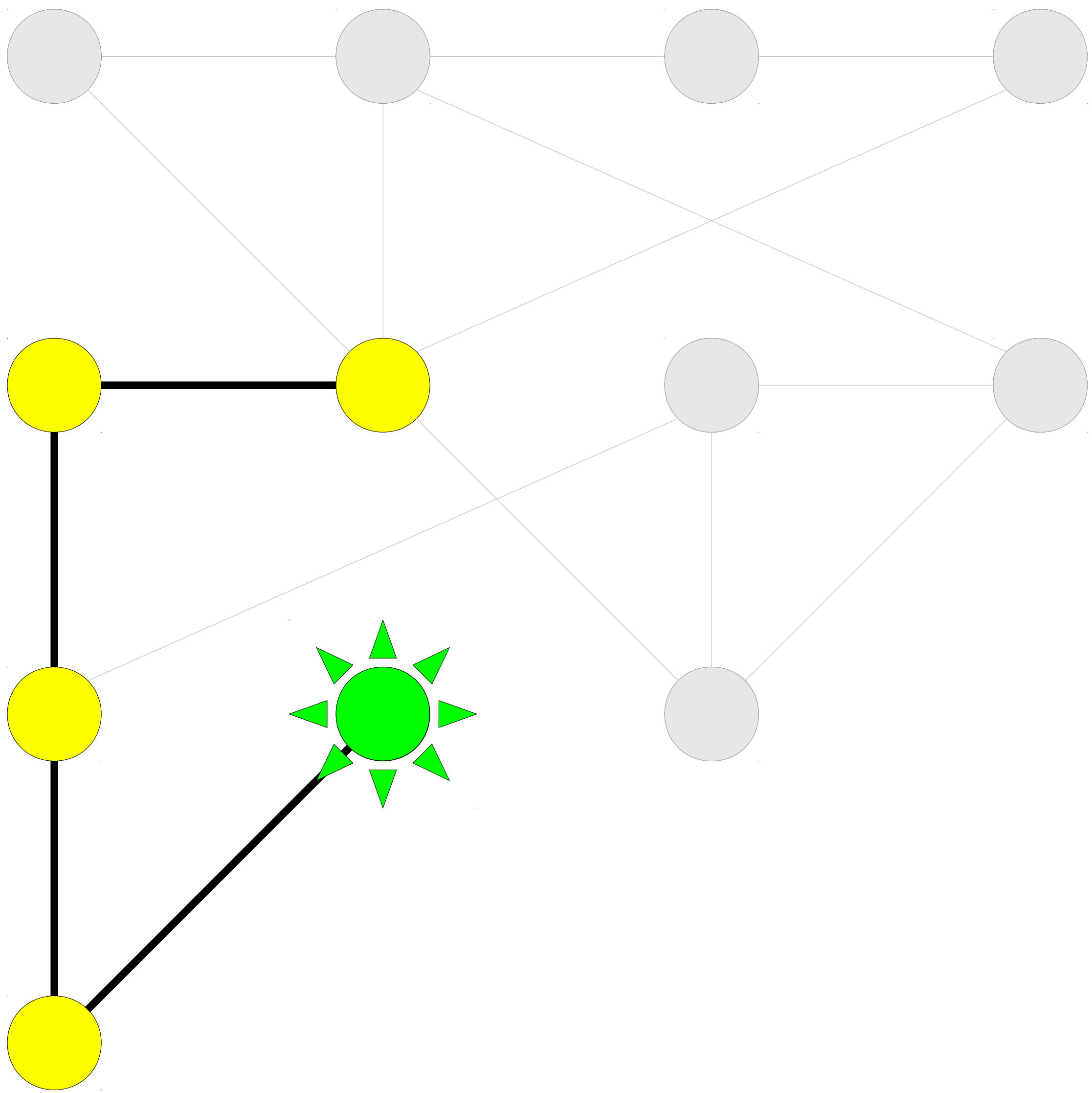

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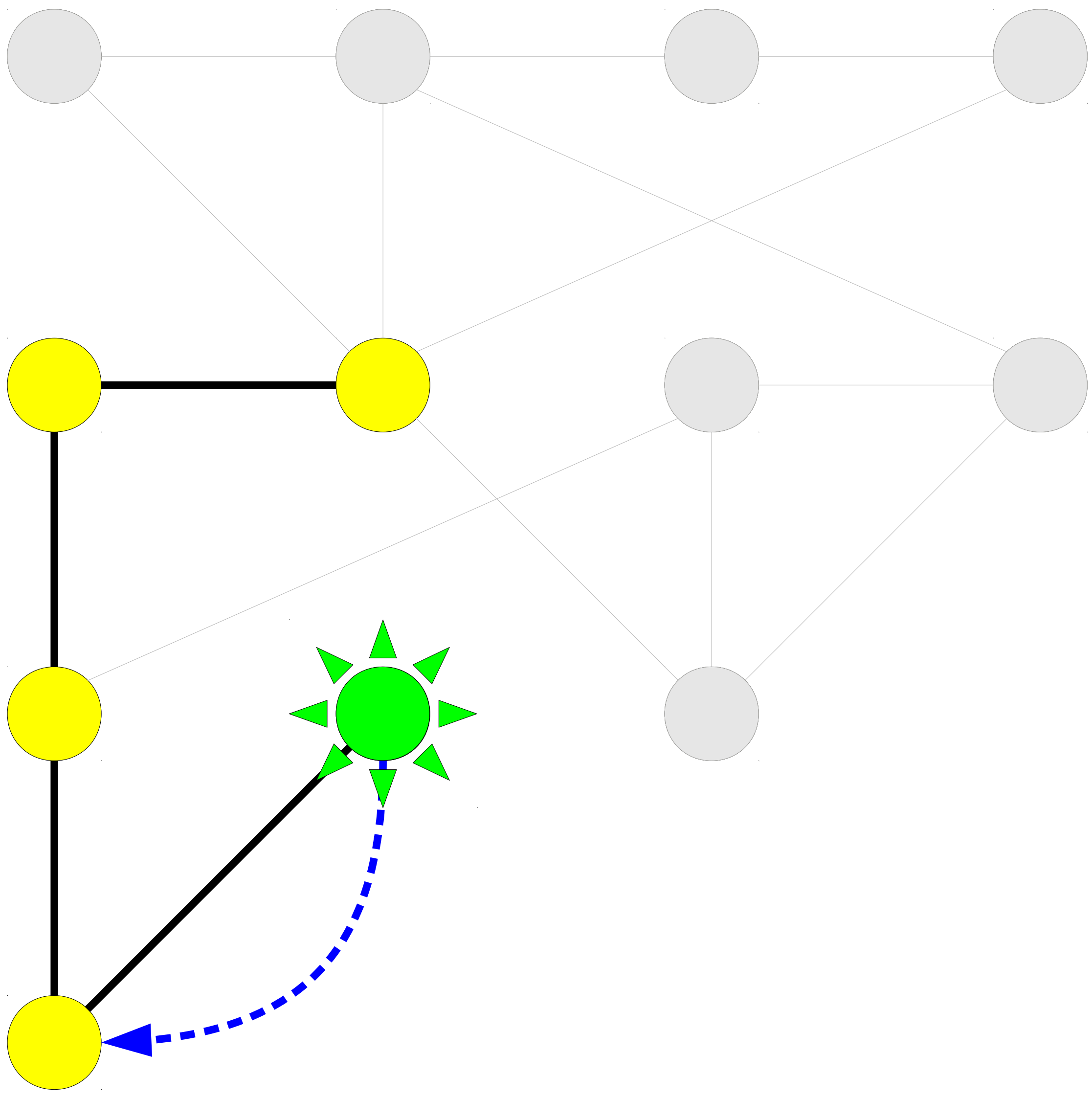

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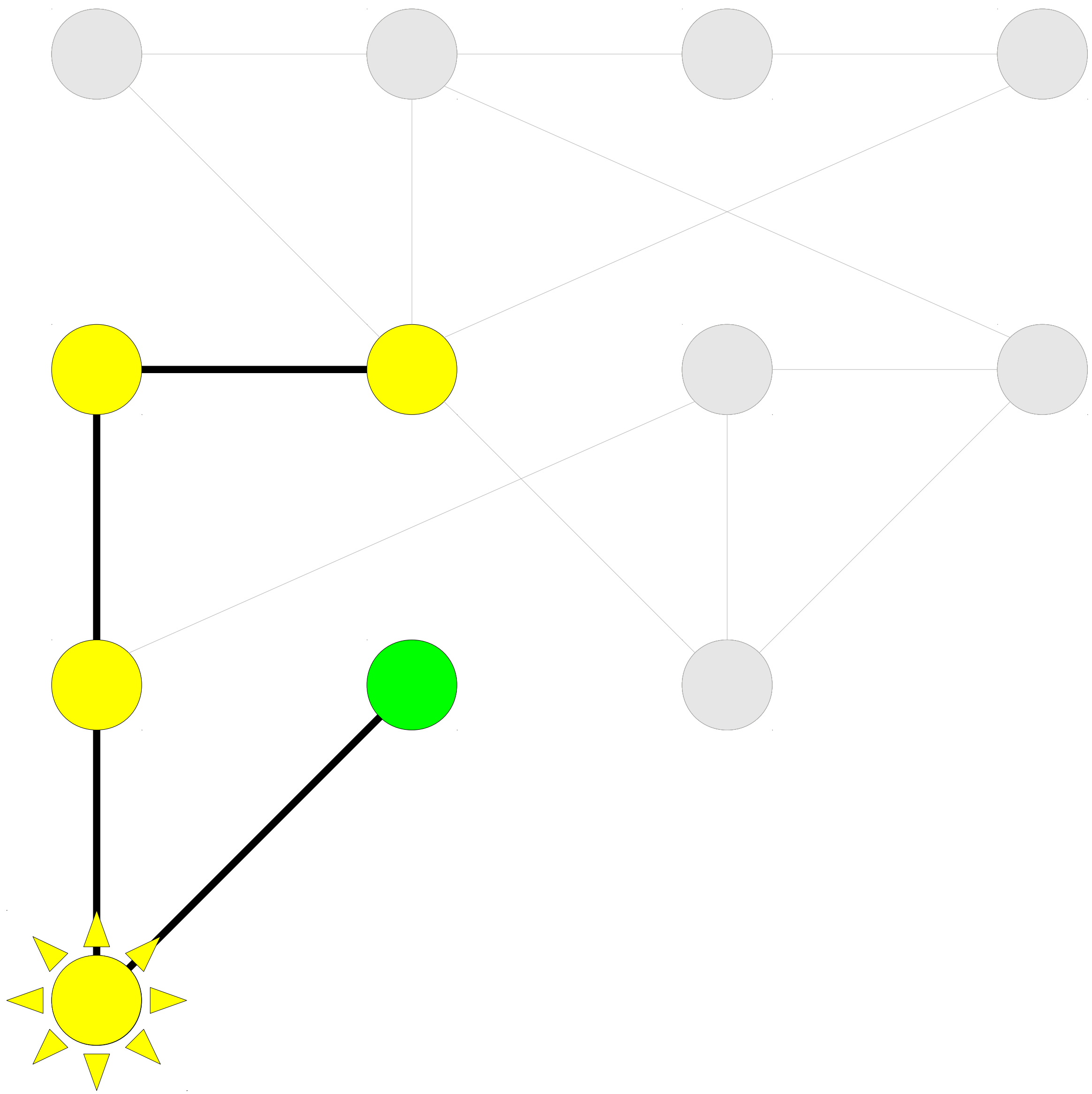

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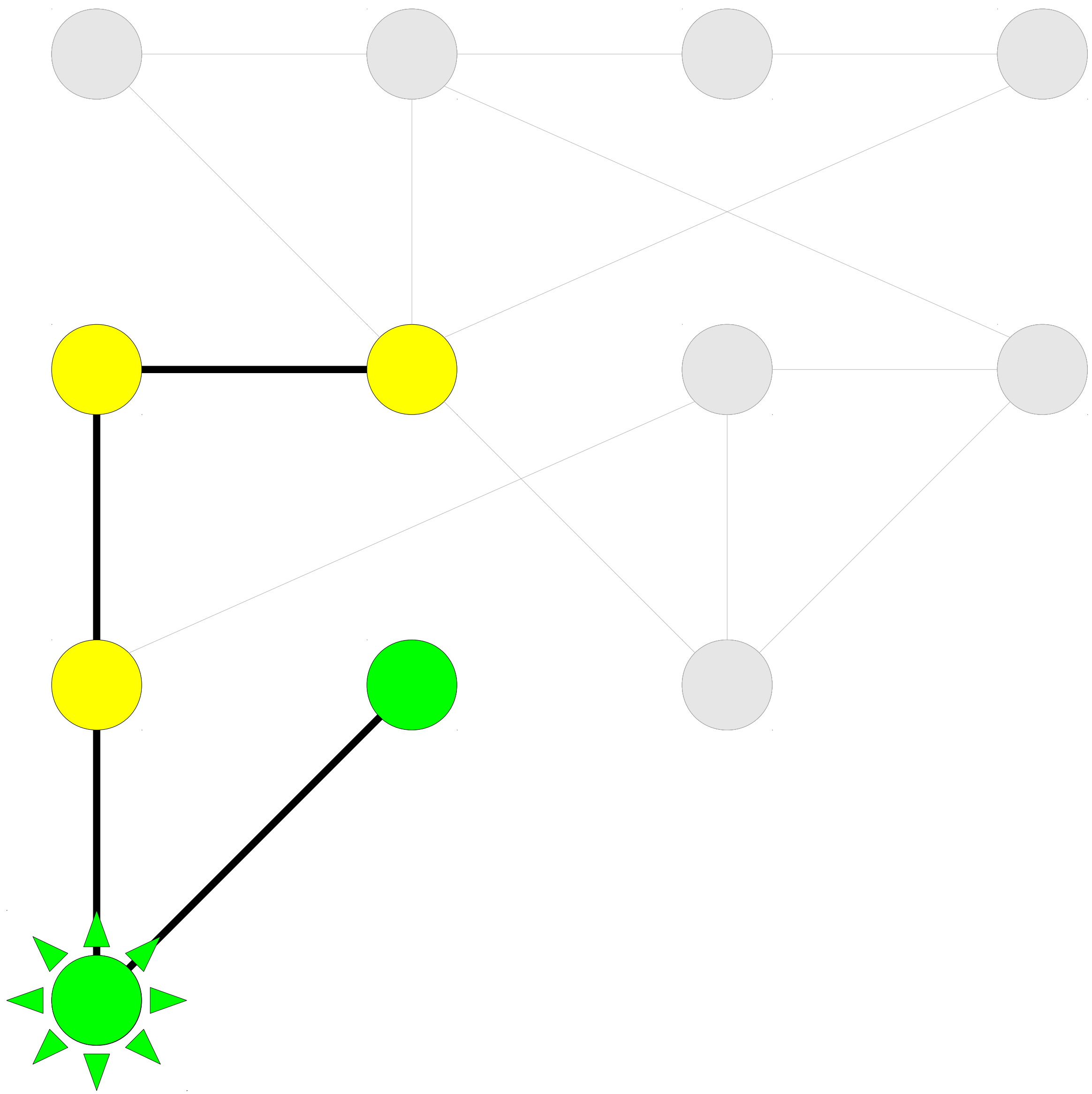

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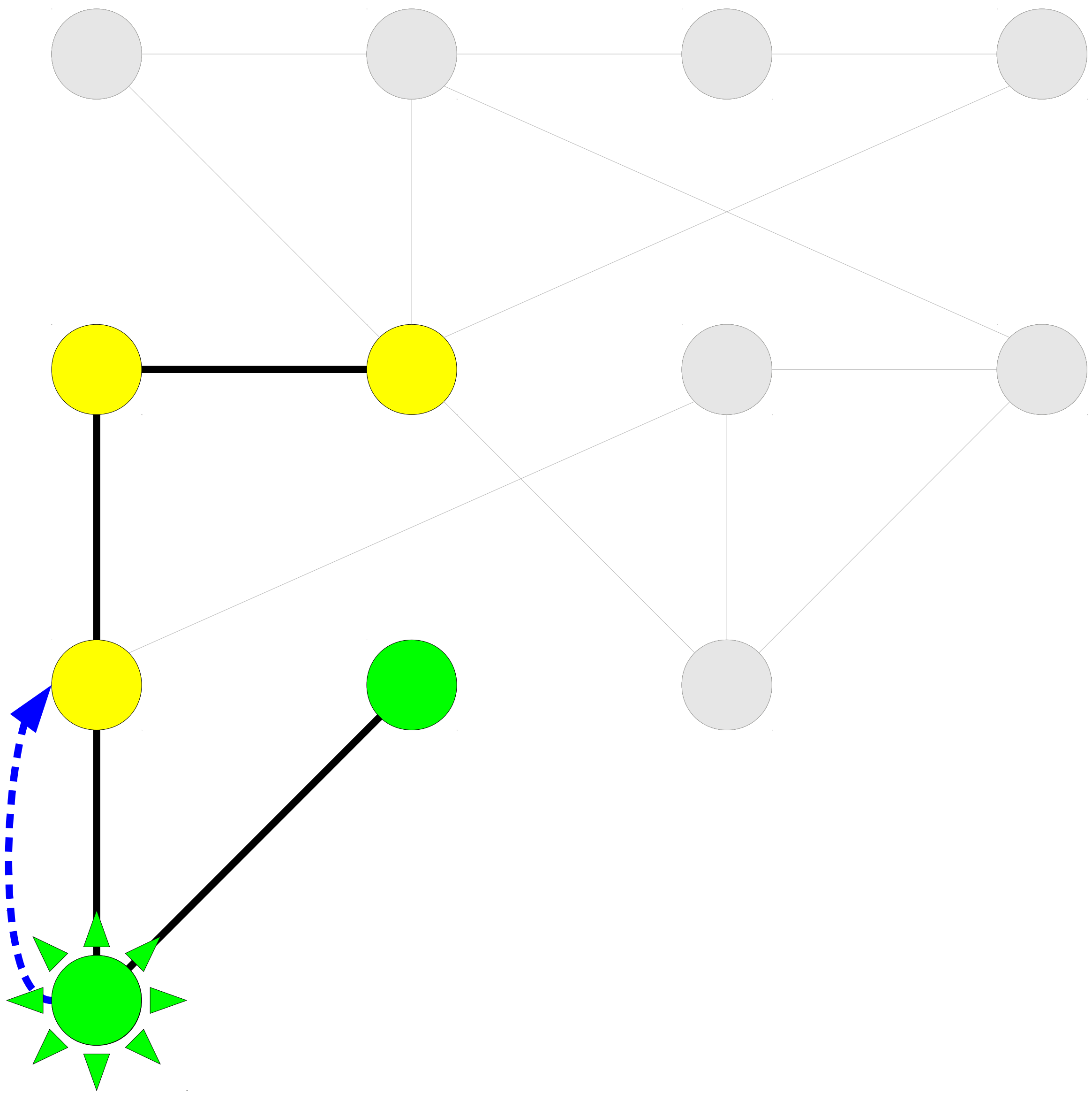

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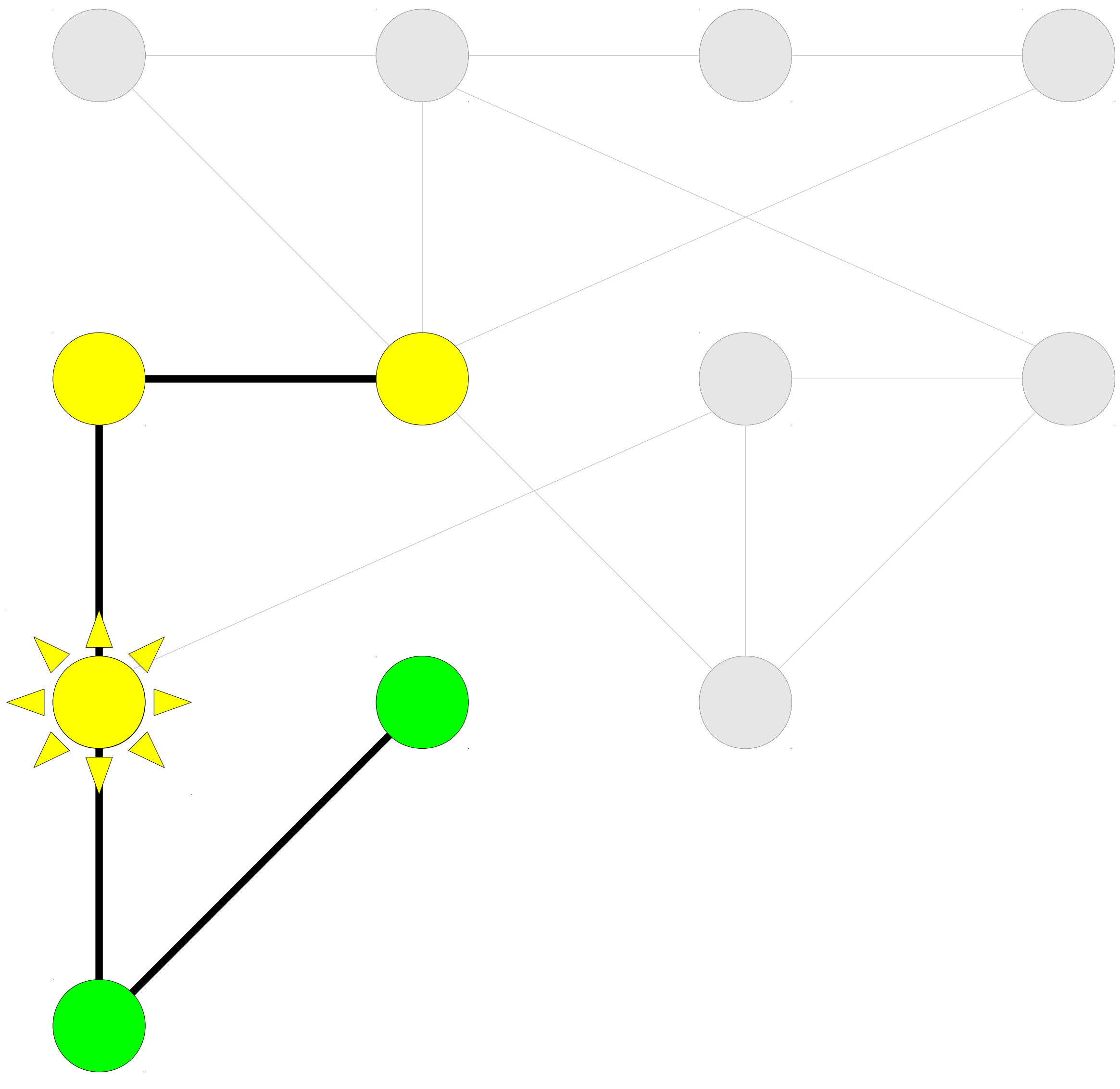

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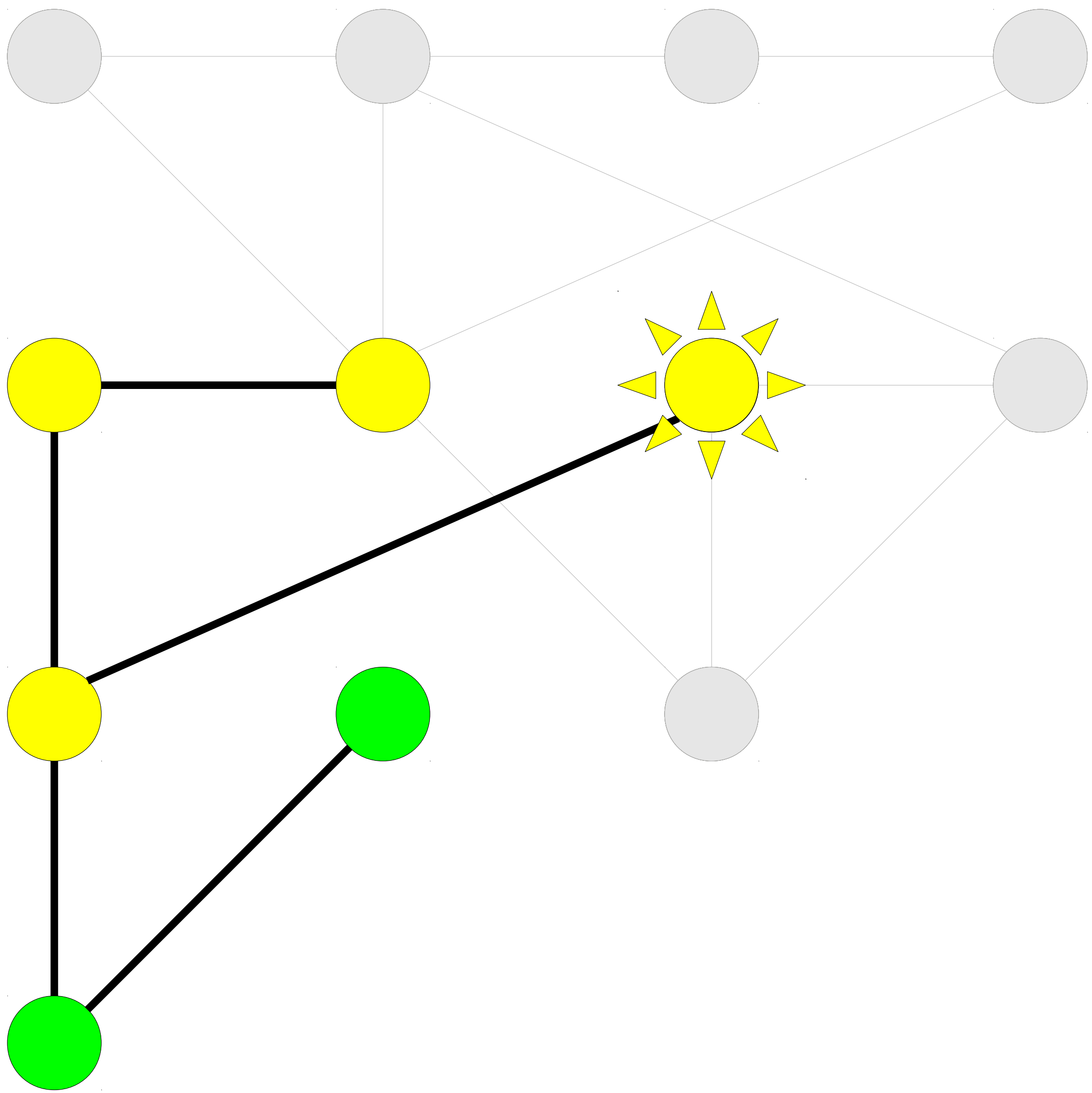

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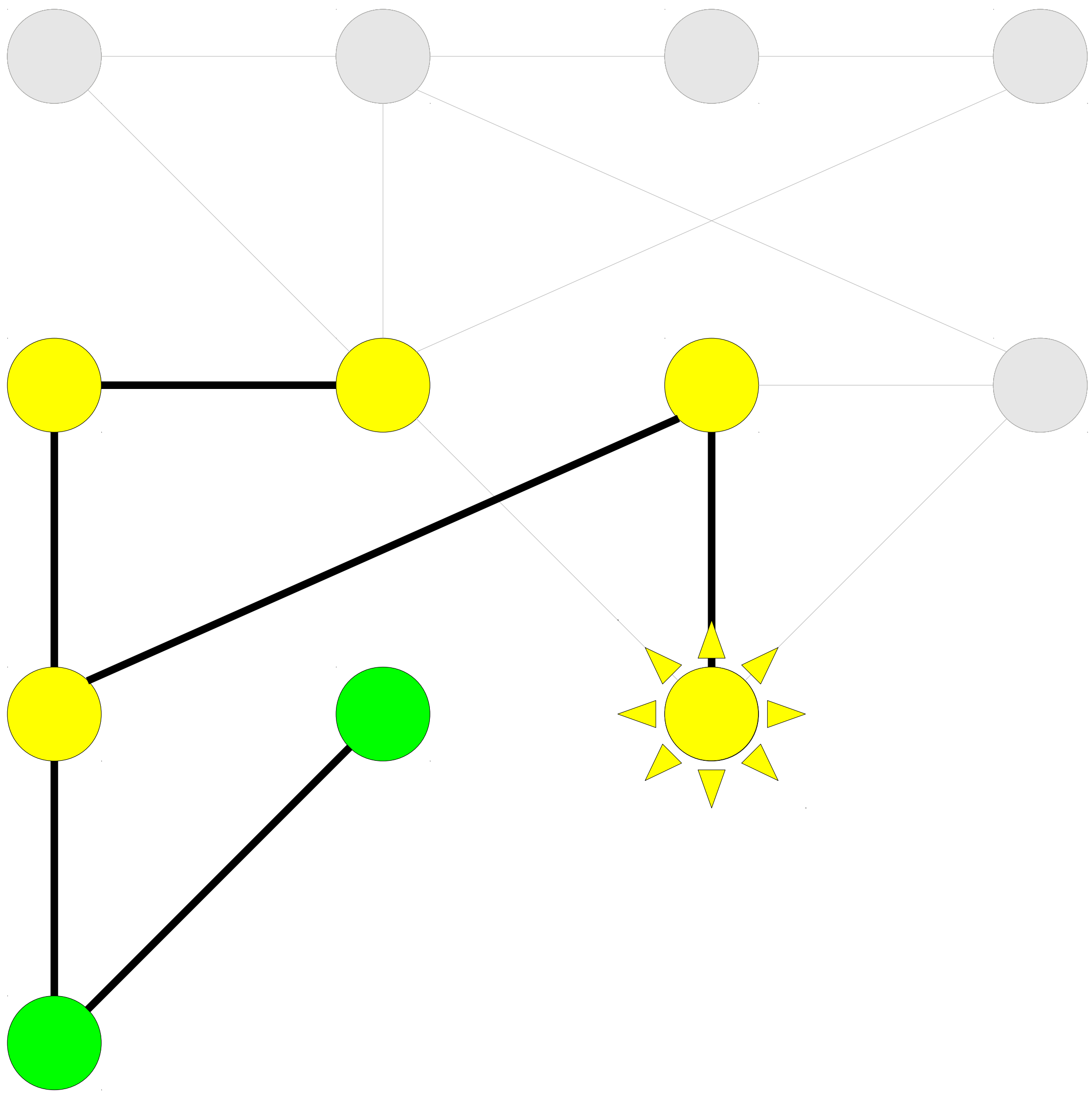

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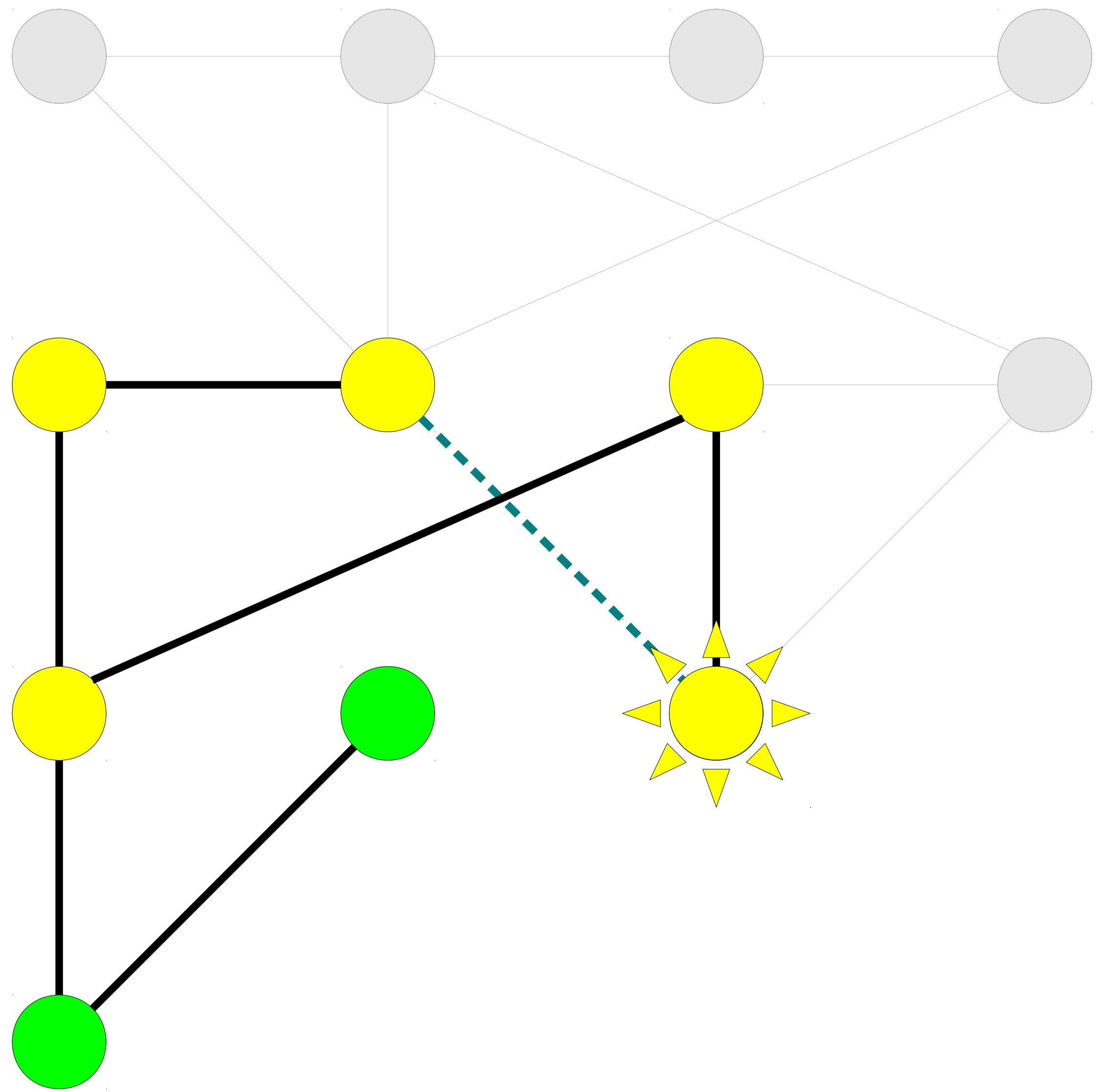

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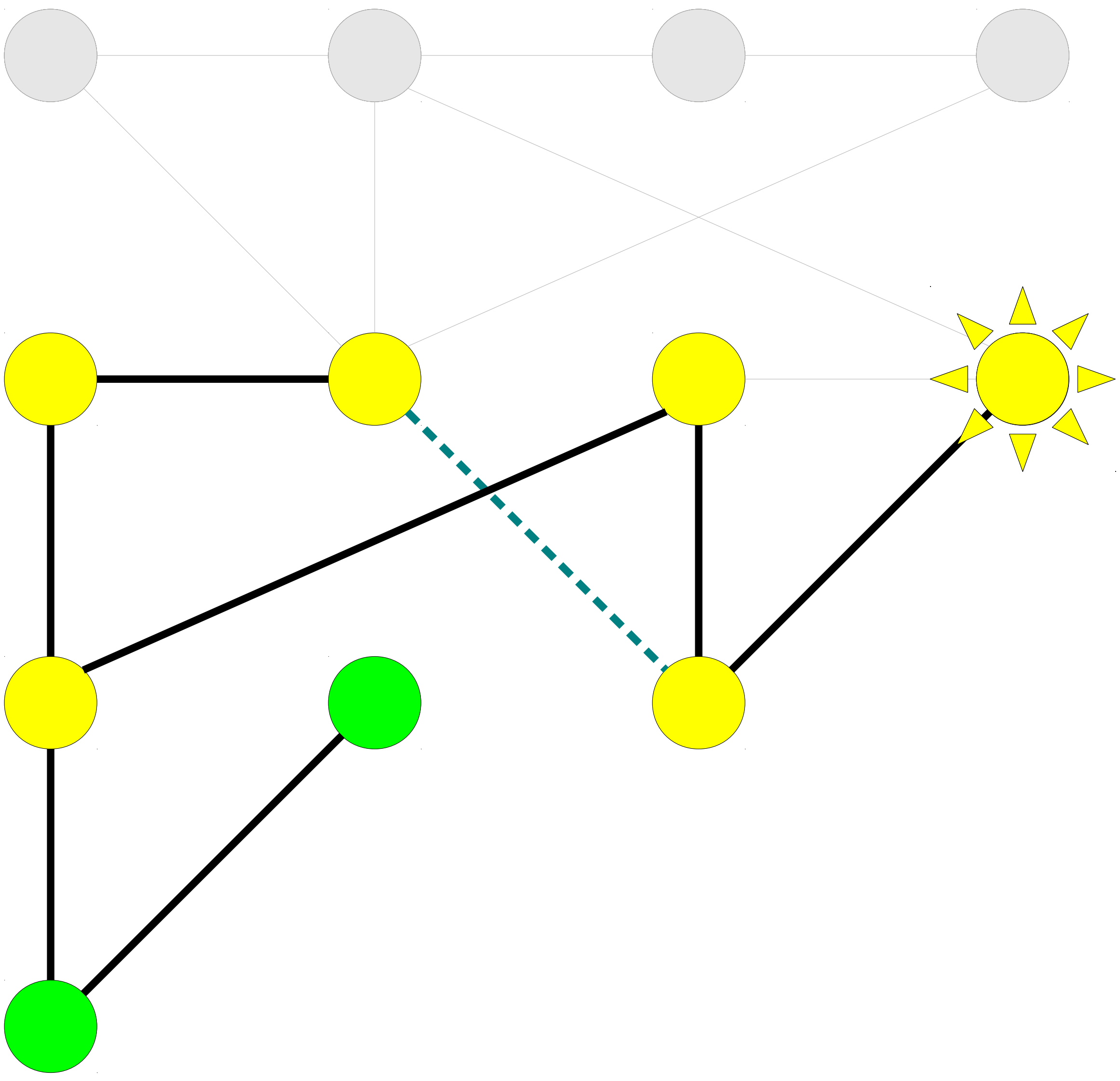

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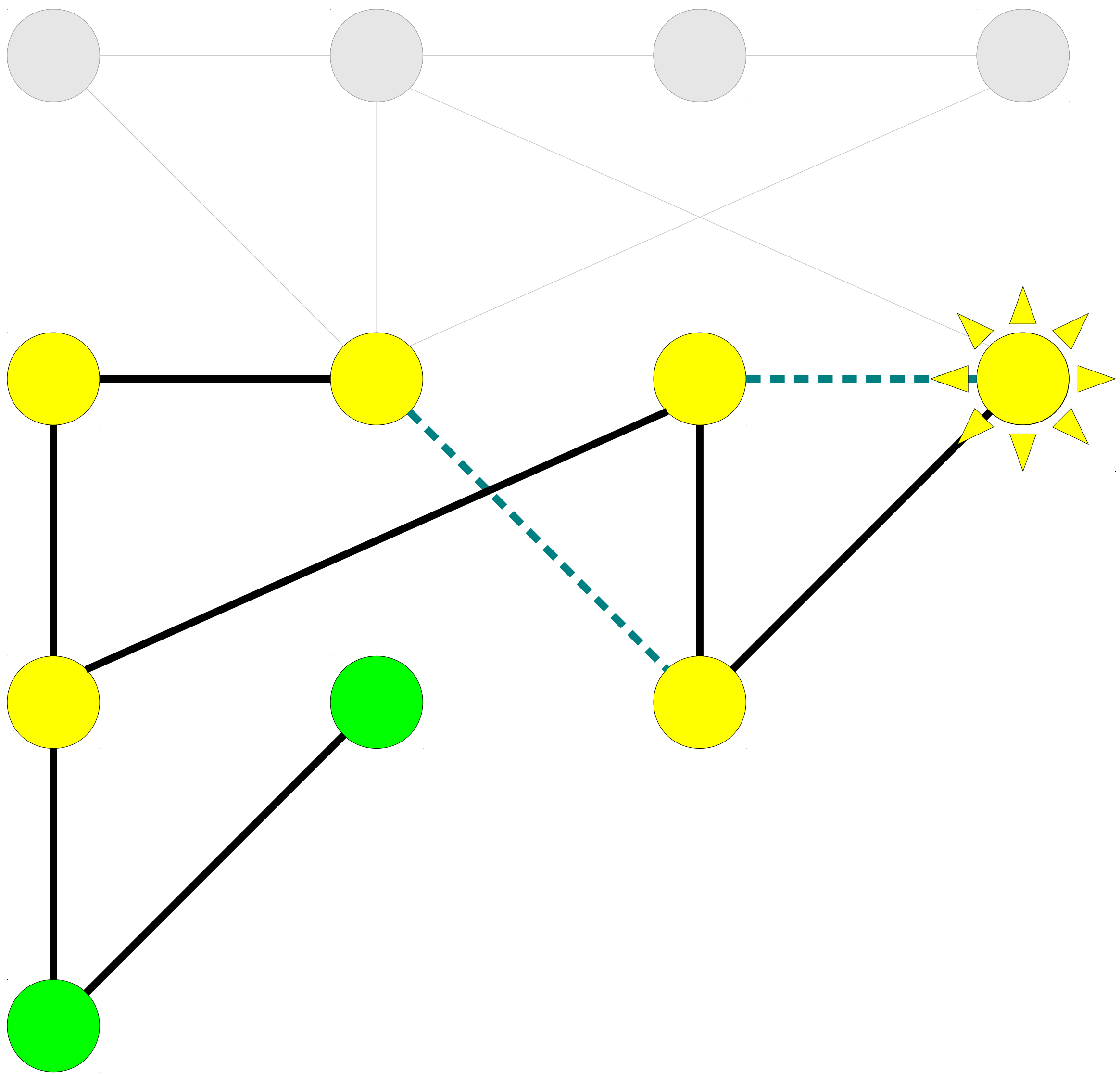

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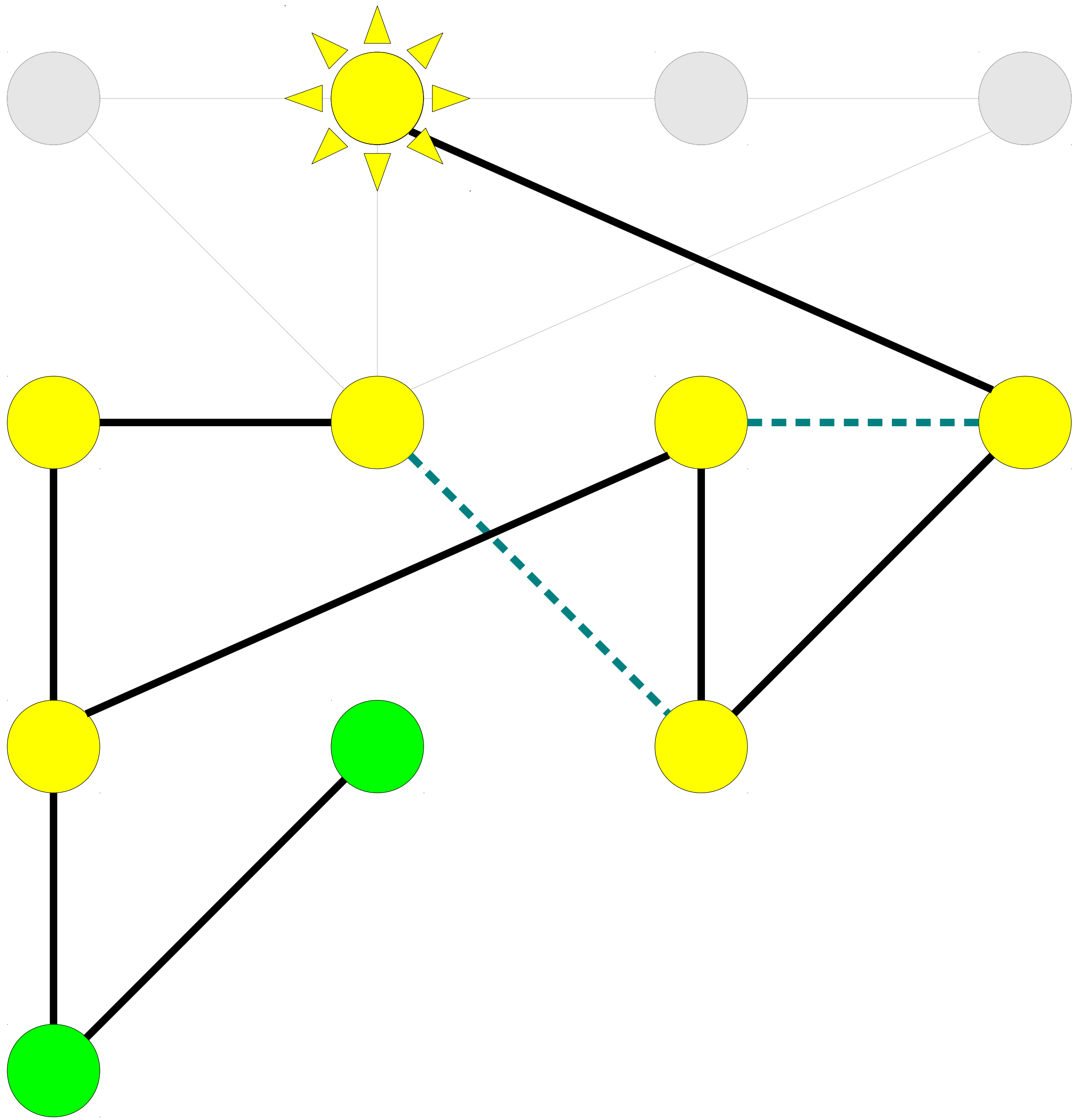

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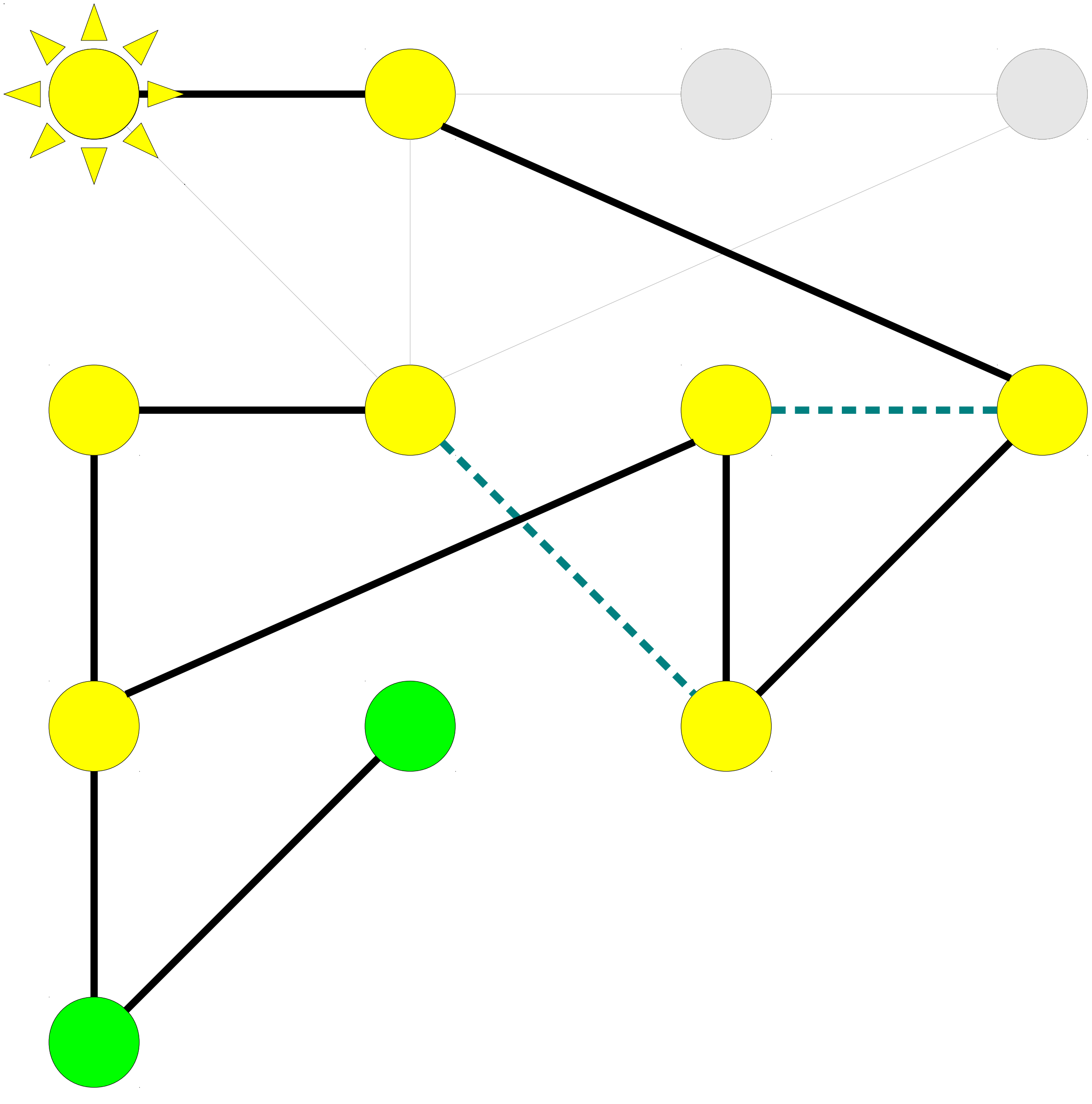

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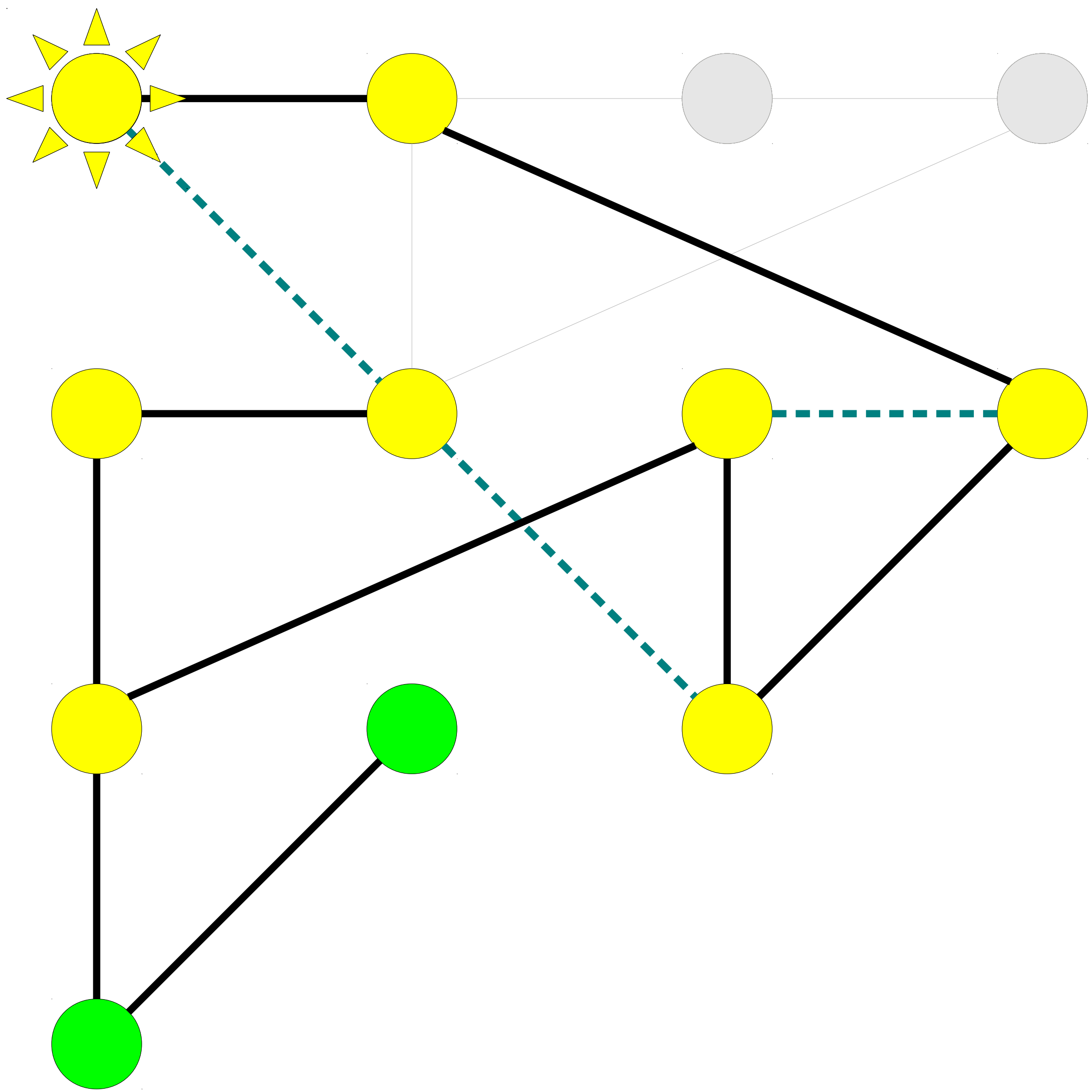

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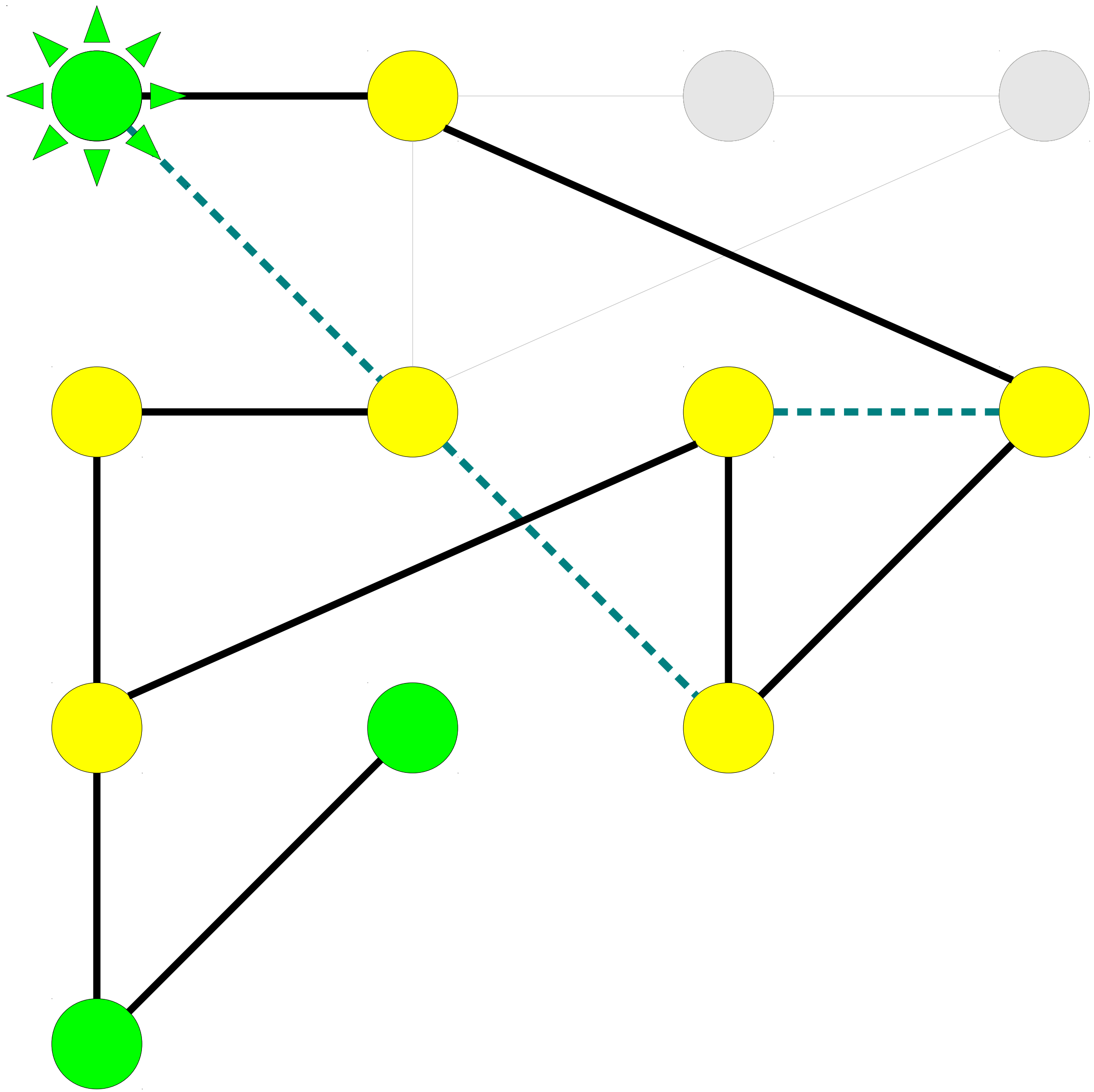

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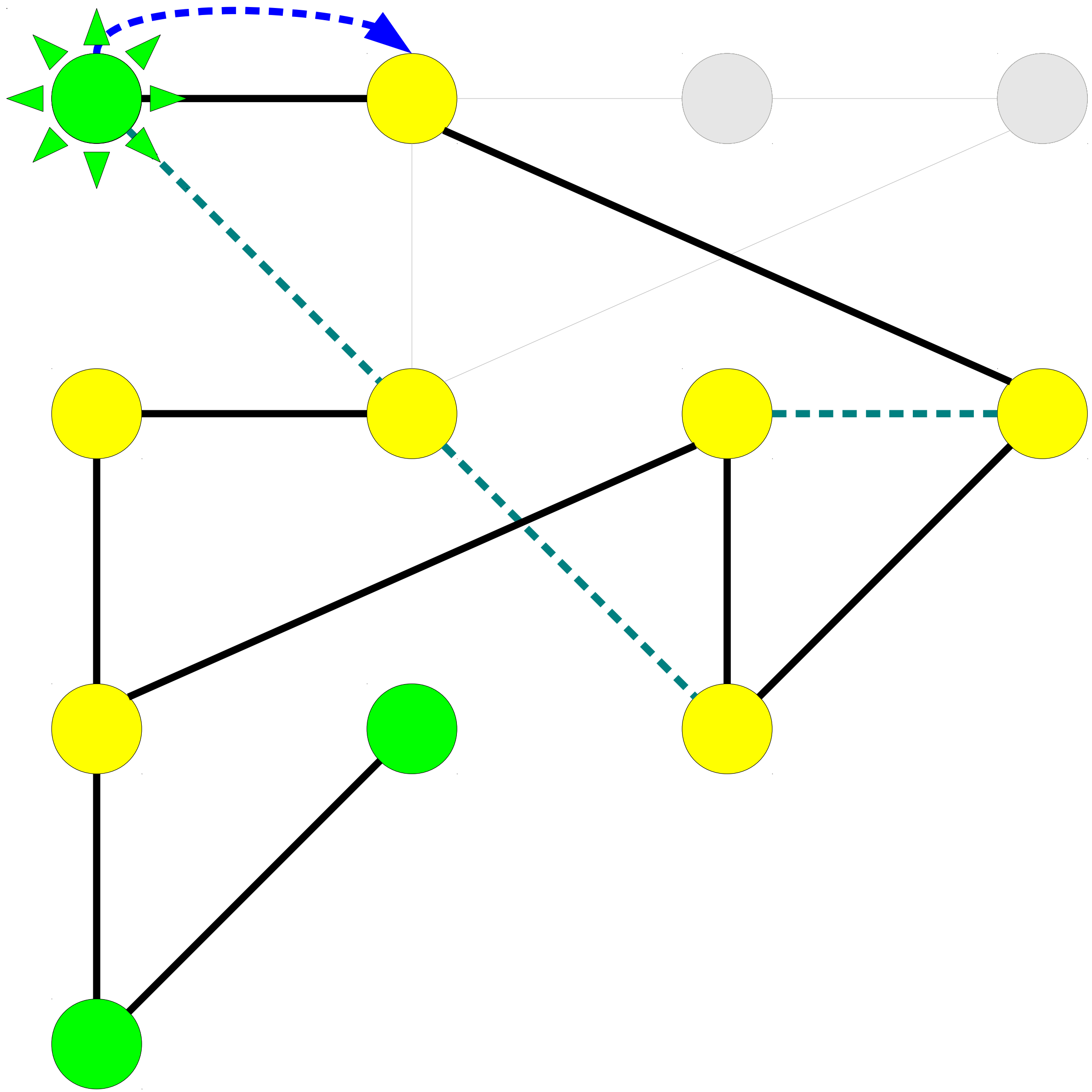

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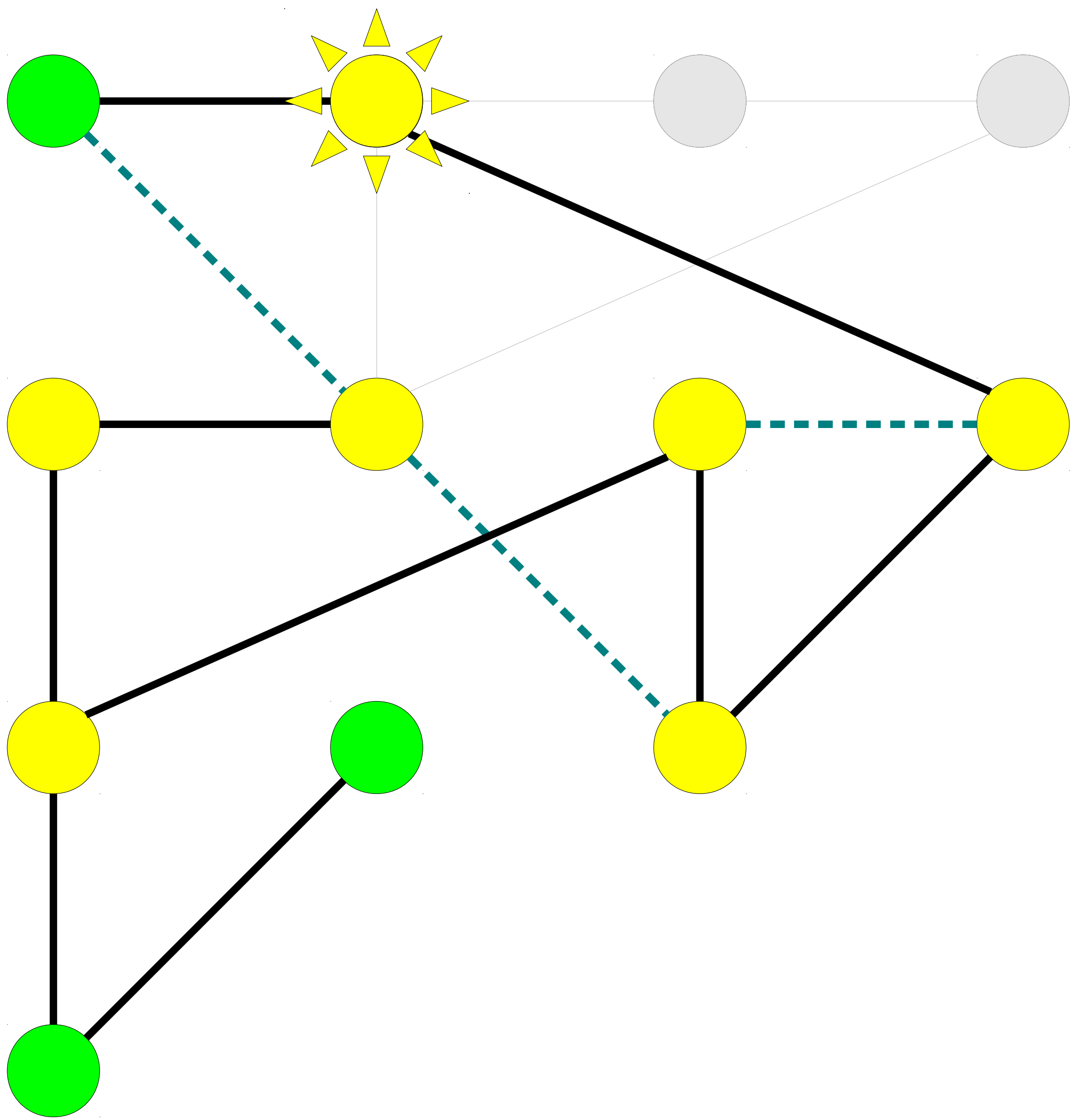

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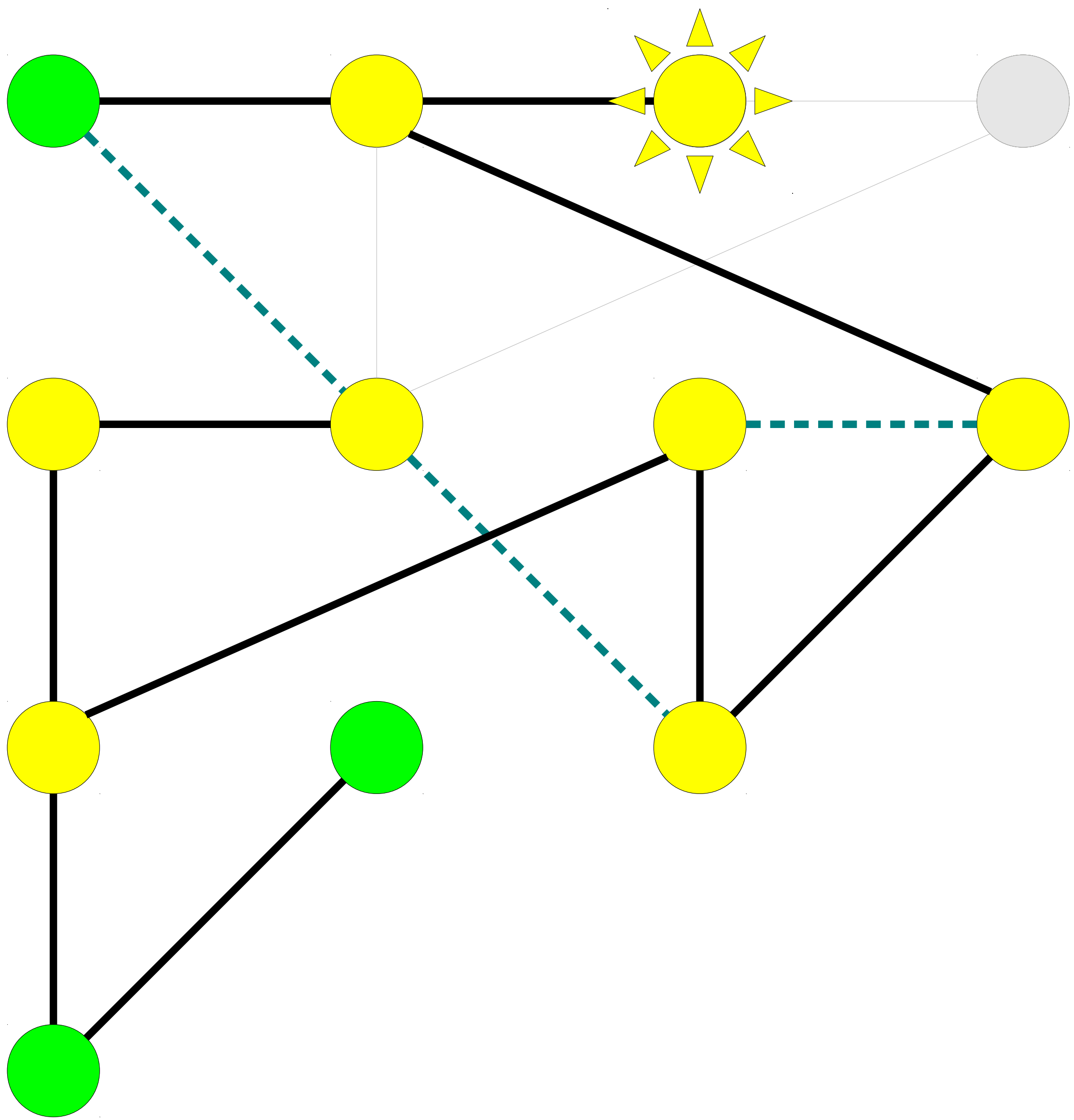

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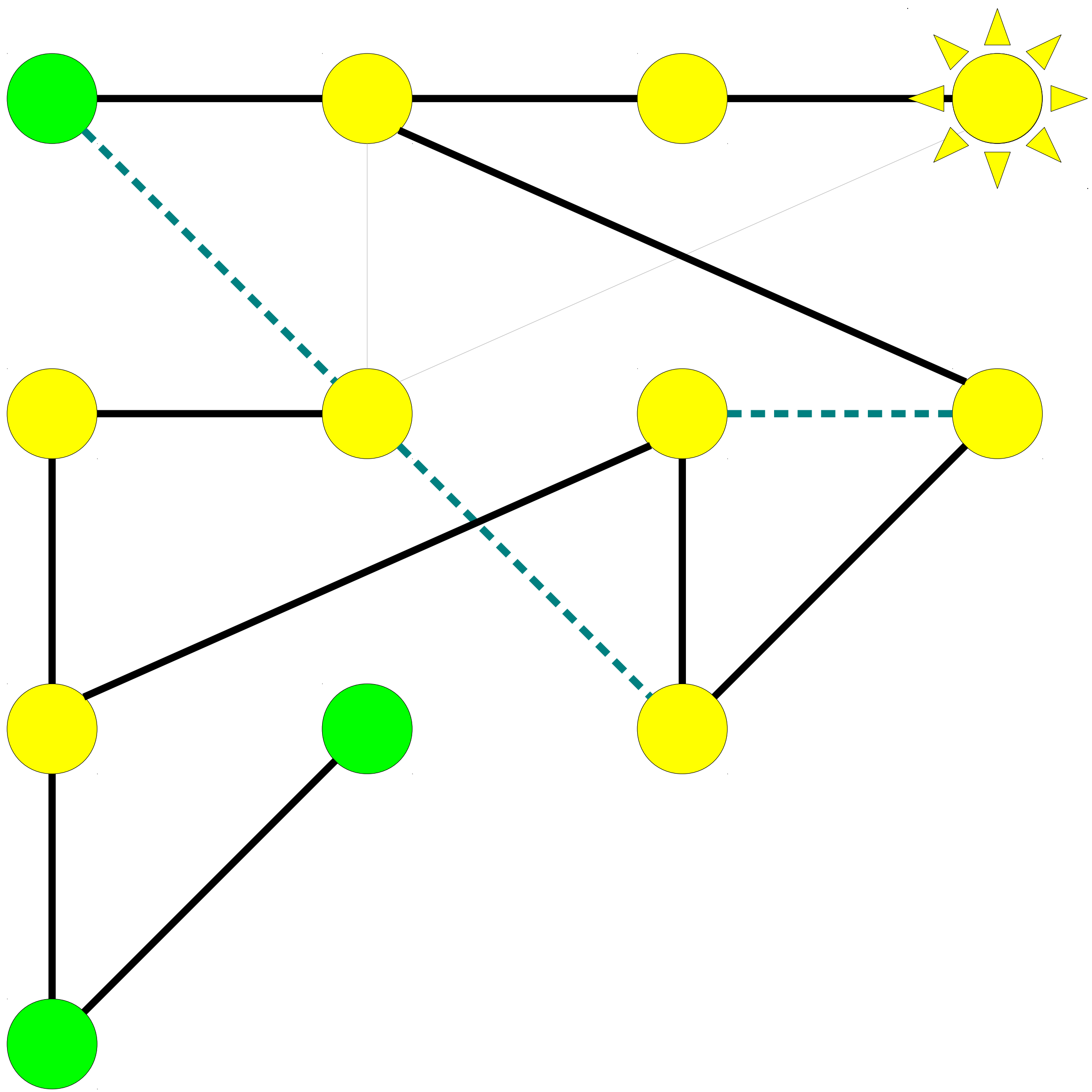

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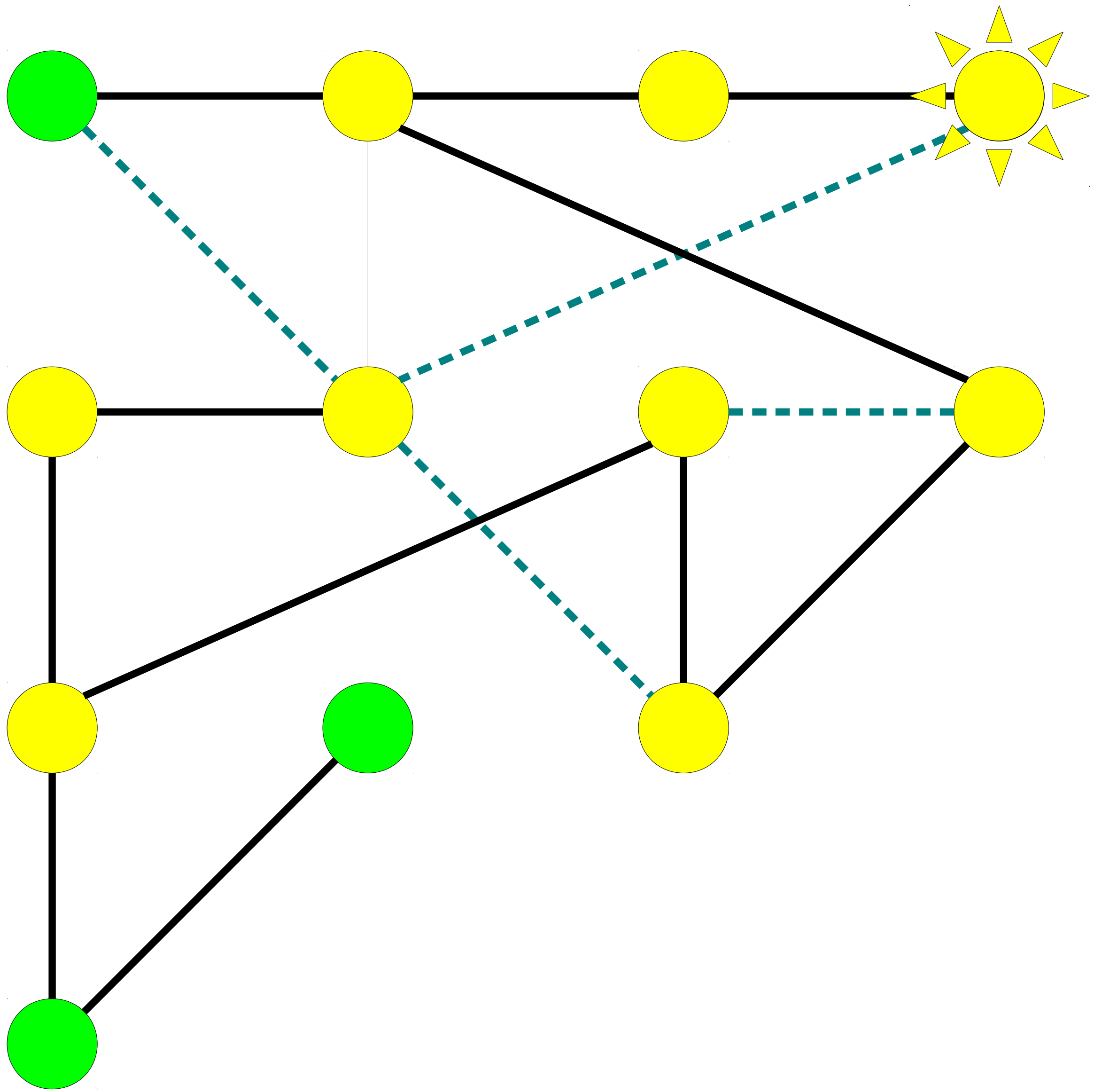

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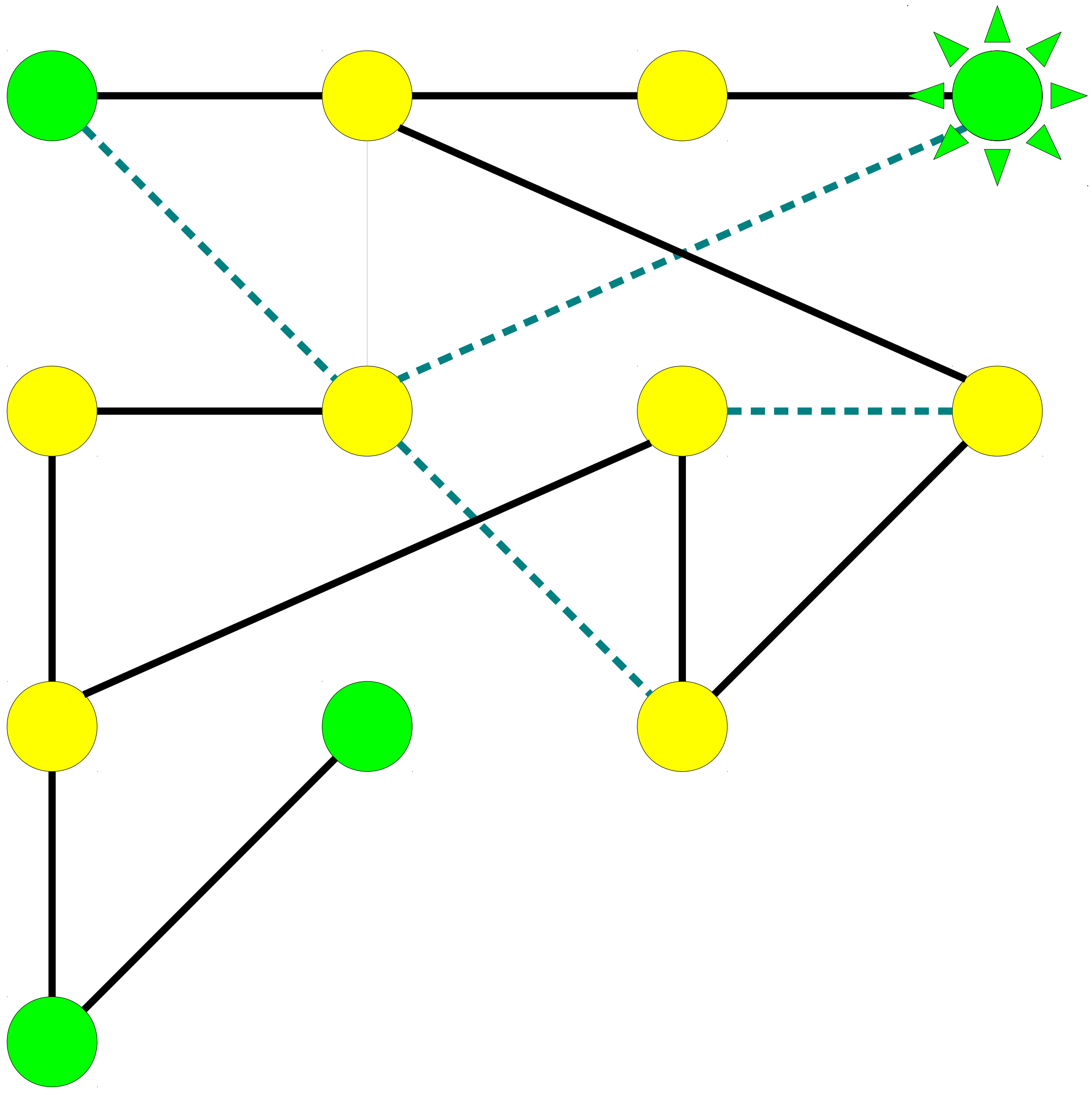

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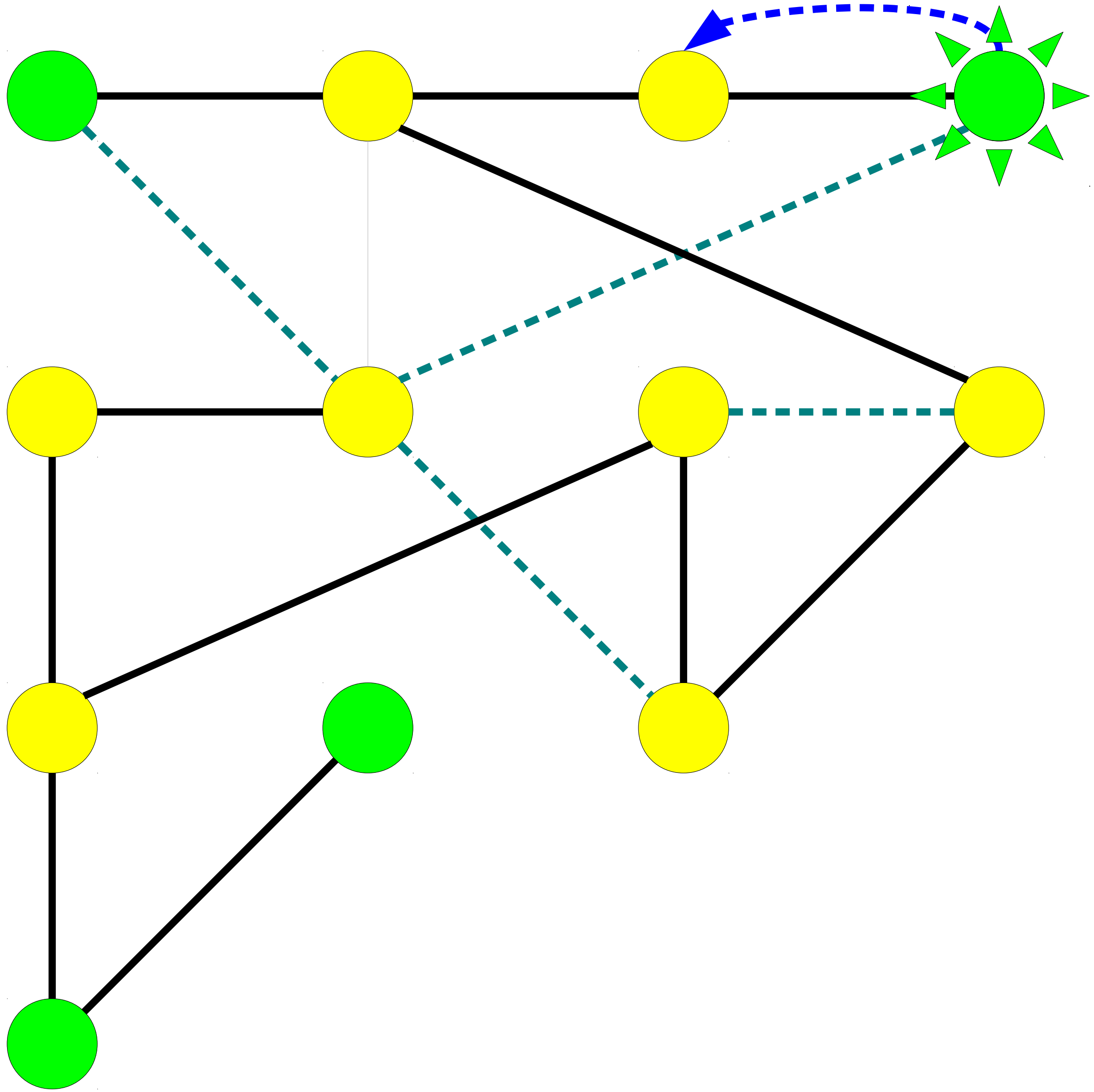

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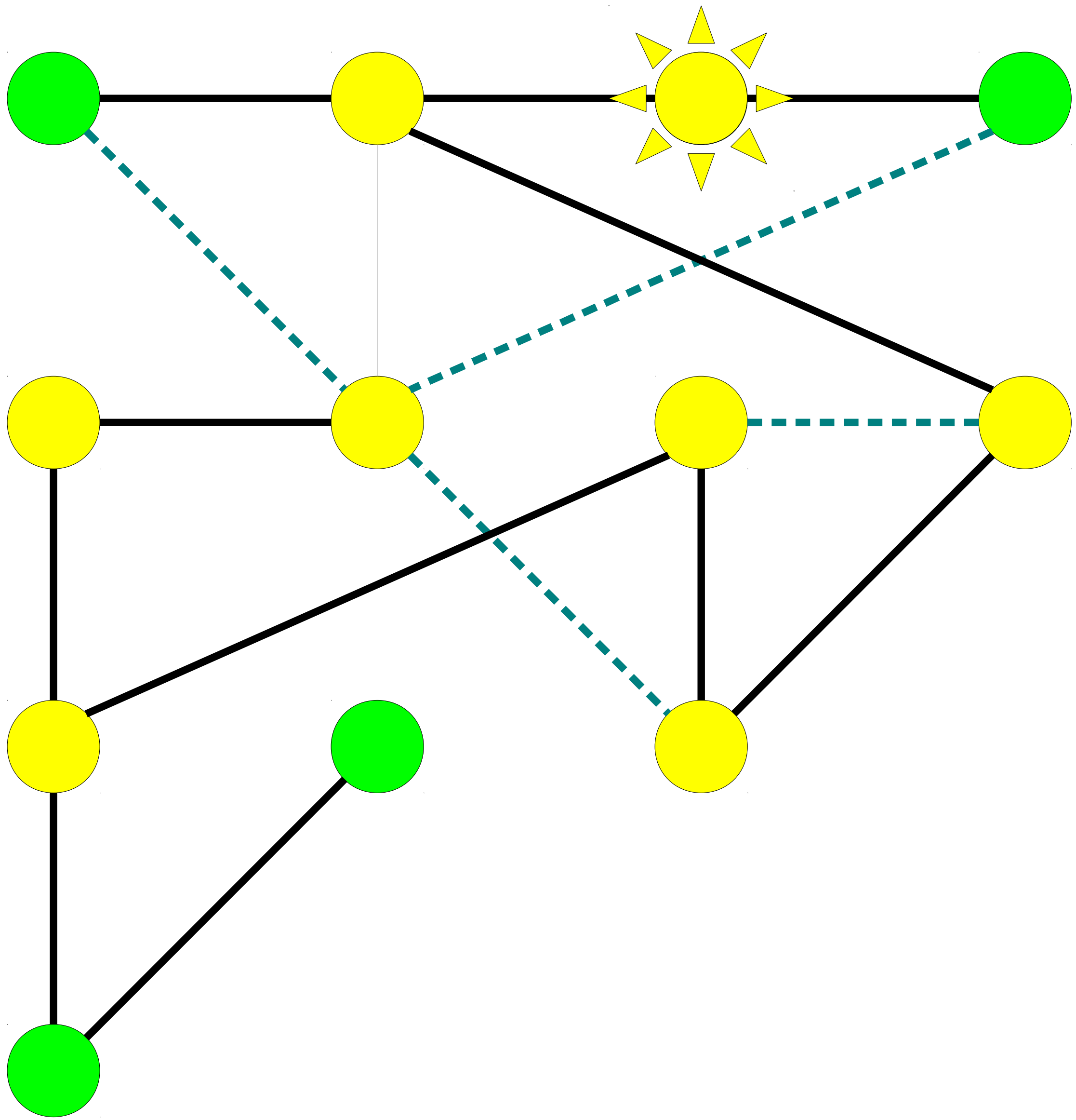

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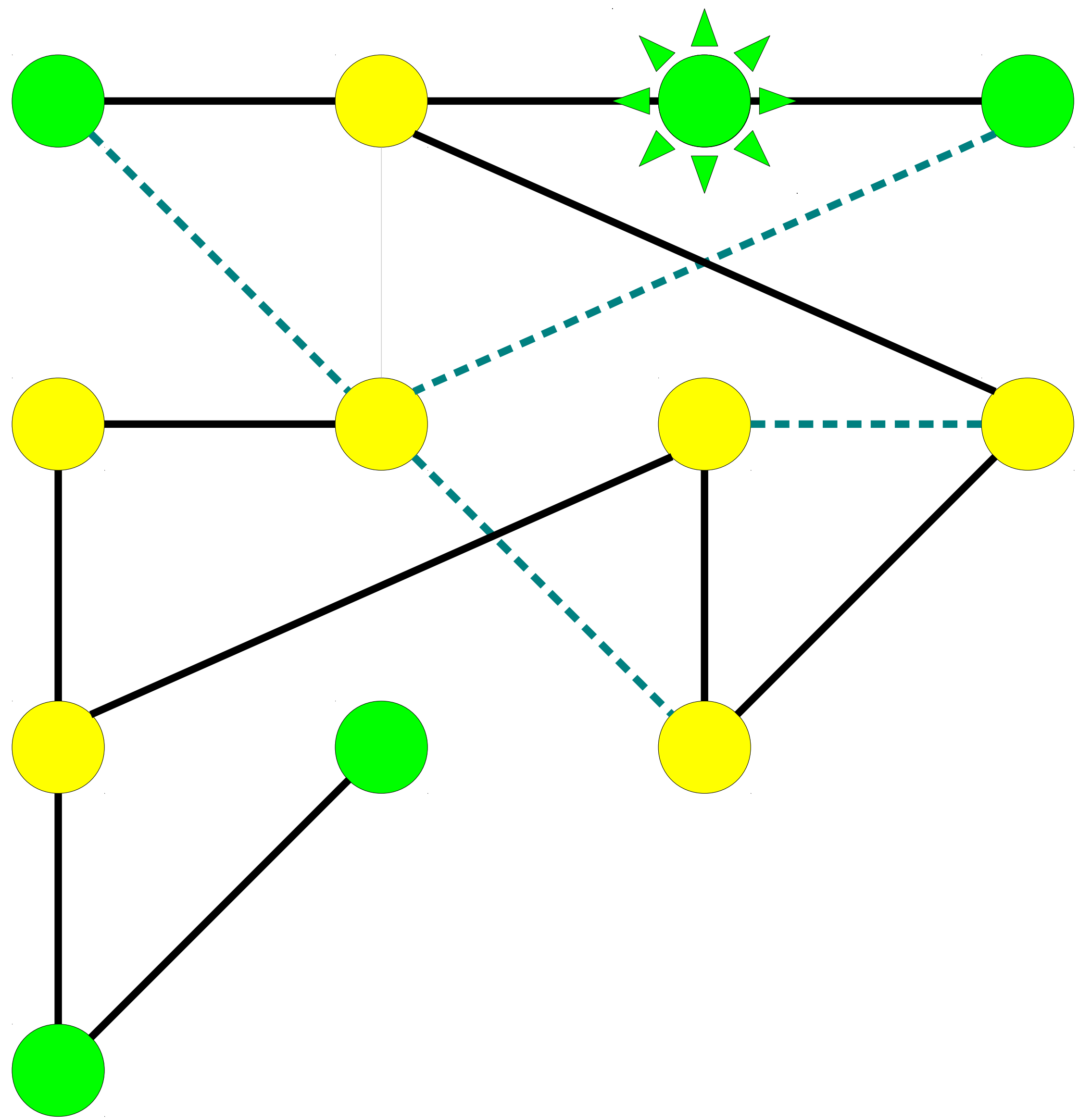

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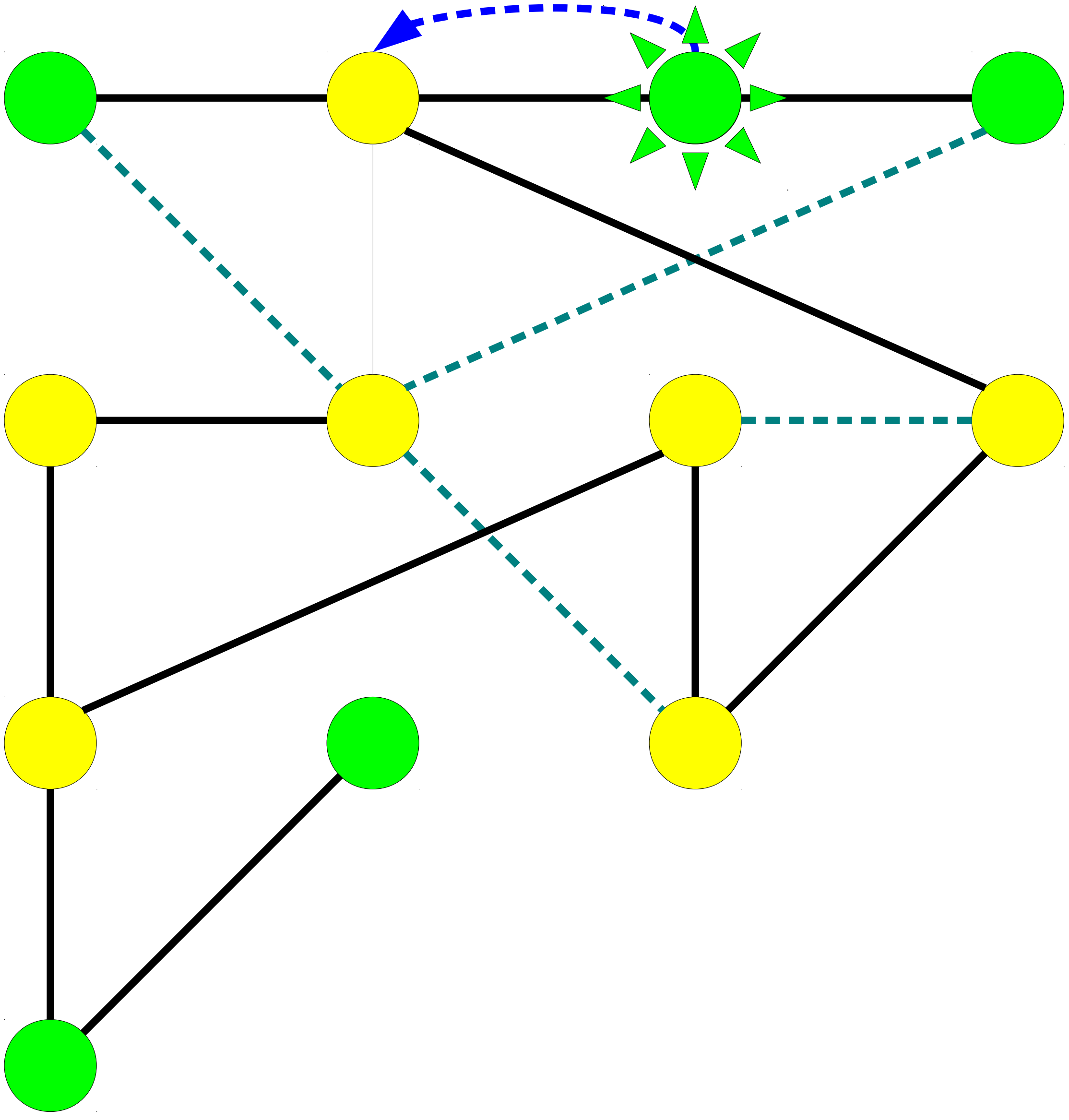

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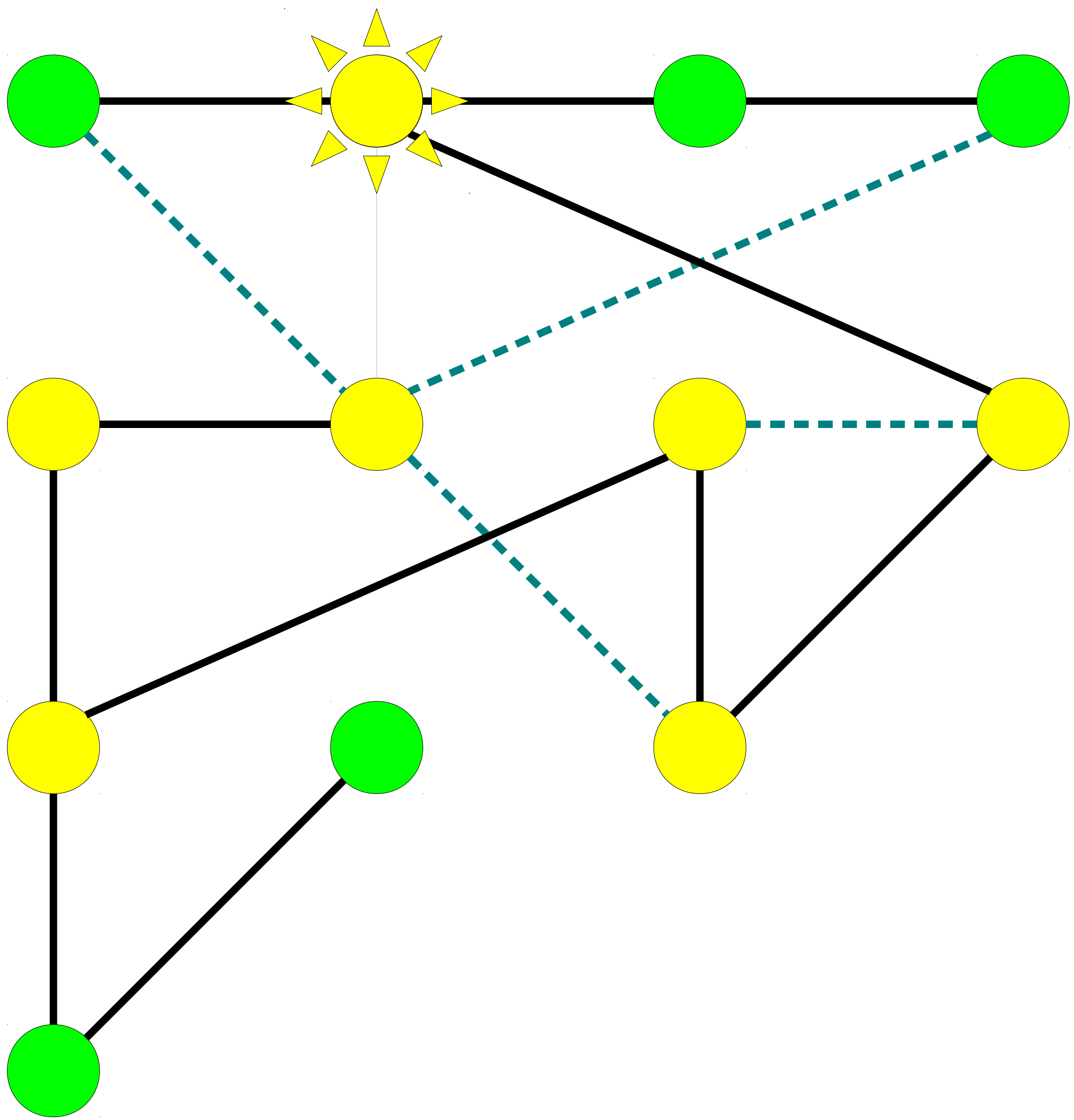

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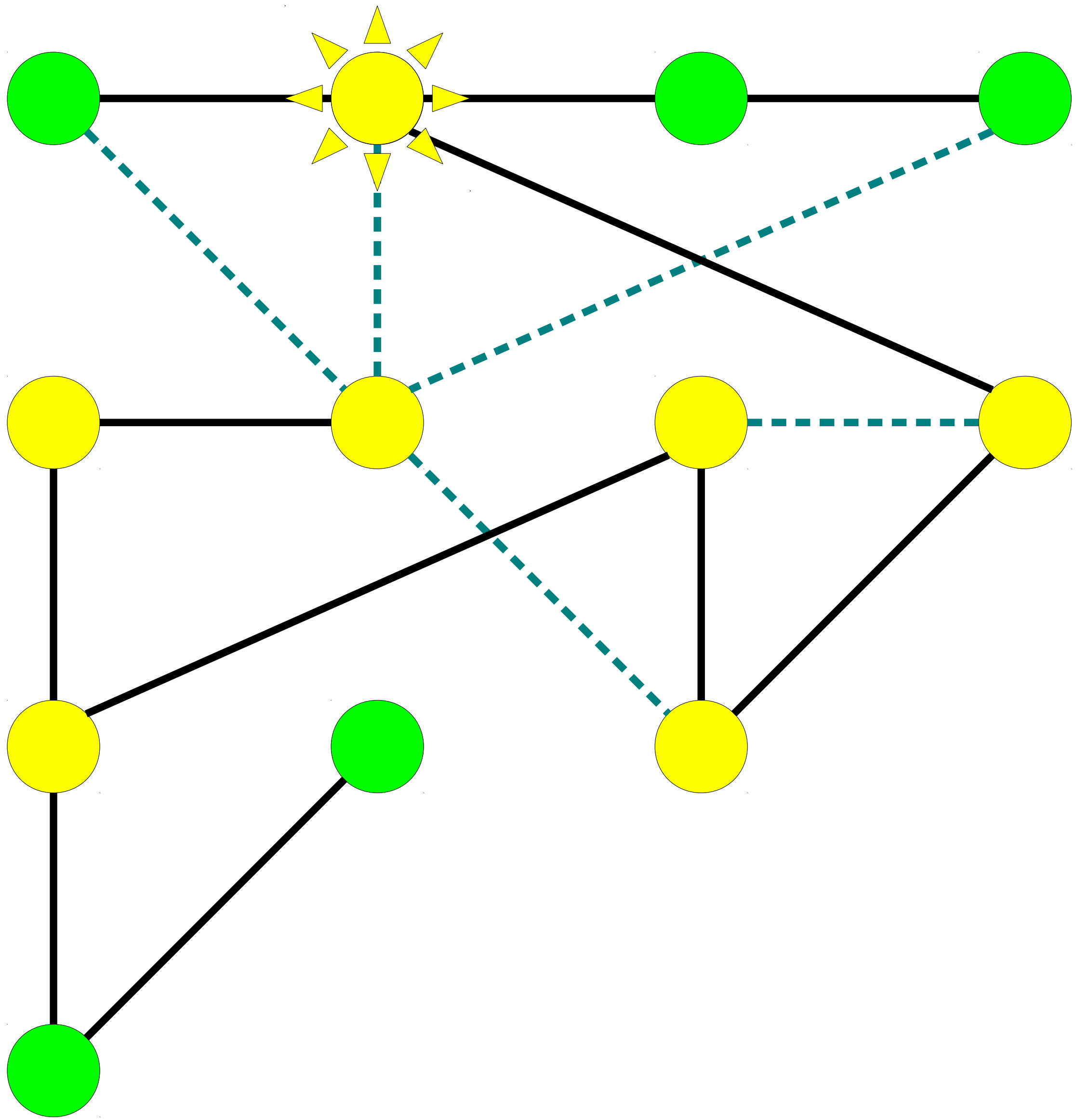

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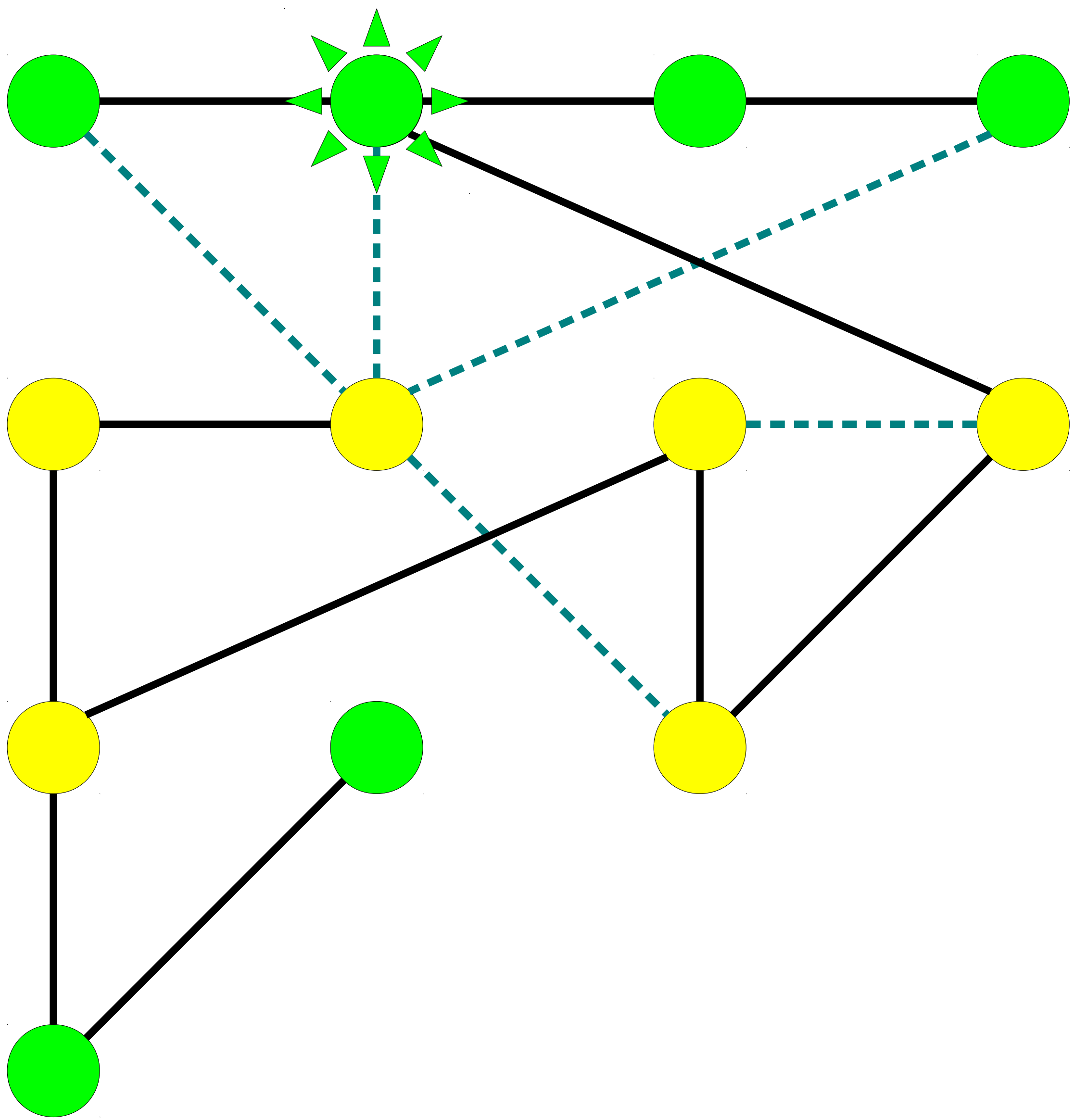

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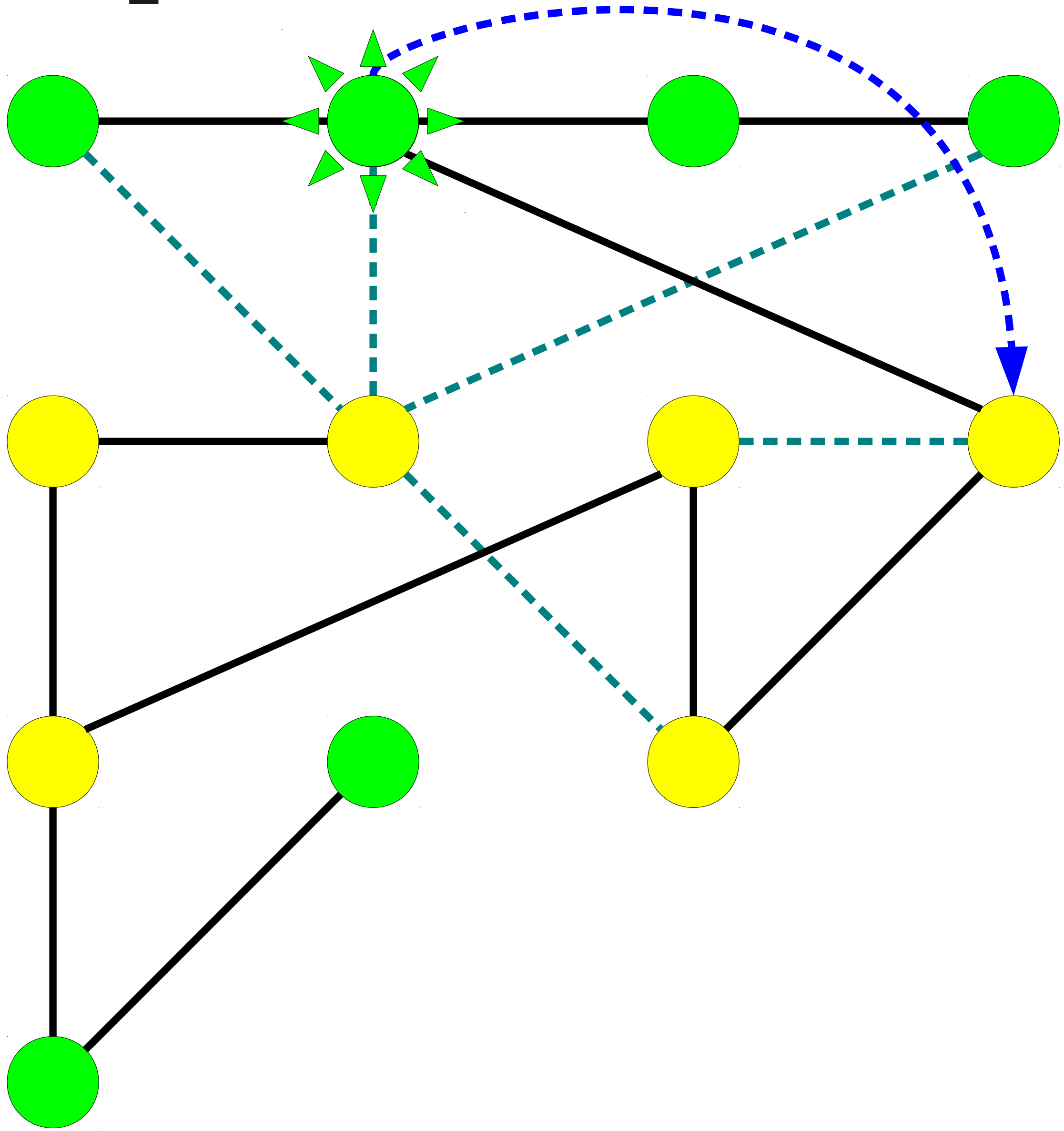

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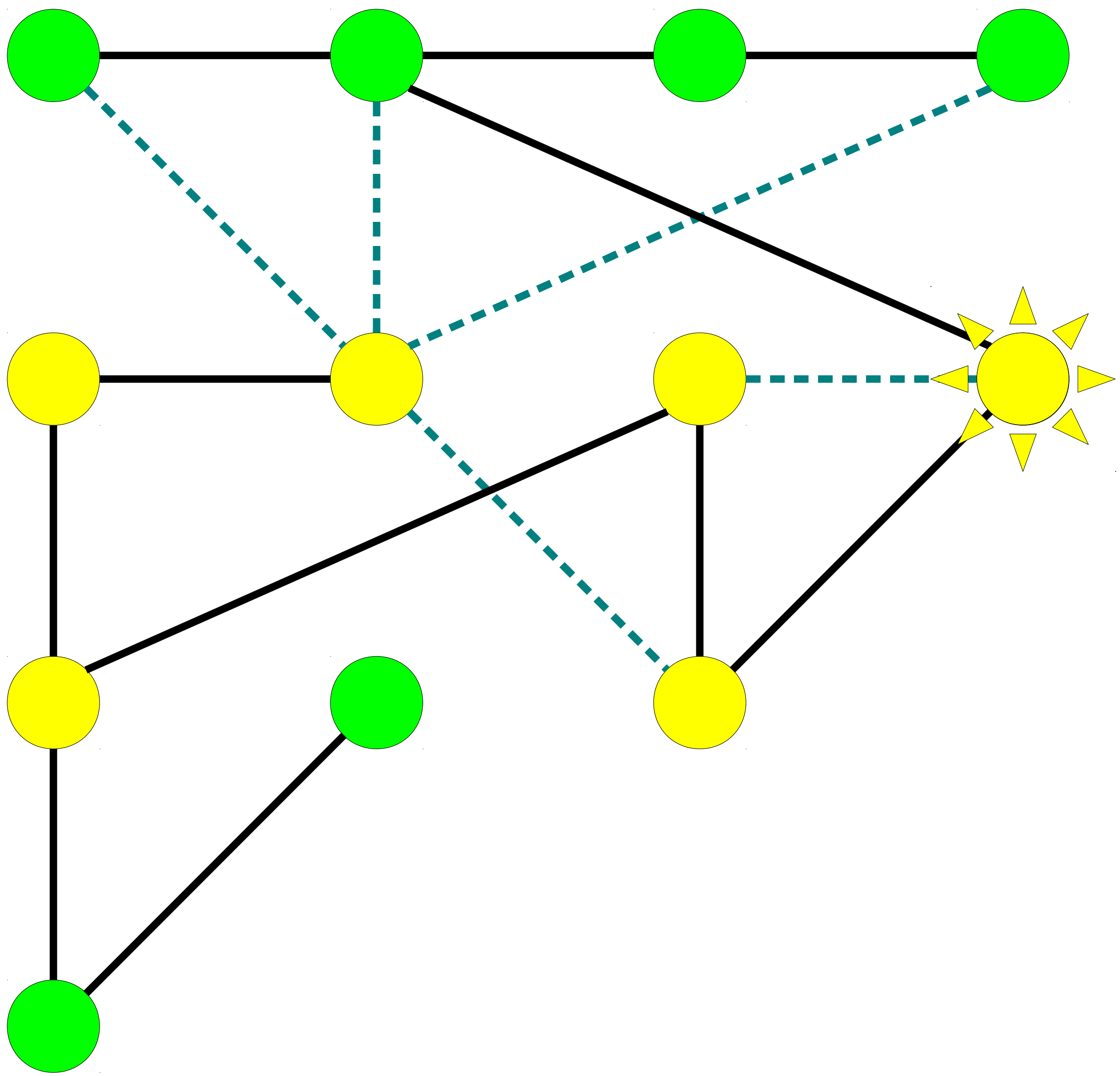

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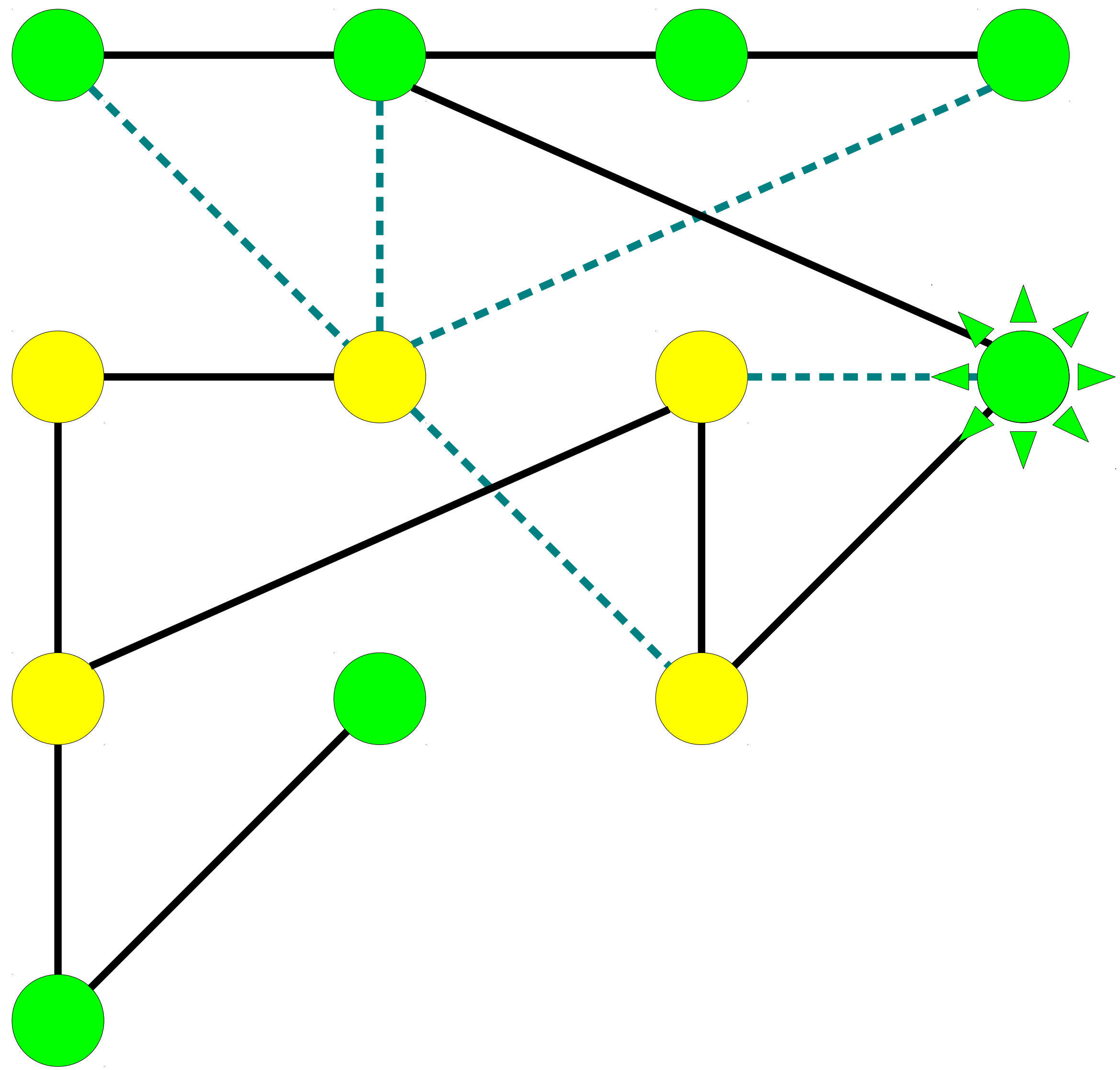

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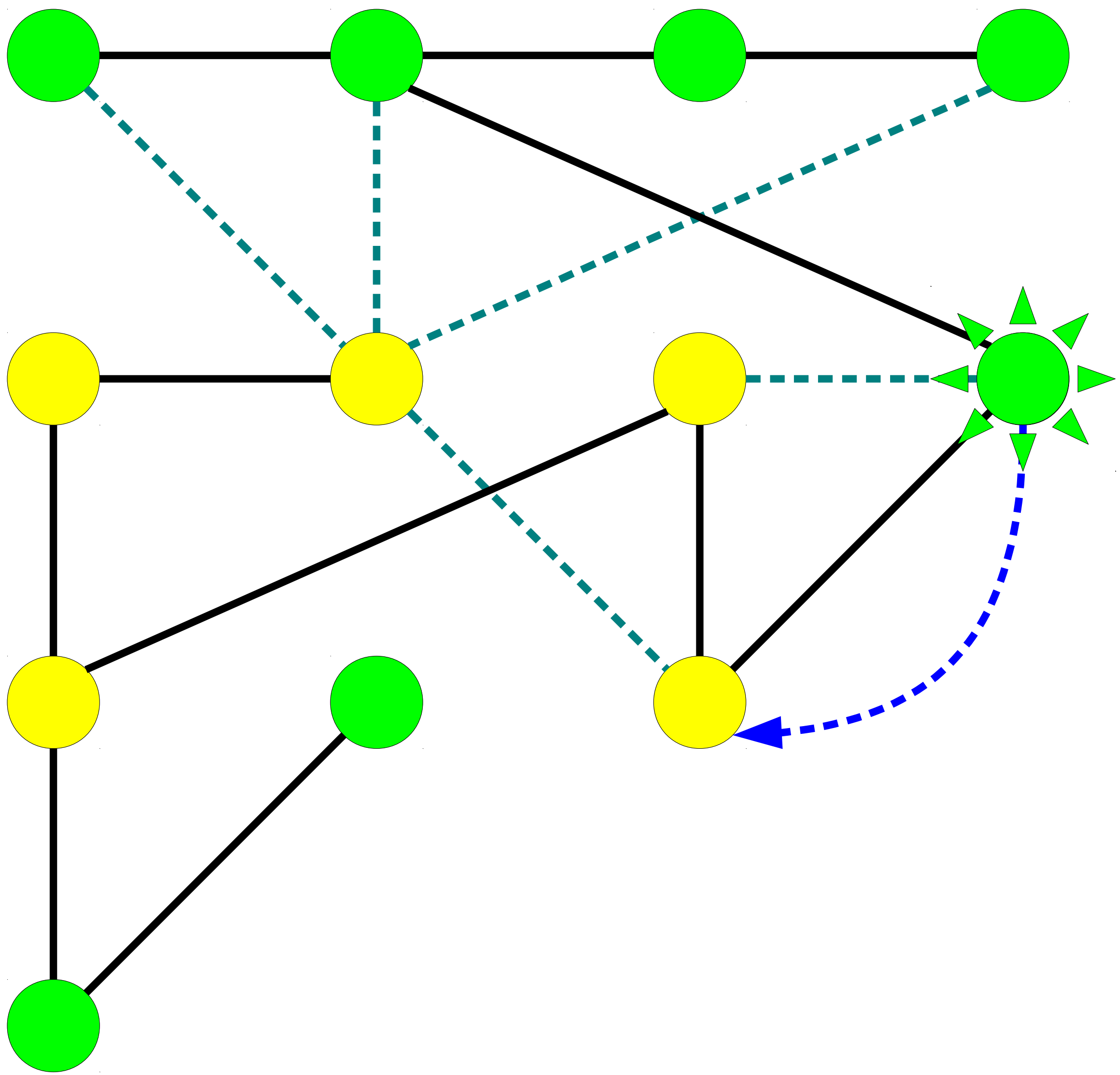

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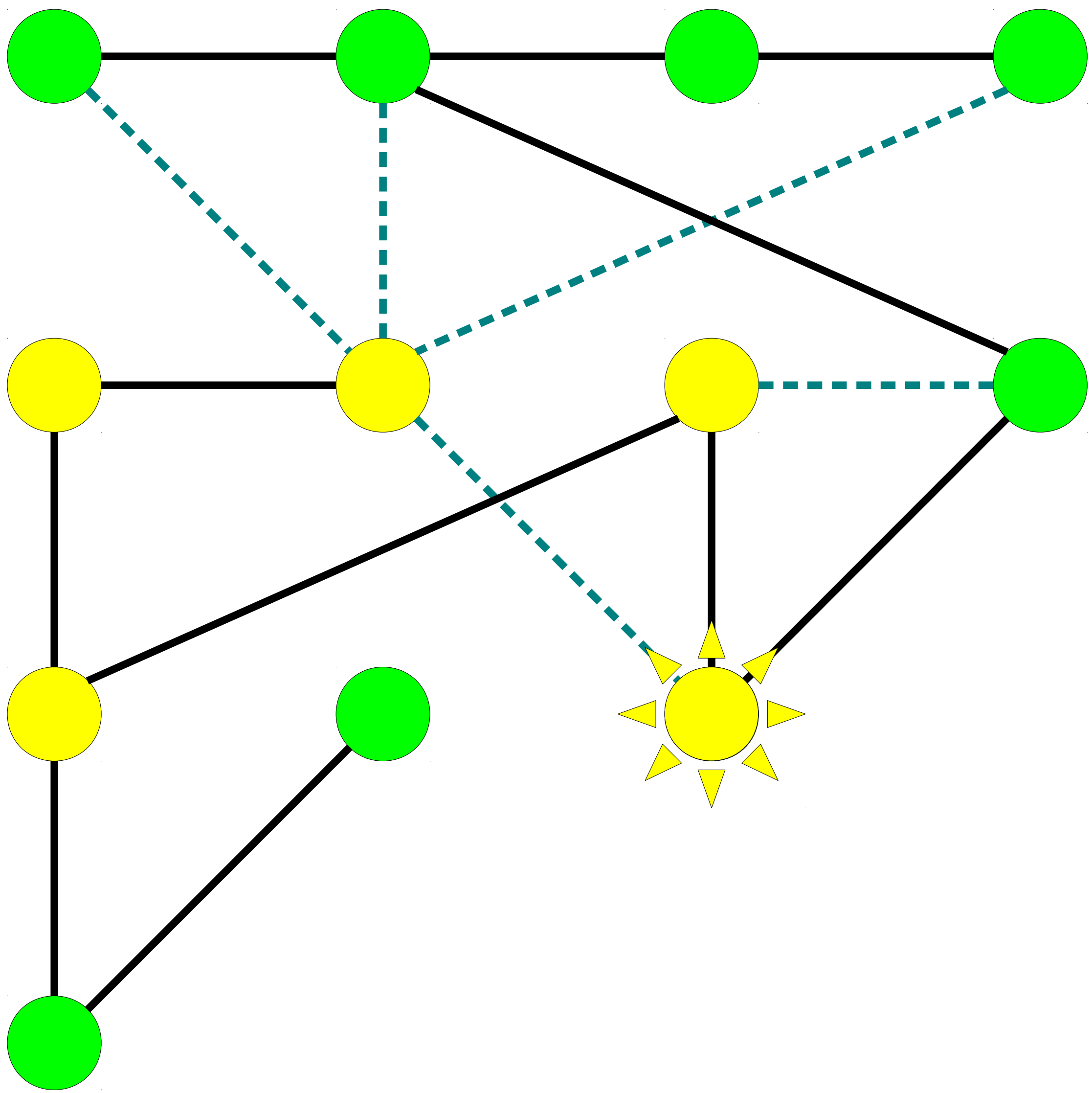

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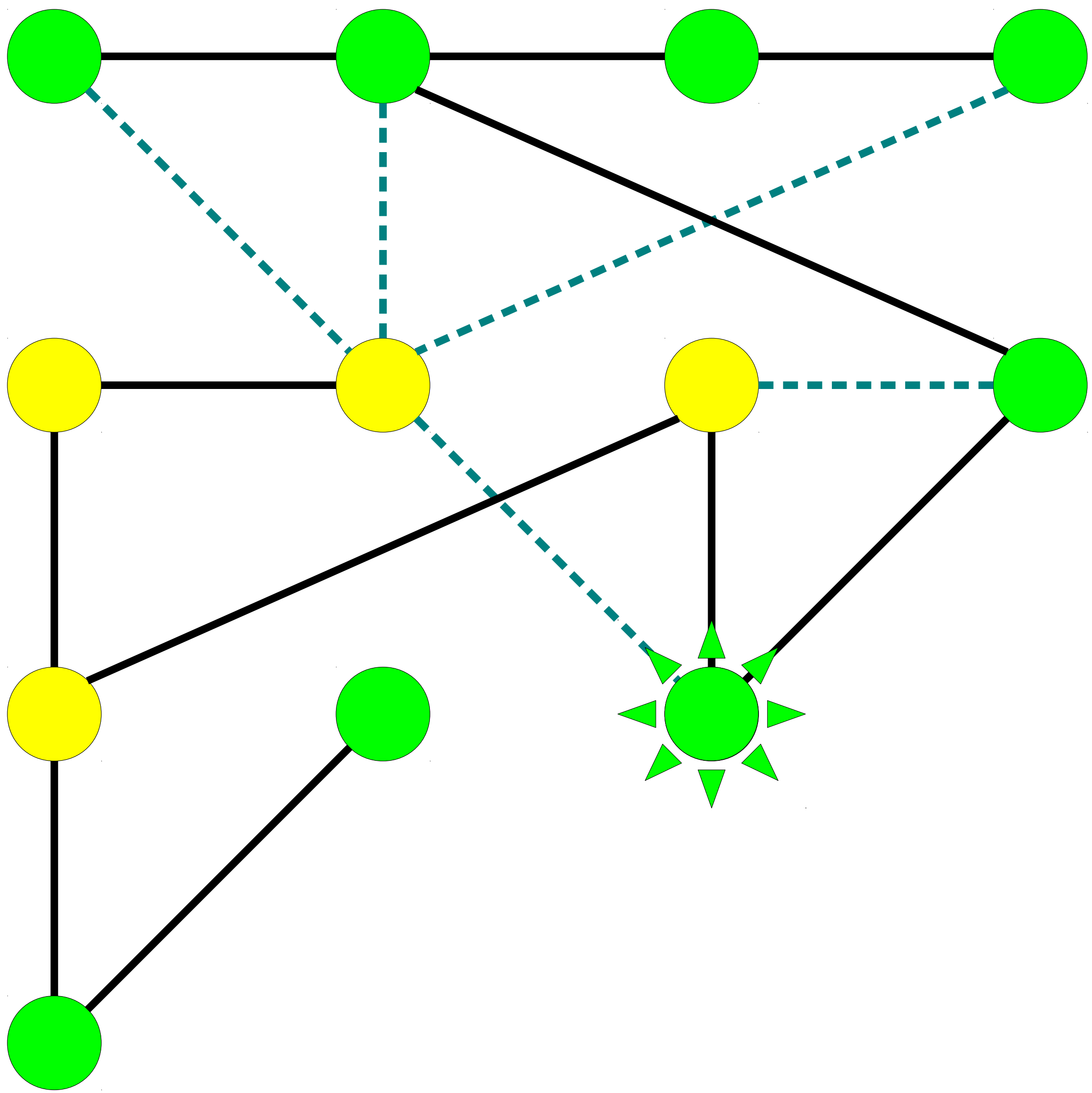

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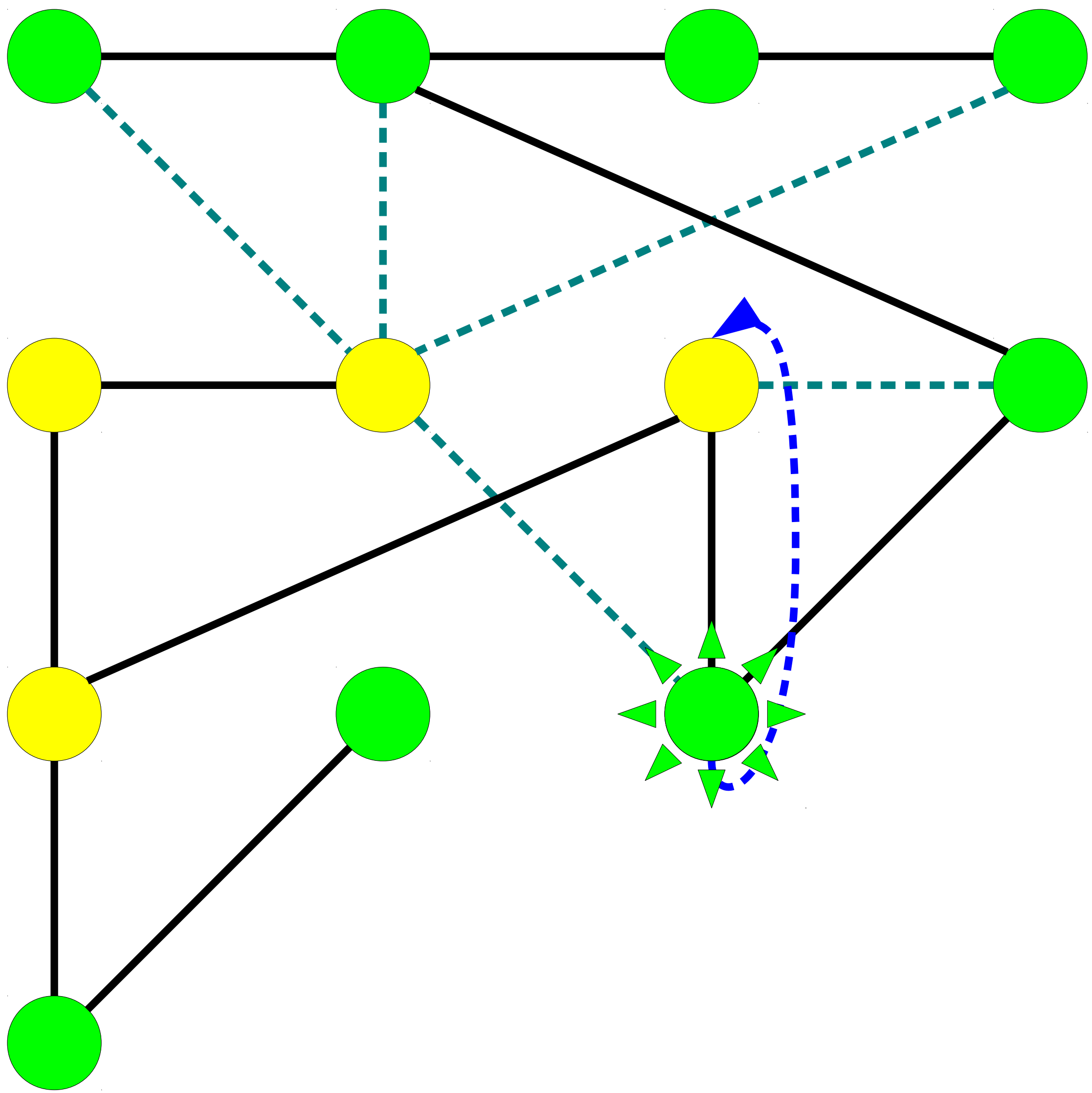

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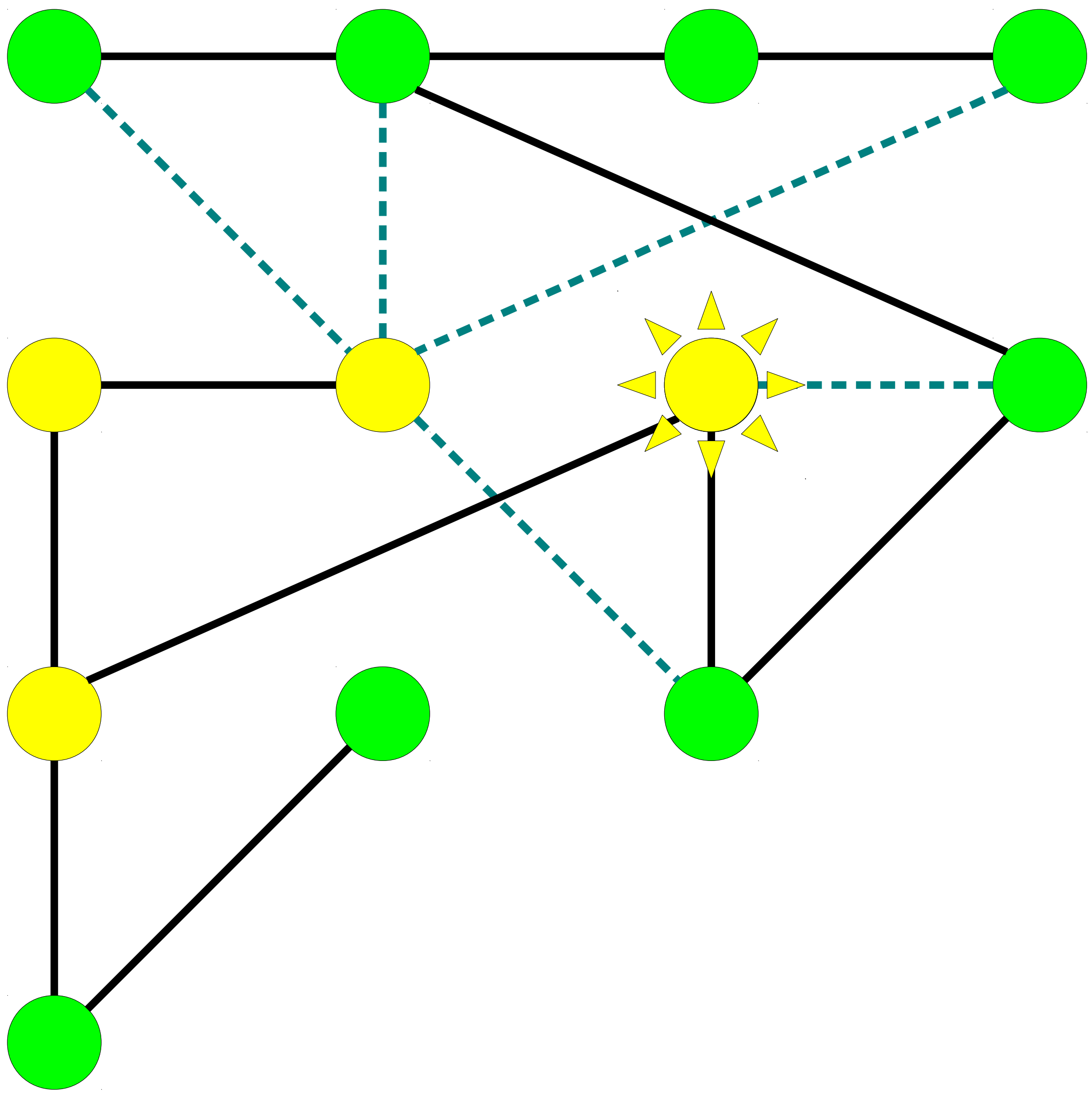

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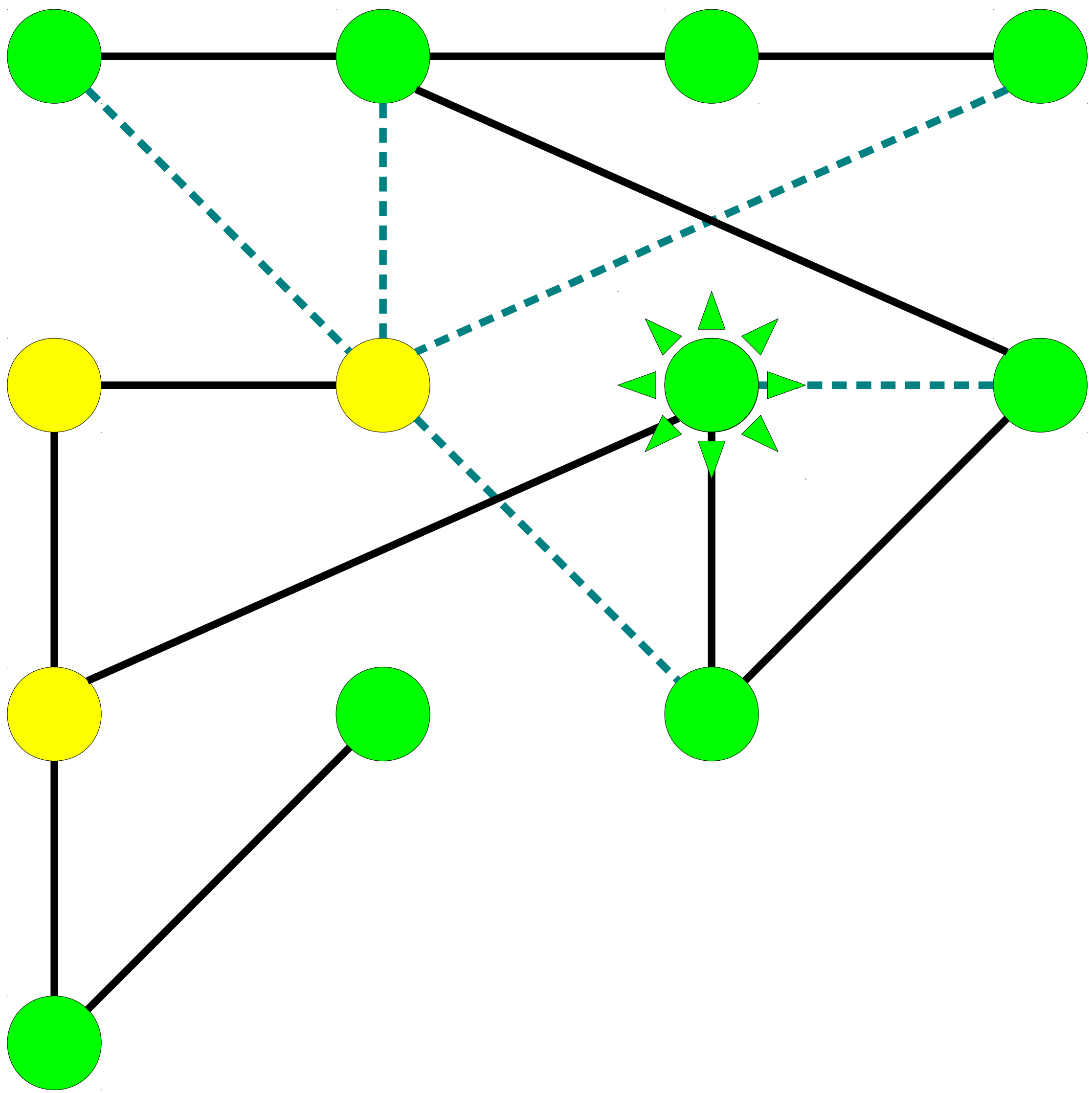

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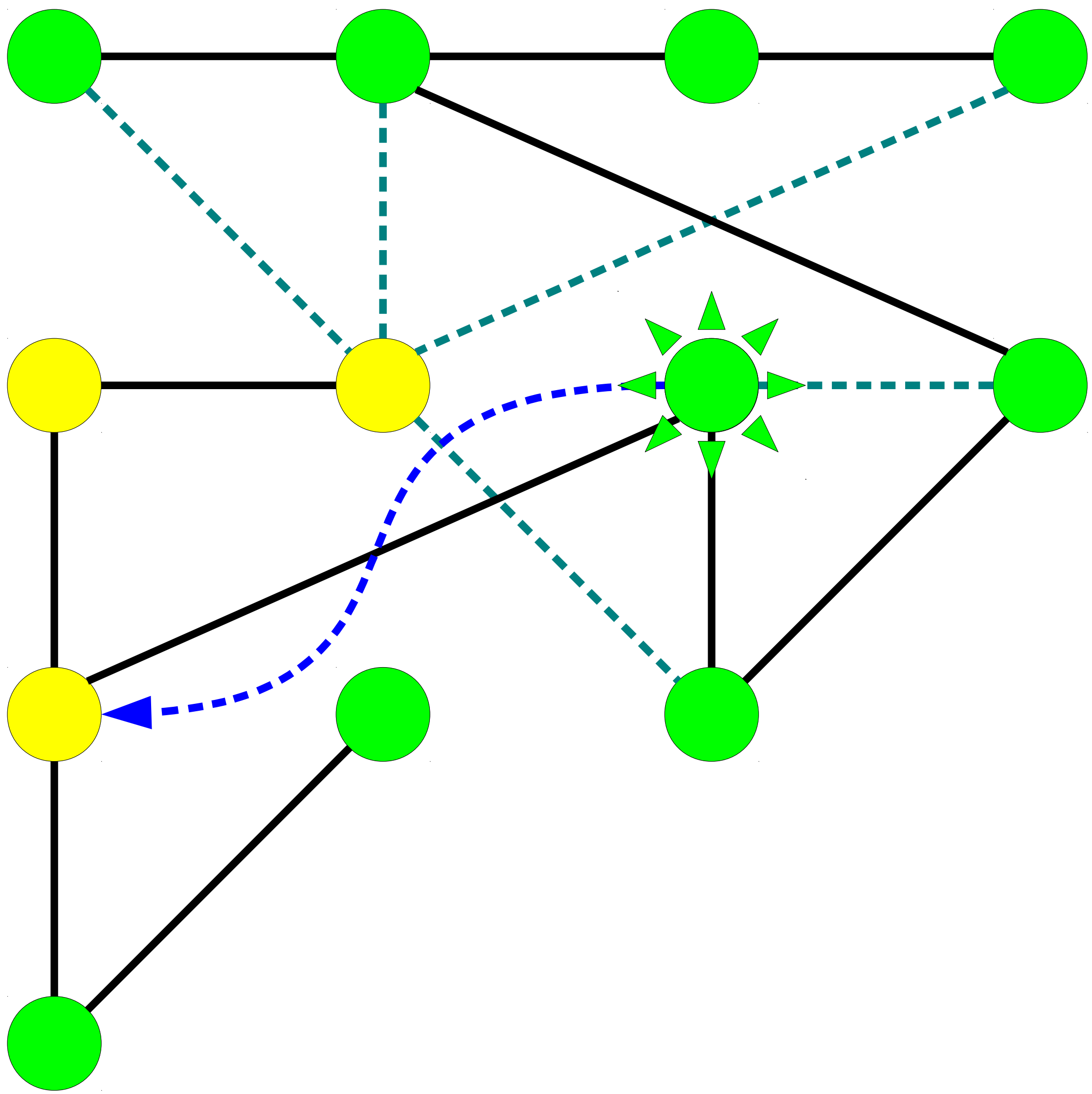

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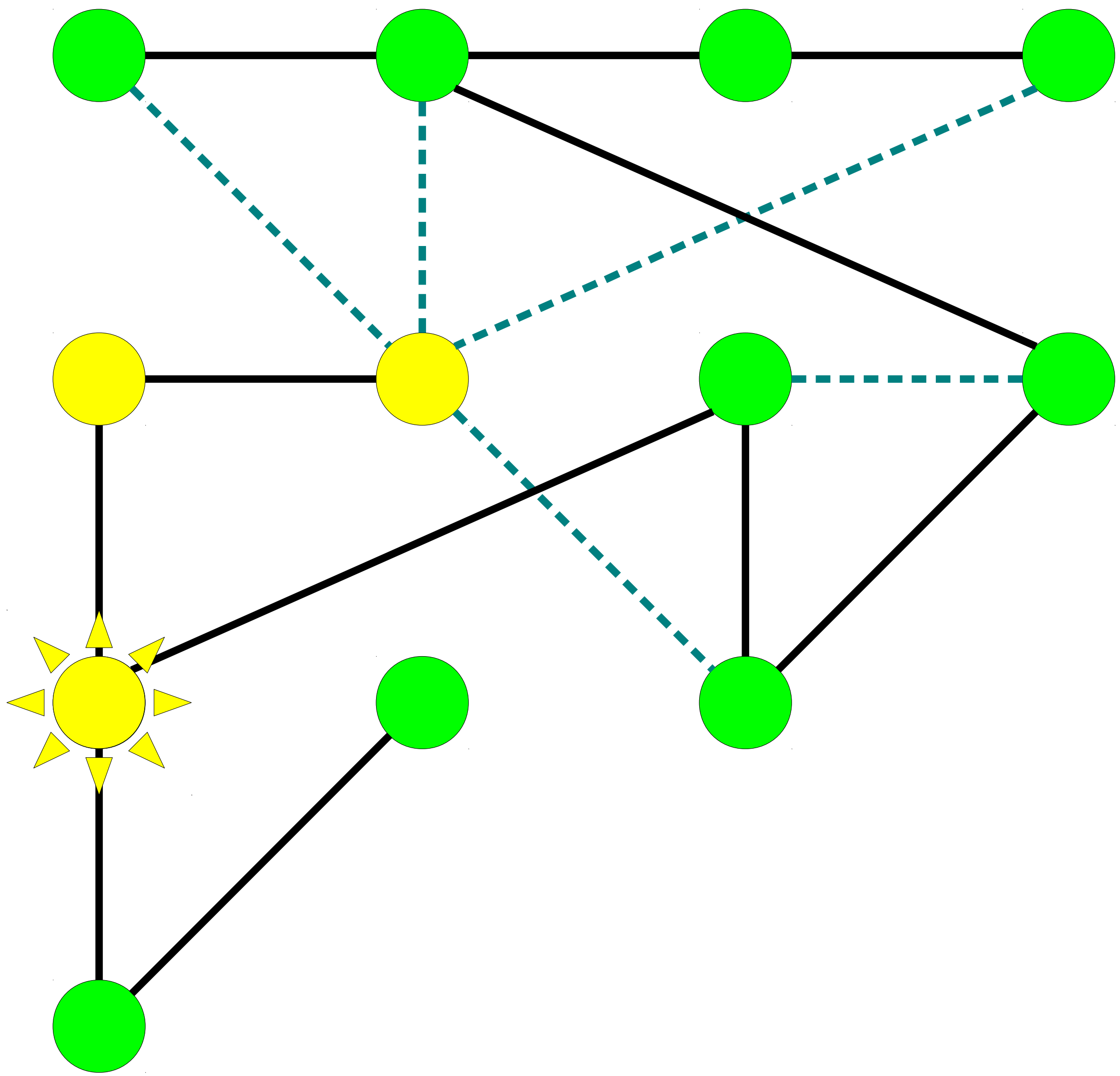

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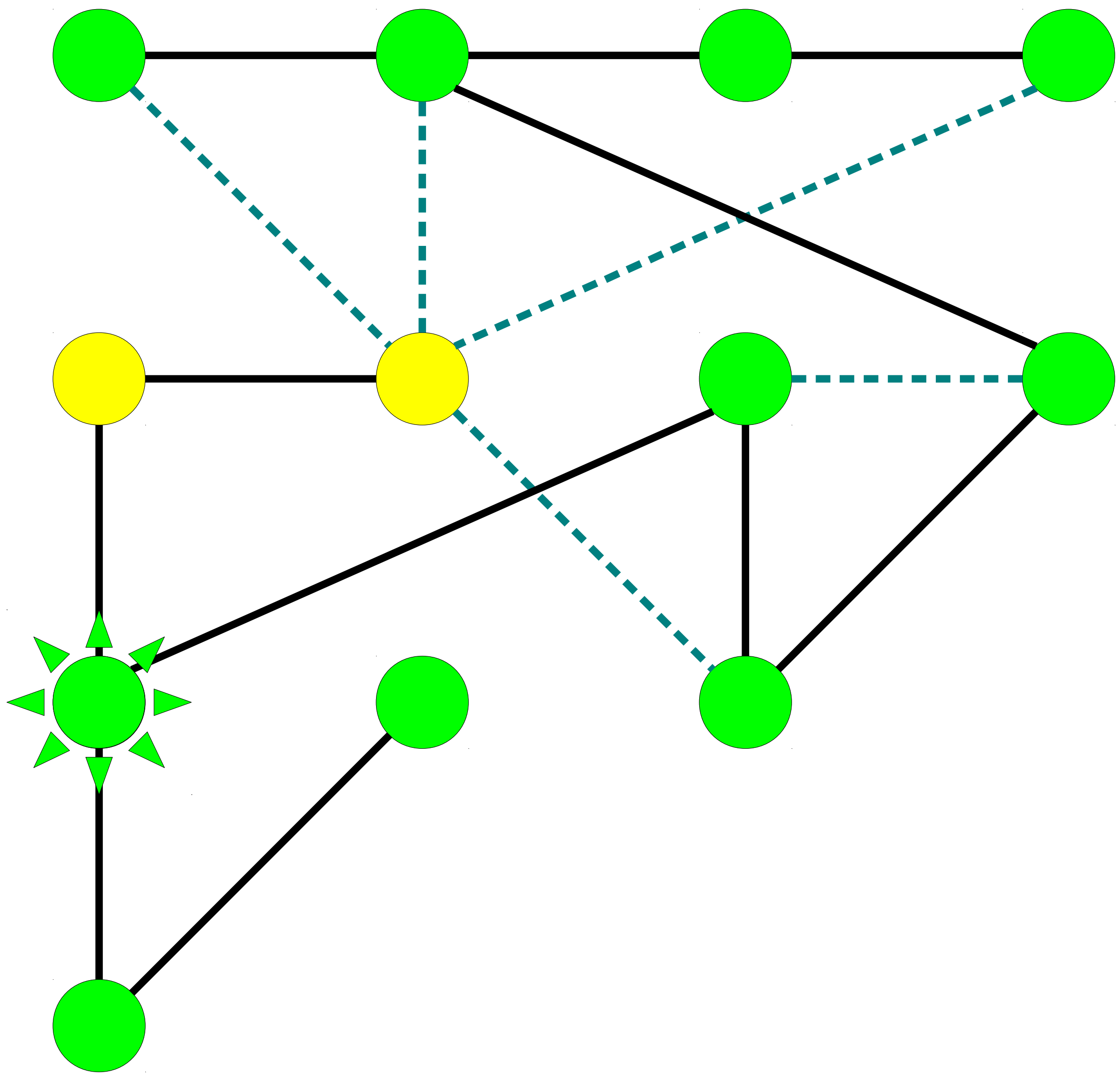

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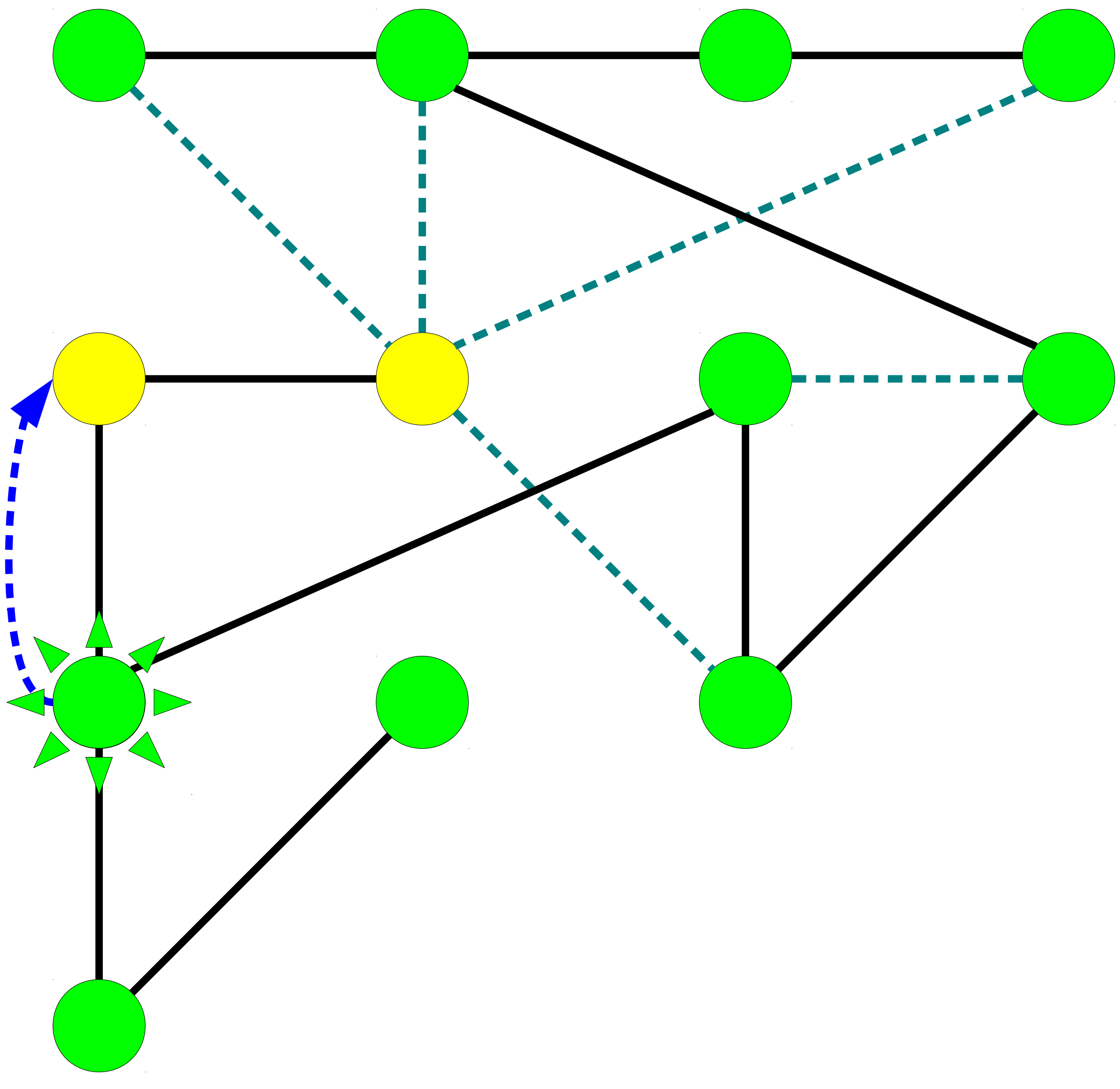

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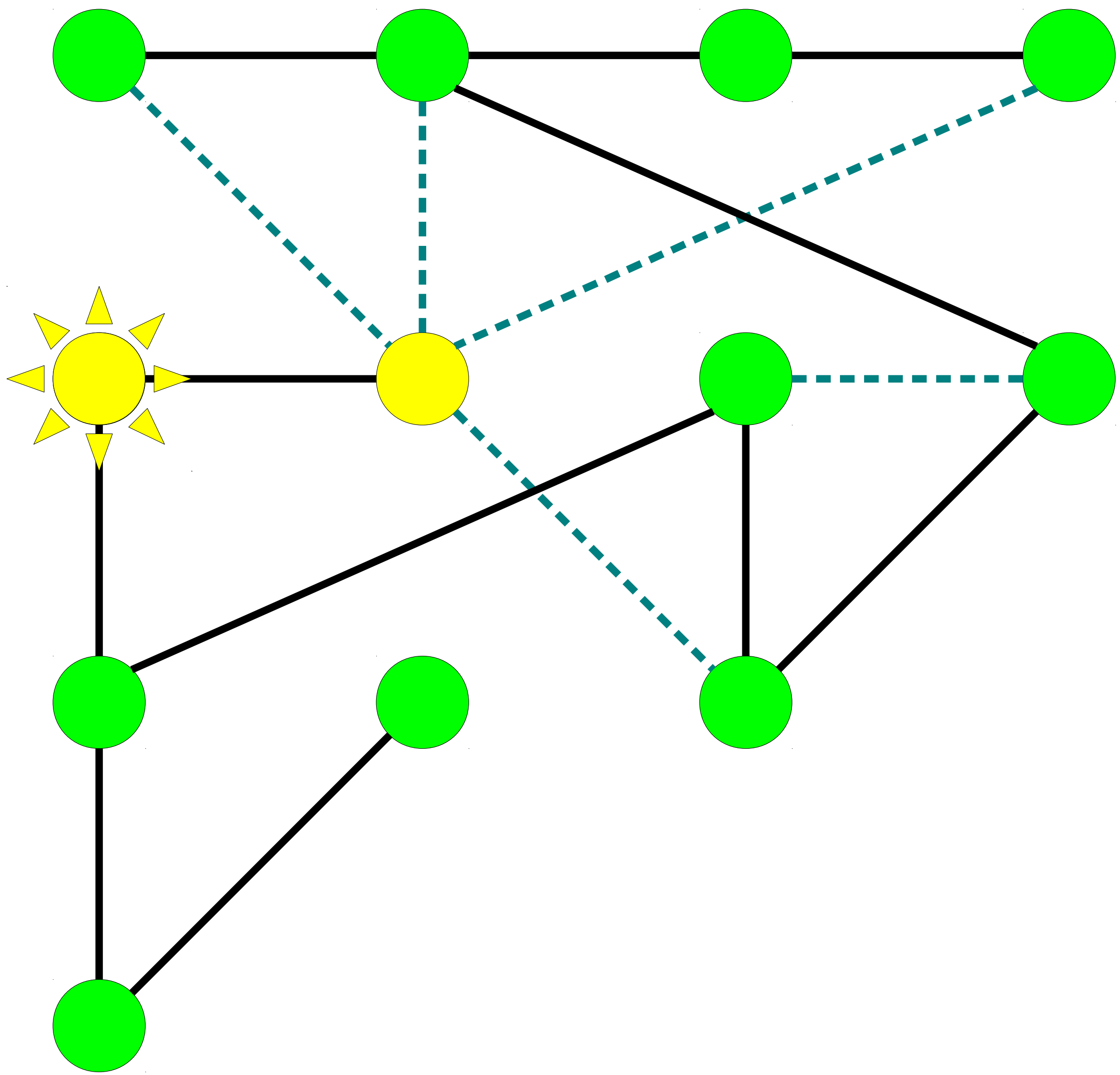

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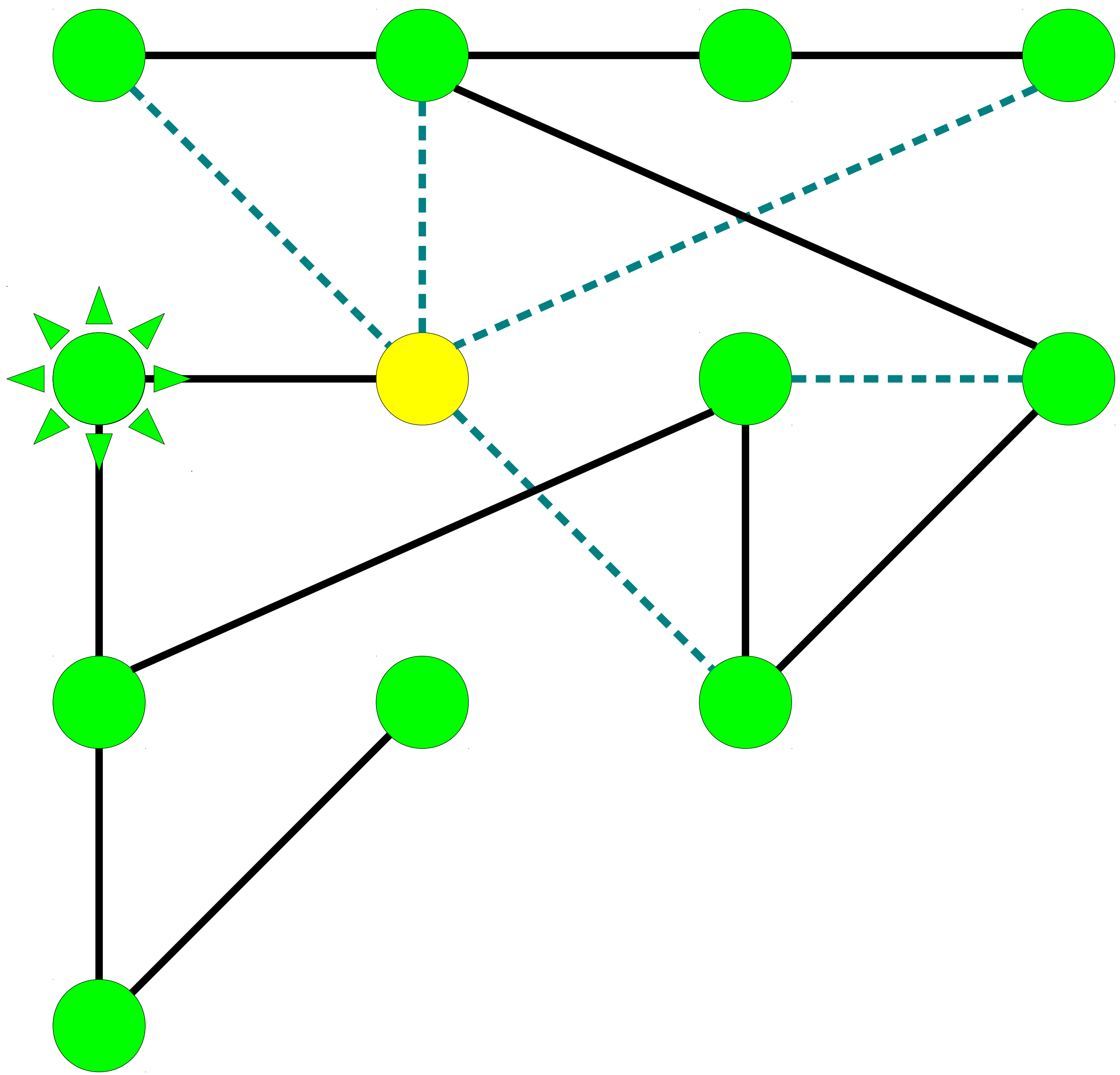

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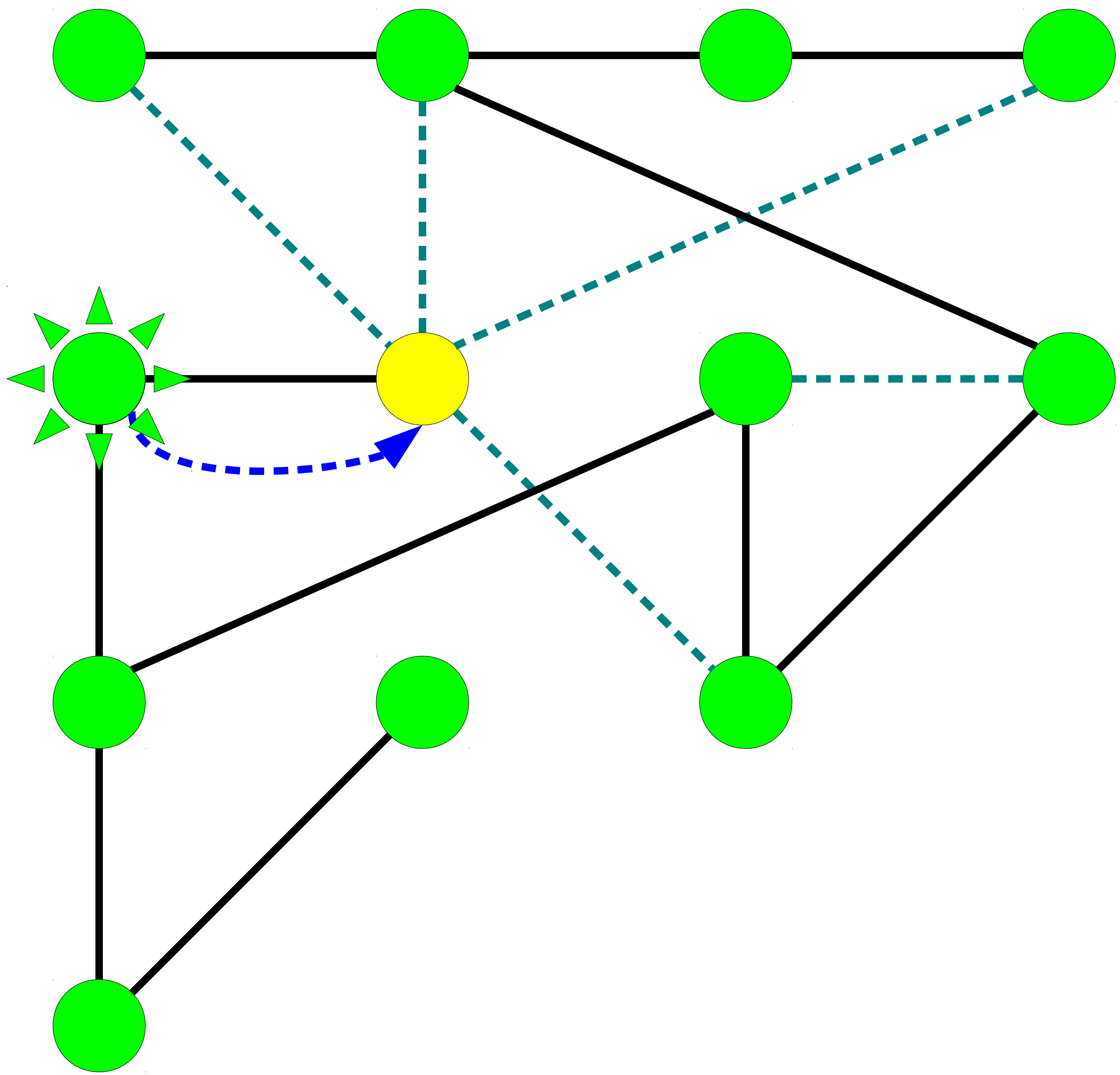

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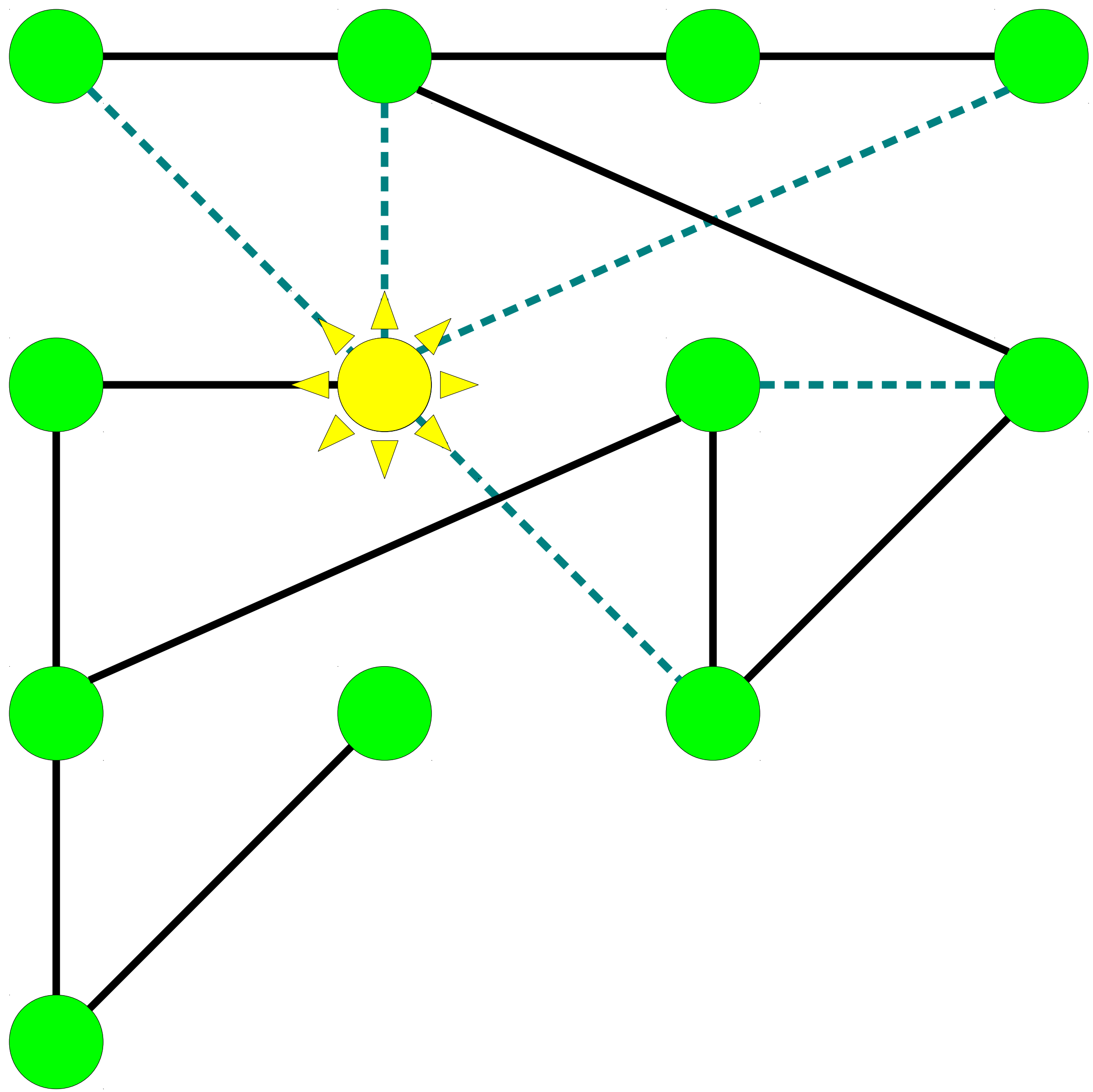

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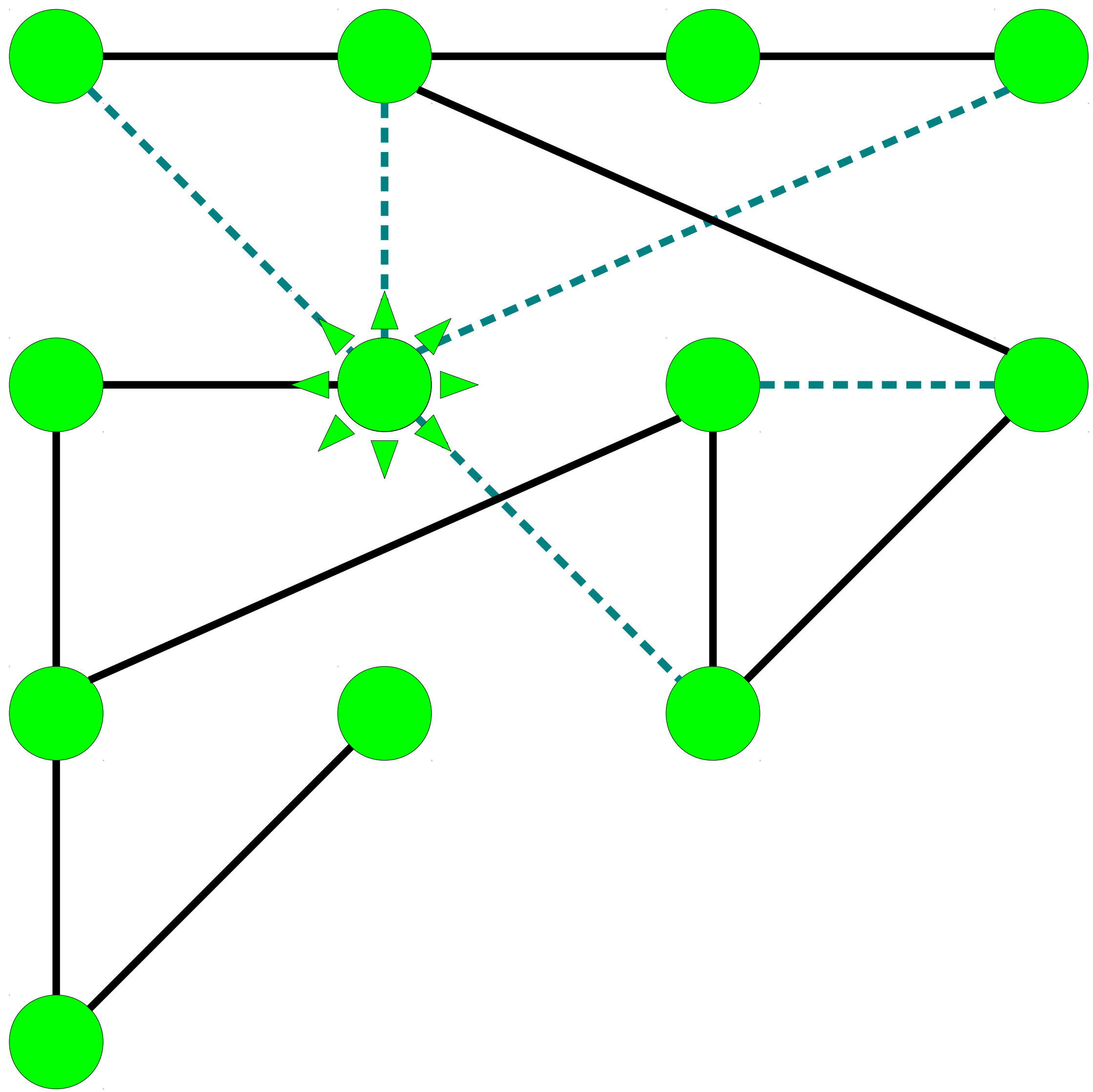

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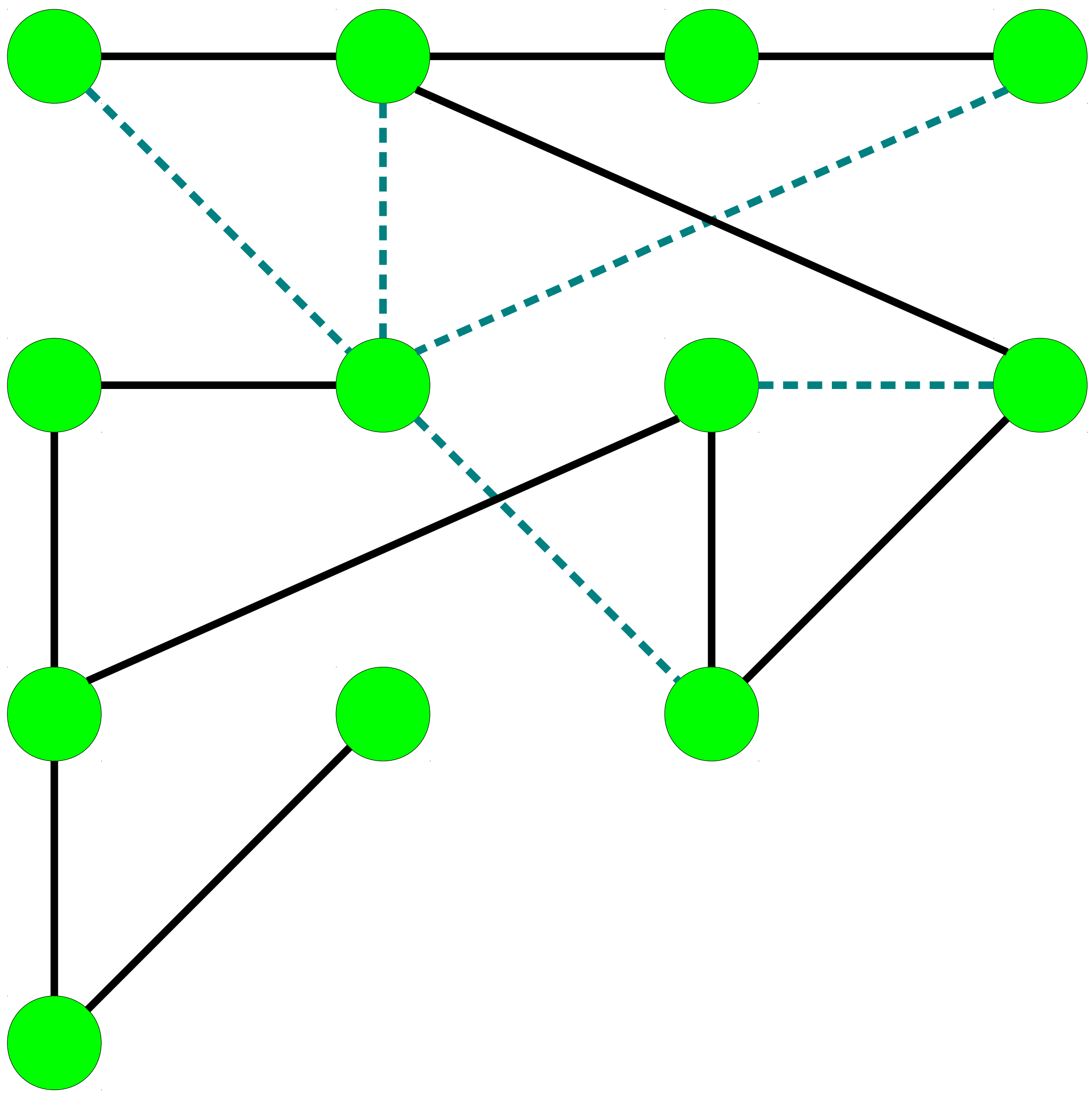

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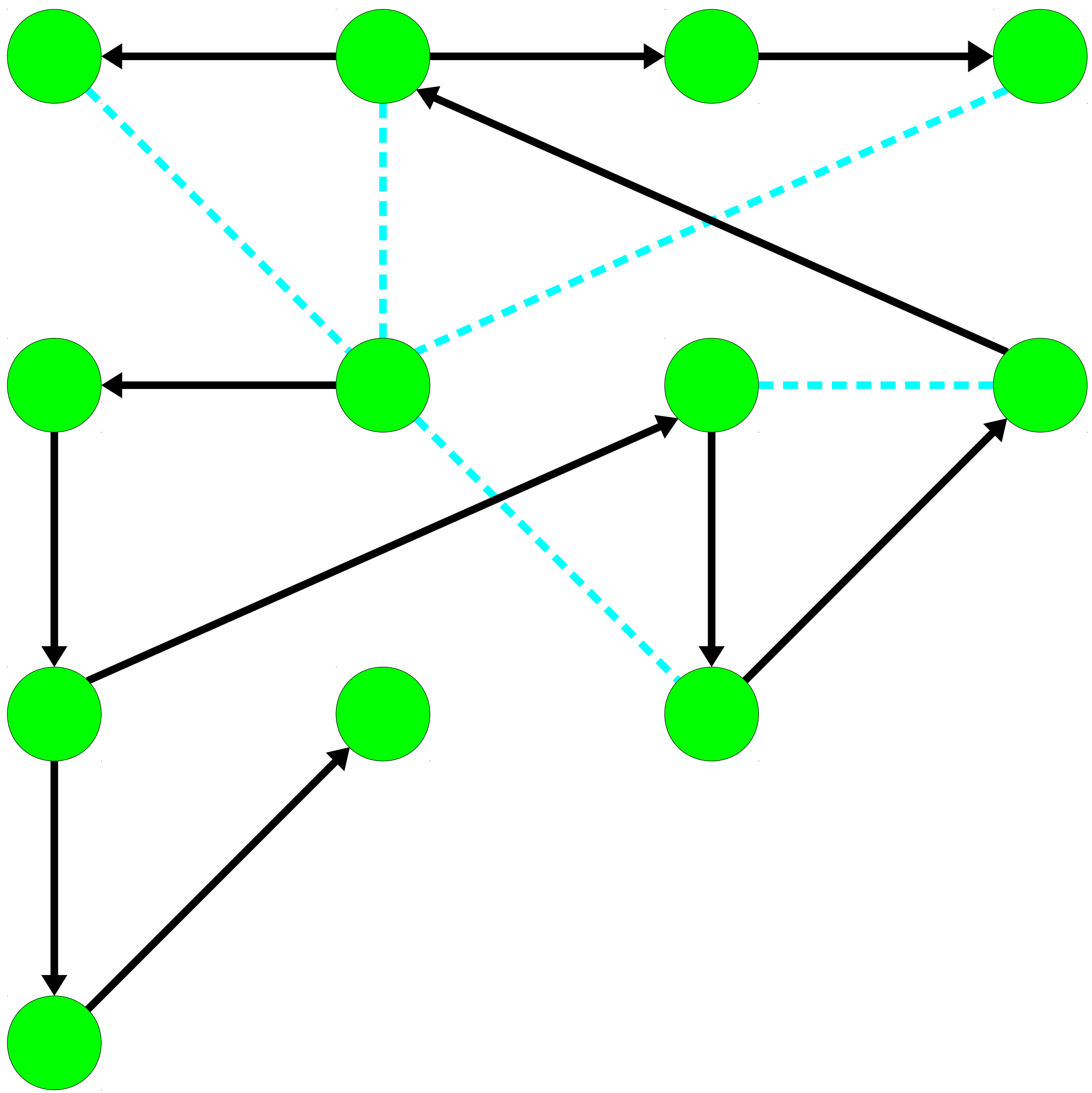

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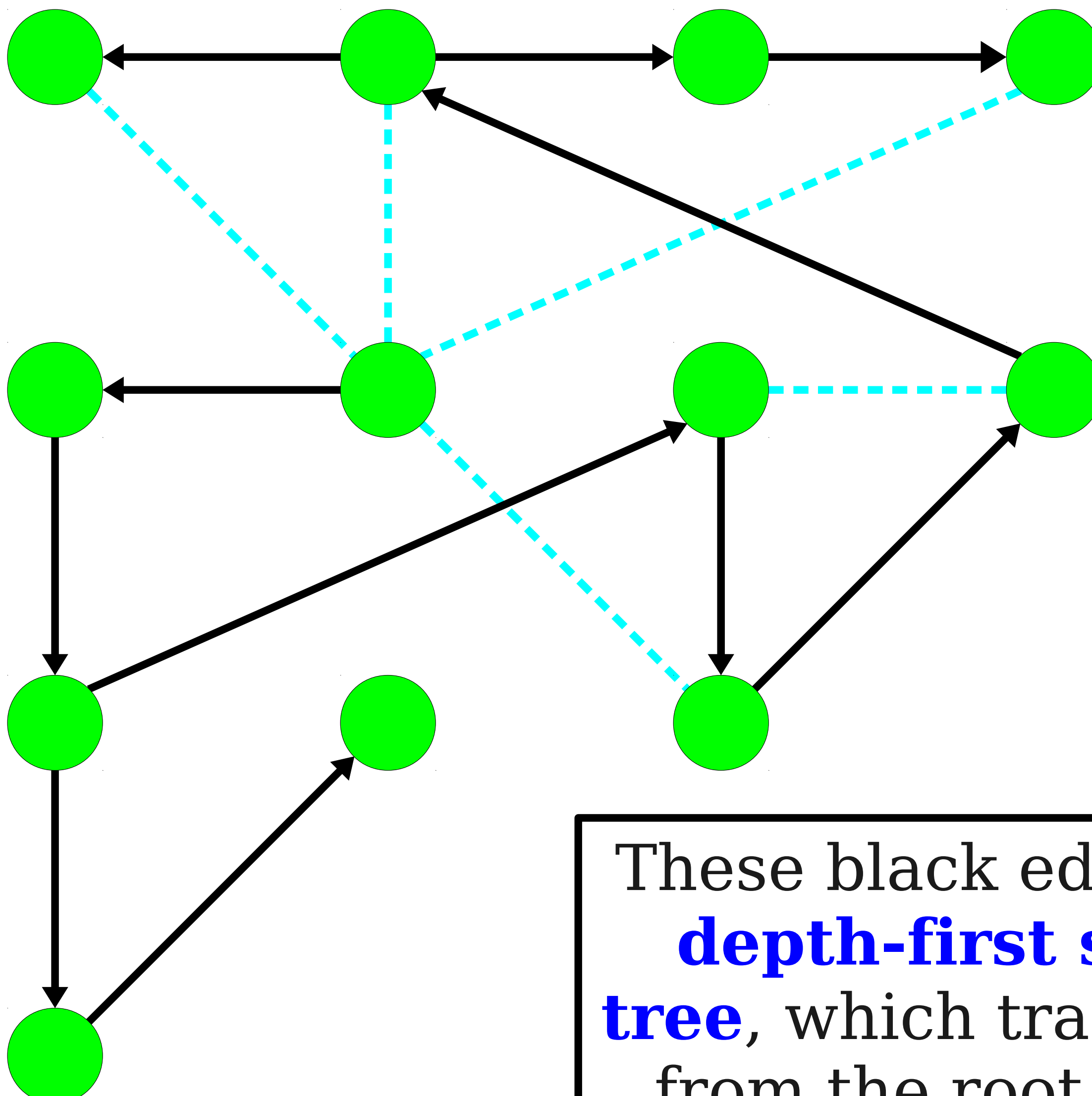
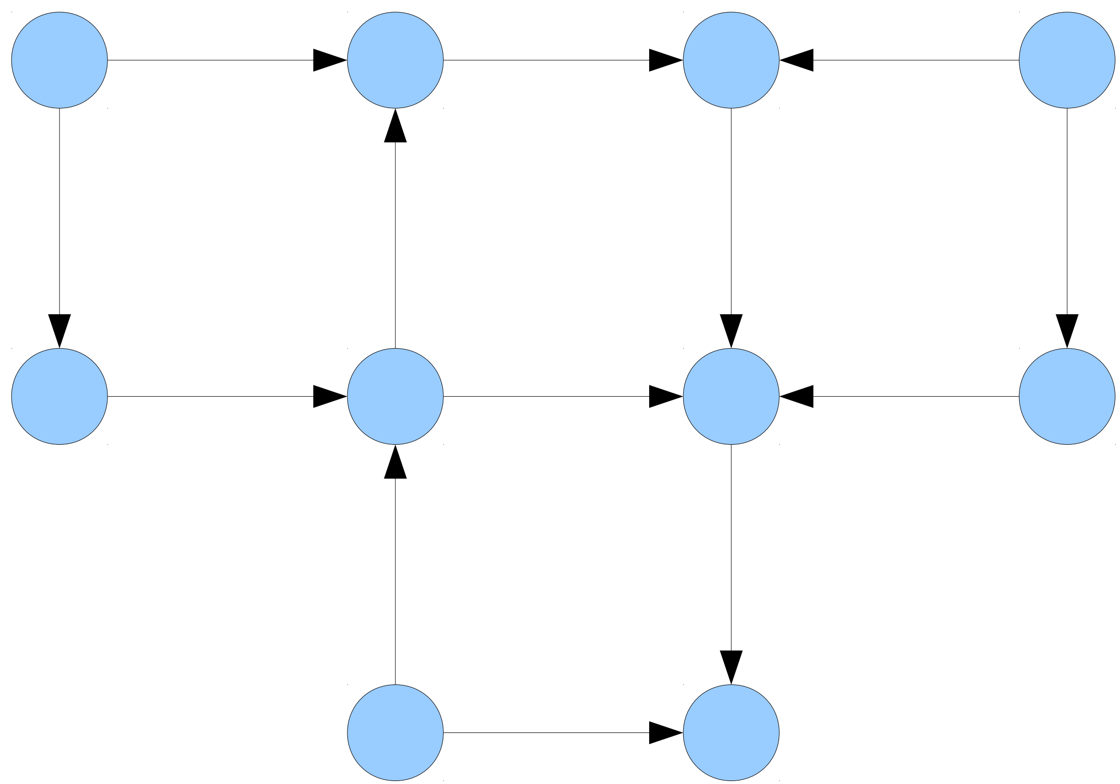


These black edges for a **depth-first search tree**, which traces paths from the root to each node in the graph.

Depth-First Search, Again



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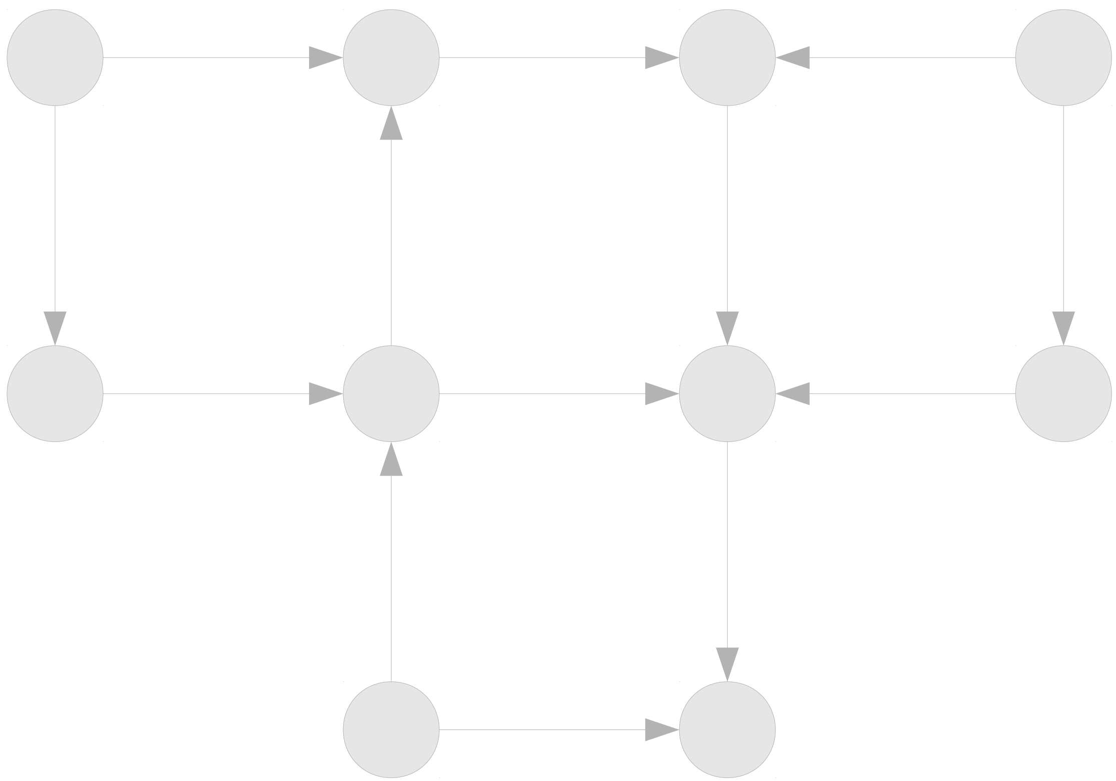

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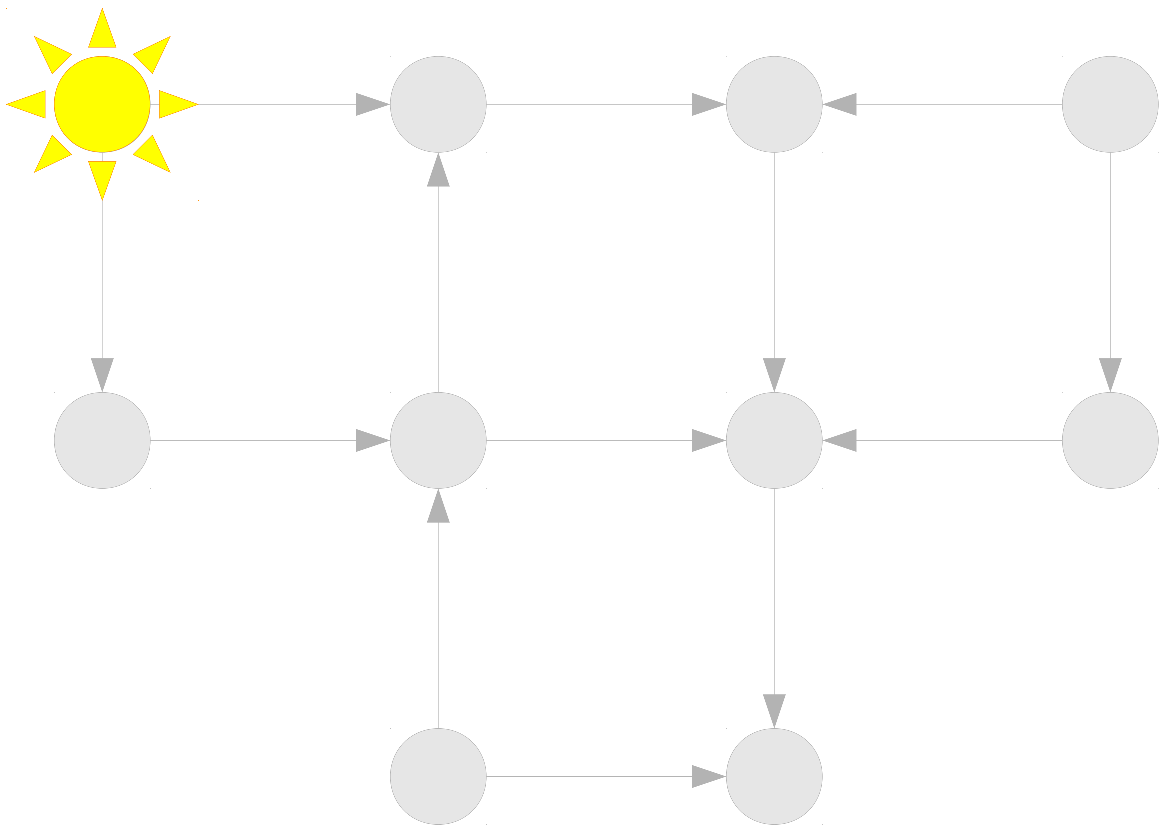

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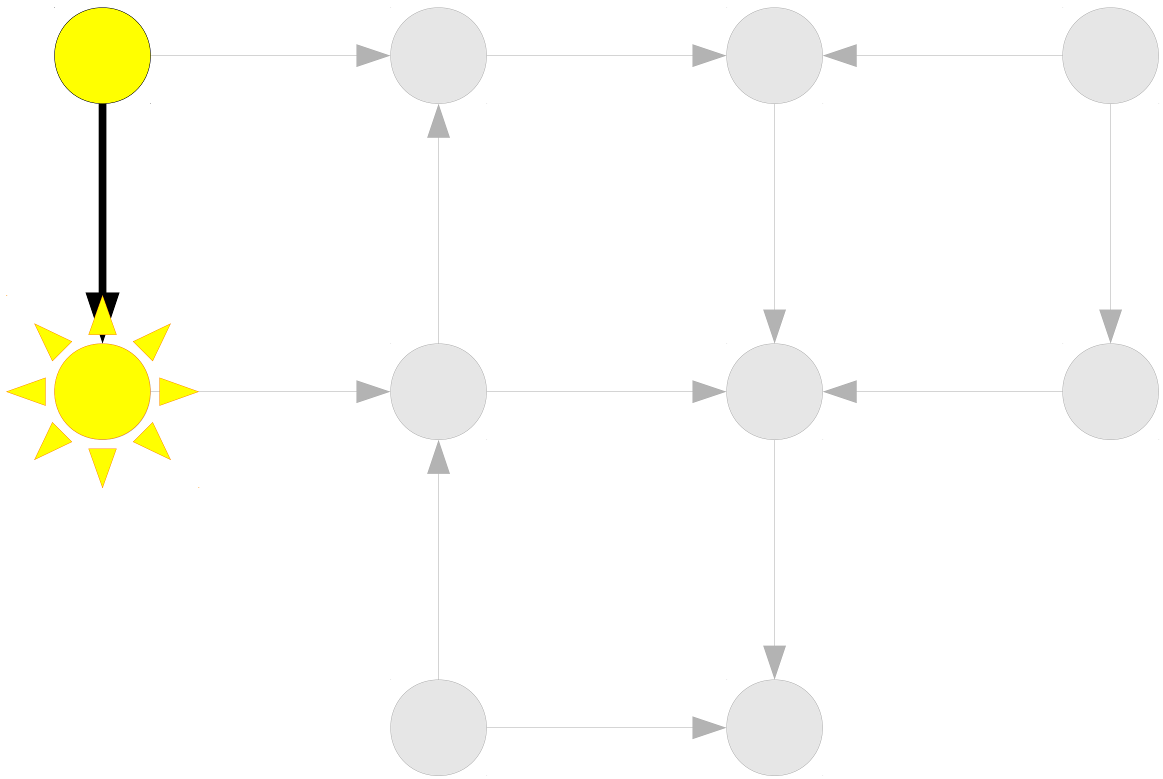

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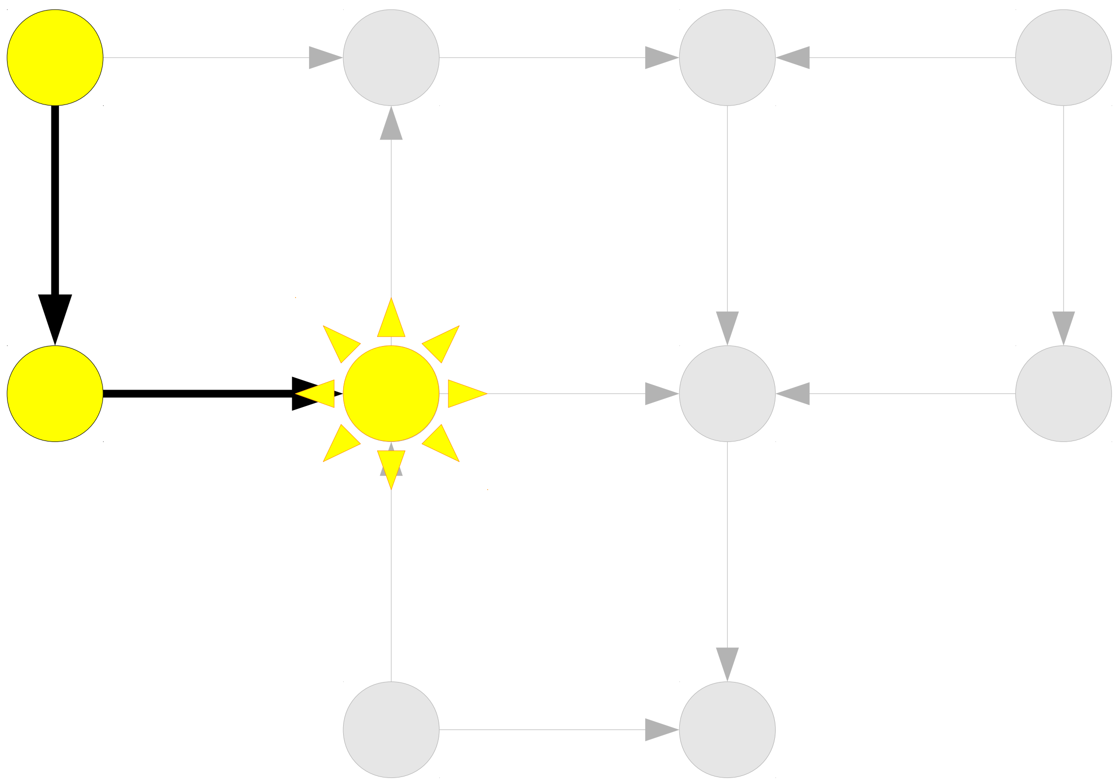

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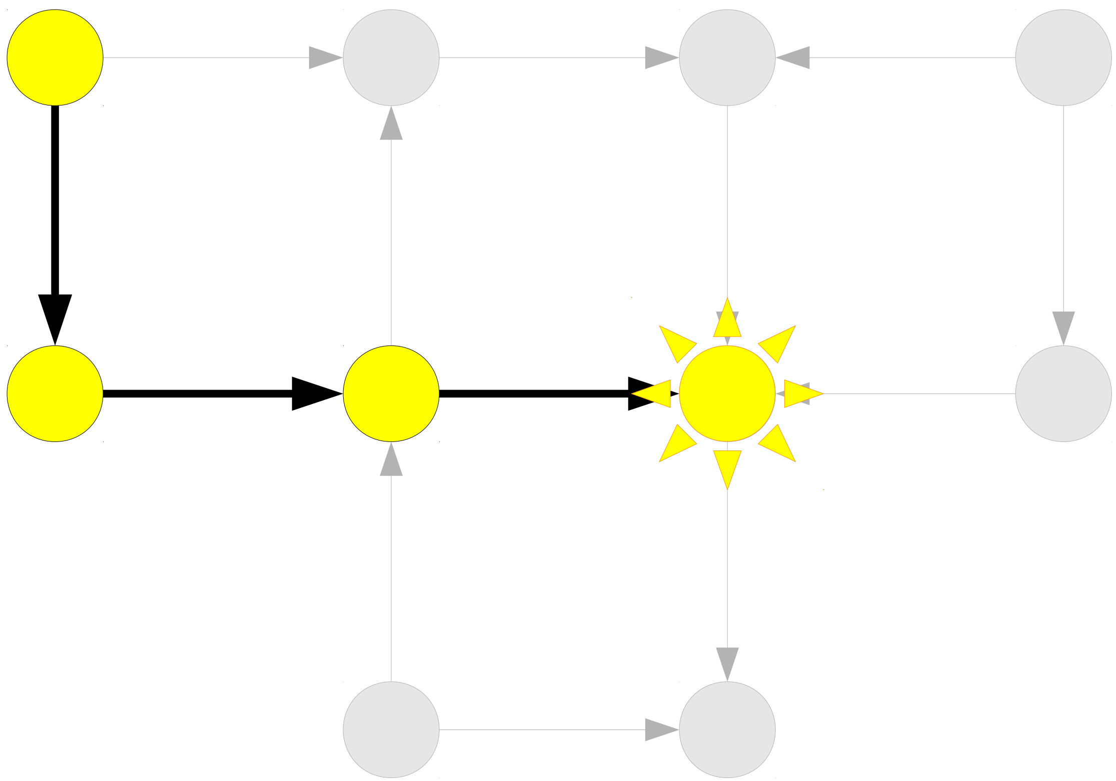

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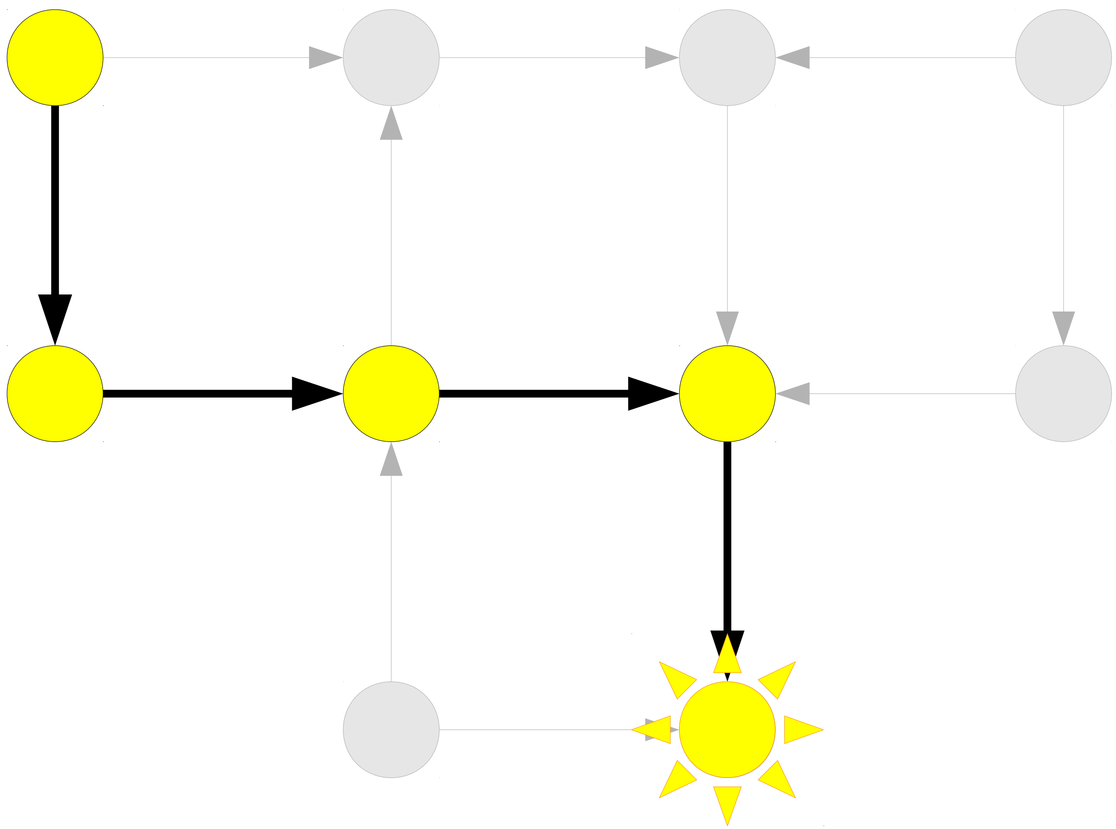

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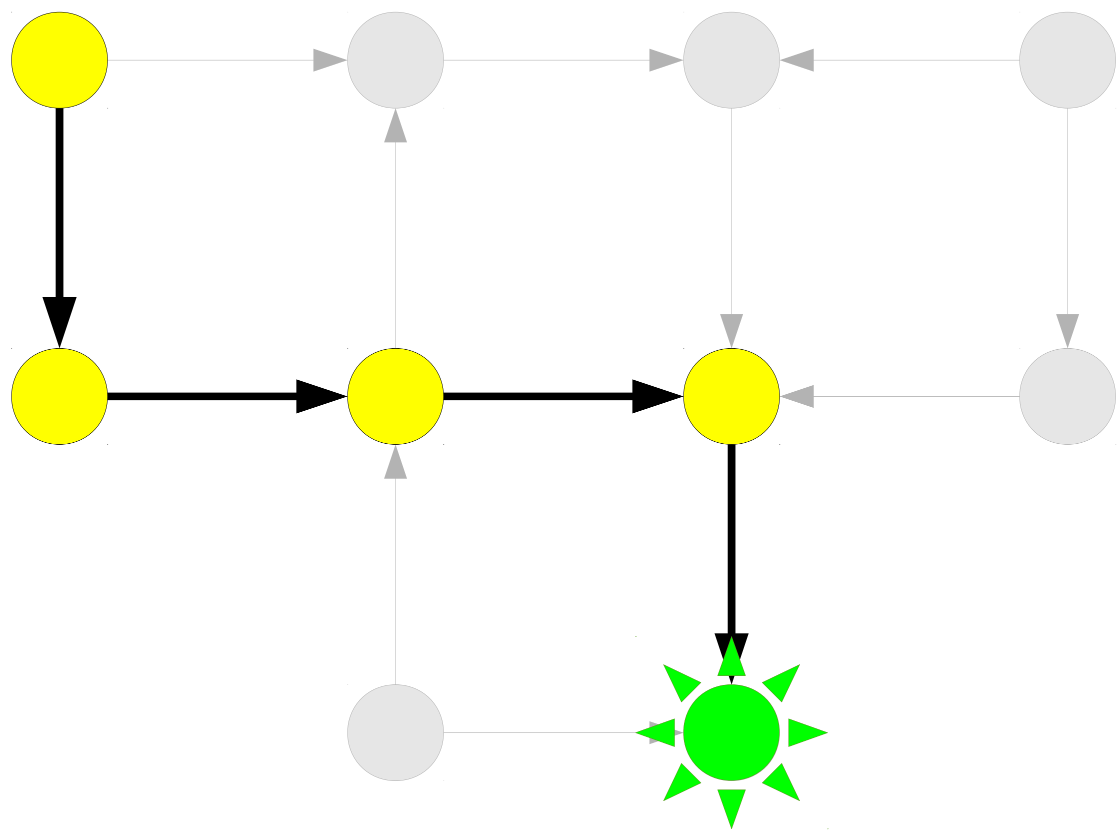

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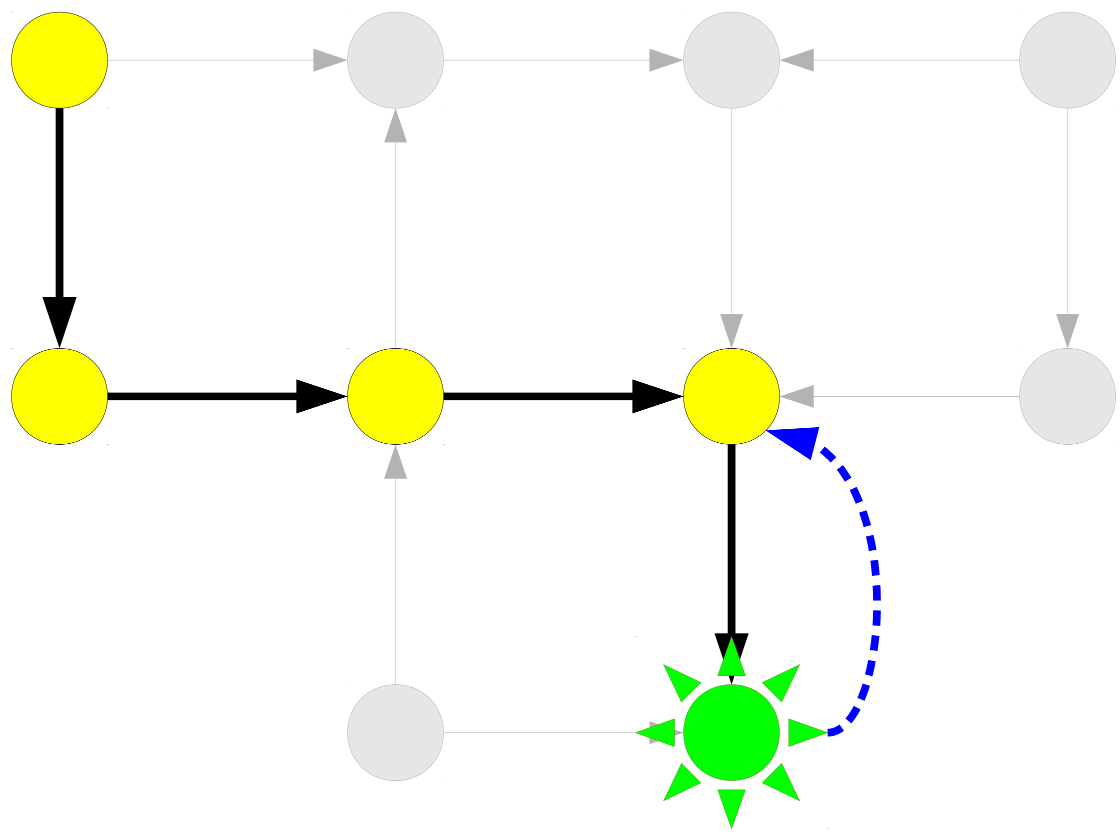

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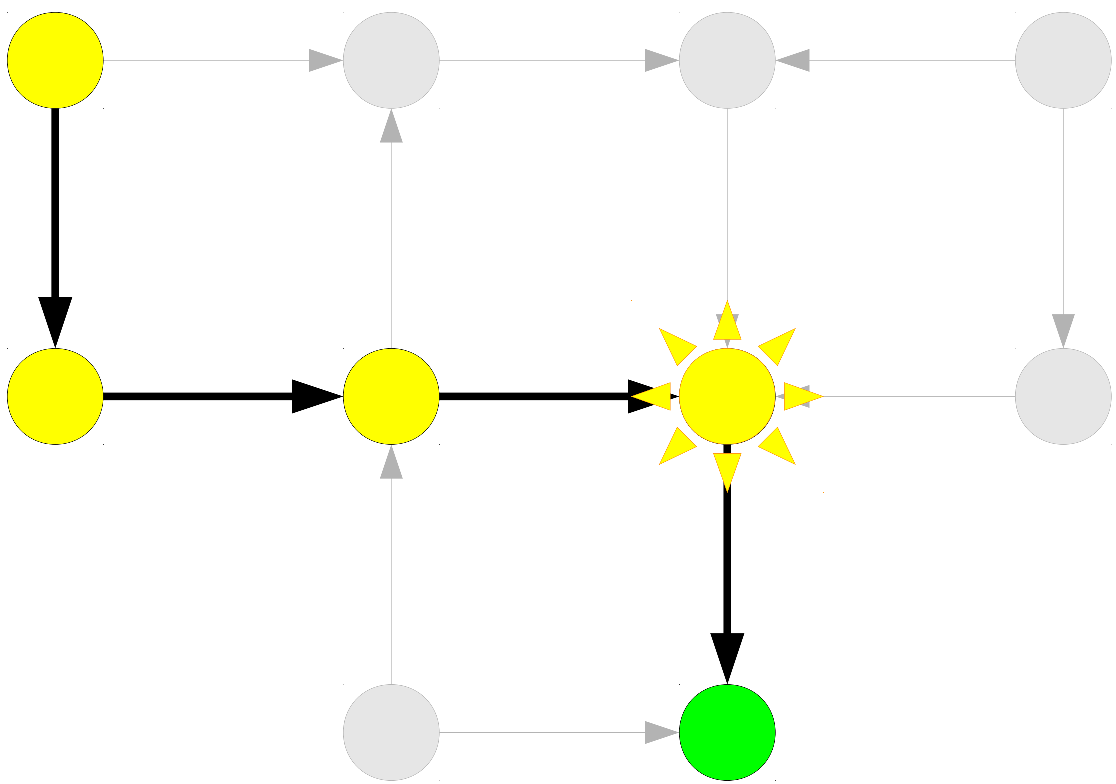

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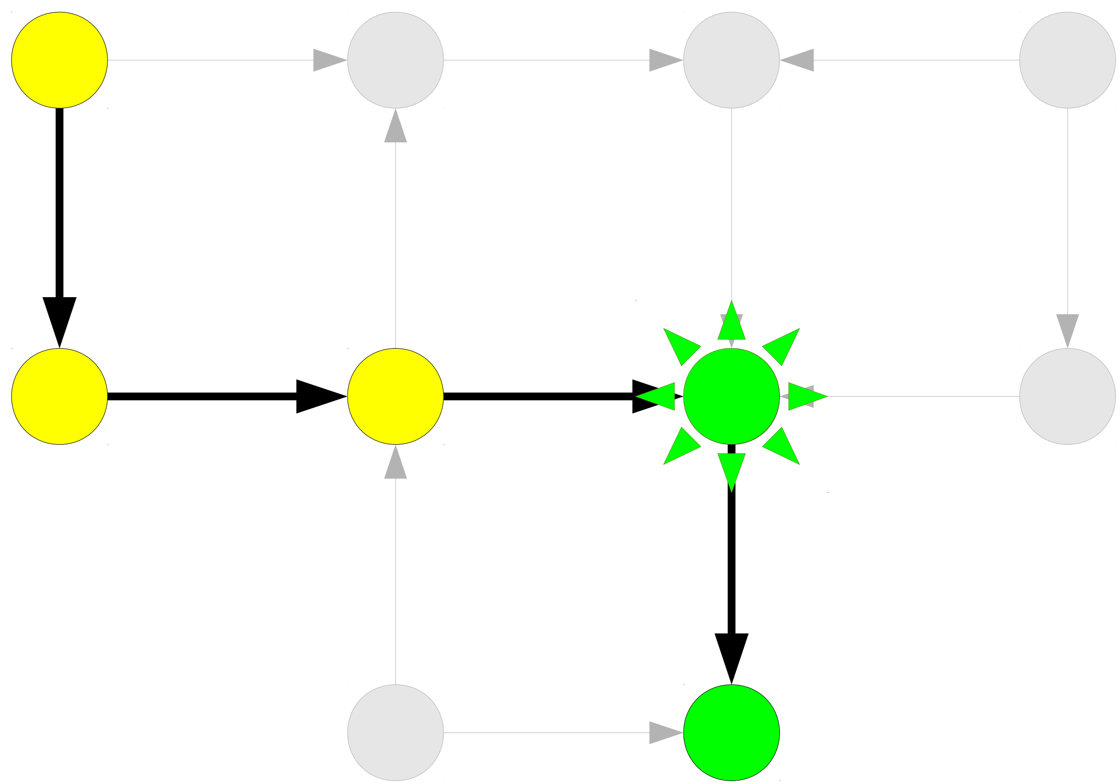

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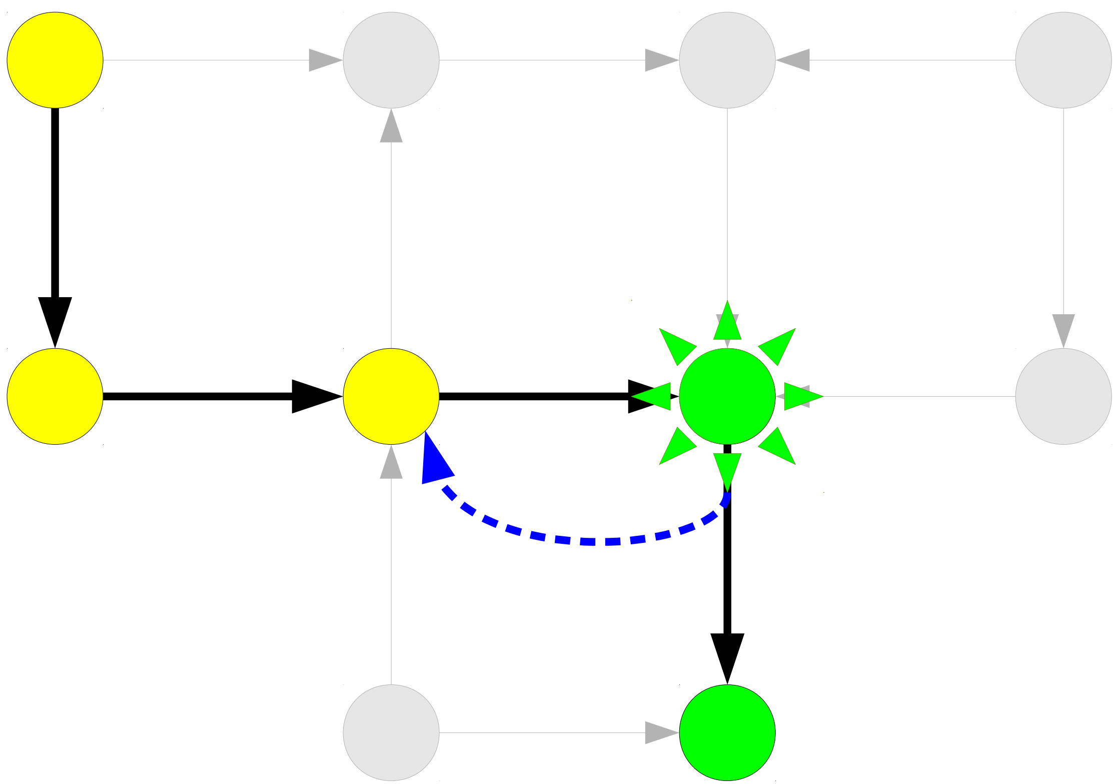

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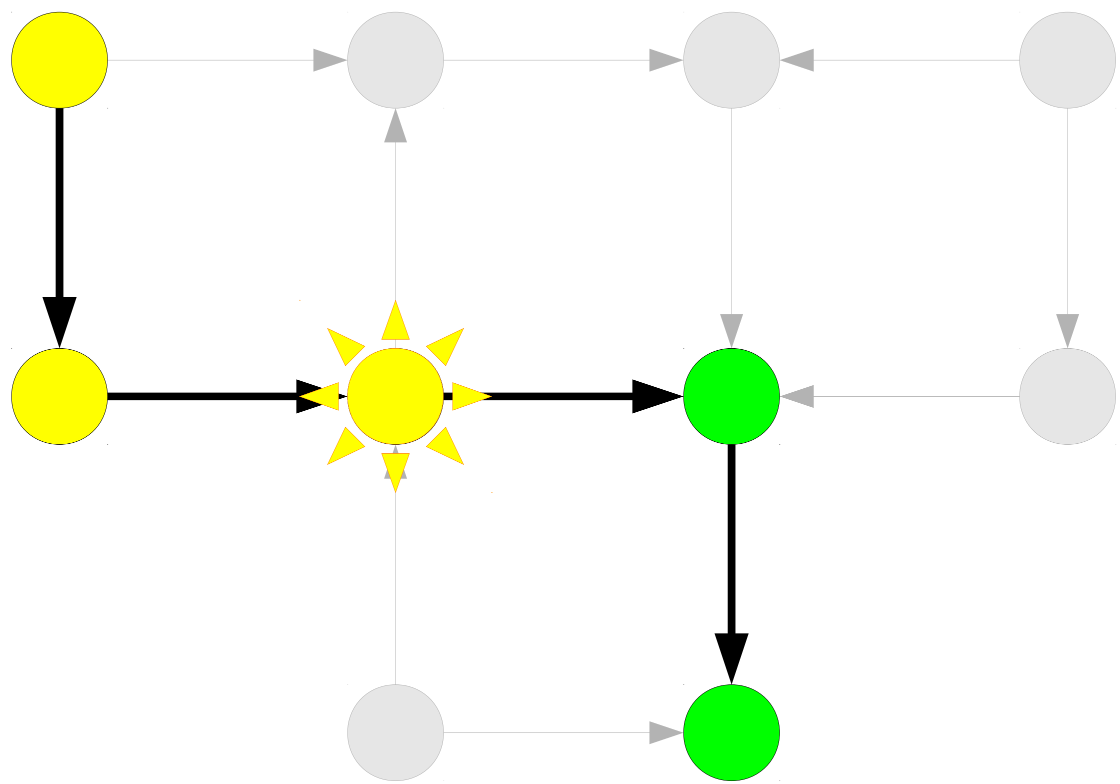

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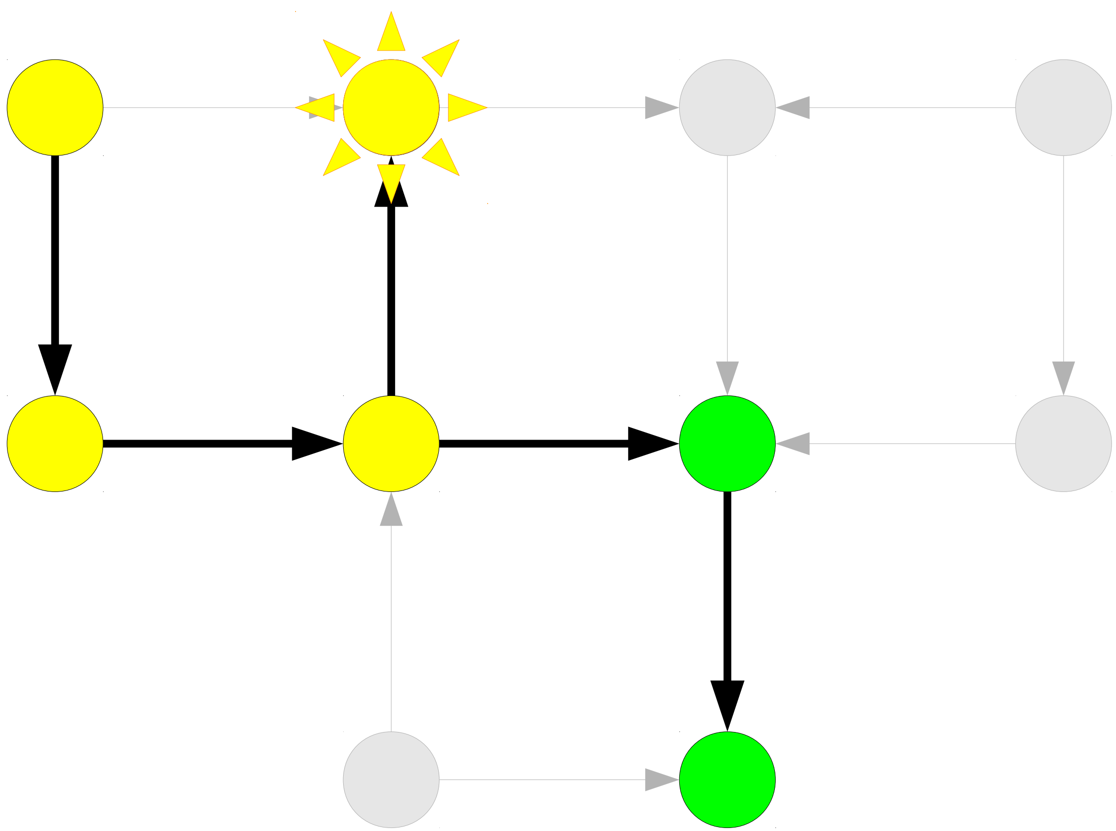

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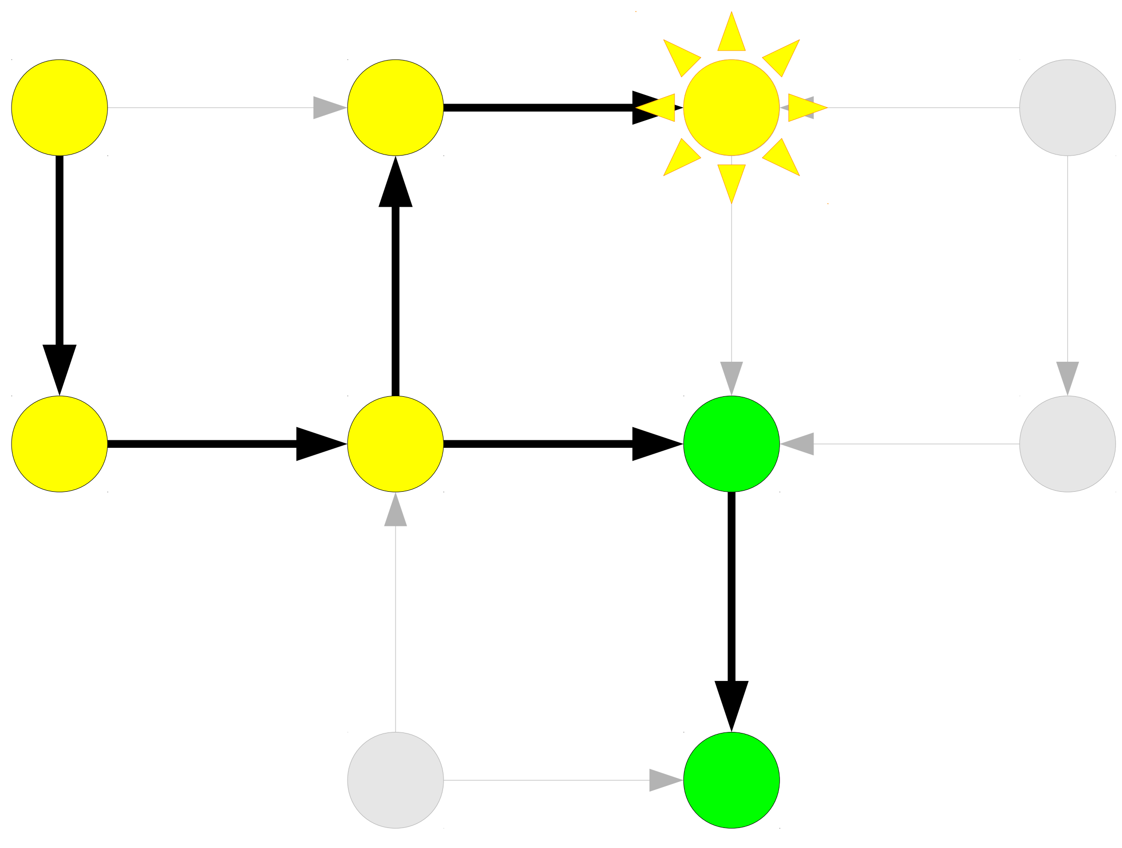

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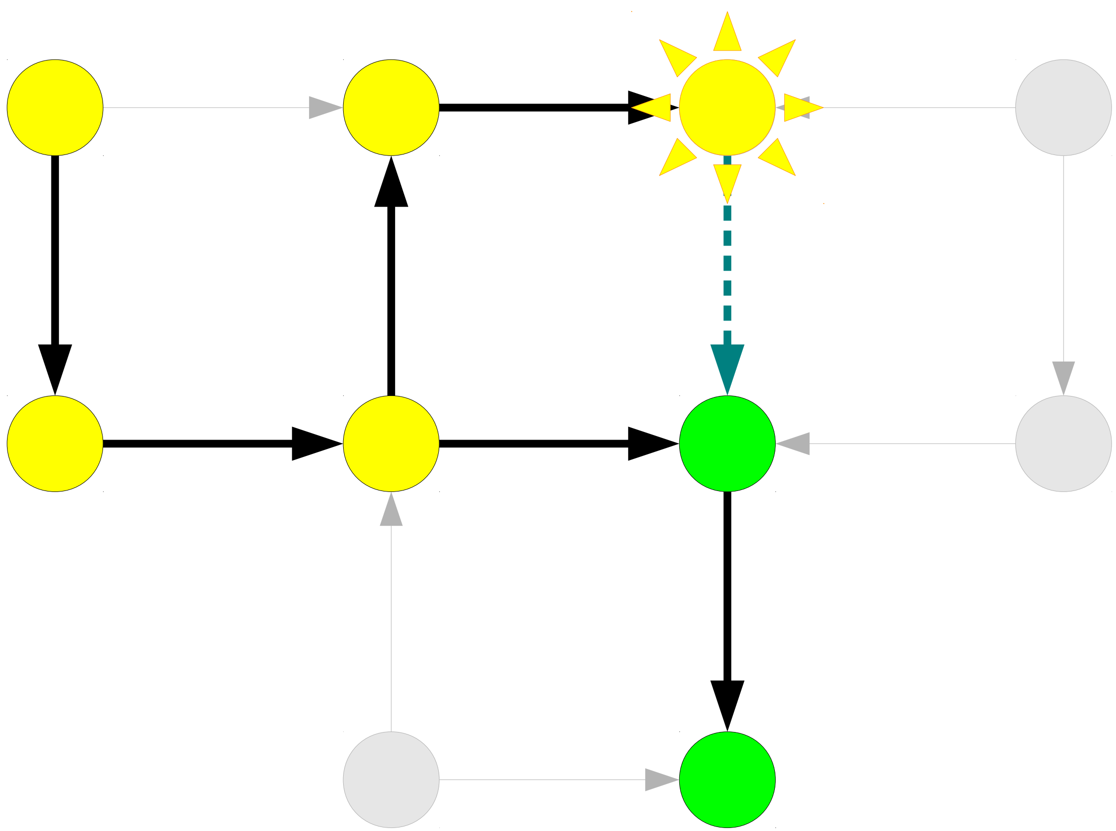

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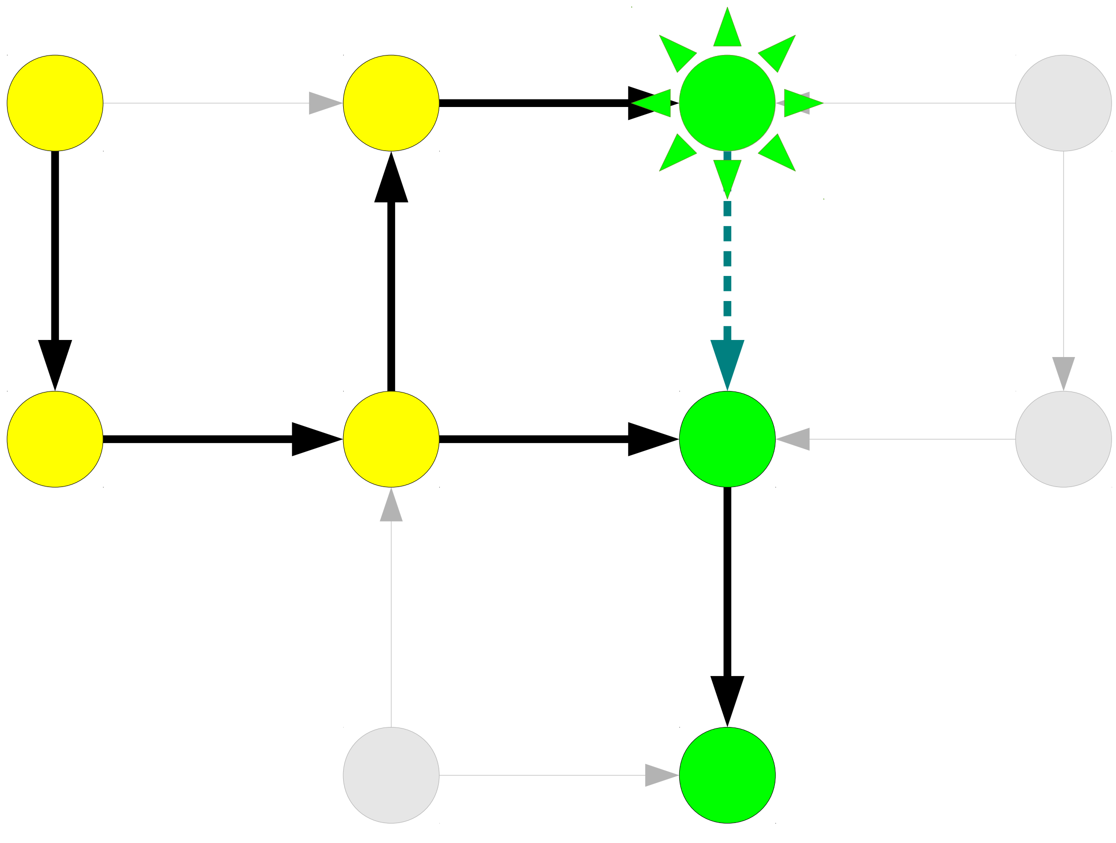

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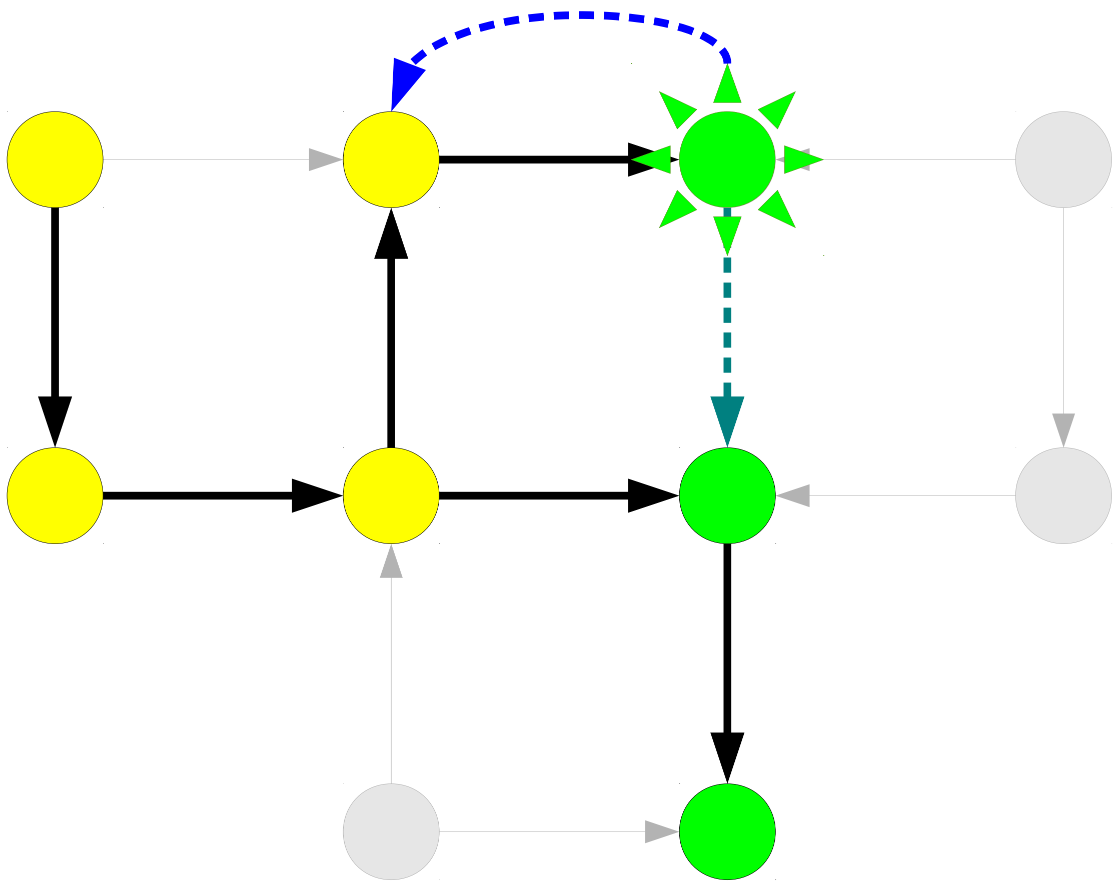
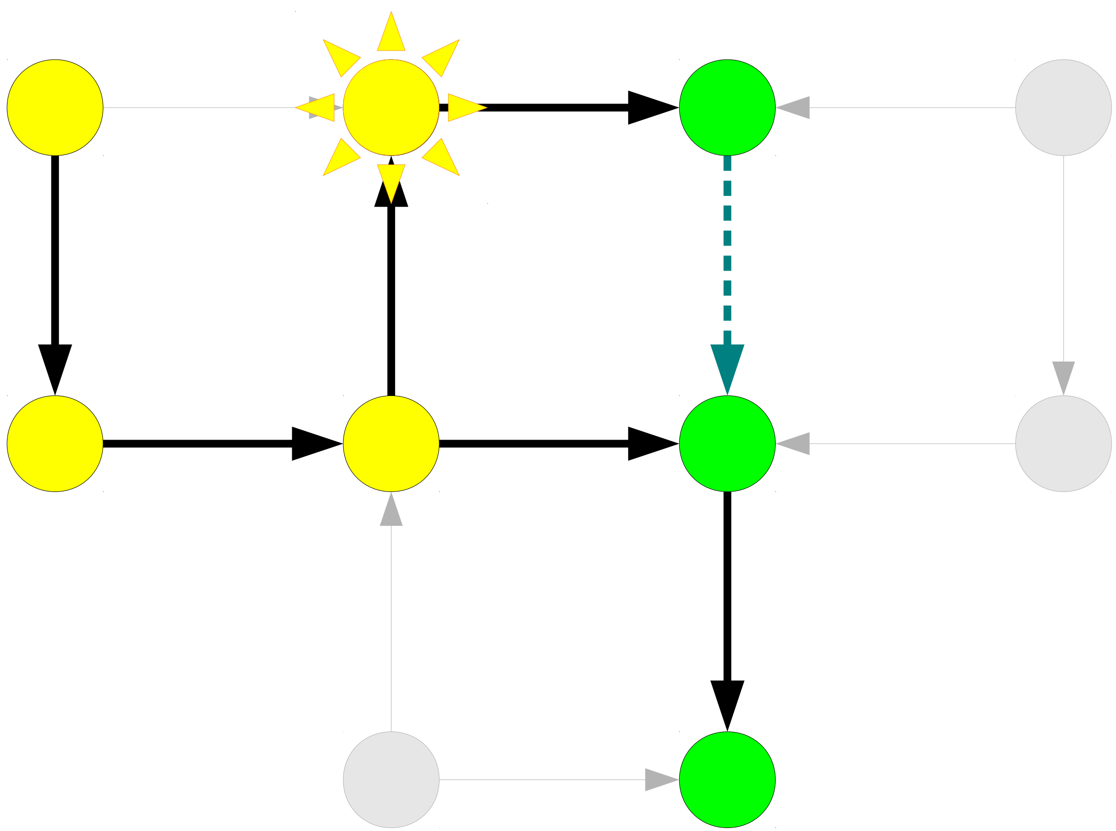


Depth-First Search, Again



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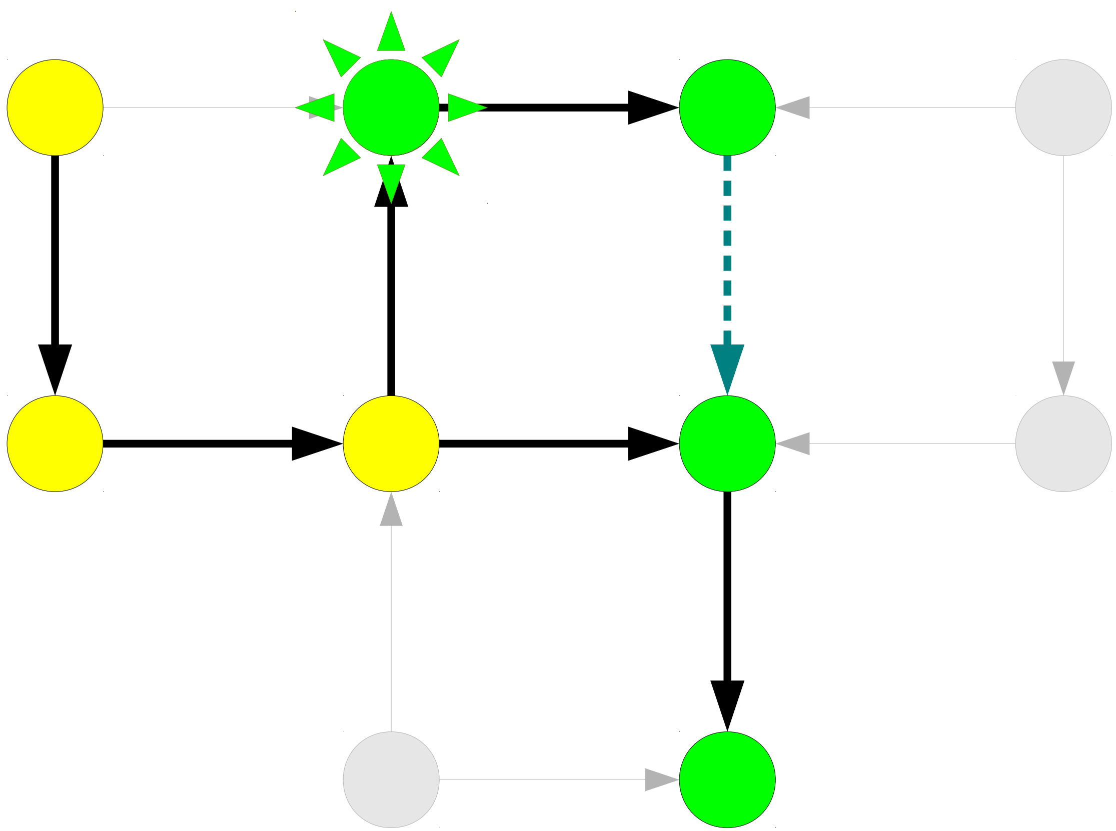

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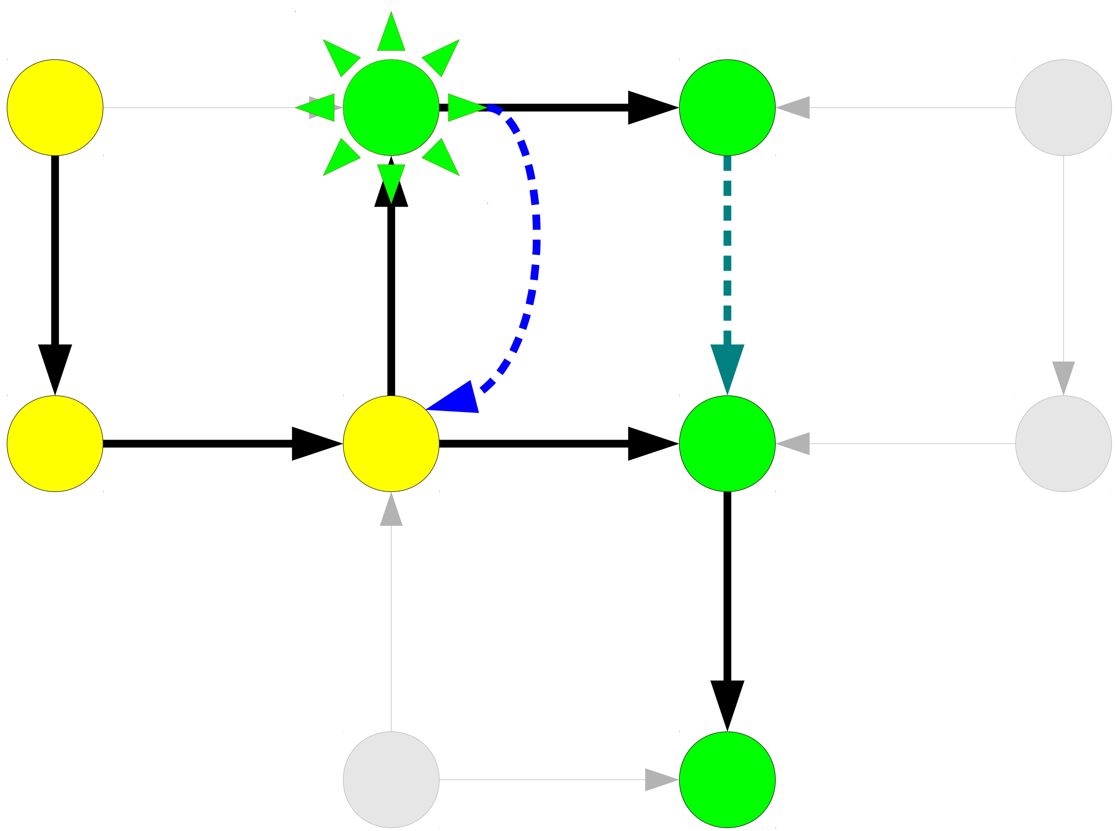

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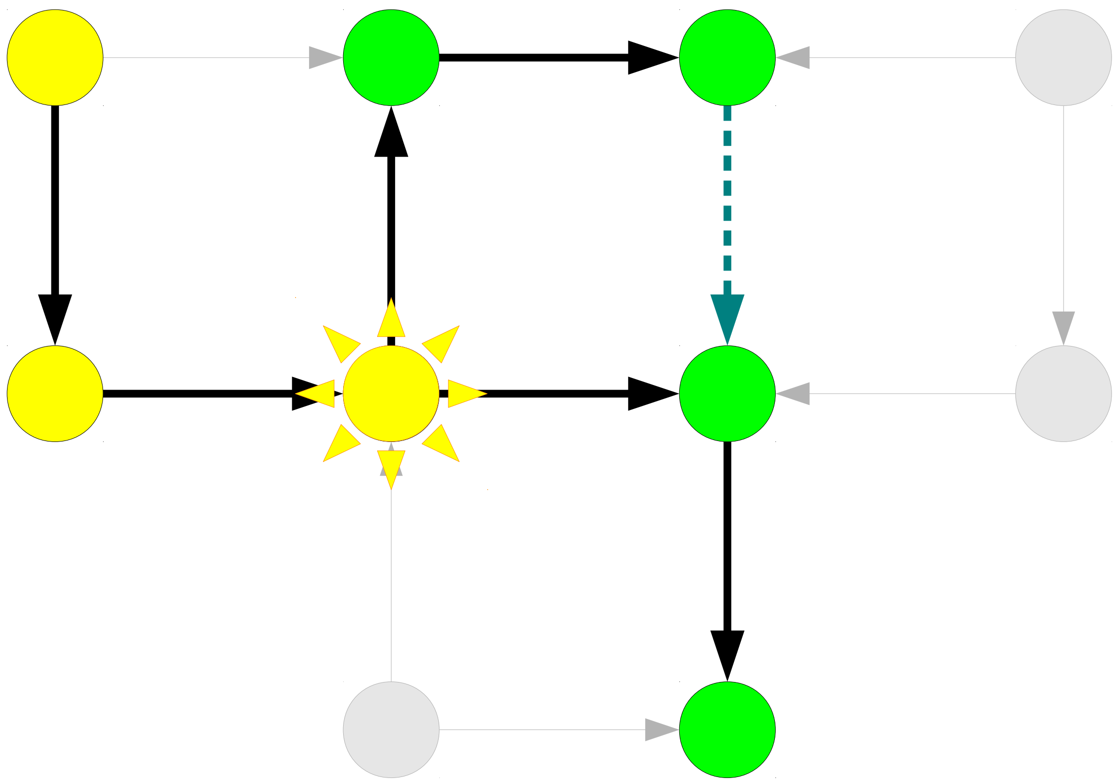

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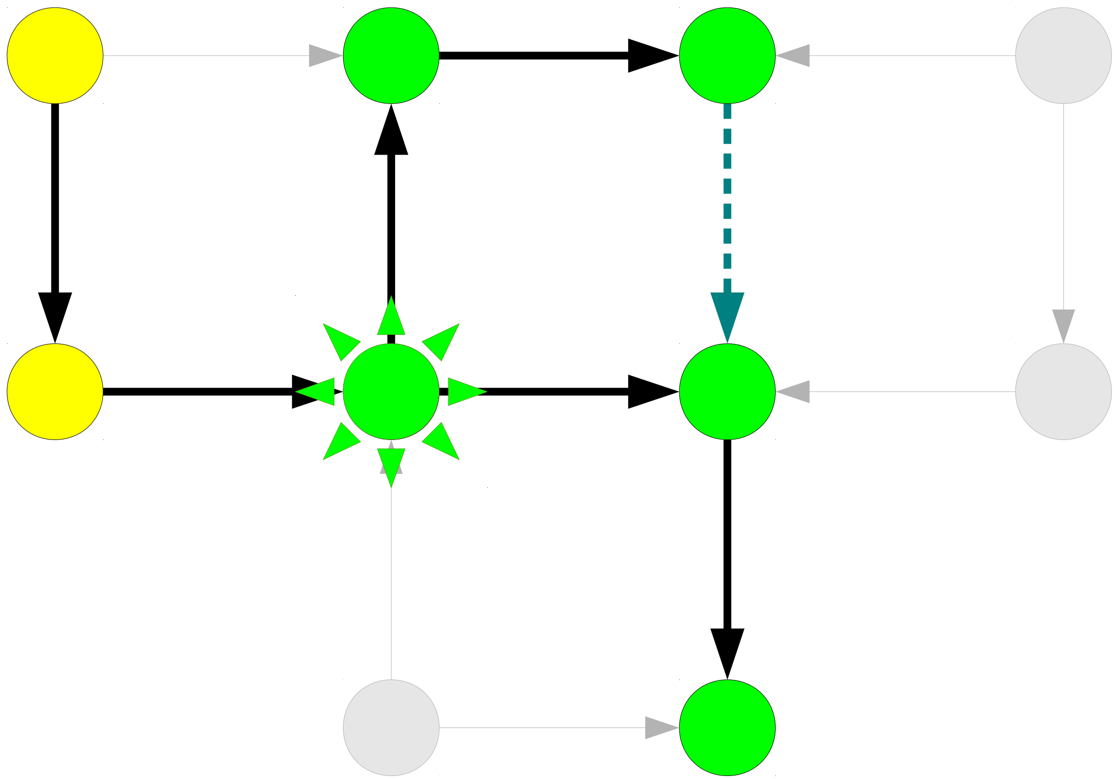

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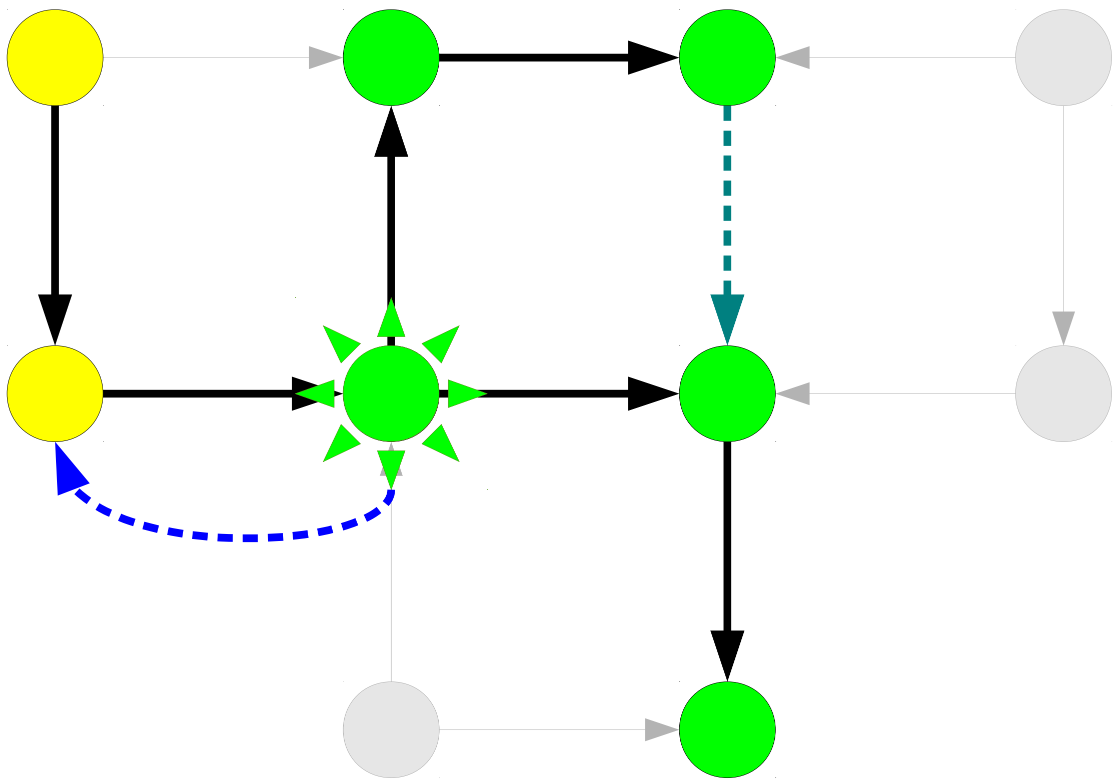

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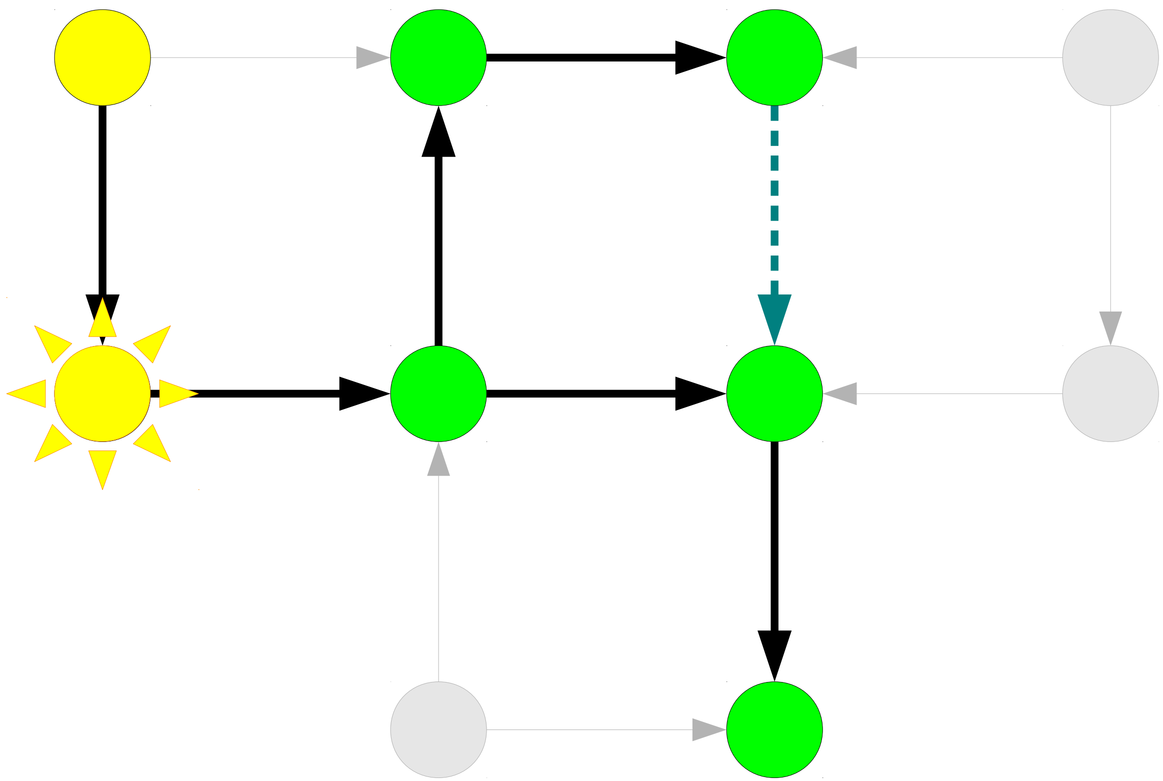

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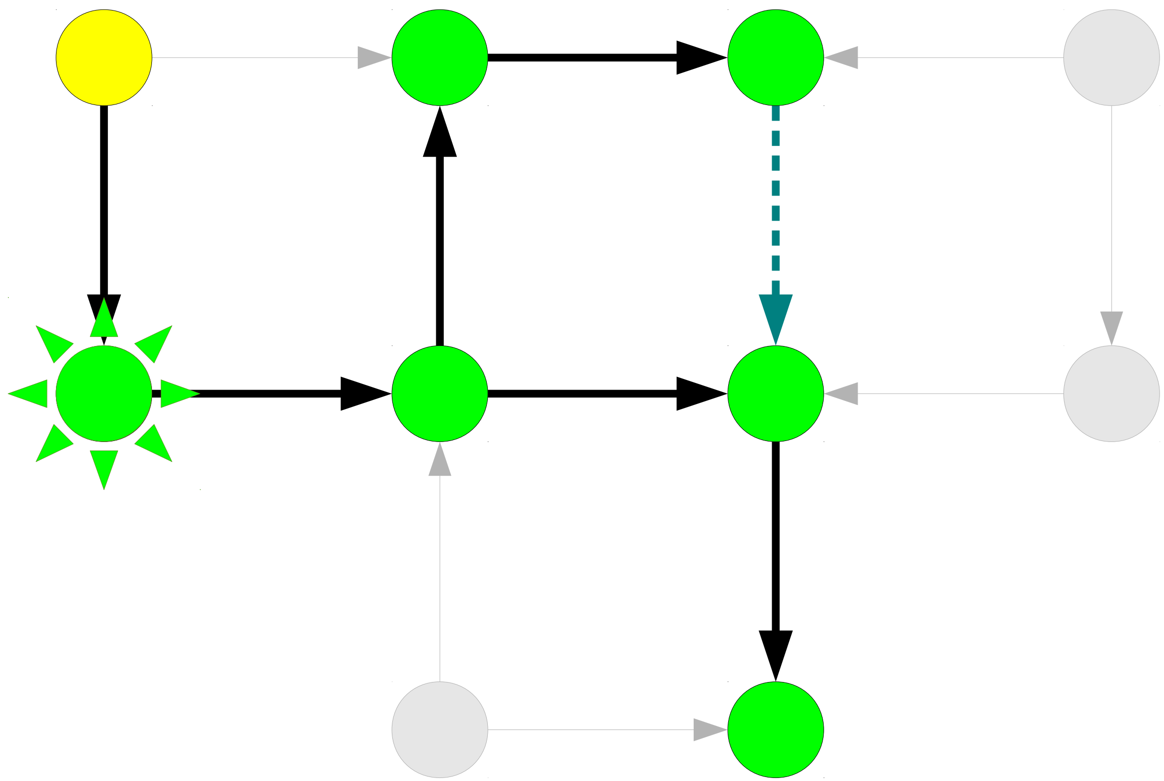

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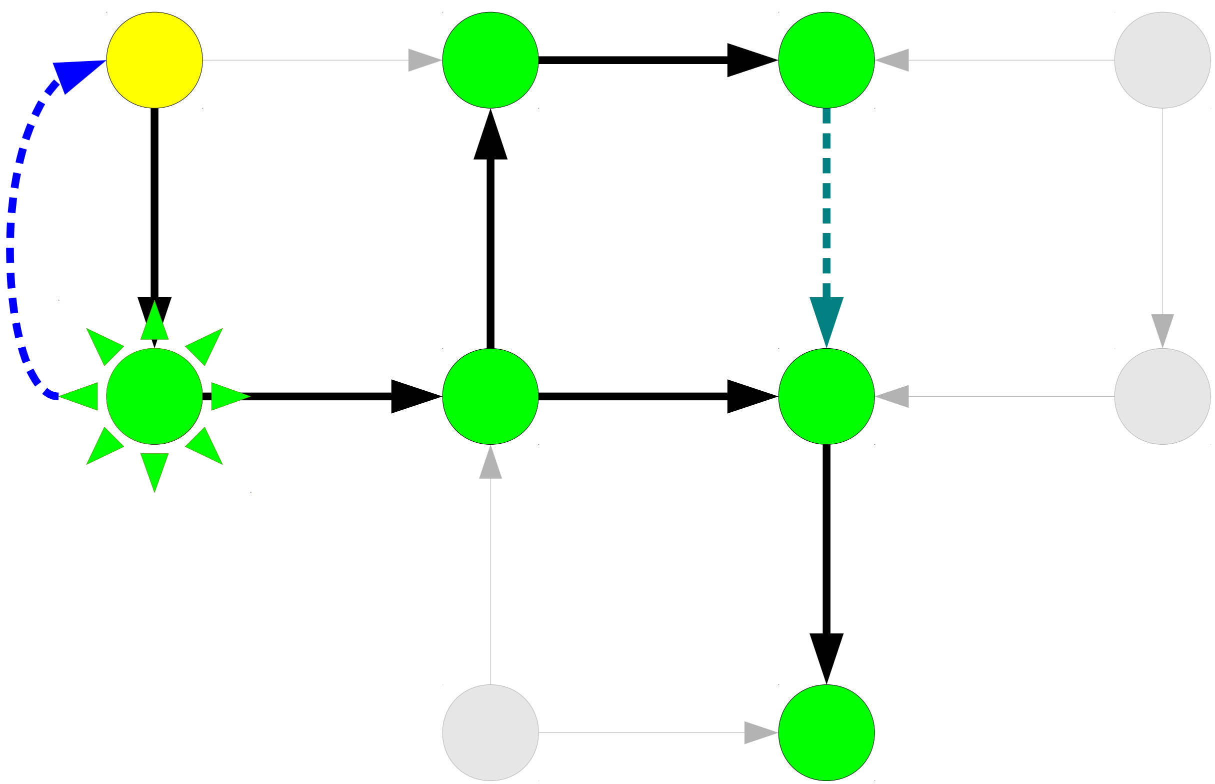

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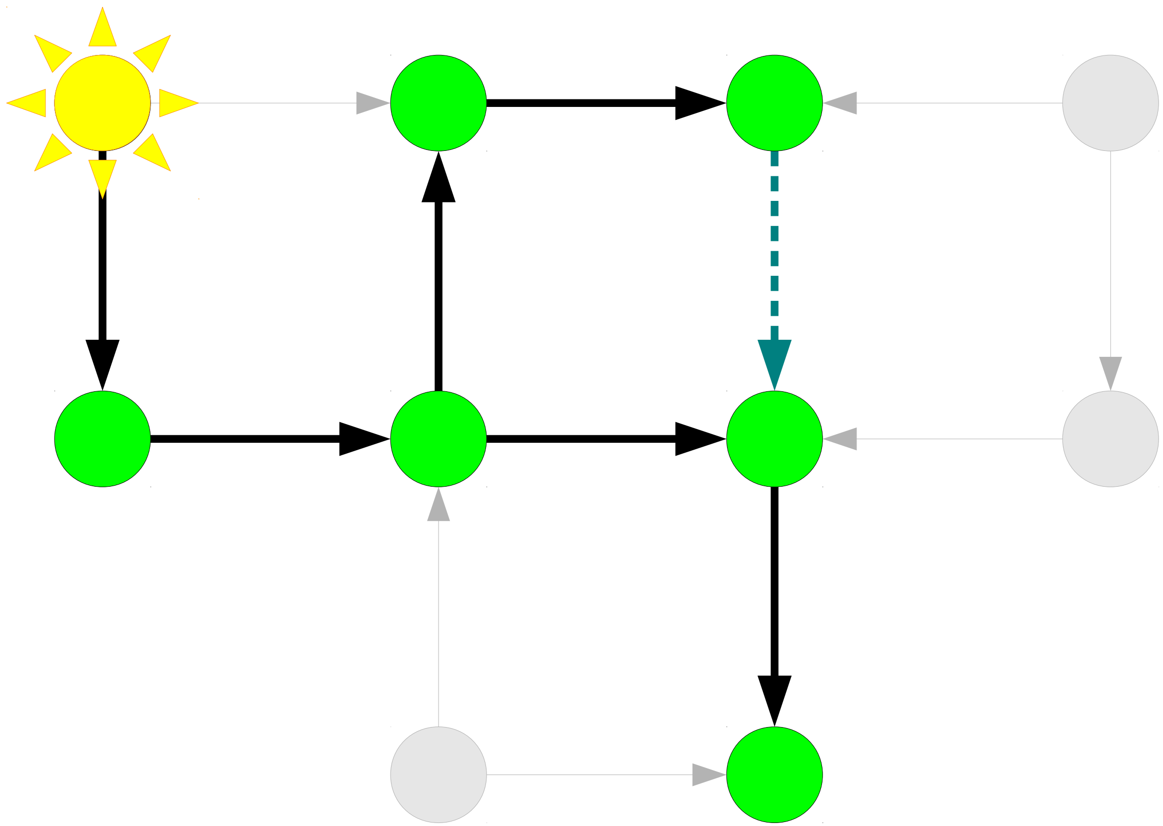

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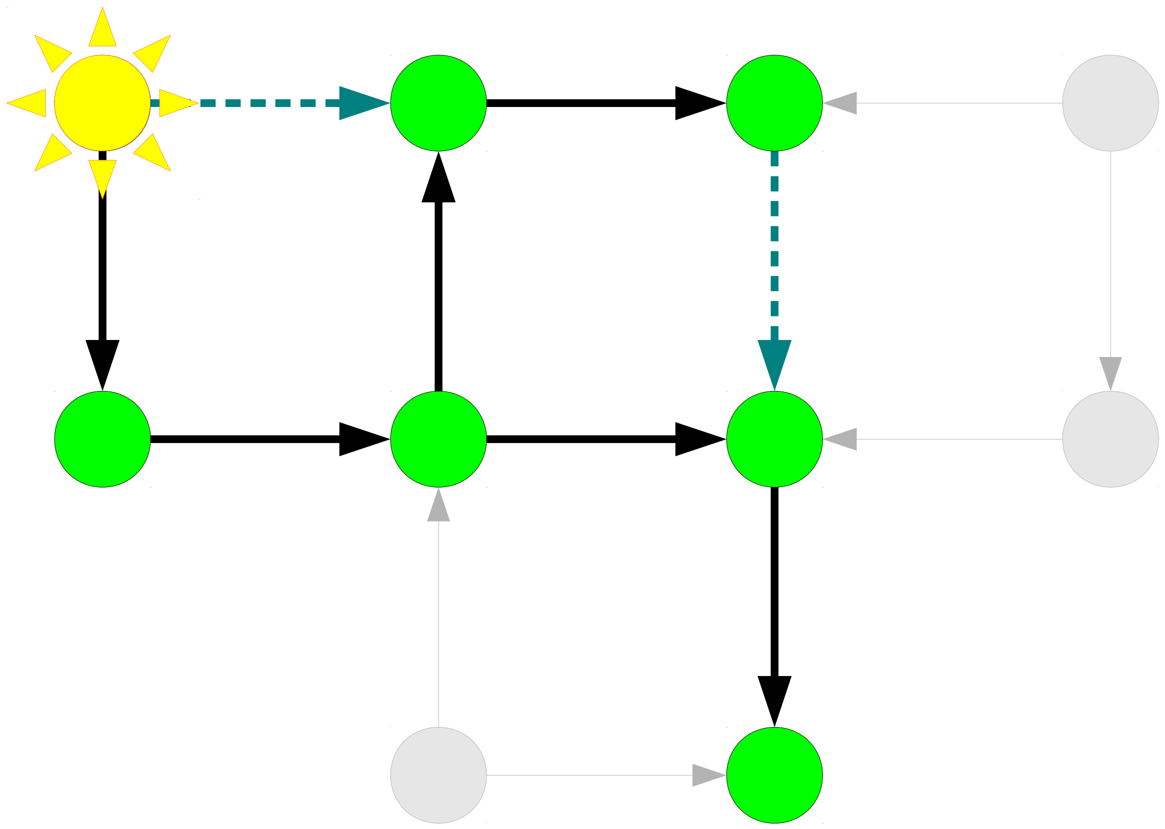

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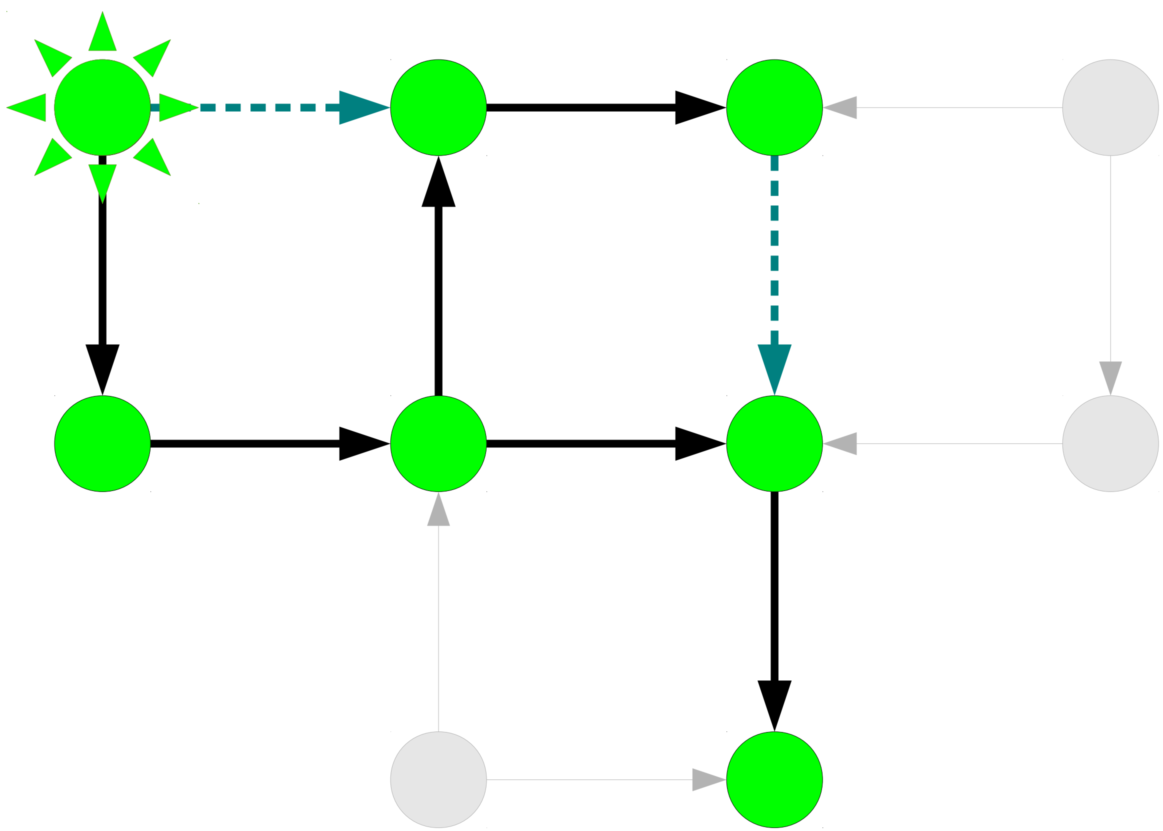

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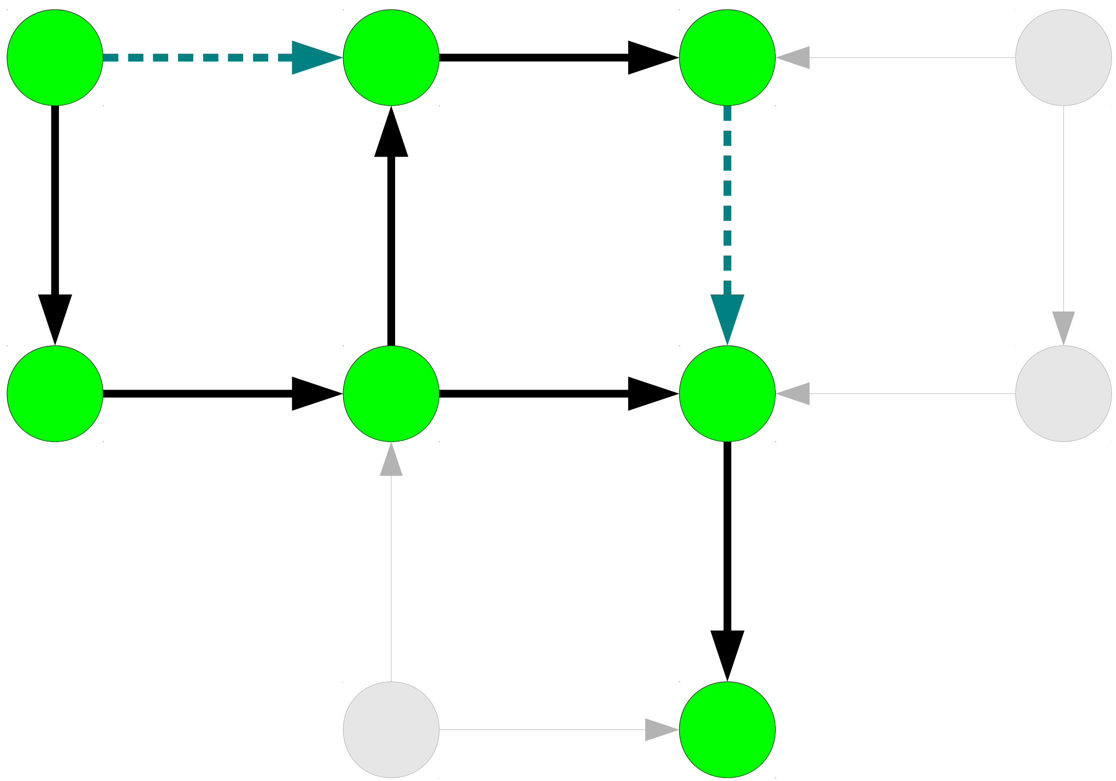

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What DFS Visits

- Taken together, the two theorems we have proven show the following:
  - When DFS(s) terminates, every node reachable from s will have had DFS called on it, though the call to DFS(s) might not have initiated those other calls.
  - When DFS(s) terminates, it will never have called DFS on a node not reachable from s.
- Thus when DFS(s) terminates, the only nodes DFS will have been called on are nodes on which DFS had already been called, plus the nodes reachable from s.

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Analyzing Recursive Functions

- In general, it can be very difficult to analyze the runtime of a recursive function.
  - We'll see some techniques for special cases later in the quarter.
- One general technique is to look at the total number of calls made and the work done at each call.

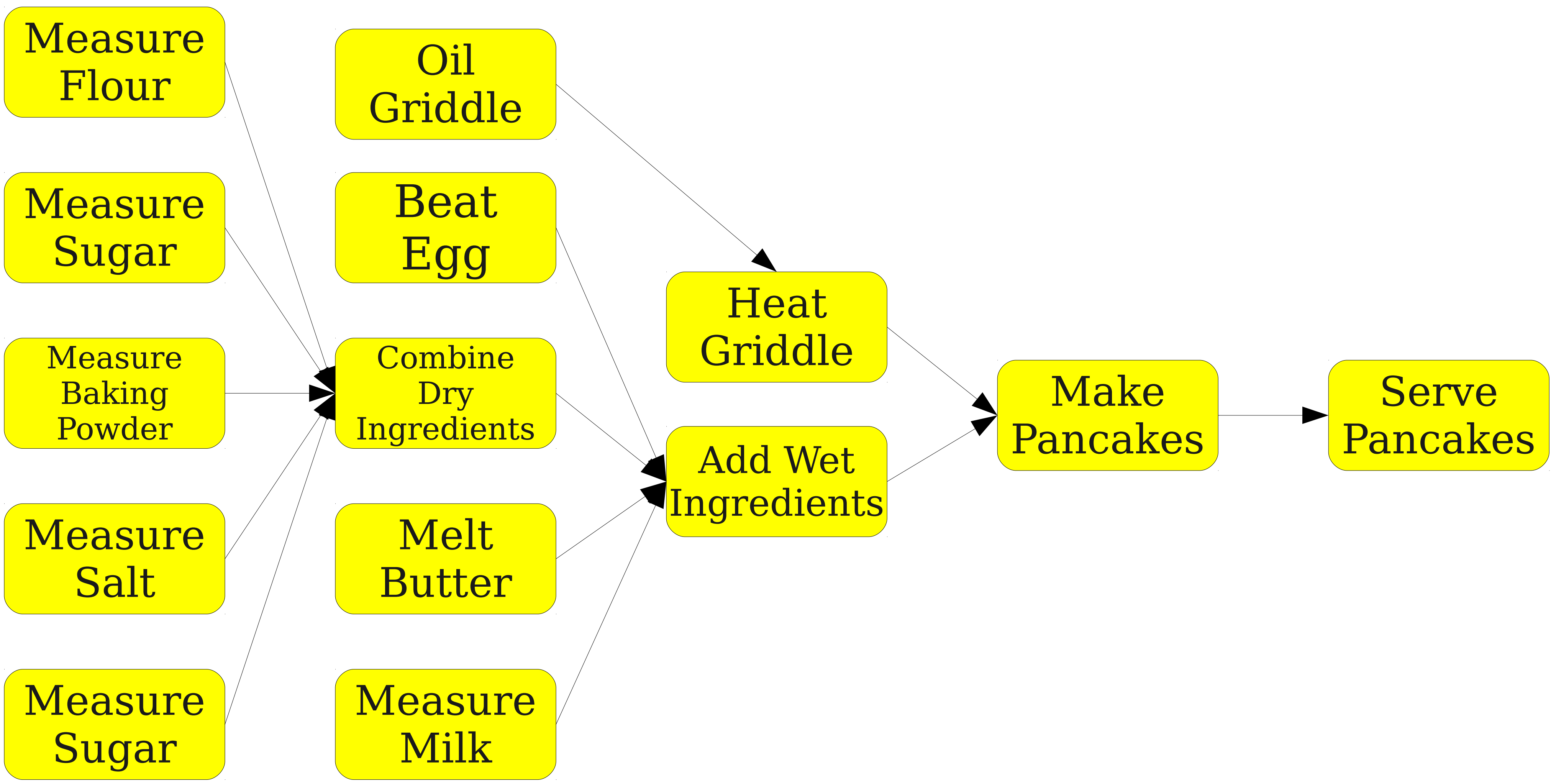
Analyzing DFS

- The maximum number of function calls made is $O(n)$, since we can't call DFS on a node twice.
- Each call to DFS on node v does $\Theta(\deg^+(v))$ work, since it visits each outgoing edge from v exactly once.
- Summing across all recursive calls:
  - $O(n)$ work done initially coloring nodes.
  - $O(n)$ work done coloring nodes yellow / green.
  - $O(m)$ work visiting edges.
  - Total work done: $\mathbf{O(m + n)}$.
- When might this not do $\Theta(m + n)$ work?

BFS and DFS

- BFS and DFS always visit the same set of nodes.
- However, BFS always finds the shortest path from the source node to each other node in the graph, while DFS might not.
- That said: the order in which DFS visits nodes is pretty important and has lots of applications. We'll see some of them soon...

Ordering Prerequisites



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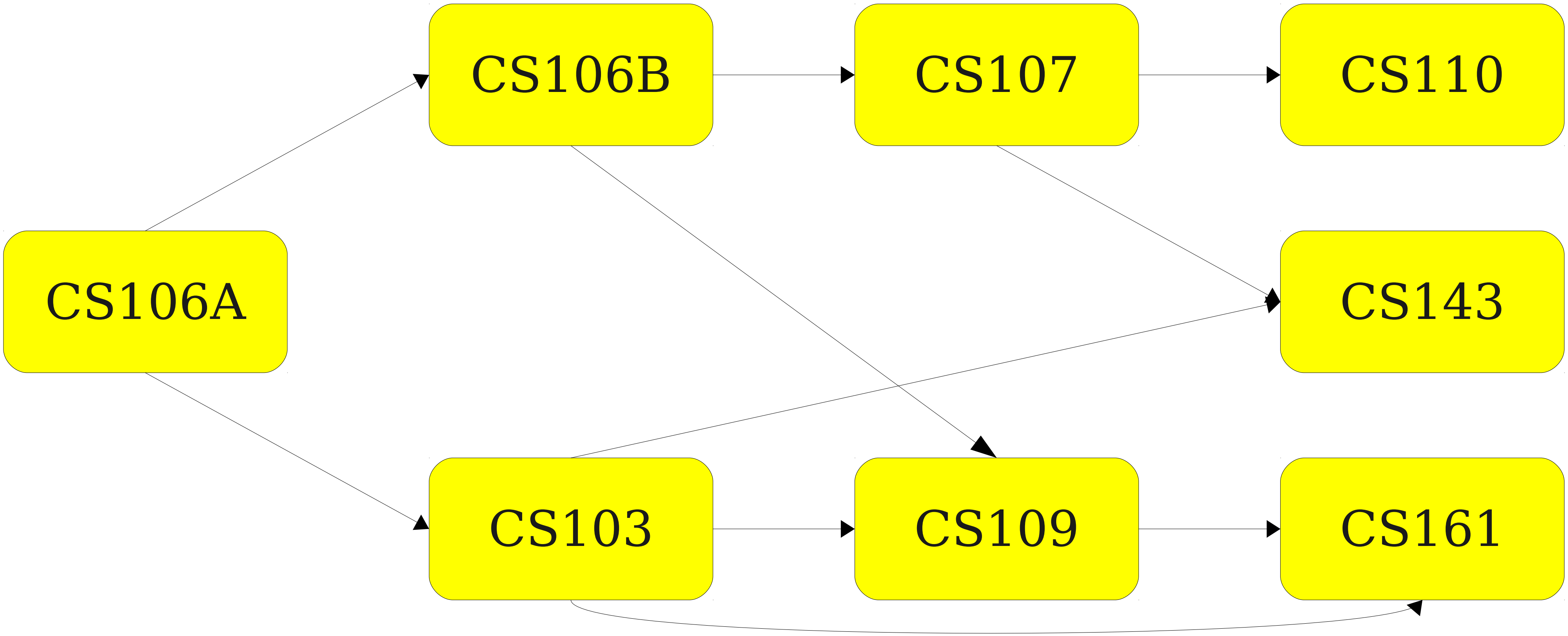
Measure Sugar
Measure Baking Powder
Measure Salt
Measure Sugar
Oil Griddle
Beat Egg
Combine Dry Ingredients
Melt Butter
Measure Milk
Heat Griddle
Add Wet Ingredients
Make Pancakes
Serve Pancakes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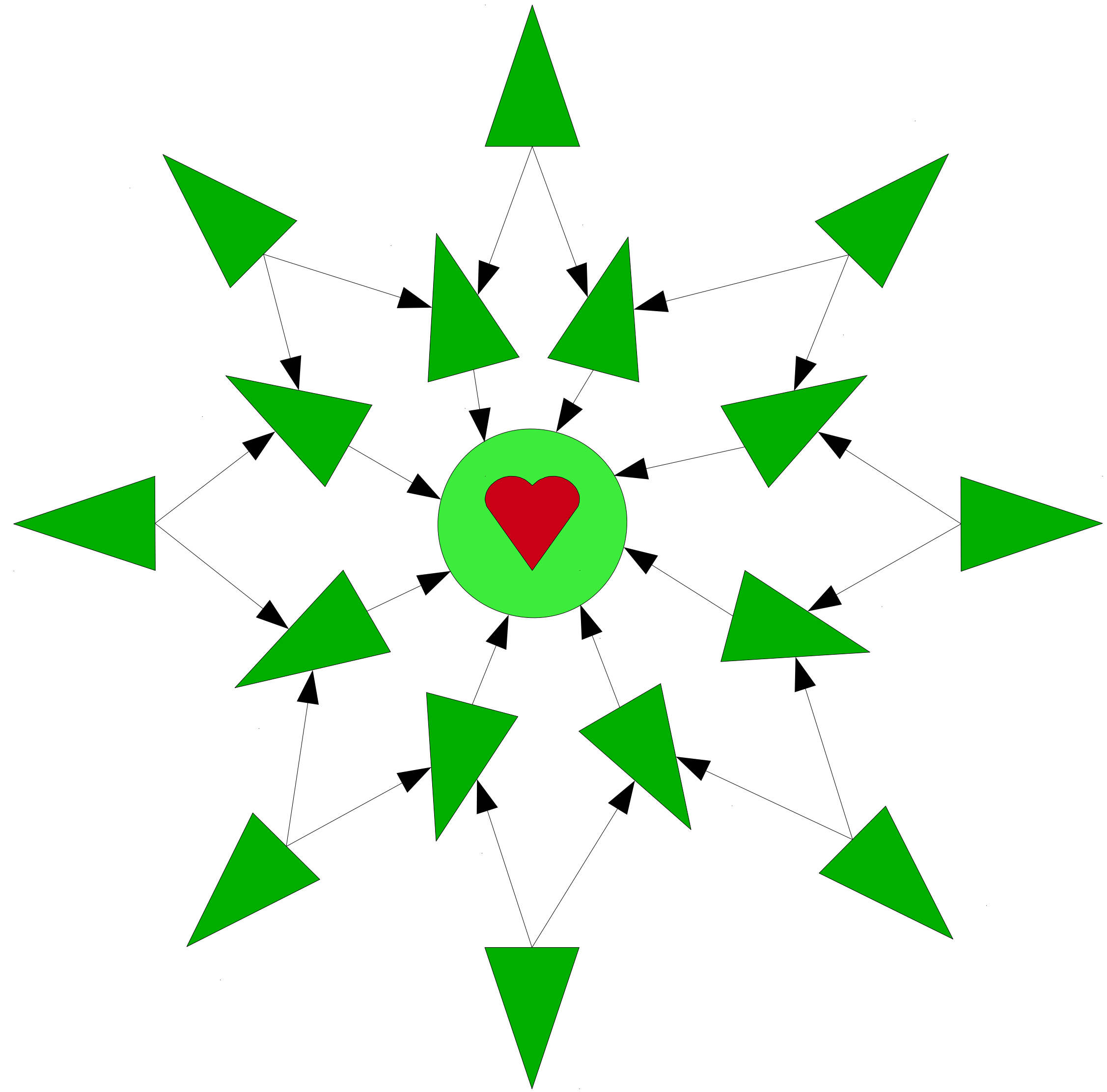
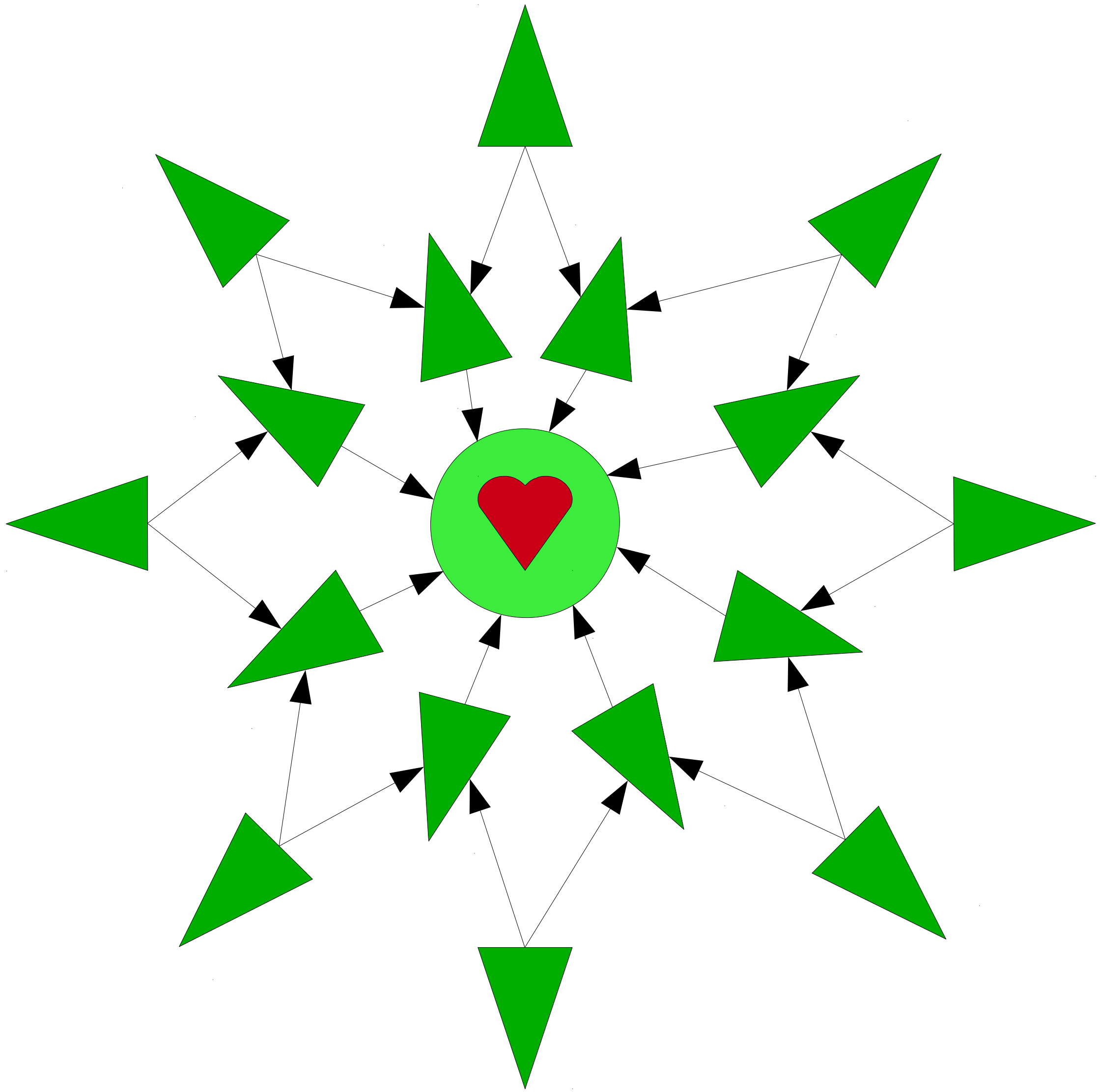


Modeling Prerequisites

- We can model prerequisites as a graph with the following properties:
  - The graph has to be **directed**, since we have to be able to distinguish “A depends on B” from “B depends on A.”
  - The graph has to be **acyclic** (containing no cycles), since otherwise there is no way to accomplish all the tasks.
- A graph with this property is called a **directed acyclic graph**, or **DAG**.

Some DAG Terminology

- A **source** node in a DAG is a node with no incoming edges.
- A **sink** node in a DAG is a node with no outgoing edges.
- DAGs can have many sources and sinks.



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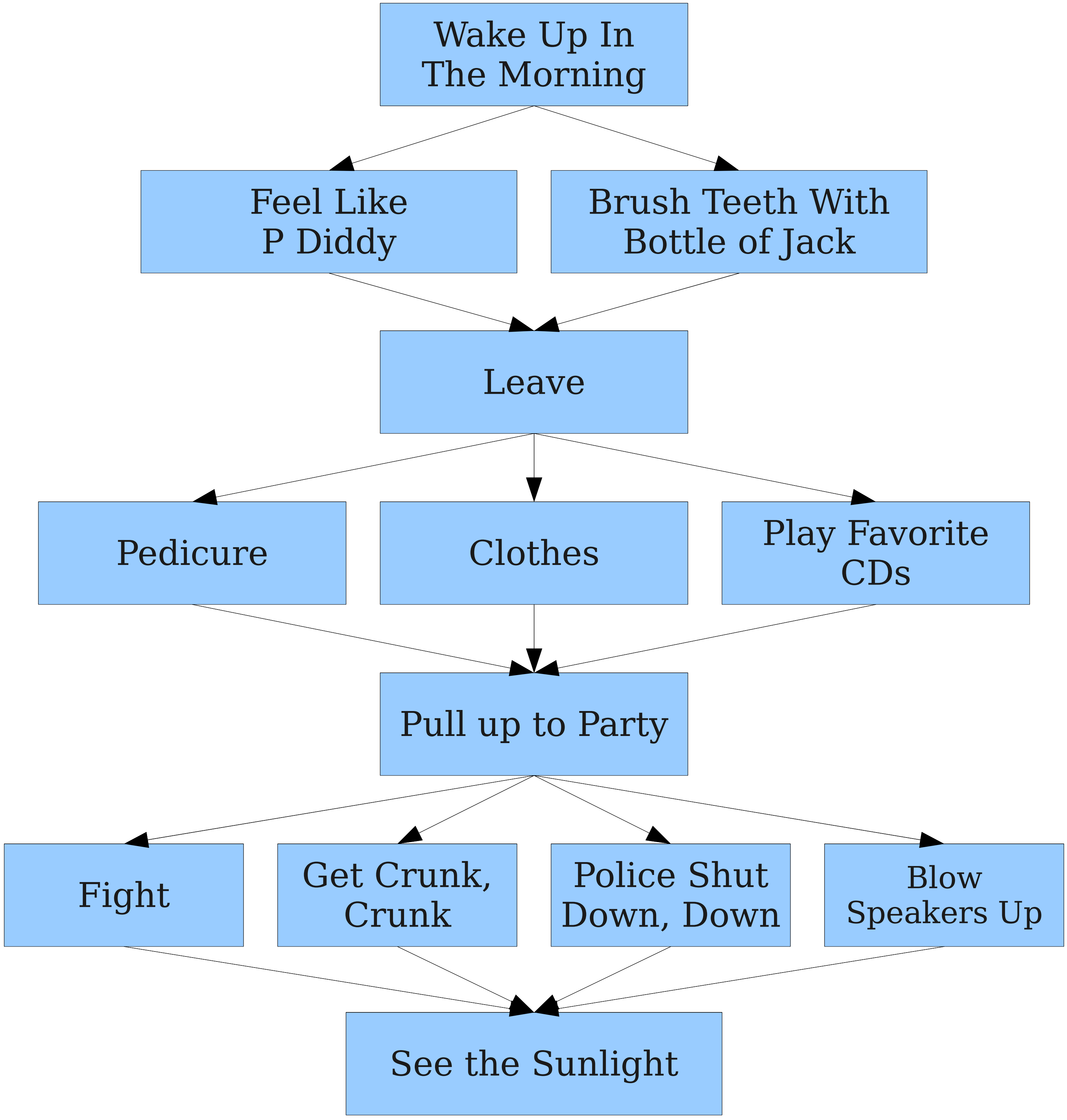
Ordering Prerequisites

- When ordering prerequisites, we want to order the tasks such that no task is placed before tasks it depends on.
- In graph-theoretic terms: given a DAG $G = (V, E)$, we want to order the nodes so that if $(u, v) \in E$, then v appears after u.
- Such an ordering is called a **topological ordering**. An algorithm for finding a topological ordering is called a **topological sort**.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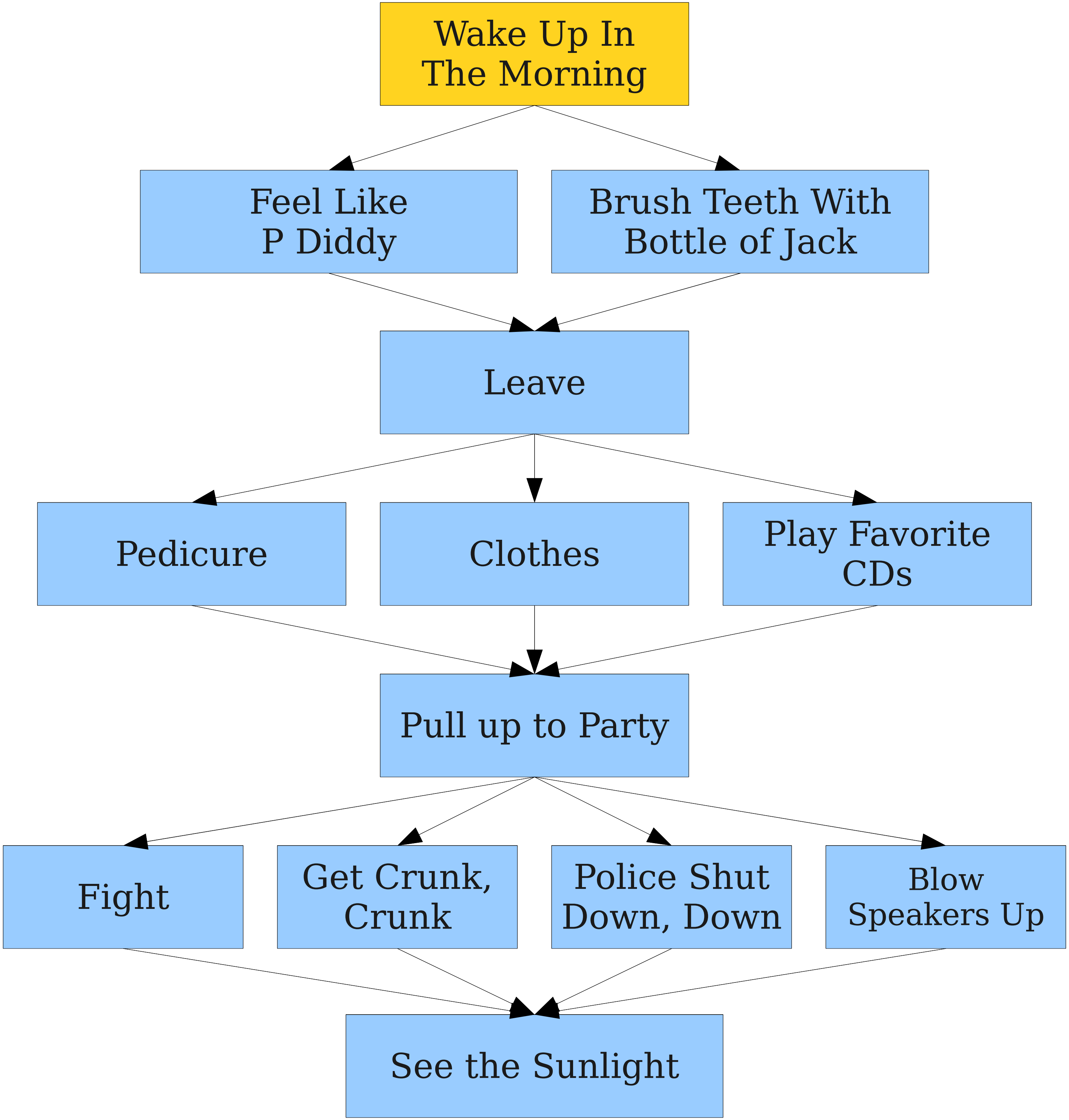
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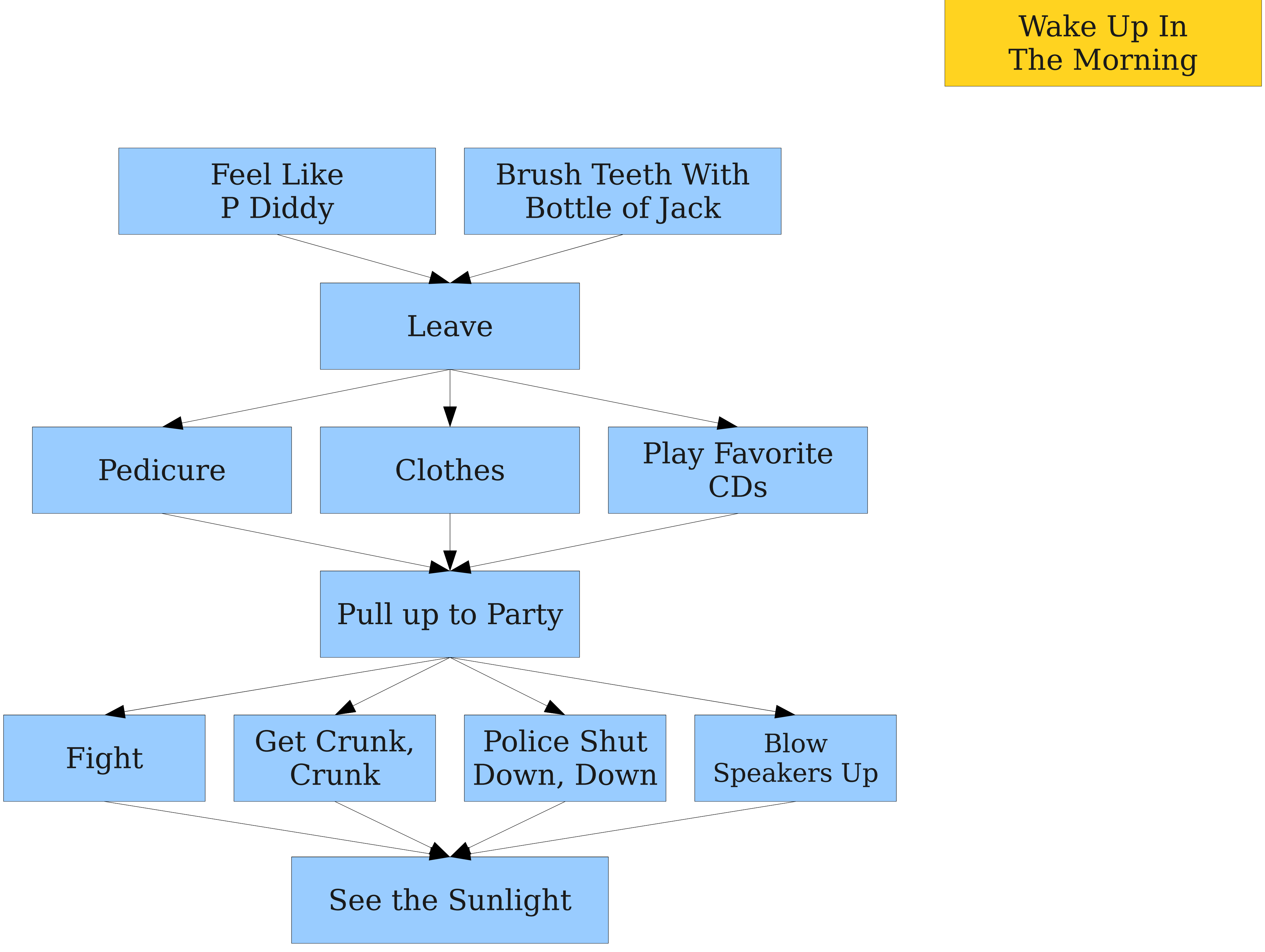
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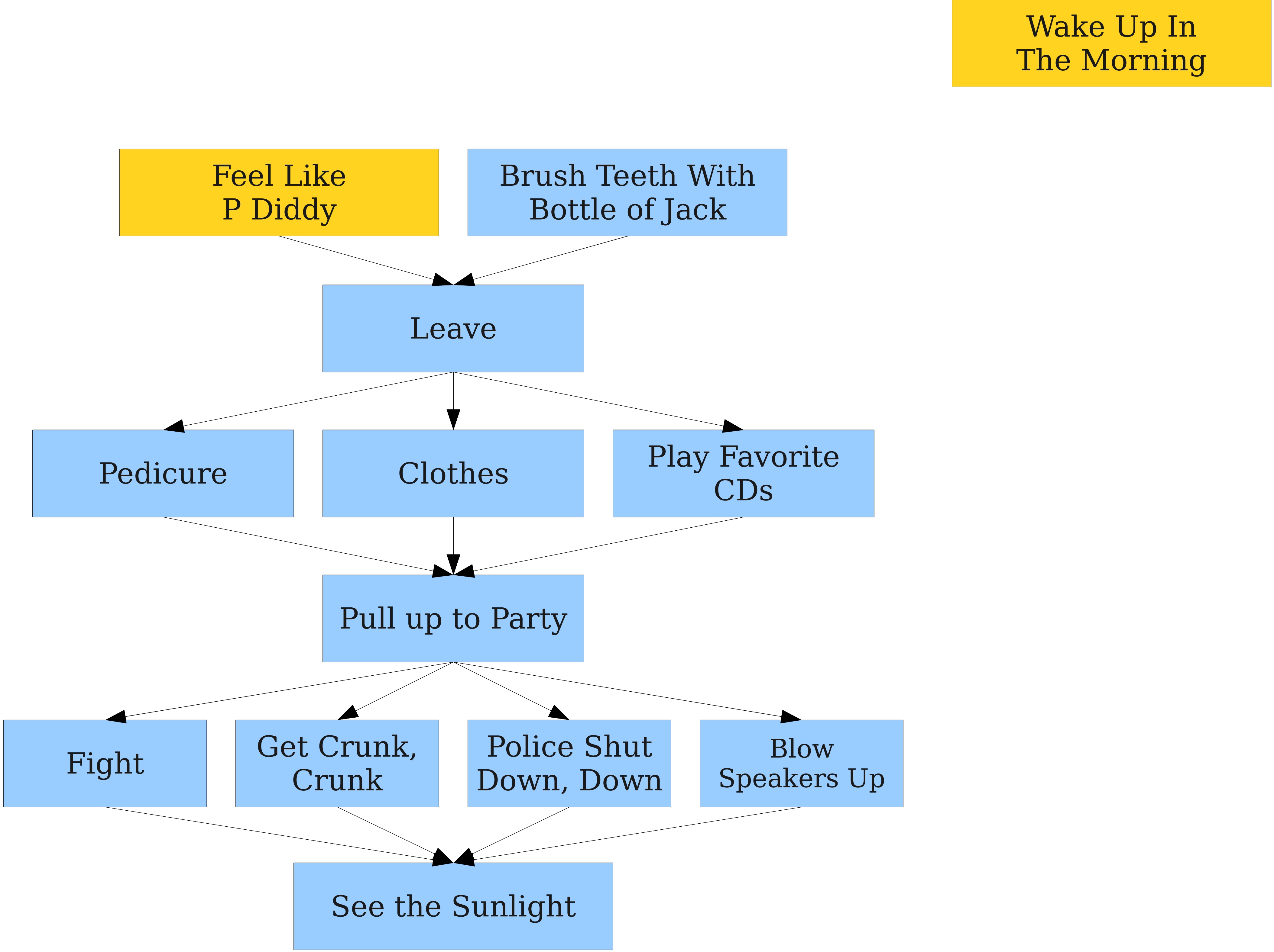
See the Sunlight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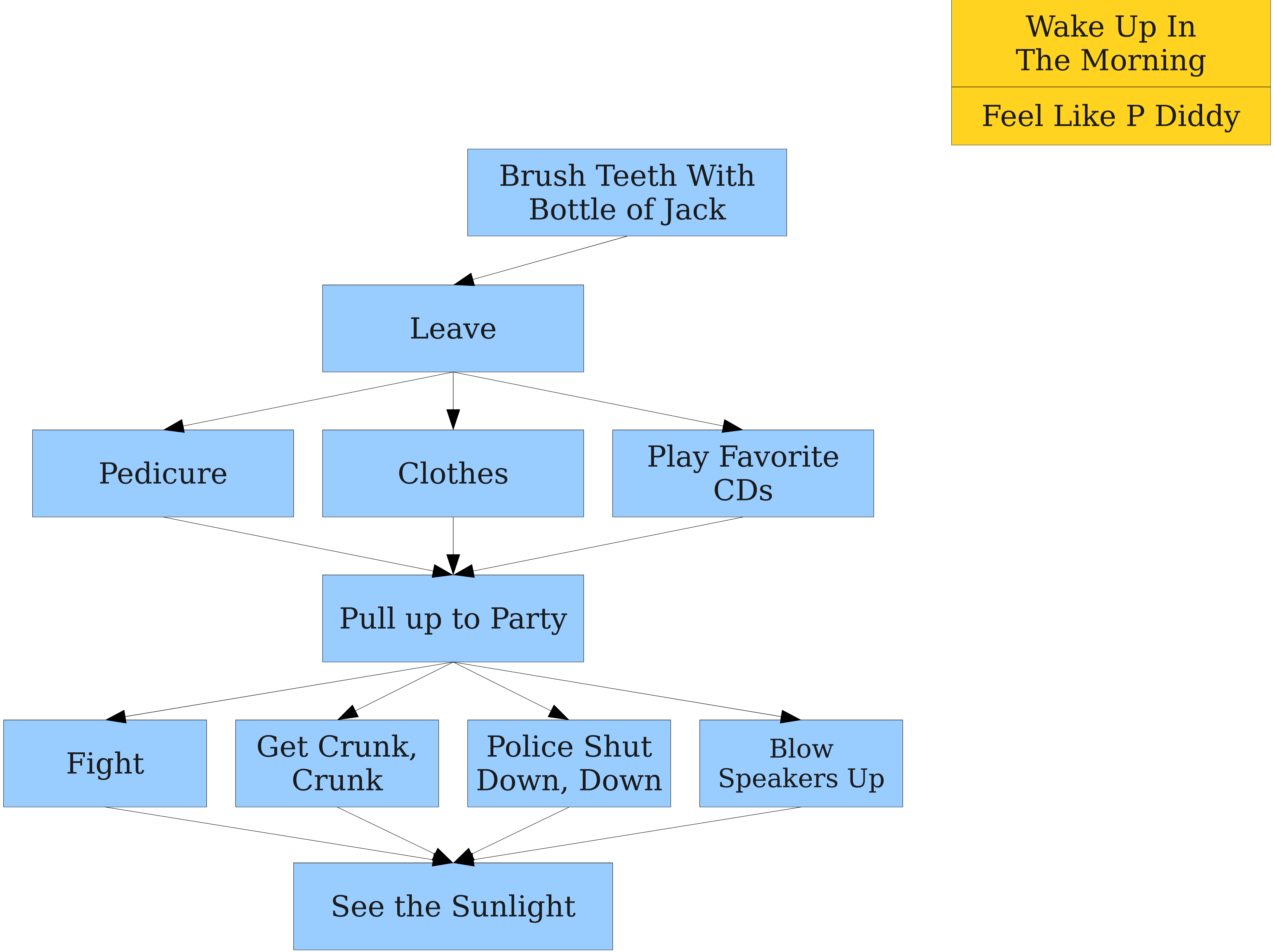
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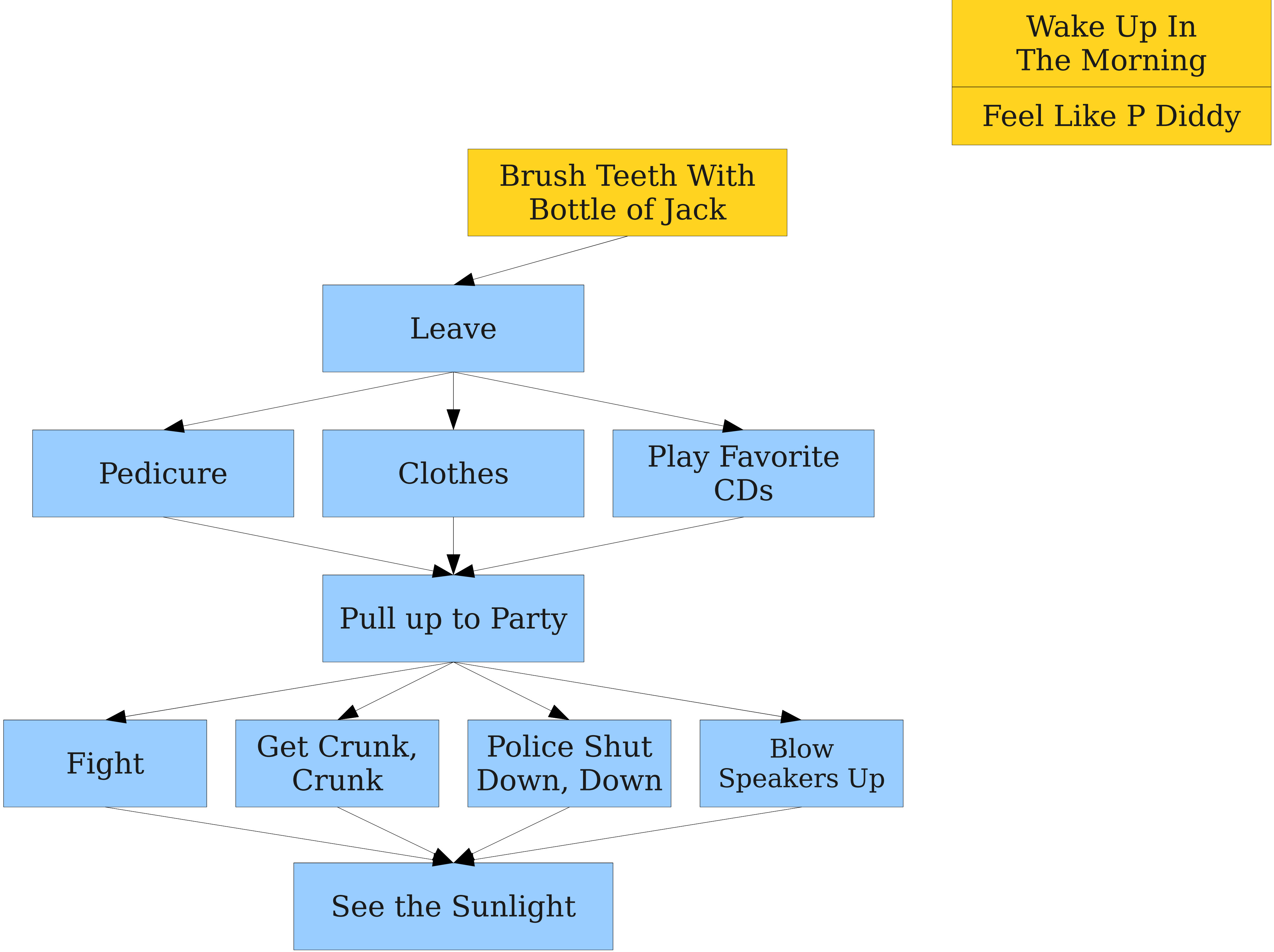
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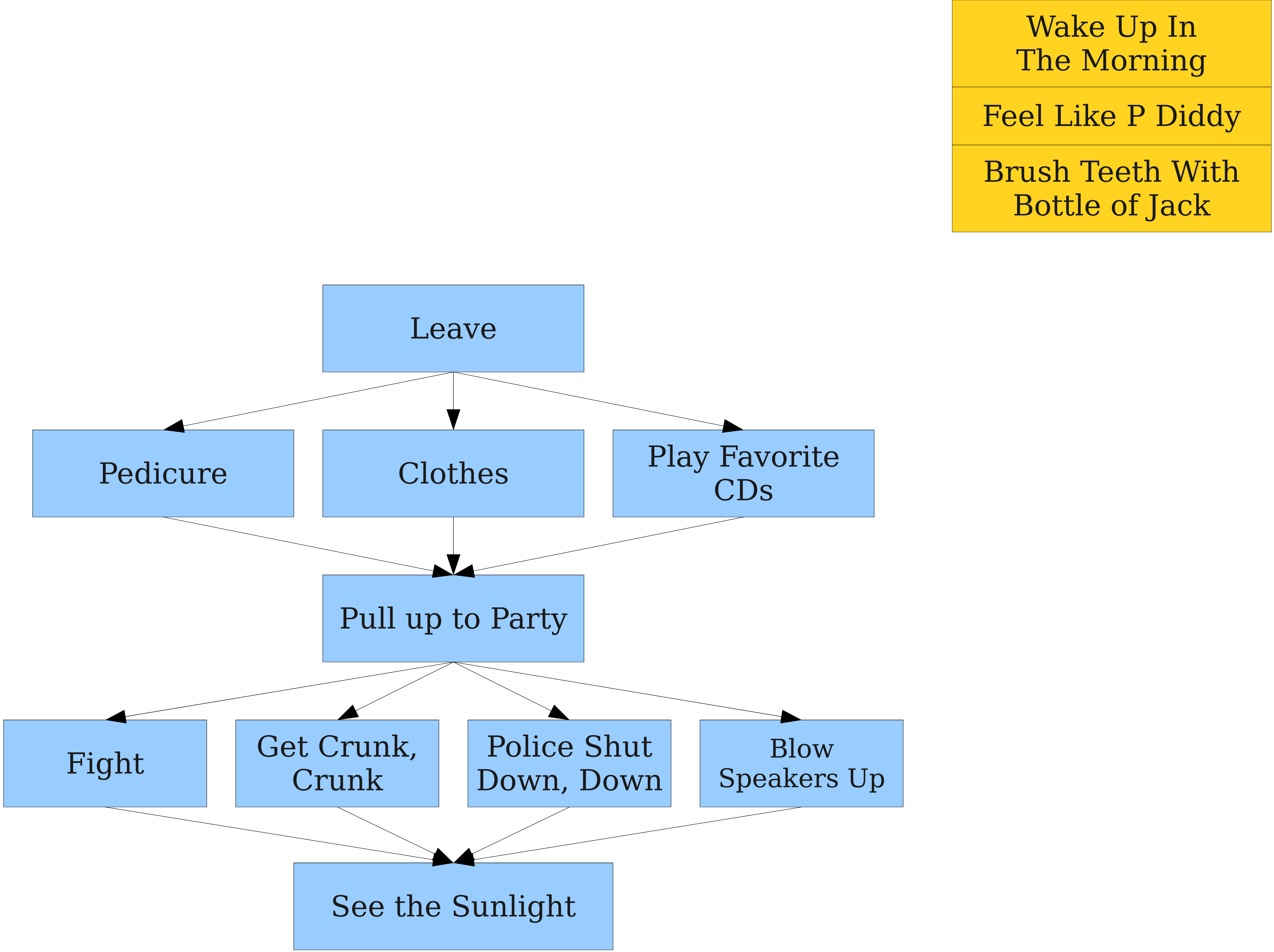
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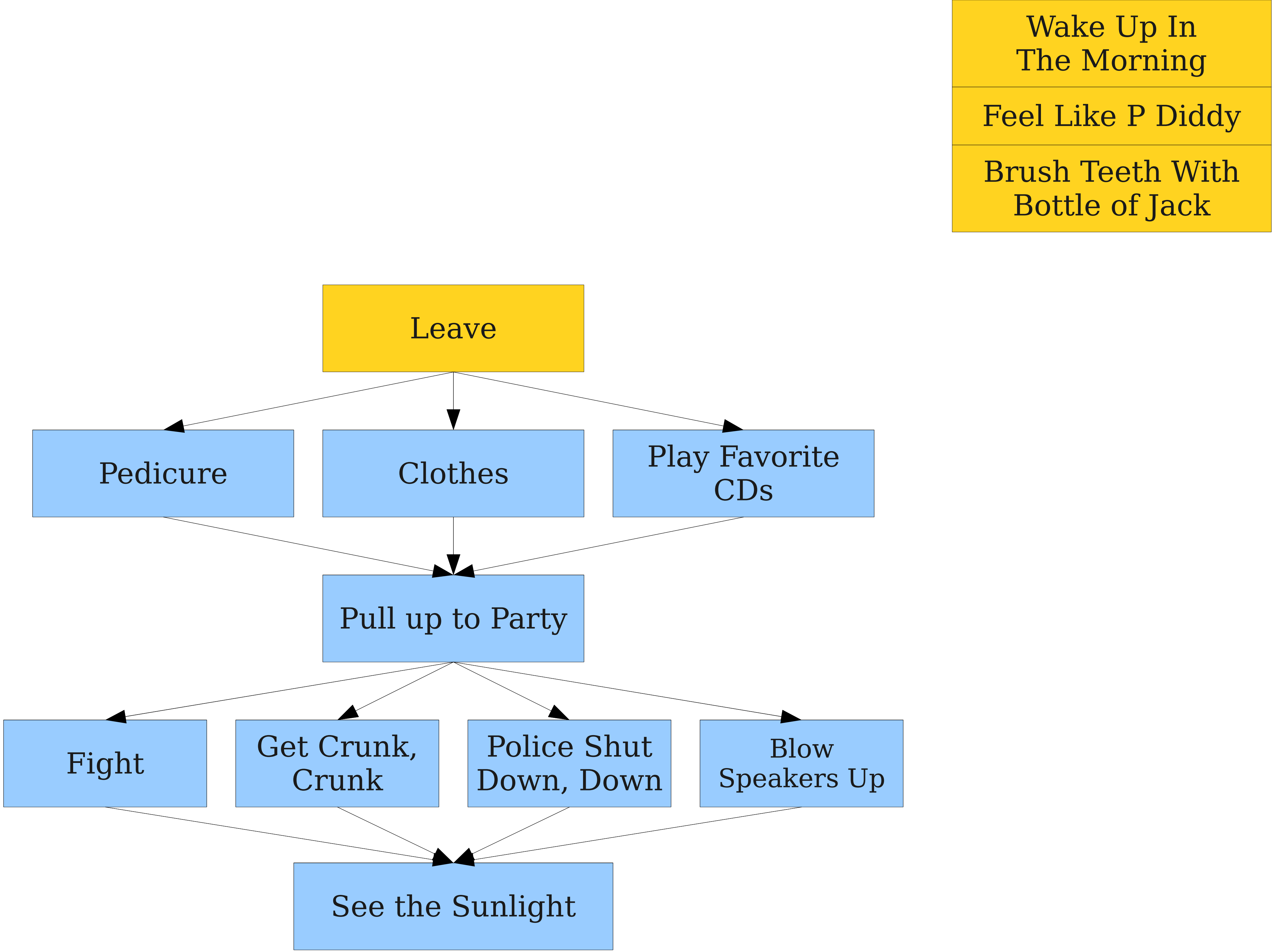

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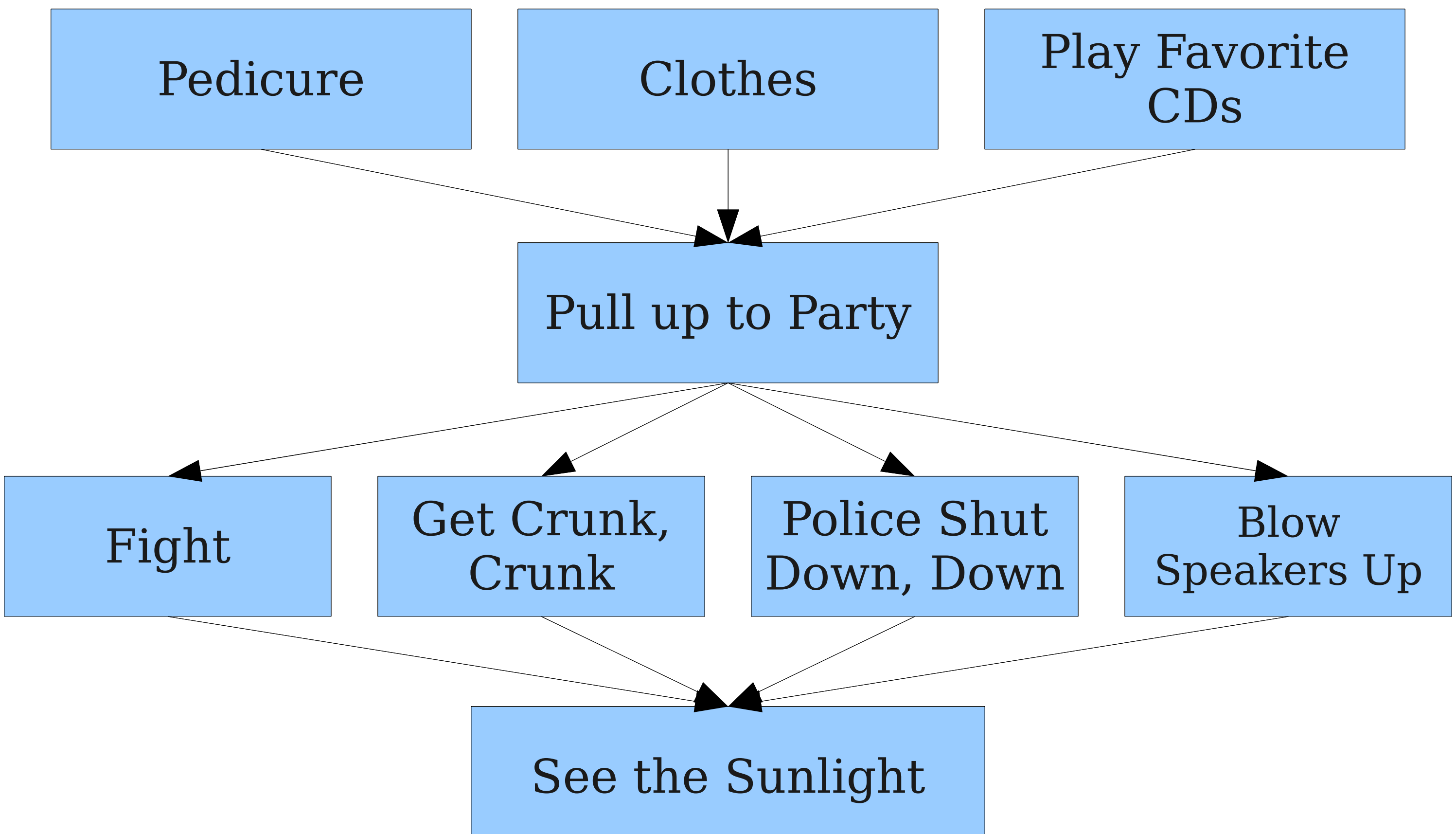
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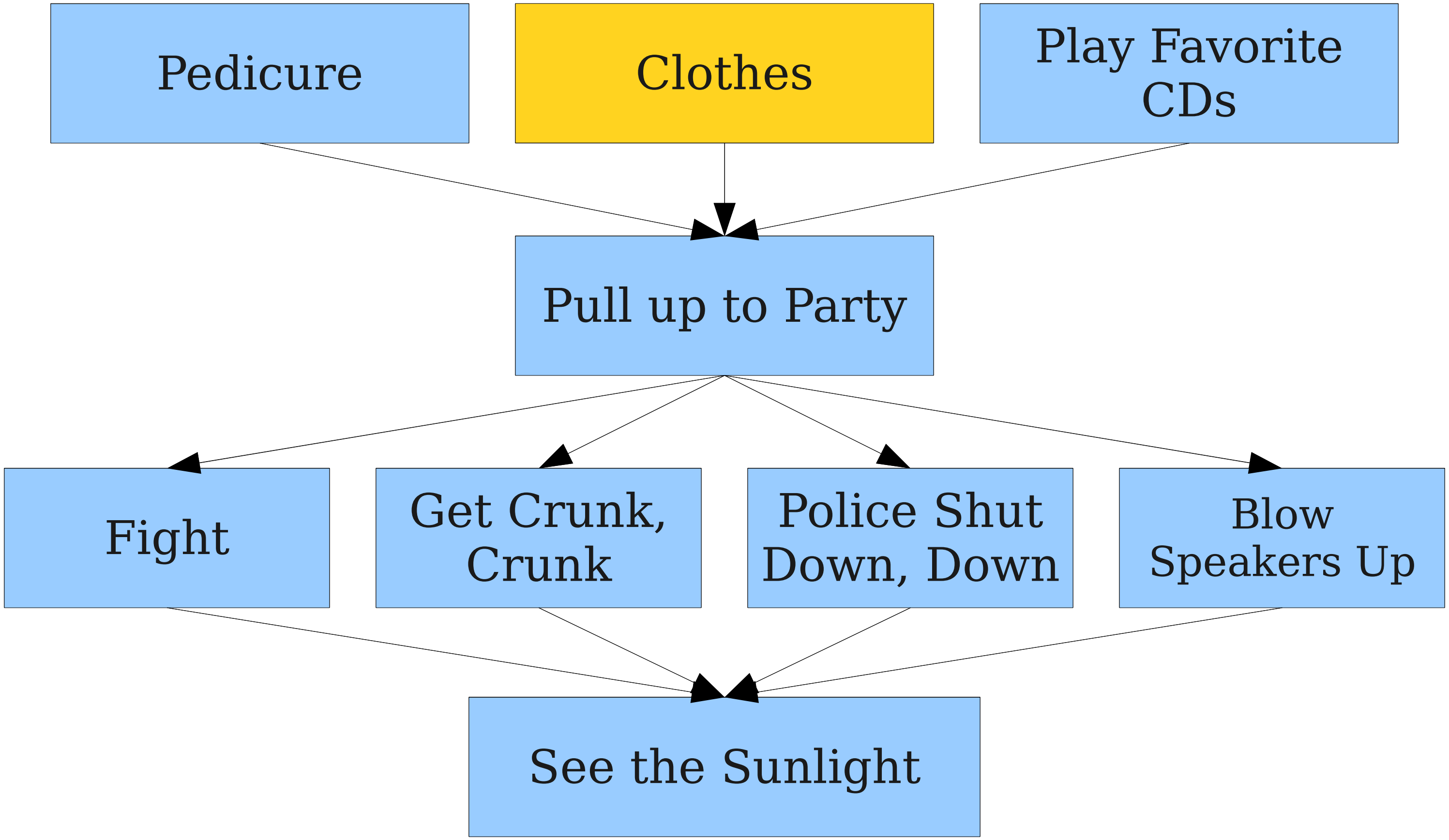

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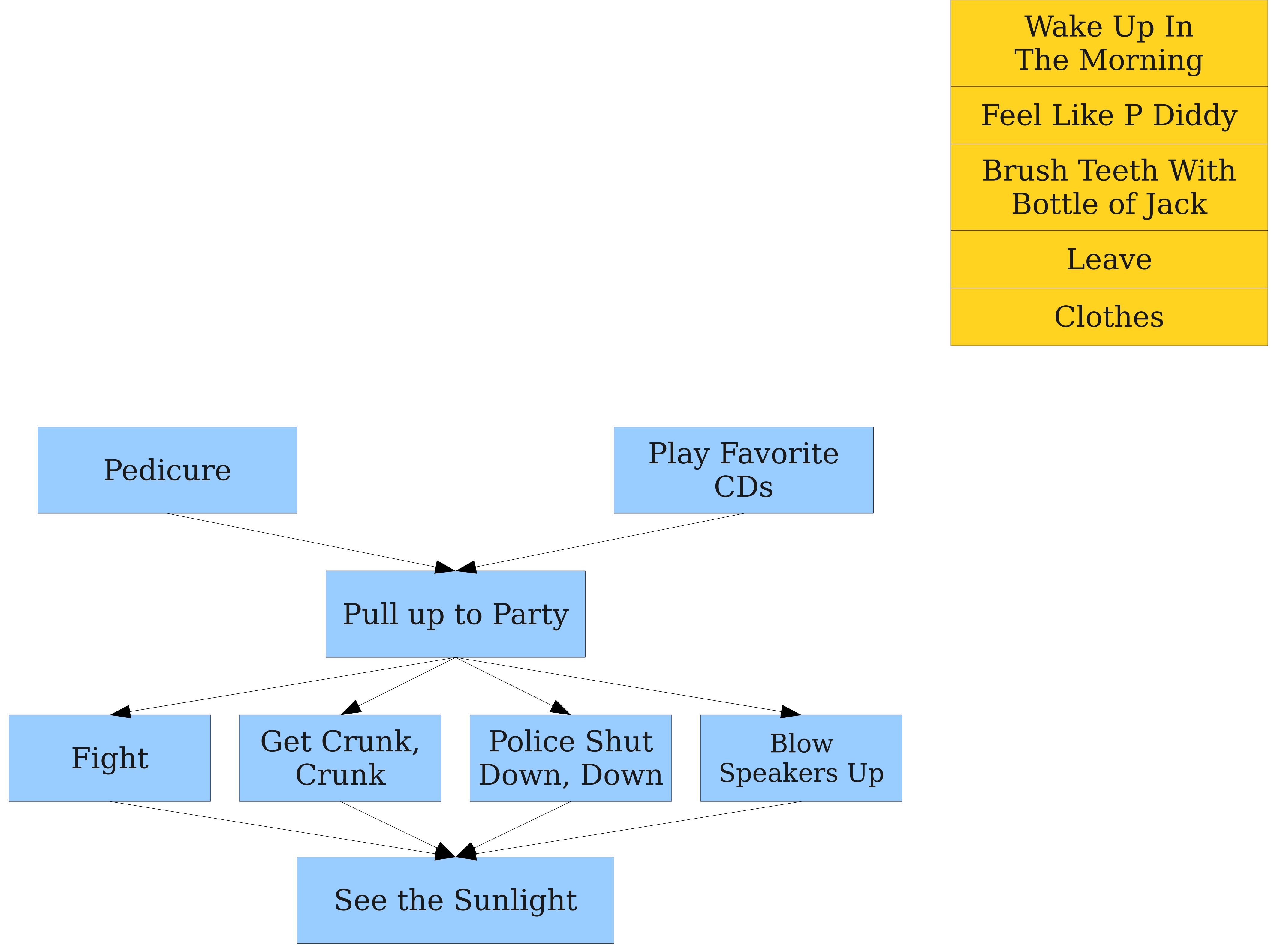
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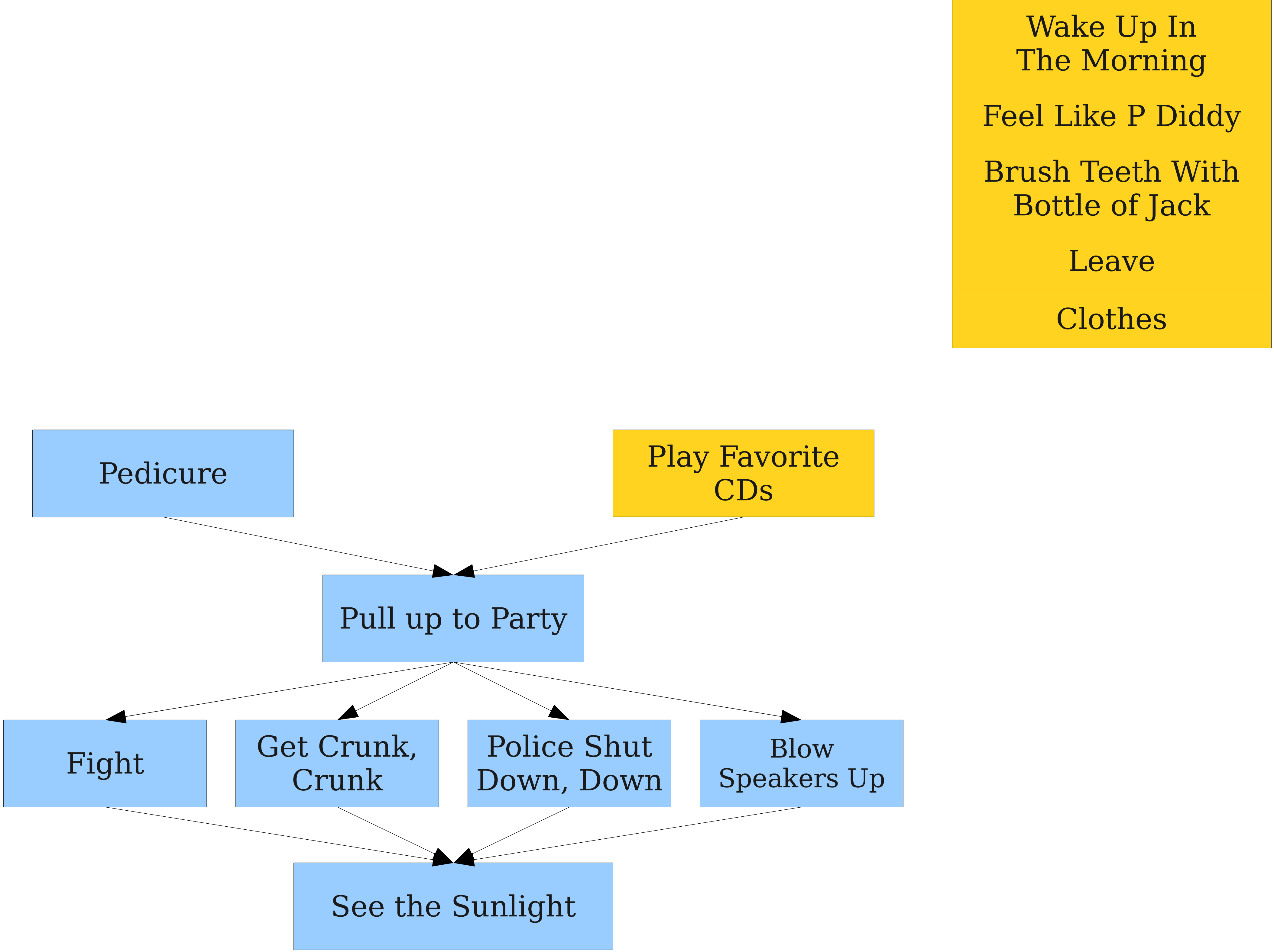
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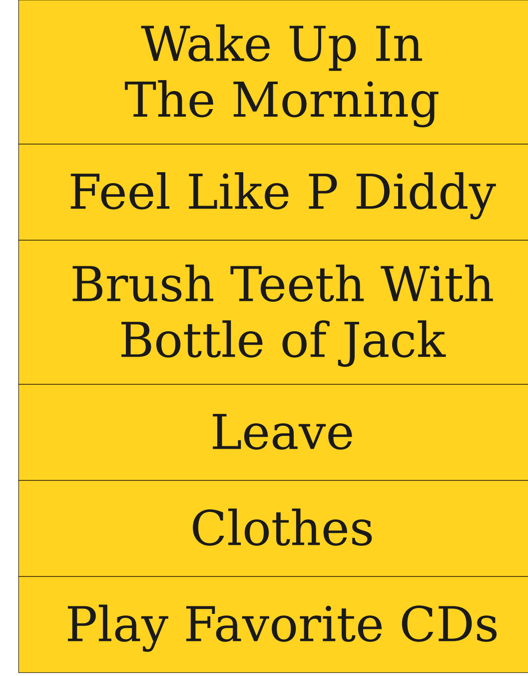
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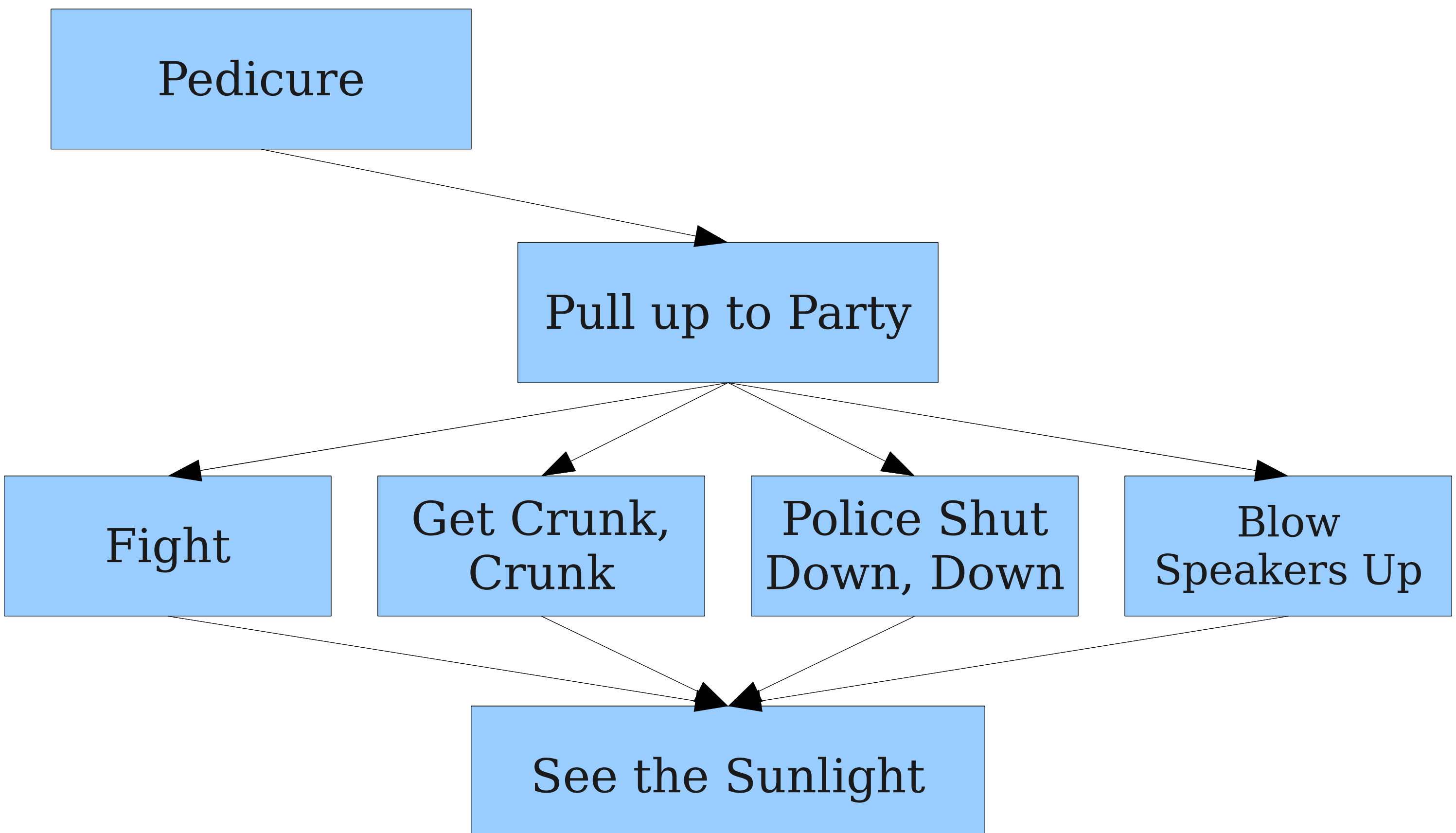
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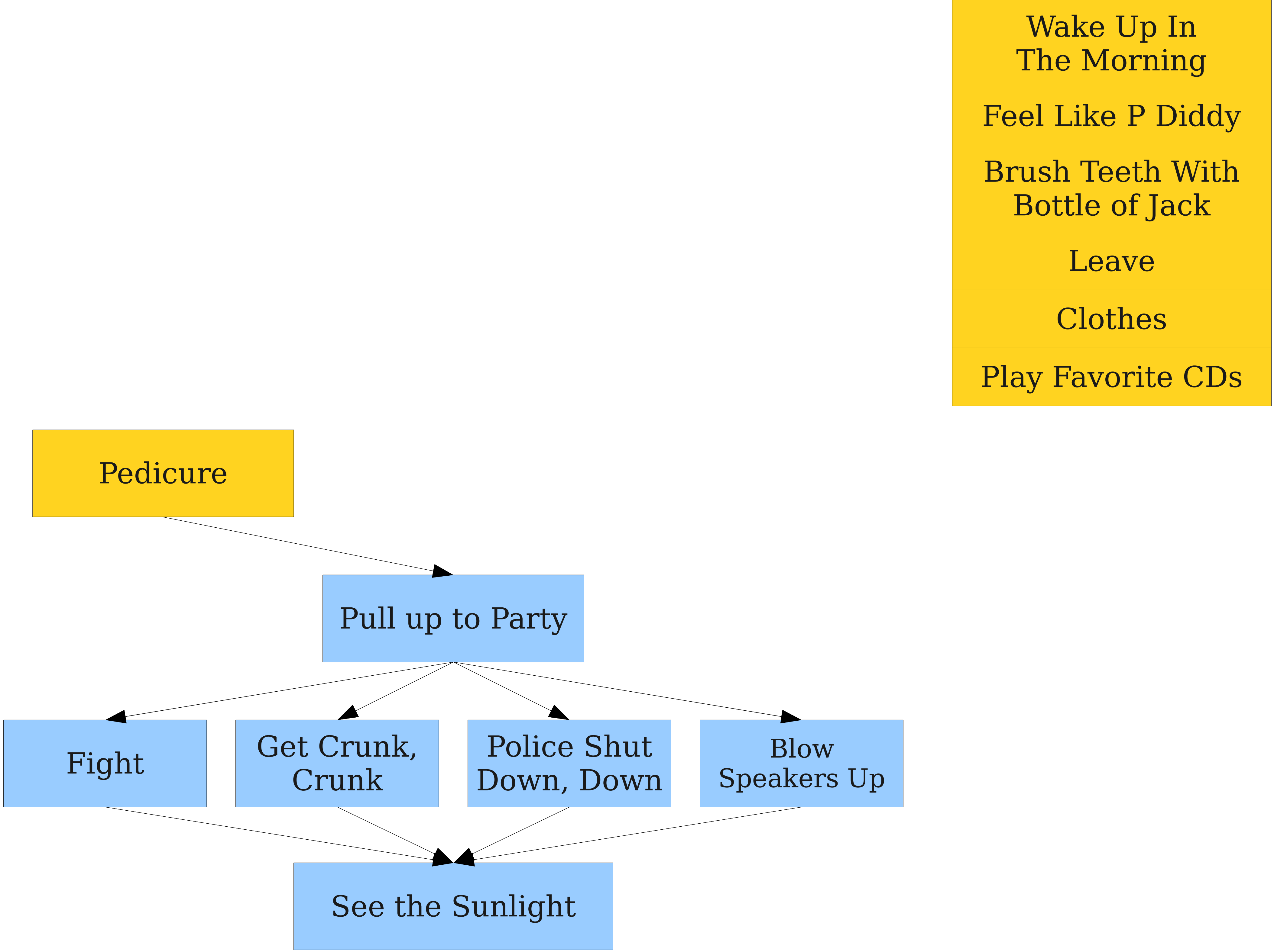

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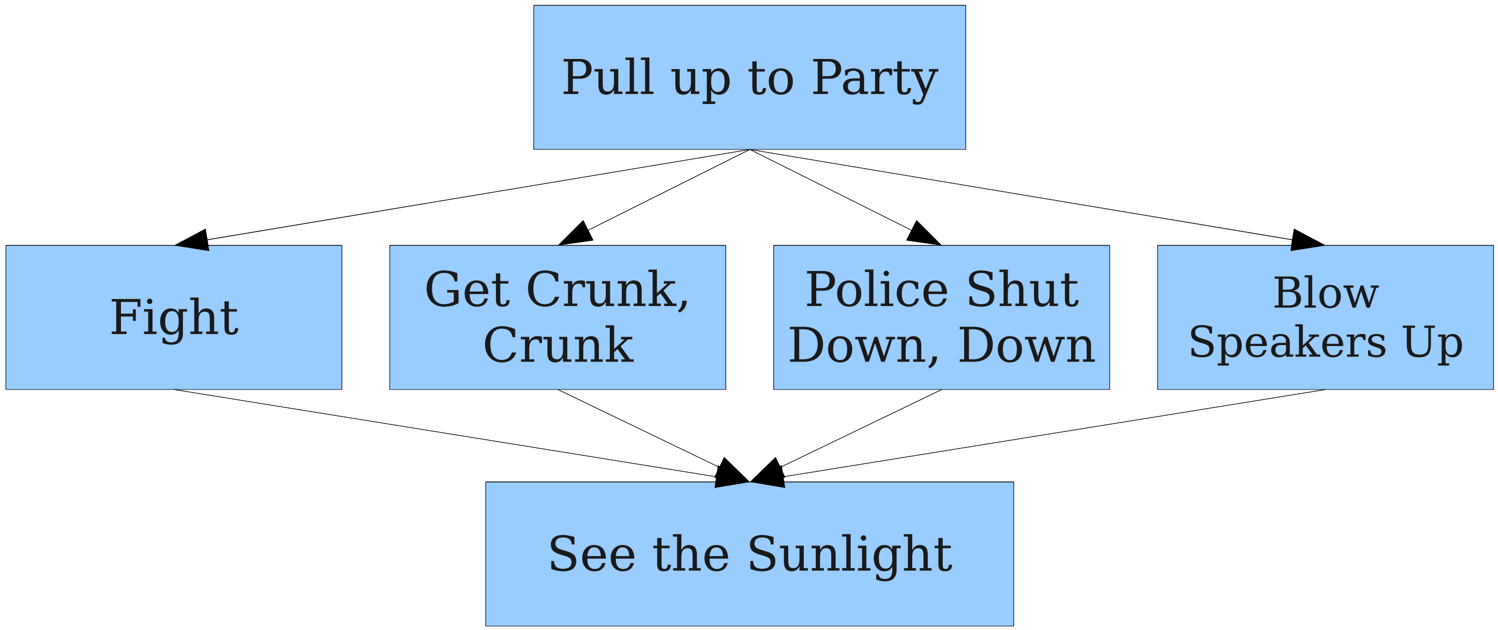
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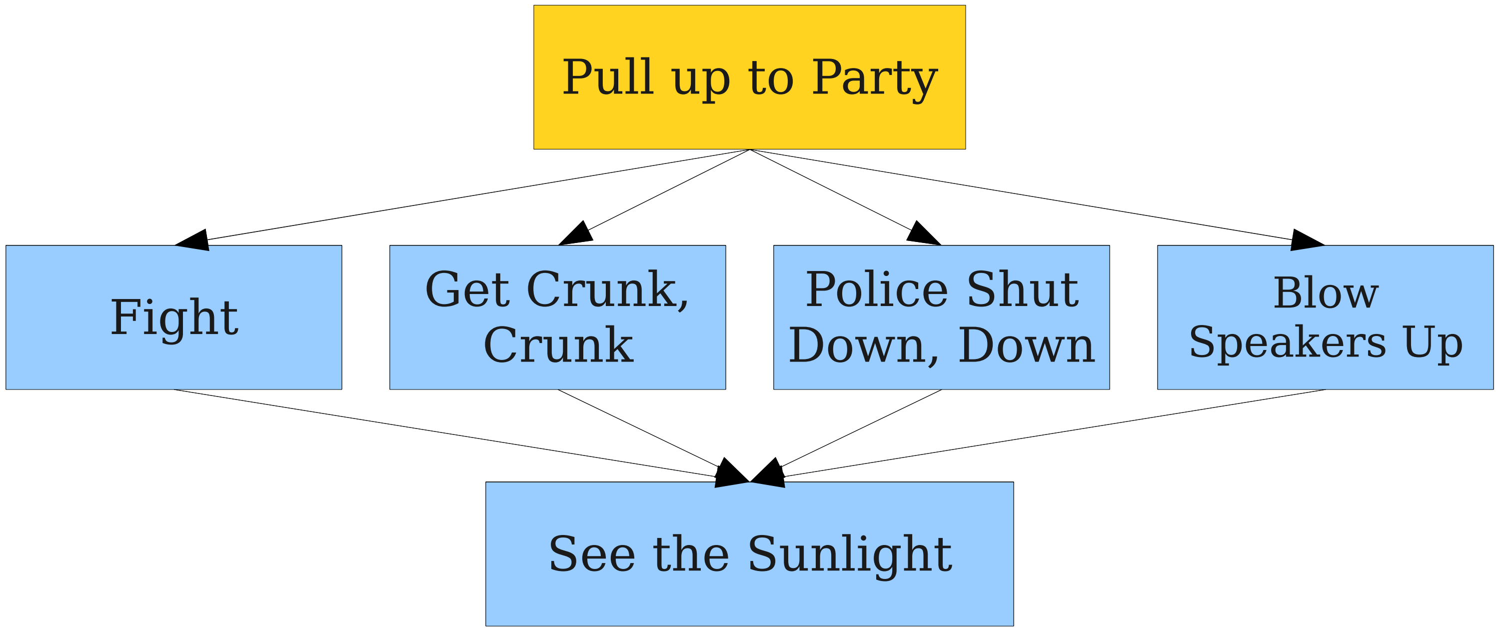
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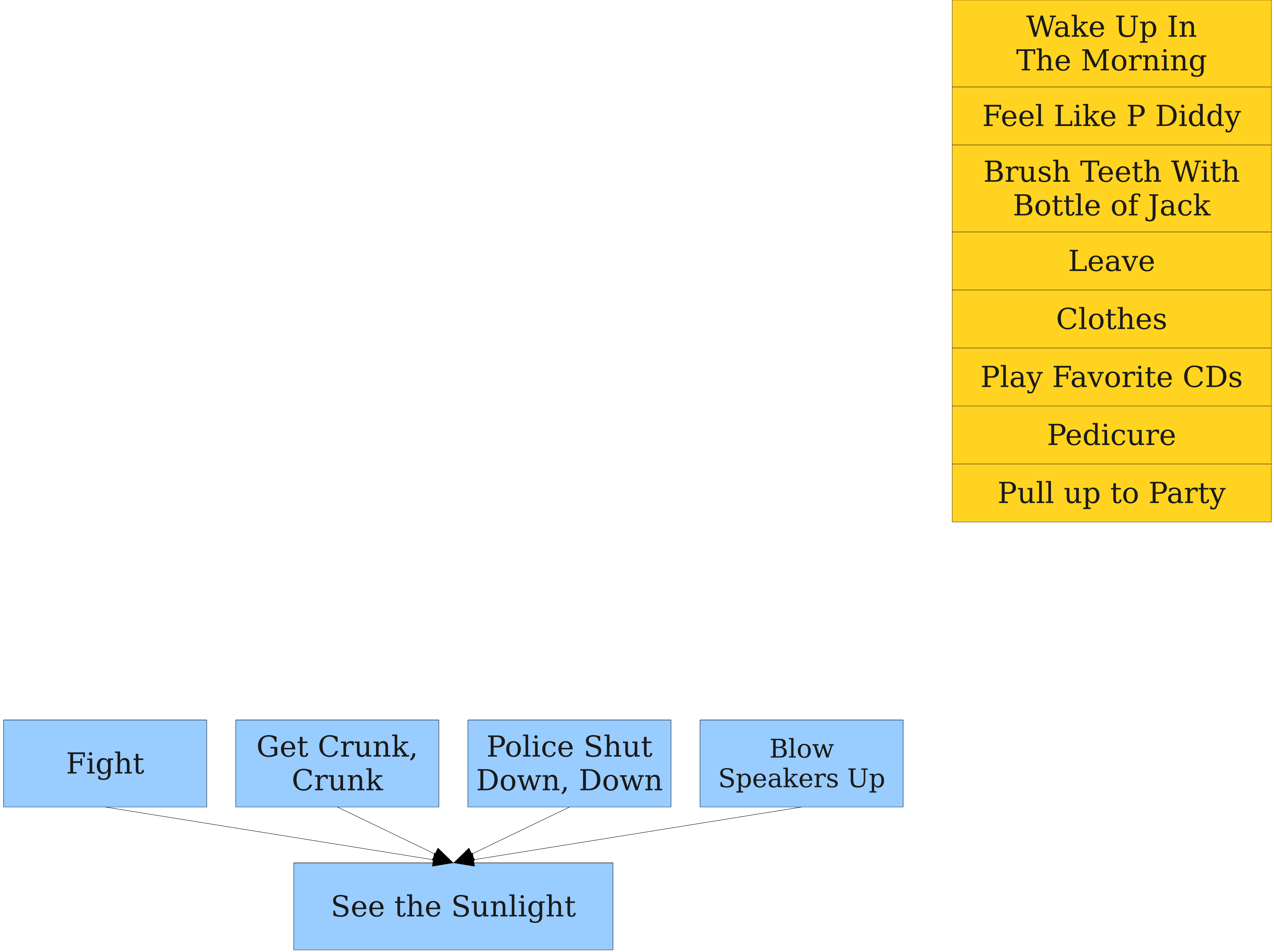
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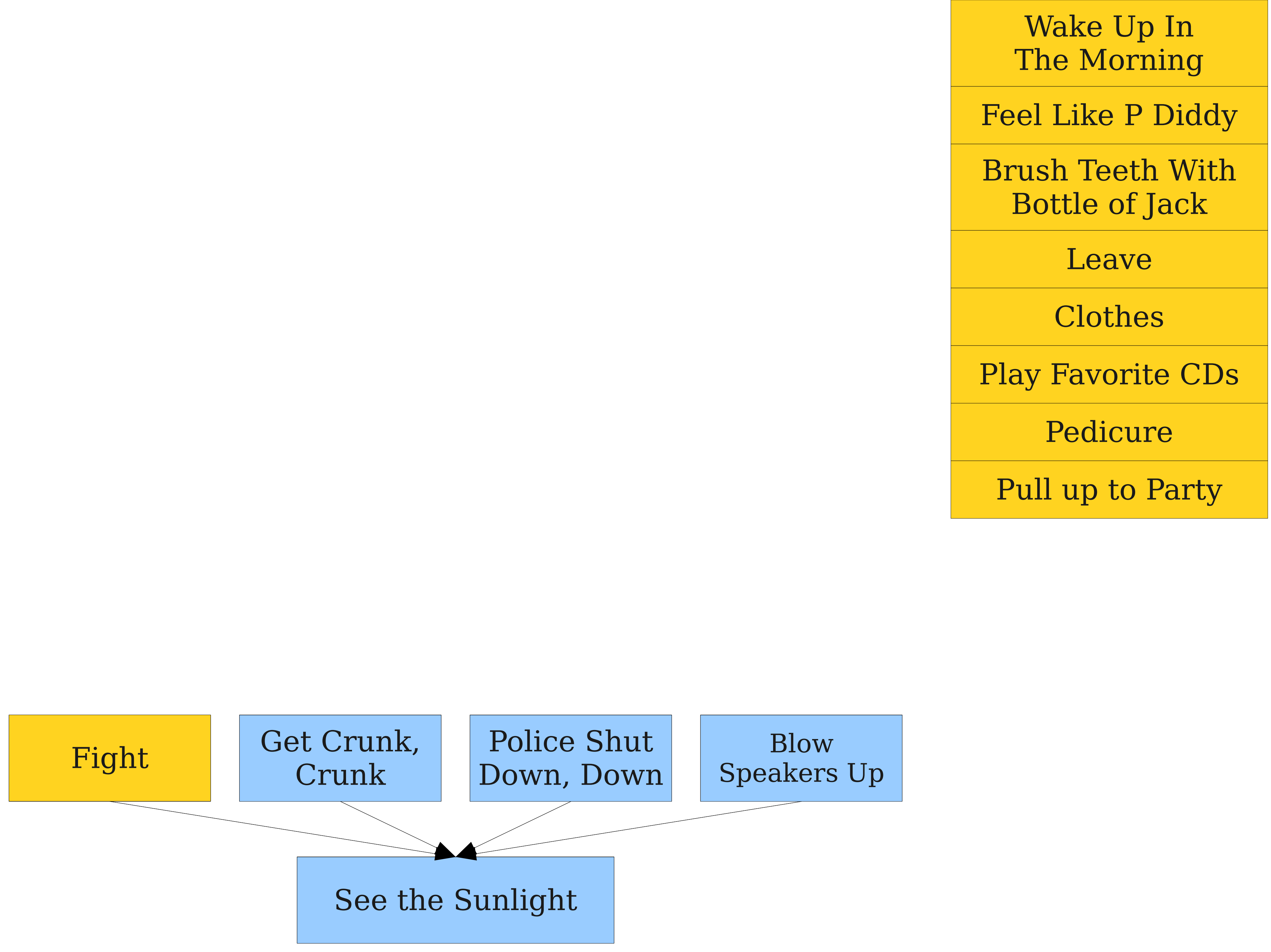
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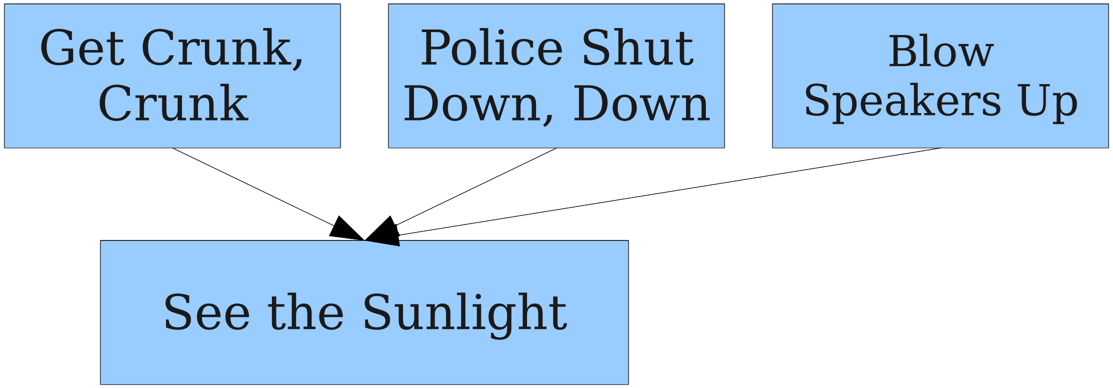
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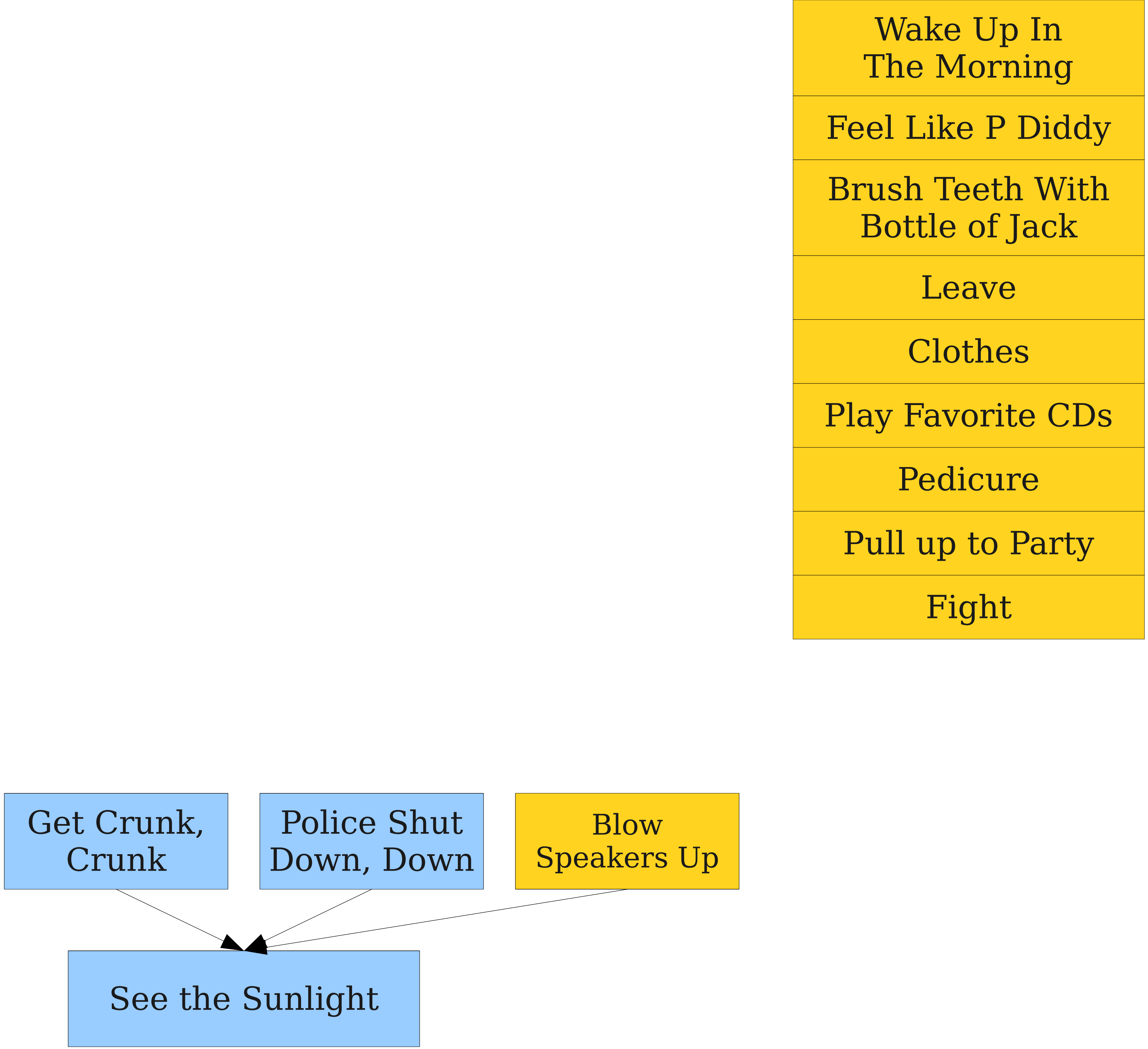
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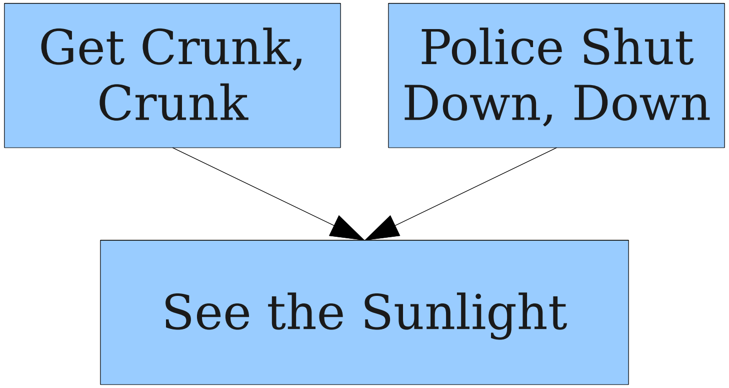
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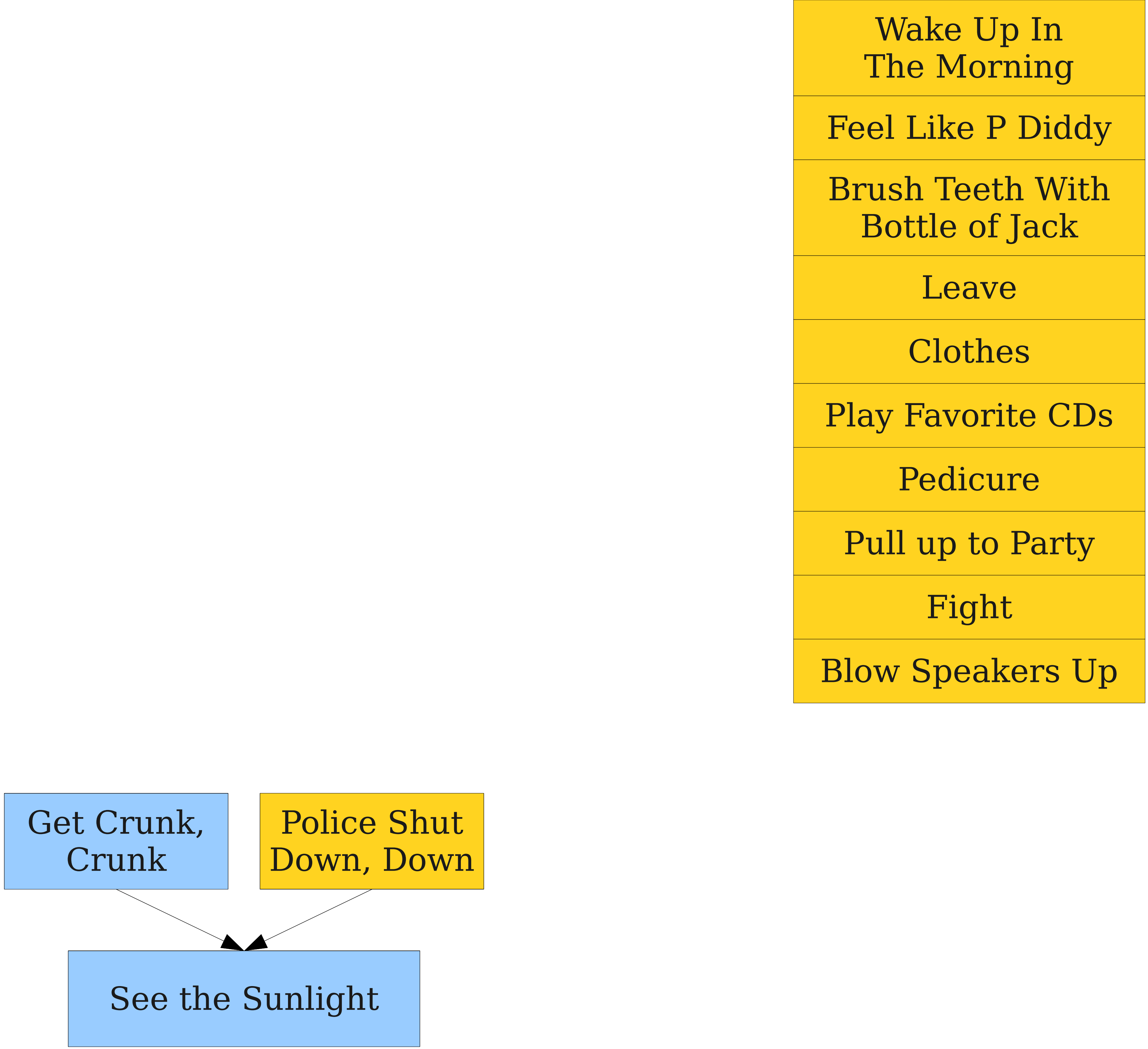

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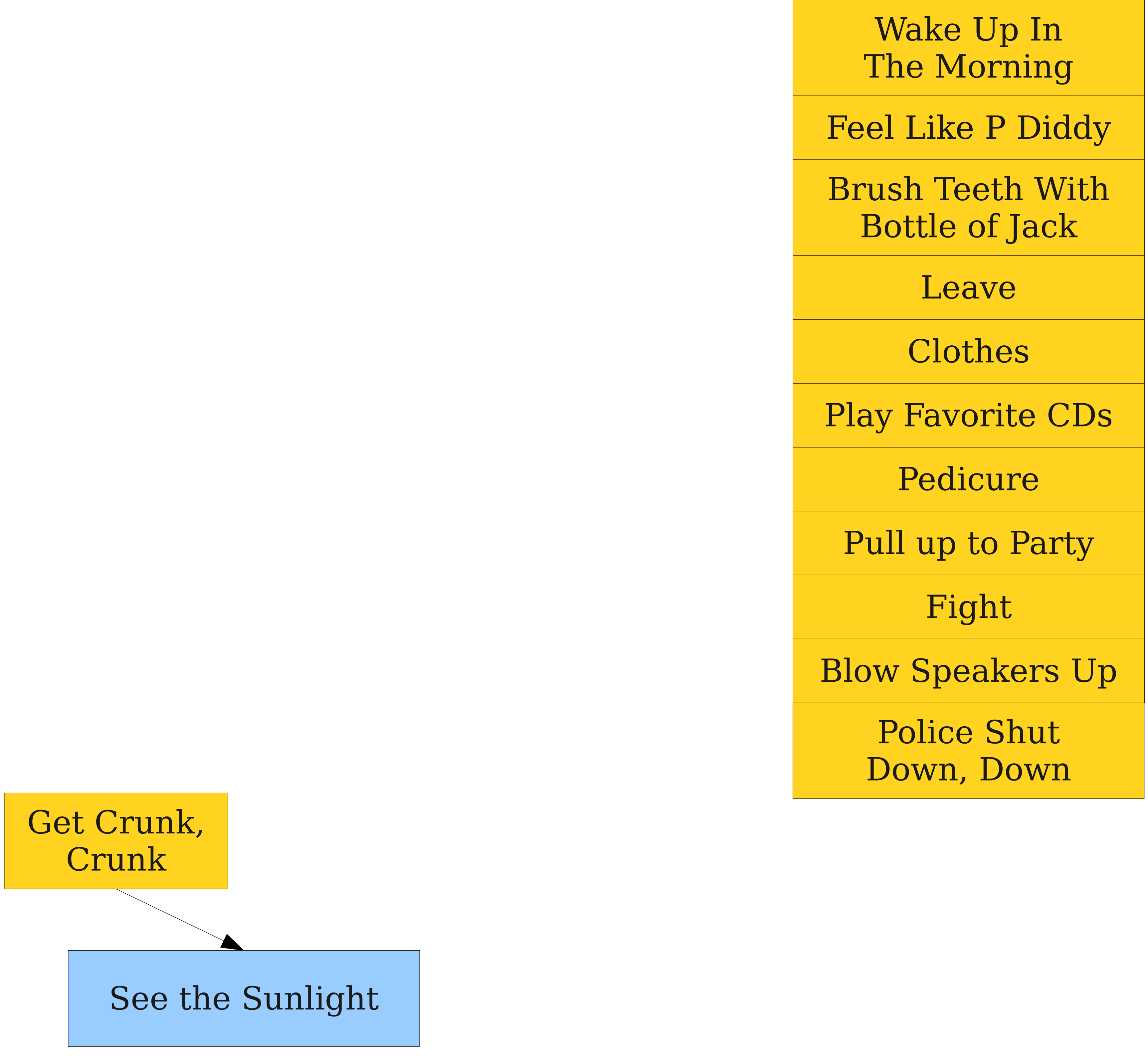
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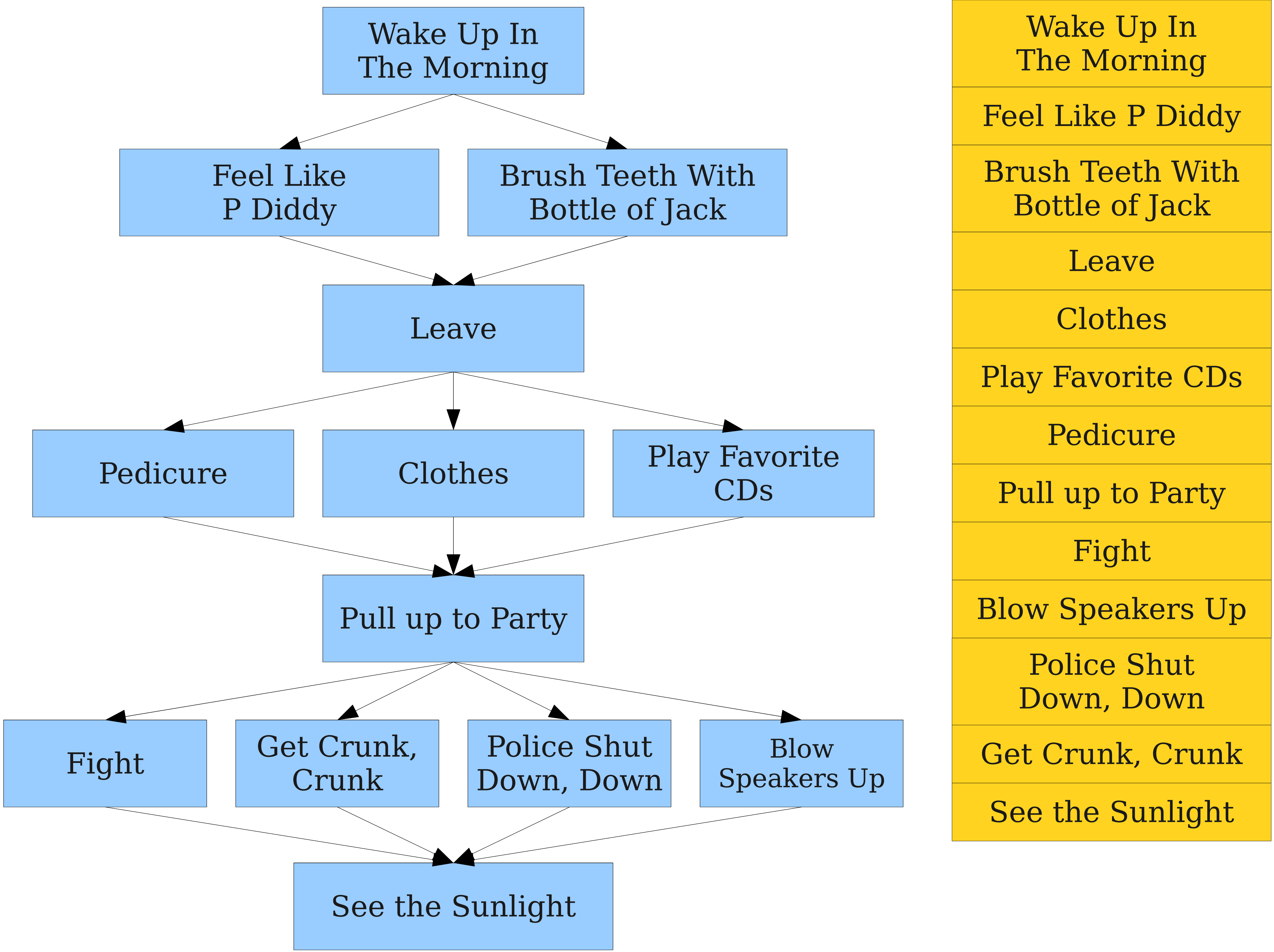
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```
procedure topologicalSort(DAG G):
    let result be an empty list.
    while G is not empty:
        let v be a node in G with indegree 0
        add v to result
        remove v from G
    return result
```

Correctness Proof Sketch

- Whenever a node *v* is added to the **`result`**, it has no incoming edges.
- Therefore, either
  - *v* never had any incoming edges, in which case adding *v* to **`result`** cannot place *v* out of order, or
  - All of *v*'s predecessors have already been placed into **`result`**, and *v* comes after all of them.
- Can't get stuck, since every nonempty DAG has at least one source.

Next Time

- Topological Sorting, Part II
- Connected Components
- Strongly-Connected Components
- Kosaraju's Algorithm I