Fundamental Graph Algorithms
Part II
Outline for Today

- **Dijkstra's Algorithm**
  - An algorithm for finding shortest paths in more realistic settings

- **Depth-First Search**
  - A different graph search algorithm.

- **Directed Acyclic Graphs**
  - Graphs for representing prerequisites.

- **(ITA) Topological Sorting**
  - Algorithms for ordering dependencies.
Recap from Last Time
Breadth-First Search

• Given an arbitrary graph $G = (V, E)$ and a starting node $s \in V$, **breadth-first search** finds shortest paths from $s$ to each reachable node $v$.

• When implemented using an adjacency list, runs in $O(m + n)$ time, which we defined to be linear time on a graph.

• One correctness proof worked in terms of “layers:” the algorithm finds all nodes at distance 0, 1, 2, ... in order.
A Second Intuition for BFS
A Second Intuition

- At each point in the execution of BFS, a node \( v \) is either
  - green, and we have the shortest path to \( v \);
  - yellow, and it is connected to some green node; or
  - gray, and \( v \) is undiscovered.
- Each iteration, we pick a yellow node with minimal distance from the start node and color it green. So what is the cost of the lowest-cost yellow node?
- If \( v \) is yellow, it is connected to a green node \( u \) by an edge.
- The cost of getting from \( s \) to \( v \) is then \( d(s, u) + 1 \).
- BFS works by picking the yellow node \( v \) minimizing
  \[
  d(s, u) + 1
  \]
  where \((u, v)\) is an edge and \( u \) is green.
Pick yellow node $v$ minimizing $d(s, u) + 1$, where $(u, v)$ is an edge and $u$ is green.

This red path can't possibly be better than the blue path we've found, since it takes a suboptimal path out of the circle!
Pick node $v \notin S$ minimizing $d(s, u) + 1$, where $(u, v)$ is an edge and $u \in S$.
**Lemma:** Suppose we have shortest paths computed for nodes $S \subseteq V$, where $s \in S$. Consider a node $v$ where $(u, v) \in E$, $u \in S$, and the quantity $d(s, u) + 1$ is minimized. Then $d(s, v) = d(s, u) + 1$.

**Proof:** There is a path to $v$ of cost $d(s, u) + 1$: follow the shortest path to $u$ (which has cost $d(s, u)$), then follow one more edge to $v$ for total cost $d(s, u) + 1$.

Now suppose for the sake of contradiction that there is a shorter path $P$ to $v$. This path must start in $S$ (since $s \in S$) and leave $S$ (since $v \notin S$). So consider when $P$ leaves $S$. When this happens, $P$ must go from $s$ to some node $x \in S$, cross an edge $(x, y)$ to some node $y$, then continue from $y$ to $v$. This means that $|P|$ is at least $d(s, x) + 1$, since the path goes from $s$ to $x$ and then follows at least one more edge.

Since $v$ was picked to minimize $d(s, u) + 1$ for any choice of $u \in S$ adjacent to an edge $(u, v)$, we know

$$d(s, u) + 1 \leq d(s, x) + 1 \leq |P|$$

contradicting the fact that $|P| < d(s, u) + 1$. We have reached a contradiction, so our assumption was wrong and no shorter path exists.

Since there is a path of length $d(s, u) + 1$ from $s$ to $v$ and no shorter path, this means that $d(s, v) = d(s, u) + 1$. ■
Why These Two Proofs Matter

- The first proof of correctness (based on layers) is based on our first observation: the nodes visited in BFS radiate outward from the start node in ascending order of distance.

- The second proof of correctness (based on picking the lowest yellow node) is based on our second observation: picking the lowest-cost yellow node correctly computes a shortest path.

- Interestingly, this second correctness proof can be generalized to a larger setting...
Edges with Costs

- In many applications, edges have an associated length (or cost, weight, etc.), denoted \(l(u, v)\).
- **Assumption**: Lengths are nonnegative. (We'll revisit this later in the quarter.)
- Let's say that the length of a path \(P\) (denoted \(l(P)\)) is the sum of all the edge lengths in the path \(P\).
- Goal: find the shortest path from \(s\) to every node in \(V\), taking costs into account.
The black edges form a **shortest-path tree**, which traces the shortest paths from each node to the source node.
procedure dijkstrasAlgorithm(s, G):
    let q be a new queue
    for each v in V:
        dist[v] = ∞
    dist[s] = 0
    enqueue(s, q)

    while q is not empty:
        let v be a node in q minimizing dist[v]
        remove(v, q)

        for each node u connected to v:
            if dist[u] > dist[v] + l(u, v):
                dist[u] = dist[v] + l(u, v)
                if u is not enqueued into q:
                    enqueue(u, q)
Dijkstra's Algorithm

- Assuming nonnegative edge lengths, finds the shortest path from $s$ to each node in $G$.
- Correctness proof sketch is based on the second argument for breadth-first search:
  - Always picks the node $v$ minimizing $d(s, u) + l(u, v)$ for yellow $v$ and green $u$.
  - If a shorter path $P$ exists to $v$, it must leave the set of green nodes through some edge $(x, y)$.
  - But then $l(P)$ is at least $d(s, x) + l(x, y)$, which is at least $d(s, u) + l(u, v)$.
  - So the “shorter” path costs at least as much as the path we found.
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  let q be a new queue
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      if dist[u] > dist[v] + l(u, v):
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        if u is not enqueued into q:
          enqueue(u, q)

  \textbf{O}(m + n)
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    while q is not empty:
        let v be a node in q minimizing dist[v]
        remove(v, q)

        for each node u connected to v:
            if dist[u] > dist[v] + l(u, v):
                dist[u] = dist[v] + l(u, v)
            if u is not enqueued into q:
                enqueue(u, q)

O(n^2)
Dijkstra Runtime

- Using a standard implementation of a queue, Dijkstra's algorithm runs in time $O(n^2)$.
  - $O(n + m)$ time processing nodes and edges, plus $O(n^2)$ time finding the lowest-cost node.
  - Since $m = O(n^2)$, $O(n + m + n^2) = O(n^2)$.
- Using a slightly fancier data structure (a binary heap), can be made to run in time $O(m \log n)$.
  - Is this necessarily more efficient?
  - More on how to do this later this quarter.
- Using a *much* fancier data structure (the *Fibonacci heap*), can be made to run in time $O(m + n \log n)$.
  - Take CS166 for details!
Shortest Path Algorithms

• If all edges have the same weight, can use breadth-first search to find shortest paths.
  • Takes time $O(m + n)$.

• If edges have nonnegative weight, can use Dijkstra's algorithm.
  • Takes time $O(n^2)$, or less using more complex data structures.

• What about the case where edges can have negative weight?
  • More on that later in the quarter...
Depth-First Search
BFS and DFS

- Last time, we saw the **breadth-first search (BFS)** algorithm, which explored a graph and found shortest paths.
- The algorithm explored outward in all directions uniformly.
- We will now see **depth-first search (DFS)**, an algorithm that explores out in one direction, backing up when necessary.
Depth-First Search

These black edges for a depth-first search tree, which traces paths from the root to each node in the graph.
Depth-First Search, Again
procedure DFS(node v):
  color v yellow.
  
  for each neighbor u of v:
    if u is gray:
      DFS(u)
  
  color v green

procedure doDFS(graph G, node s):
  for each node v in G:
    color v gray
  DFS(s)
**Question 1:** What nodes will DFS reach?

**Question 2:** How *efficiently* will DFS reach those nodes?
**Theorem:** When DFS\((s)\) is called on node \(s\), no nodes reachable from \(s\) will be gray when DFS\((s)\) and all ancestor calls return.

**Proof:** By induction on the distance of nodes from \(s\). As a base case, consider all nodes at distance 0 from \(s\). This is just \(s\) itself. When DFS\((s)\) is called, DFS\((s)\) will color \(s\) yellow, then green.

Suppose the claim holds for all nodes at distance \(n\) from \(s\); we'll prove it holds for all nodes at distance \(n + 1\) from \(s\). Take any node \(v\) at distance \(n + 1\) from \(s\); \(v\) is adjacent to some node \(u\) at distance \(n\) from \(s\). By our IH, \(u\) will not be gray when DFS\((s)\) and its ancestor calls return, so DFS\((u)\) must have been called at some point. This call must have called DFS on each of \(u\)'s gray neighbors. If \(v\) was gray at this time, DFS\((v)\) must have been called on \(v\), coloring \(v\) yellow and then green. Otherwise, \(v\) was already not colored gray.

Since our choice of \(v\) was arbitrary, no nodes at distance \(n + 1\) will be gray when DFS\((s)\) and its ancestor calls return, completing the induction. ■
Theorem: When DFS(s) is called on a node s, no recursive calls will be made on nodes not reachable from s.

Proof: By contradiction; assume a recursive call is made on at least one node not reachable from s. There must be a first node visited this way; call it v. v can't be s, since s is trivially reachable from itself. Thus DFS(v) must have been recursively invoked by DFS(u) for some node u ≠ v, which in turn called DFS(v). This means edge (u, v) must exist. Now, we consider two cases:

- Case 1: u is reachable from s. But then v is reachable from s, because we can take the path from s to u and follow edge (u, v).

- Case 2: u is not reachable from s. But then v was not the first node not reachable from s to have DFS called on it.

In either case, we reach a contradiction, so our assumption was wrong. Thus DFS(s) never makes recursive calls on nodes not reachable from s. ■
What DFS Visits

• Taken together, the two theorems we have proven show the following:
  • When DFS(s) terminates, every node reachable from s will have had DFS called on it, though the call to DFS(s) might not have initiated those other calls.
  • When DFS(s) terminates, it will never have called DFS on a node not reachable from s.
• Thus when DFS(s) terminates, the only nodes DFS will have been called on are nodes on which DFS had already been called, plus the nodes reachable from s.
Question 1: What nodes will DFS reach?

Question 2: How efficiently will DFS reach those nodes?
procedure DFS(node v):
    color v yellow.
    for each neighbor u of v:
        if u is gray:
            DFS(u)
    color v green

procedure doDFS(graph G, node s):
    for each node v in G:
        color v gray
    DFS(s)
Analyzing Recursive Functions

• In general, it can be very difficult to analyze the runtime of a recursive function.
  • We'll see some techniques for special cases later in the quarter.
• One general technique is to look at the total number of calls made and the work done at each call.
Analyzing DFS

- The maximum number of function calls made is \( O(n) \), since we can't call DFS on a node twice.
- Each call to DFS on node \( v \) does \( \Theta(\text{deg}^+(v)) \) work, since it visits each outgoing edge from \( v \) exactly once.
- Summing across all recursive calls:
  - \( O(n) \) work done initially coloring nodes.
  - \( O(n) \) work done coloring nodes yellow / green.
  - \( O(m) \) work visiting edges.
  - Total work done: \( O(m + n) \).
- When might this not do \( \Theta(m + n) \) work?
BFS and DFS

- BFS and DFS always visit the same set of nodes.
- However, BFS always finds the shortest path from the source node to each other node in the graph, while DFS might not.
- That said: the order in which DFS visits nodes is pretty important and has lots of applications. We'll see some of them soon...
Ordering Prerequisites
Measure Flour
Measure Sugar
Measure Baking Powder
Measure Salt
Measure Sugar

Oil Griddle
Beat Egg
Combine Dry Ingredients
Melt Butter
Measure Milk

Heat Griddle
Add Wet Ingredients

Make Pancakes
Serve Pancakes
Modeling Prerequisites

• We can model prerequisites as a graph with the following properties:
  • The graph has to be **directed**, since we have to be able to distinguish “A depends on B” from “B depends on A.”
  • The graph has to be **acyclic** (containing no cycles), since otherwise there is no way to accomplish all the tasks.

• A graph with this property is called a **directed acyclic graph**, or **DAG**.
Some DAG Terminology

- A **source** node in a DAG is a node with no incoming edges.
- A **sink** node in a DAG is a node with no outgoing edges.
- DAGs can have many sources and sinks.
**Theorem:** Every nonempty DAG has at least one source node.

**Proof:** Suppose for the sake of contradiction that there is a nonempty DAG $G$ where each node has at least one incoming edge. Start at any node $v_1 \in G$ and repeatedly follow an edge entering $v_1$ in reverse. This gives a sequence of nodes $v_1, v_2, v_3, \ldots$

Since there are only finitely many nodes in the DAG, this process eventually must revisit a node $v_i$. But then we have that $v_i, v_{i+1}, v_{i+2}, \ldots, v_i$ is a cycle in $G$ traced in reverse order, contradicting the fact that $G$ is a DAG. We have reached a contradiction, so our assumption was wrong and every DAG must contain at least one node with no incoming edges. ■
Ordering Prerequisites

• When ordering prerequisites, we want to order the tasks such that no task is placed before tasks it depends on.

• In graph-theoretic terms: given a DAG $G = (V, E)$, we want to order the nodes so that if $(u, v) \in E$, then $v$ appears after $u$.

• Such an ordering is called a topological ordering. An algorithm for finding a topological ordering is called a topological sort.
Wake Up In The Morning

Feel Like P Diddy
Brush Teeth With Bottle of Jack

Leave

Pedicure
Clothes
Play Favorite CDs

Pull up to Party

Fight
Get Crunk, Crunk
Police Shut Down, Down
Blow Speakers Up

See the Sunlight
procedure topologicalSort(DAG G):
    let result be an empty list.
    while G is not empty:
        let v be a node in G with indegree 0
        add v to result
        remove v from G
    return result
Correctness Proof Sketch

• Whenever a node $v$ is added to the result, it has no incoming edges.

• Therefore, either
  • $v$ never had any incoming edges, in which case adding $v$ to result cannot place $v$ out of order, or
  • All of $v$'s predecessors have already been placed into result, and $v$ comes after all of them.

• Can't get stuck, since every nonempty DAG has at least one source.
Next Time

- Topological Sorting, Part II
- Connected Components
- Strongly-Connected Components
- Kosaraju's Algorithm I