Divide-and-Conquer Algorithms

Part Two

Recap from Last Time

Divide-and-Conquer Algorithms

- A **divide-and-conquer** algorithm is one that works as follows:
  - **(Divide)** Split the input apart into multiple smaller pieces, then recursively invoke the algorithm on those pieces.
  - **(Conquer)** Combine those solutions back together to form the overall answer.
- Can be analyzed using **recurrence relations**.

Two Important Recurrences

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Solves to $O(n \log n)$

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lfloor n / 2 \rfloor) + \Theta(1)$$

Solves to $O(\log n)$

Outline for Today

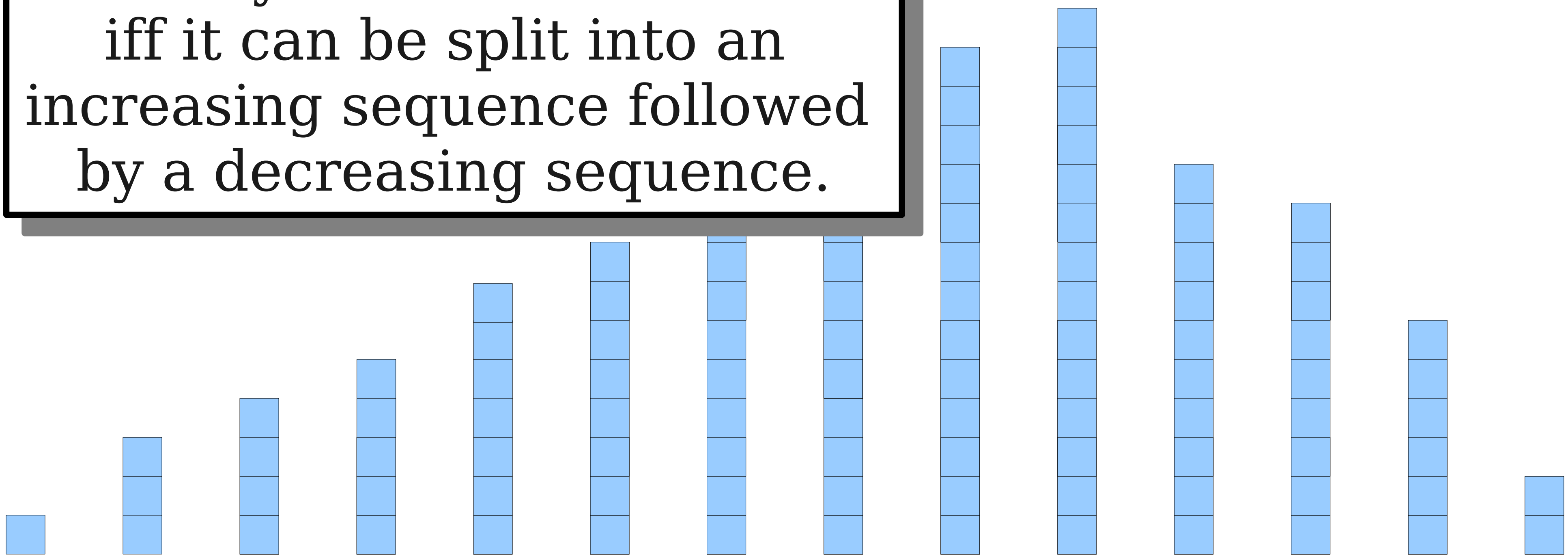
- **More Recurrences**
  - Other divide-and-conquer relations.
- **Algorithmic Lower Bounds**
  - Showing that certain problems cannot be solved within certain limits.
- **Binary Heaps**
  - A fast data structure for retrieving elements in sorted order.

Another Algorithm:

Maximizing Unimodal Arrays

Unimodality

An array is called **unimodal** iff it can be split into an increasing sequence followed by a decreasing sequence.



1	3	4	5	7	8	10	12	13	14	10	9	6	2

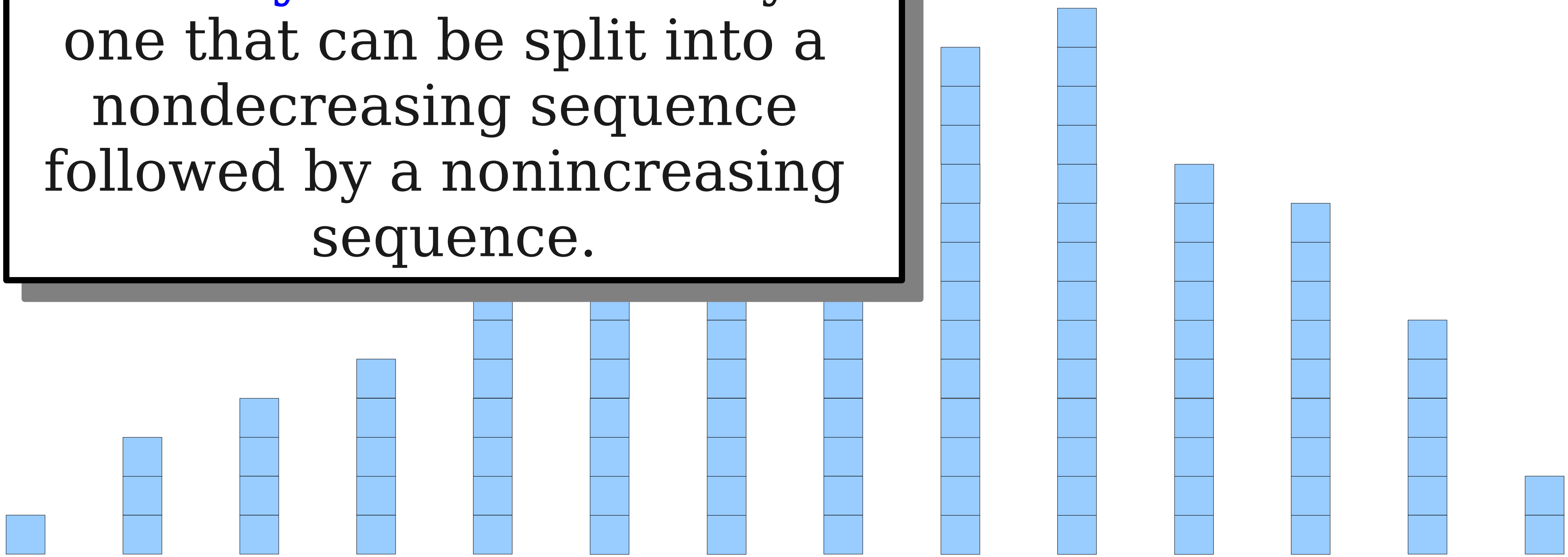
```
procedure unimodalMax(list A, int low, int high):
  if low = high - 1:
     return A[low]

  let mid = ⌊(high + low) / 2⌋
  if A[mid] < A[mid + 1]
     return unimodalMax(A, mid + 1, high)
  else:
     return unimodalMax(A, low, mid + 1)
```

$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lceil n / 2 \rceil) + \Theta(1)$$

O(log *n*)

Unimodality II

A **weakly unimodal** array is one that can be split into a nondecreasing sequence followed by a nonincreasing sequence.



1	3	4	5	7	8	10	10	13	14	10	9	6	2

```
procedure weakUnimodalMax(list A, int low, int high):
  if low = high - 1:
     return A[low]

  let mid = ⌊(high + low) / 2⌋
  if A[mid] < A[mid + 1]
     return weakUnimodalMax(A, mid + 1, high)
  else if A[mid] > A[mid + 1]
     return weakUnimodalMax(A, low, mid + 1)
  else
     return max(weakUnimodalMax(A, low, mid + 1)
                weakUnimodalMax(A, mid + 1, high))
```

$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(1)$$

```
procedure weakUnimodalMax(list A, int low, int high):
  if low = high - 1:
     return A[low]

  let mid = ⌊(high + low) / 2⌋
  if A[mid] < A[mid + 1]
     return weakUnimodalMax(A, mid + 1, high)
  else if A[mid] > A[mid + 1]
     return weakUnimodalMax(A, low, mid + 1)
  else
     return max(weakUnimodalMax(A, low, mid + 1)
                weakUnimodalMax(A, mid + 1, high))
```

$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + c$$

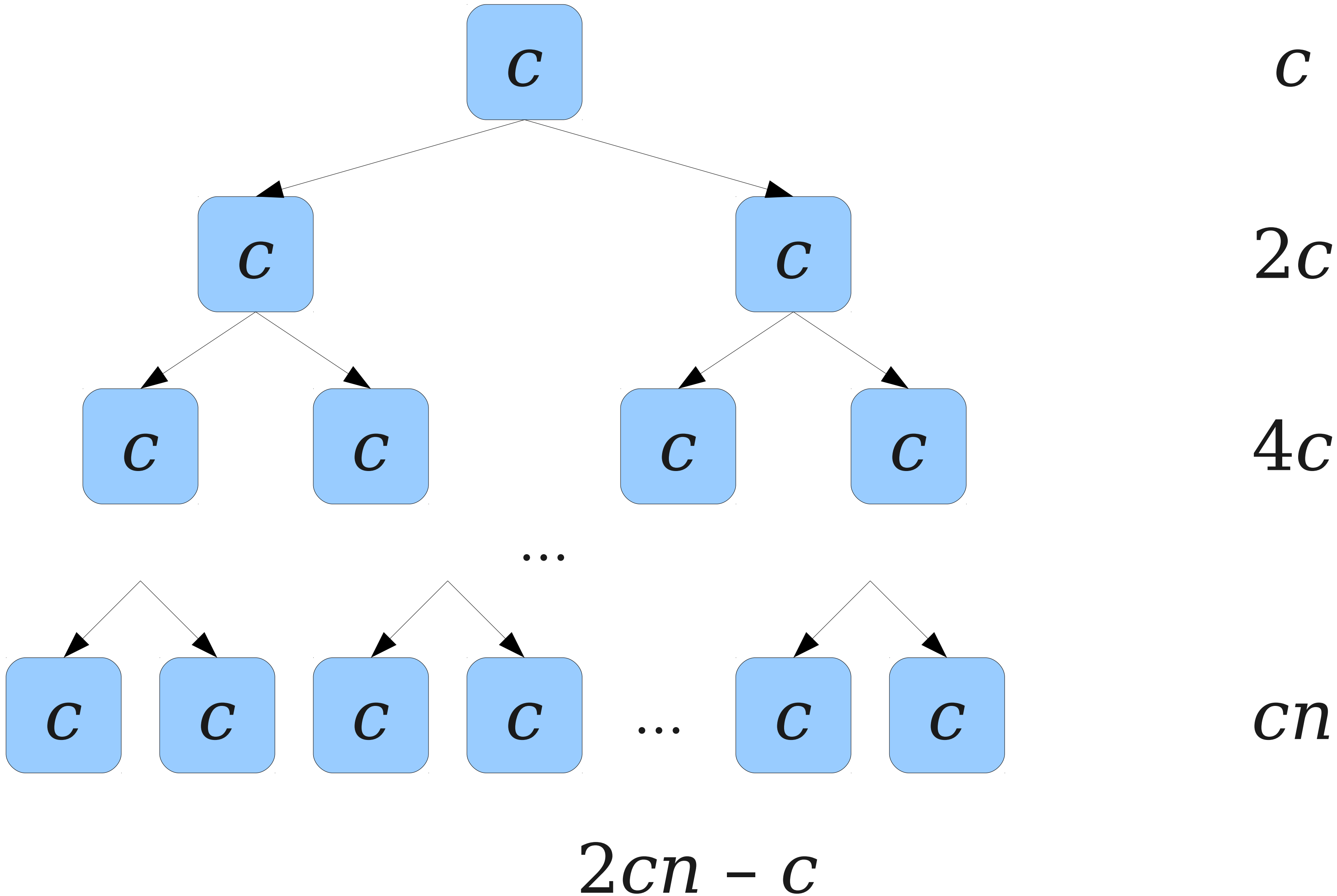
$T(1) \le c$
$T(n) \le 2T(n/2) + c$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + c \\
&\le 2\left(2T\left(\frac{n}{4}\right) + c\right) + c \\
&\le 4T\left(\frac{n}{4}\right) + 2c + c \\
&= 4T\left(\frac{n}{4}\right) + 3c \\
&\le 4\left(2T\left(\frac{n}{8}\right) + c\right) + 3c \\
&= 8T\left(\frac{n}{8}\right) + 4c + 3c \\
&= 8T\left(\frac{n}{8}\right) + 7c \\
&\dots \\
&\le 2^k T\left(\frac{n}{2^k}\right) + (2^k - 1)c
\end{aligned}$$

$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + c$$

$$\begin{aligned}
T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + (2^k - 1)c \\
&\le 2^{\log_2 n} T(1) + (2^{\log_2 n} - 1)c \\
&= nT(1) + c(n-1) \\
&\le cn + c(n-1) \\
&= 2cn - c \\
&= O(n)
\end{aligned}$$

$T(1) \le c$
$T(n) \le 2T(n / 2) + c$



Another Recurrence Relation

- The recurrence relation

$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(1)$$

  solves to $T(n) = $ **$O(n)$**

- Intuitively, the recursion tree is "bottomheavy:" the bottom of the tree accounts for almost all of the work.

Unimodal Arrays

- Our recurrence shows that the work done is O(n), but this might not be a tight bound.
- Does our algorithm ever do Ω(n) work?
- **Yes:** What happens if all array values are equal to one another?
- Can we do better?

A Lower Bound

- **Claim**: Every correct algorithm for finding the maximum value in a unimodal array must do $\Omega(n)$ work in the worst-case.
- Note that this claim is over *all possible algorithms*, so the argument had better be watertight!

A Lower Bound

- We will prove that any algorithm for finding the maximum value of a unimodal array must, on at least one input, inspect all n locations.
- *Proof idea*: Suppose that the algorithm didn't do this.

0	0	0	0	0	0	?	0	0	0	0	0	0	0

A Lower Bound

- We will prove that any algorithm for finding the maximum value of a unimodal array must, on at least one input, inspect all *n* locations.
- *Proof idea*: Suppose that the algorithm didn't do this.

0	0	0	0	0	0	**10**	0	0	0	0	0	0	0

A Lower Bound

- We will prove that any algorithm for finding the maximum value of a unimodal array must, on at least one input, inspect all *n* locations.
- *Proof idea*: Suppose that the algorithm didn't do this.

0	0	0	0	0	0	**0**	0	0	0	0	0	0	0

Algorithmic Lower Bounds

- The argument we just saw is called an **adversarial argument** and is often used to establish algorithmic lower bounds.
- Idea: Show that if an algorithm doesn't do enough work, then it cannot distinguish two different inputs that require different outputs.
- Therefore, the algorithm cannot always be correct.

o Notation

- Let f, $g : \mathbb{N} \to \mathbb{N}$.
- We say that ***$f(n) = o(g(n))$*** (f is ***little-o*** of g) iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

- In other words, f grows strictly slower than g.
- Often used to describe impossibility results.
- For example: There is no $o(n)$-time algorithm for finding the maximum element of a weakly unimodal array.

What Does This Mean?

- In the worst-case, our algorithm must do $\Omega(n)$ work.
- That's the same as a linear scan over the input array!
- Is our algorithm even worth it?
- **Yes**: In most cases, the runtime is $\Theta(\log n)$ or close to it.

Binary Heaps

Data Structures Matter

- We have seen two instances where a better choice of data structure improved the runtime of an algorithm:
  - Using adjacency lists instead of adjacency matrices in graph algorithms.
  - Using a double-ended queue in 0/1 Dijkstra's algorithm.
- Today, we'll explore a data structure that is useful for improving algorithmic efficiency.
- We'll come back to this structure in a few weeks when talking about Prim's algorithm and Kruskal's algorithm.

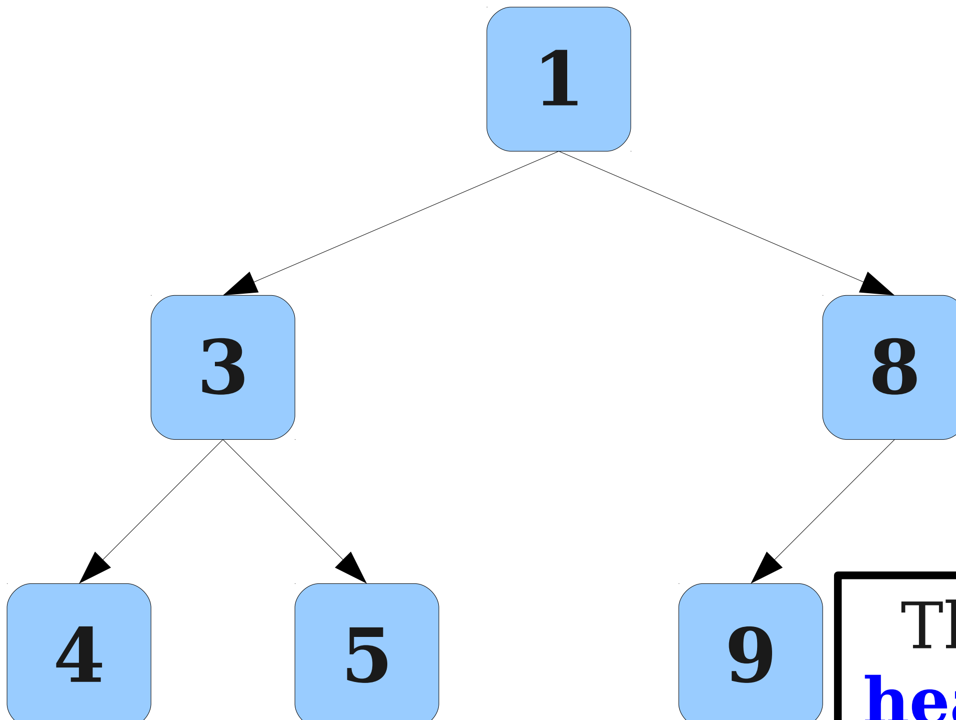
Priority Queues

- A **priority queue** is a data structure for storing elements associated with *priorities* (often called **keys**).
- Optimized to find the element that currently has the smallest key.
- Supports the following operations:
  - **enqueue**(k, v) which adds element v to the queue with key k.
  - **is-empty**, which returns whether the queue is empty.
  - **dequeue-min**, which removes the element with the least priority from the queue.
- Many implementations are possible with varying tradeoffs.

A Naive Implementation

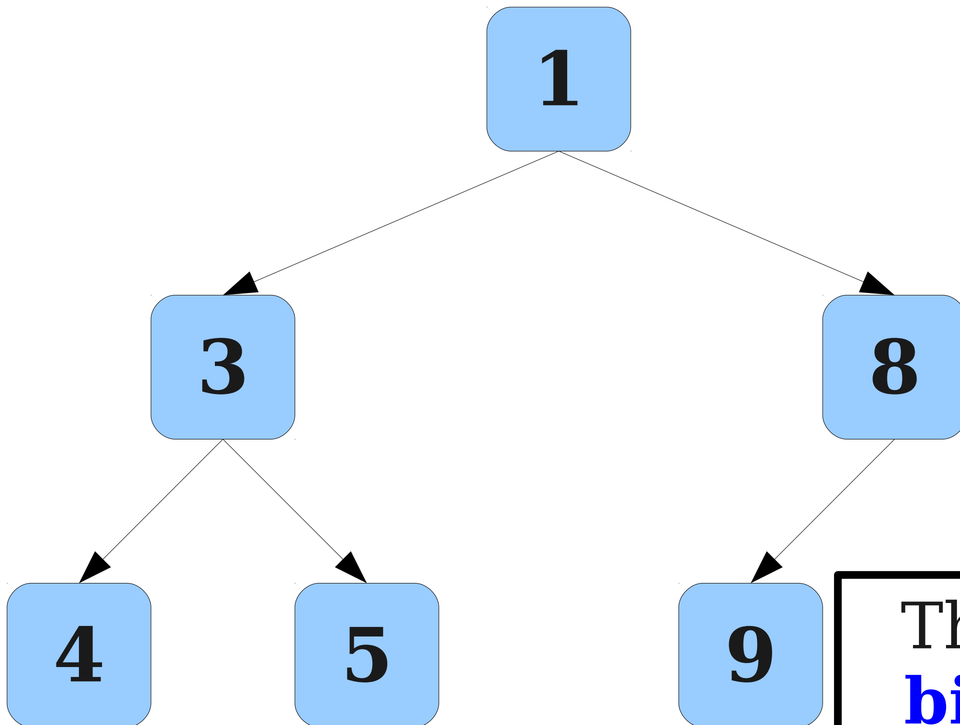
- One simple way to implement a priority queue is with an unsorted array key/value pairs.
- To enqueue v with key k, append (k, v) to the array in time O(1).
- To check whether the priority queue is empty, check whether the underlying array is empty in time O(1).
- To dequeue-min, scan across the array to find an element with minimum key, then remove it in time $O(n)$.
- Doing n enqueues and n dequeues takes time $O(n^2)$.

A Better Implementation



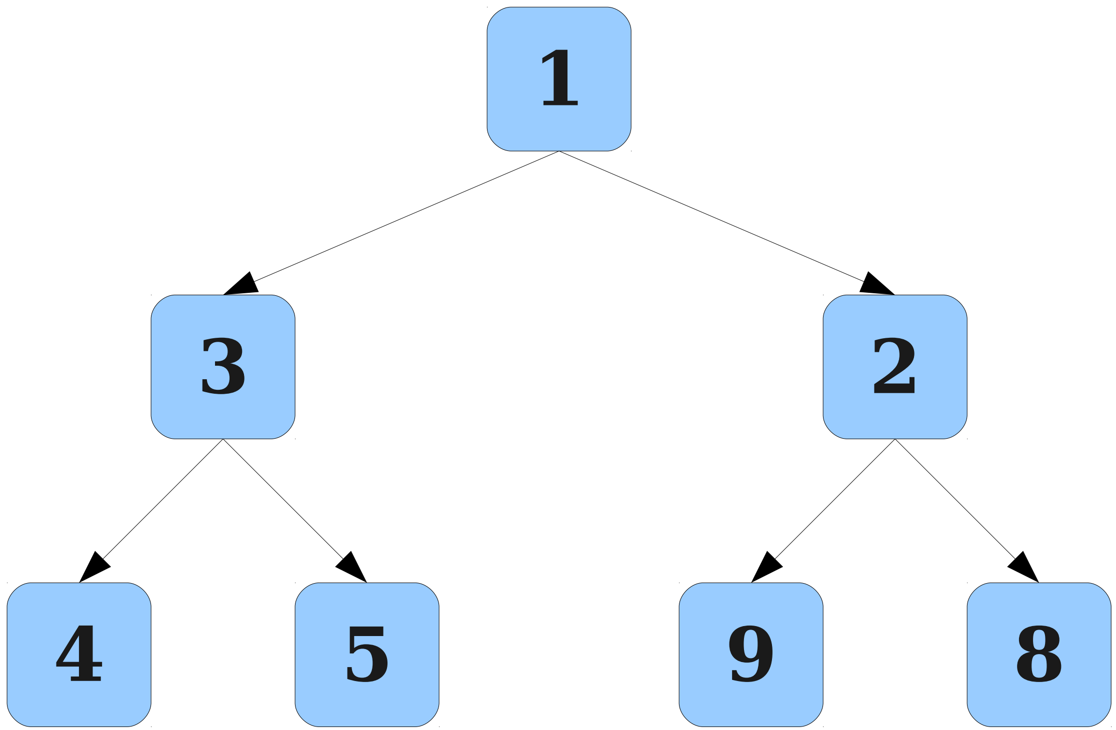
This tree obeys the **heap property**: each node's key is less than or equal to all its descendants' keys.

A Better Implementation

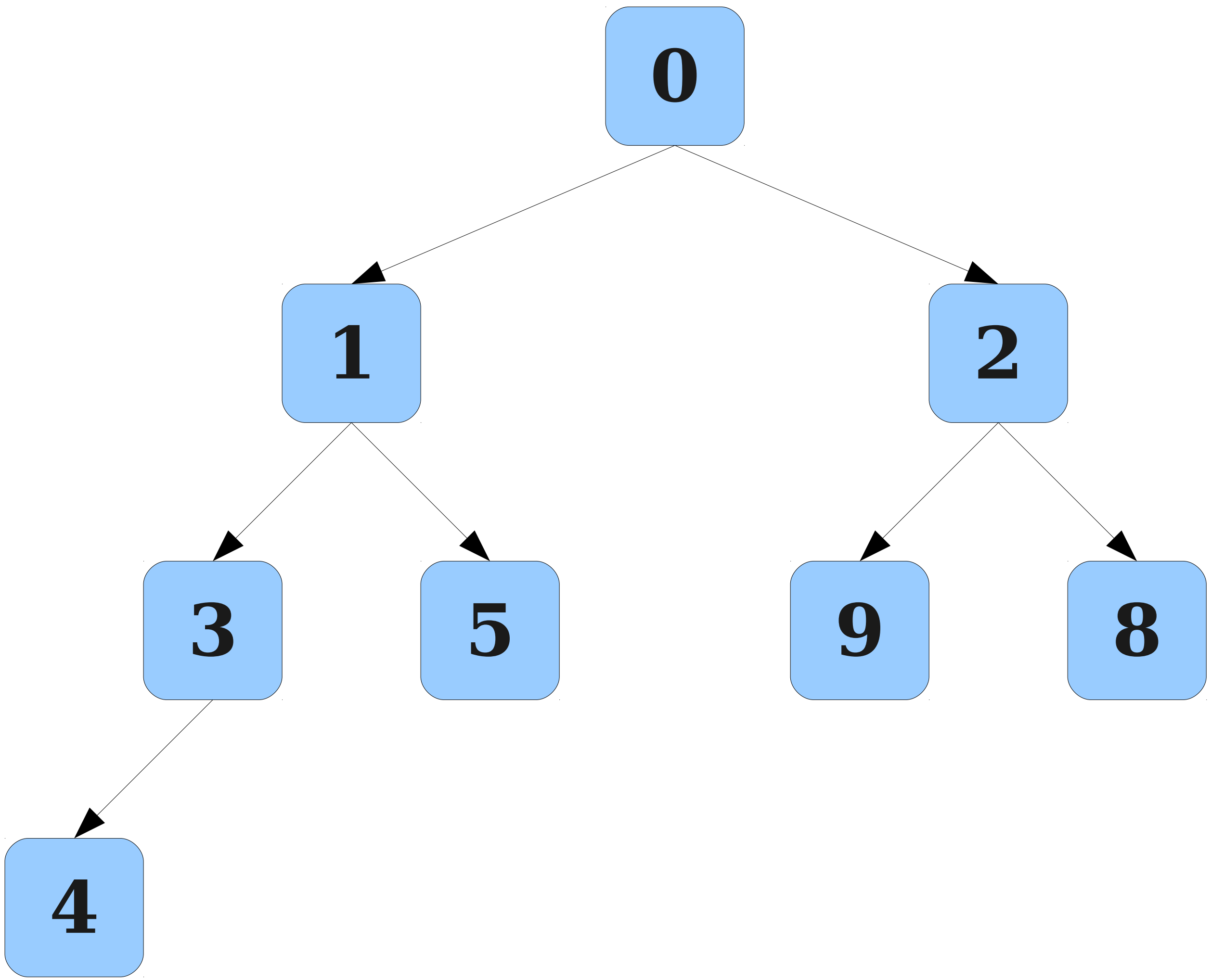


This is a **complete binary tree**: every level except the last one is filled in completely.

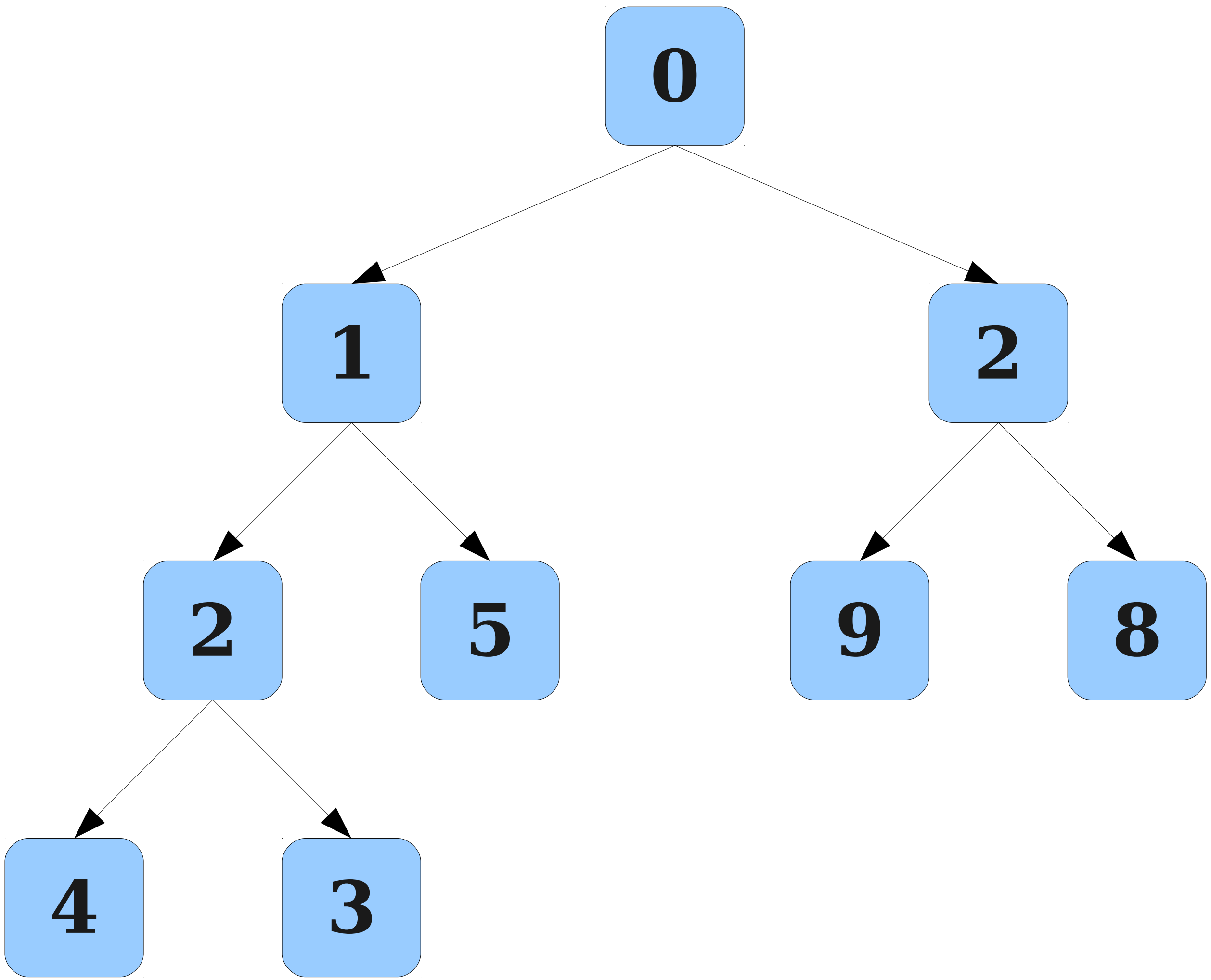
A Better Implementation



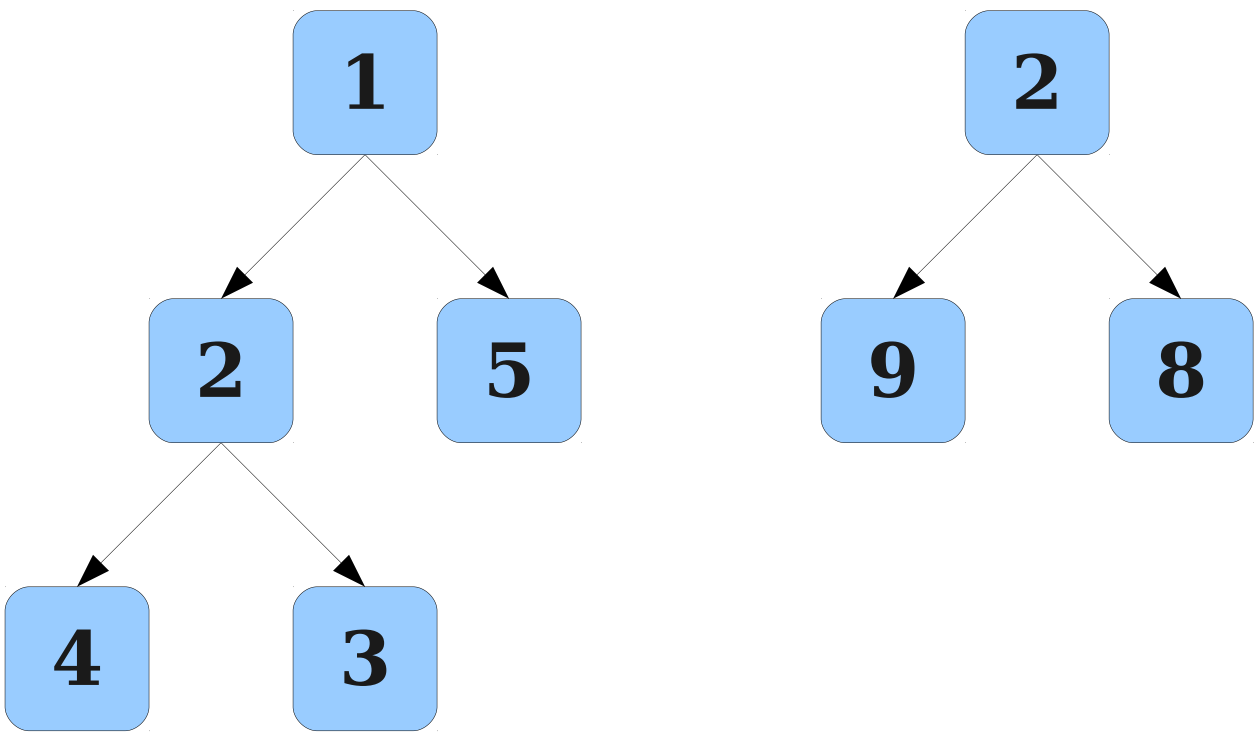
A Better Implementation



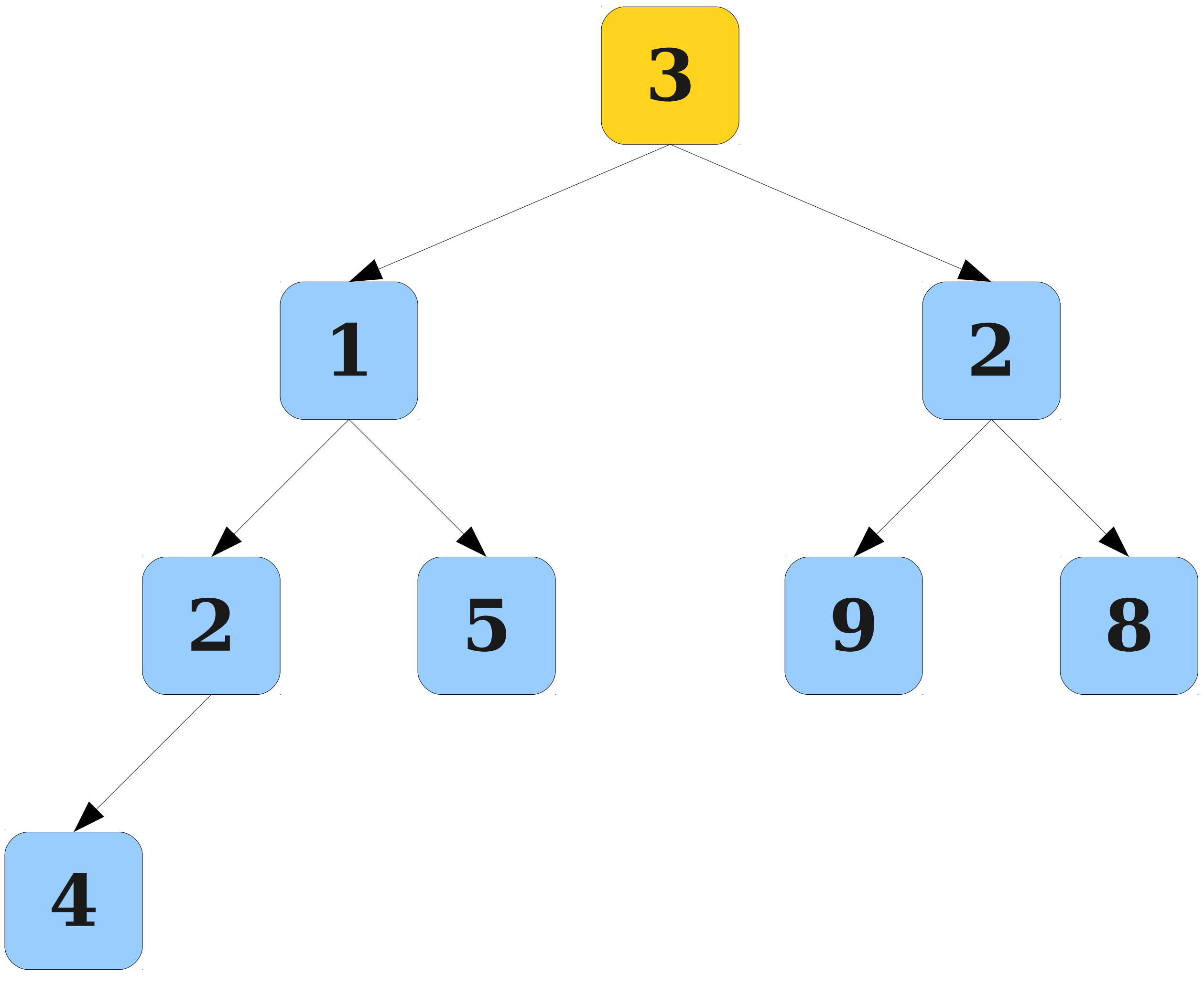
A Better Implementation



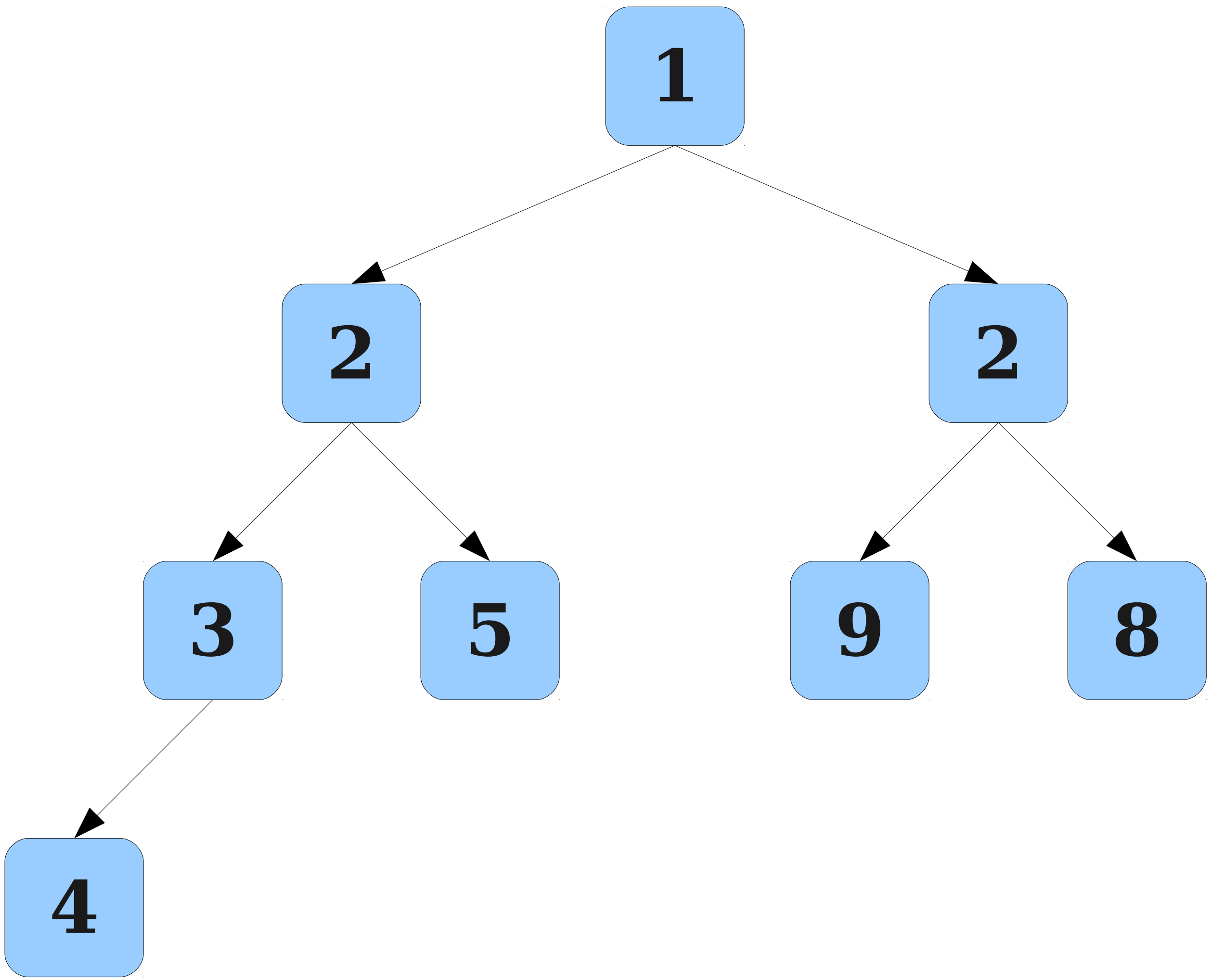
A Better Implementation



A Better Implementation



A Better Implementation

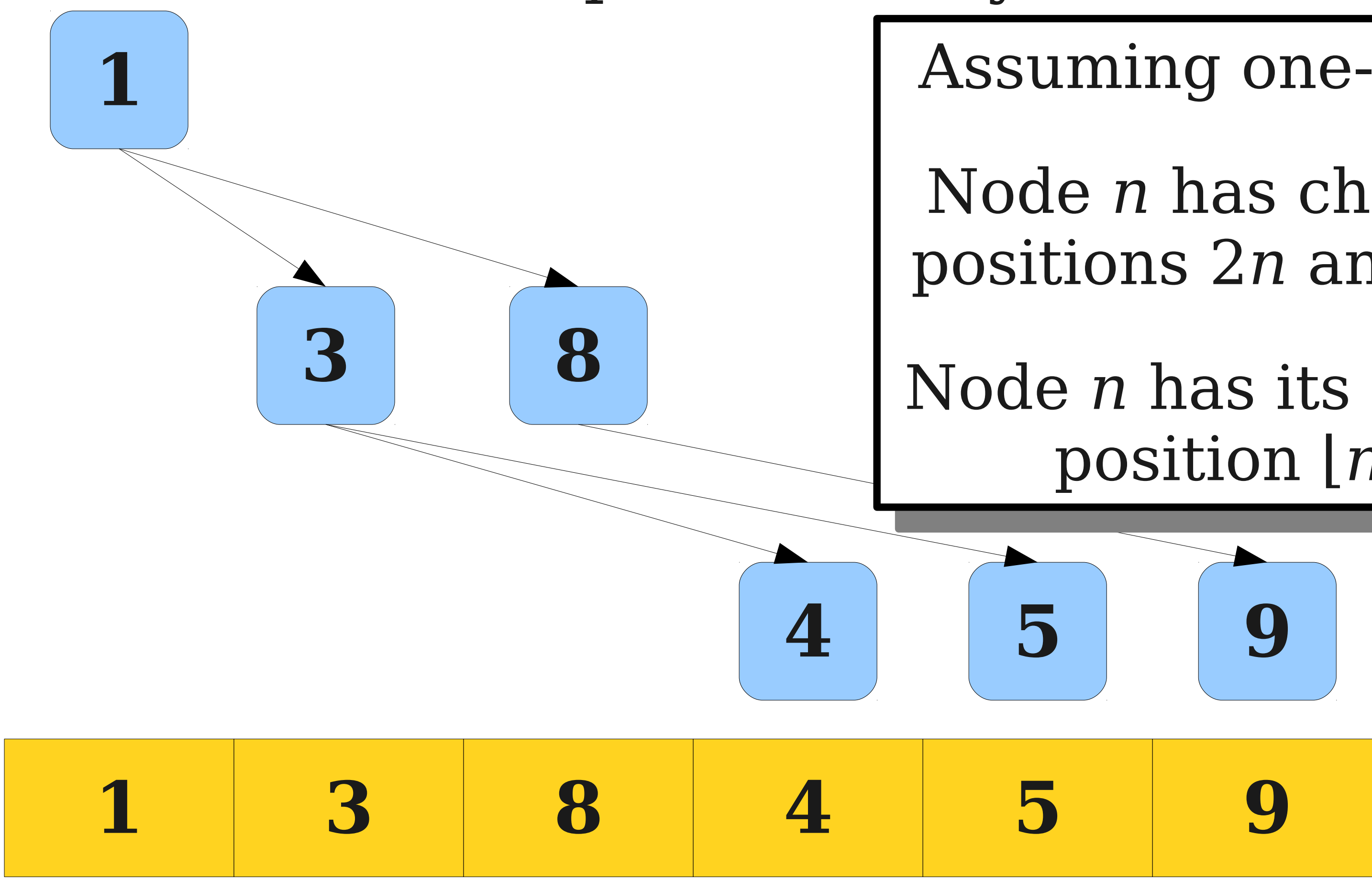


Binary Heap Efficiency

- The enqueue and dequeue operations on a binary heap all run O(h), where h is the height of the tree.
- In a perfect binary tree of height h, there are $1 + 2 + 4 + 8 + \ldots + 2^h = 2^{h+1} - 1$ nodes.
- If there are n nodes, the maximum height would be found by setting $n = 2^{h+1} - 1$.
- Solving, we get that $h = \log_2 (n + 1) - 1$
- Thus $h = \Theta(\log n)$, so enqueue and dequeue take time $O(\log n)$.

Implementing Binary Heaps

- It is extremely rare to actually implement a binary heap as a tree structure.
- Can encode the heap as an array:



Assuming one-indexing:

Node n has children at positions $2n$ and $2n + 1$.

Node n has its parent at position $\lfloor n / 2 \rfloor$

Application: **Heapsort**

Heapsort

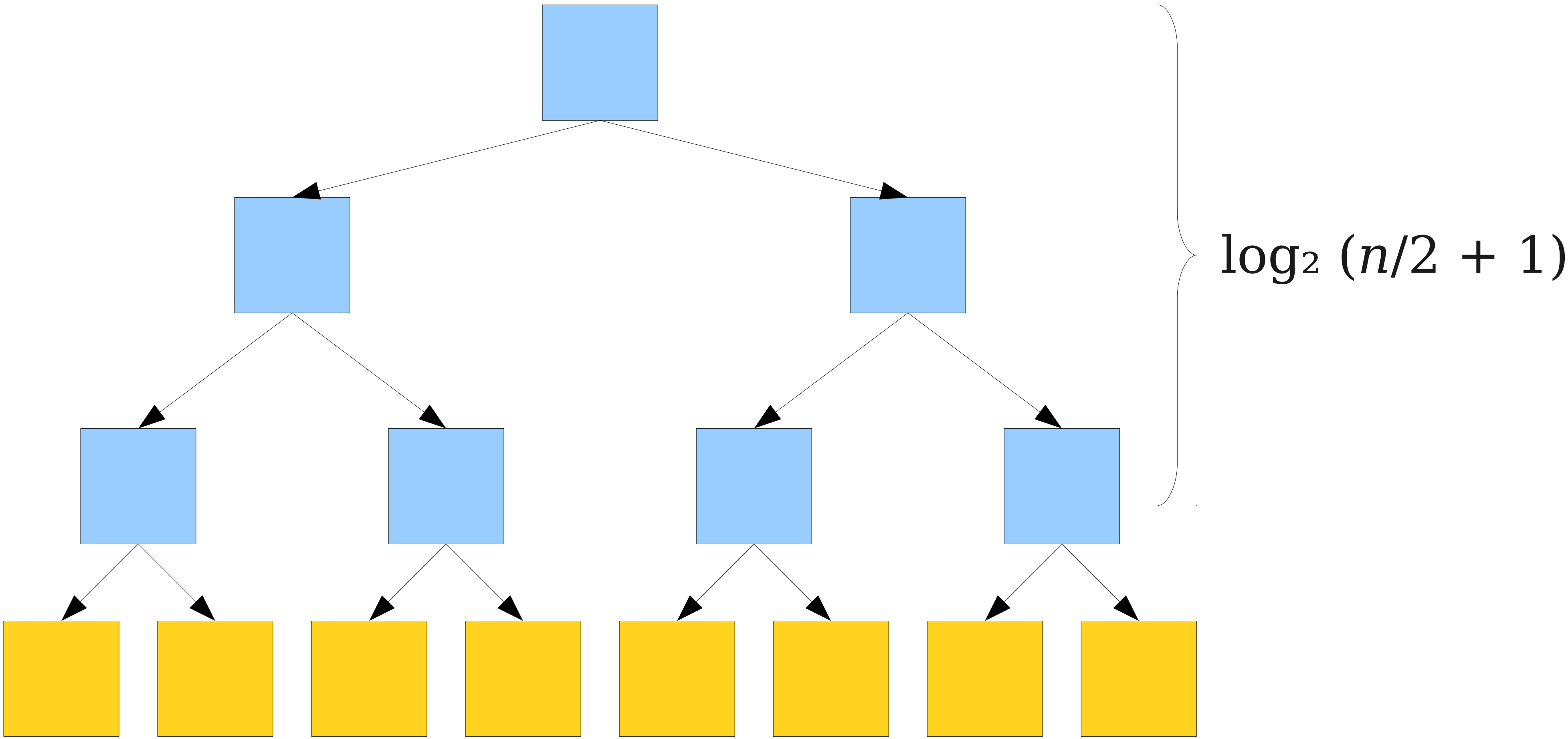
- The **heapsort** algorithm is as follows:
  - Build a max-heap from the array elements, using the array itself to represent the heap.
  - Repeatedly dequeue from the heap until all elements are placed in sorted order.
- This algorithm runs in time O(n log n), since it does n enqueues and n dequeues.
- Only requires O(1) auxiliary storage space, compared with O(n) space required in mergesort.

An Optimization: **Heapify**

Making a Binary Heap

- Suppose that you have *n* elements and want to build a binary heap from them.
- One way to do this is to enqueue all of them, one after another, into the binary heap.
- We can upper-bound the runtime as *n* calls to an O(log *n*) operation, giving a total runtime of O(*n* log *n*).
- Is that a tight bound?

Making a Binary Heap



Total Runtime: $\Theta(n \log n)$

Quickly Making a Binary Heap

- Here is a slightly different algorithm for building a binary heap out of a set of data:
  - Put the nodes, in any order, into a complete binary tree of the right size. (Shape property holds, but heap property might not.)
  - For each node, starting at the bottom layer and going upward, run a bubble-down step on that node.

Analyzing the Runtime

- At most half of the elements start one layer above that and can move down at most once.
- At most a quarter of the elements start one layer above that and can move down at most twice.
- At most an eighth of the elements start two layers above that and can move down at most thrice.
- More generally: At most n / 2^k of the elements can move down k steps.
- Can upper-bound the runtime with the sum

$$\mathrm{T}(n) \leq \sum_{i=0}^{\lceil \log_2 n \rceil} \frac{ni}{2^i} = n \sum_{i=0}^{\lceil \log_2 n \rceil} \frac{i}{2^i}$$

Simplifying the Summation

- We want to simplify the sum

$$\sum_{i=0}^{\lceil \log_2 n \rceil} \frac{i}{2^i}$$

- Let's introduce a new variable x, then evaluate the sum when $x = ½$:

$$\sum_{i=0}^{\lceil \log_2 n \rceil} i x^i$$

- If $x < 1$, each term is less than the previous, so

$$\sum_{i=0}^{\lceil \log_2 n \rceil} i x^i < \sum_{i=0}^{\infty} i x^i$$

Solving the Summation

$$
\begin{aligned}
\sum_{i=0}^{\infty} i\,x^{i} &= x\sum_{i=0}^{\infty} i\,x^{i-1} \\
&= x\sum_{i=0}^{\infty} \frac{d}{dx}\,x^{i} \\
&= x\,\frac{d}{dx}\left(\sum_{i=0}^{\infty} x^{i}\right) \\
&= x\,\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= x\,\frac{1}{(1-x)^{2}} \\
&= \frac{x}{(1-x)^{2}}
\end{aligned}
$$

The Finishing Touches

- We know know that

$$T(n) \leq n \sum_{i=0}^{\lceil \log_2 n \rceil} i x^i < n \sum_{i=0}^{\infty} i x^i = \frac{nx}{(1-x)^2}$$

- Evaluating at x = ½, we get

$$T(n) \leq \frac{n(1/2)}{(1-(1/2))^2} = \frac{n(1/2)}{(1/2)^2} = 2n$$

- So at most 2n swaps are performed!
- We visit each node once and do at most O(n) swaps, so the runtime is **Θ(*n*)**.