Divide-and-Conquer Algorithms

Part Three

Announcements

- Problem Set One graded; will be returned at the end of lecture.
  - If you submitted by email, let us know if you don't hear back by 5PM today.
  - If you submitted through the SCPD office, we'll return your problem set through the SCPD office.
- Handout: "Mathematical Terms and Identities."
  - Covers useful mathematical definitions, terms, and identities that we'll be using over the rest of the quarter.
  - Let us know if there's anything you'd like us to add for future quarters!

Outline for Today

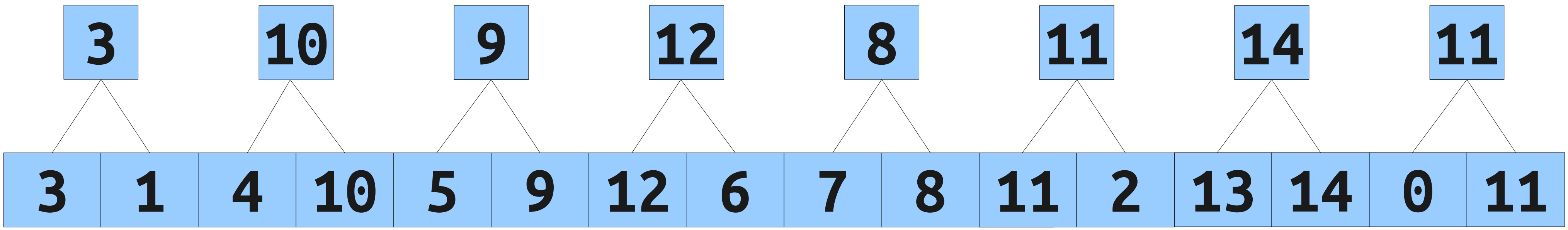
- **The Master Theorem**
  - A powerful tool for solving recurrences.
- **Applications of the Master Theorem**
  - Rapidly solving a variety of recurrence relations!

One More Recurrence Relation

Finding the Maximum Value

3	1	4	10	5	9	12	6	7	8	11	2	13	14	0	11

Finding the Maximum Value



Finding the Maximum Value

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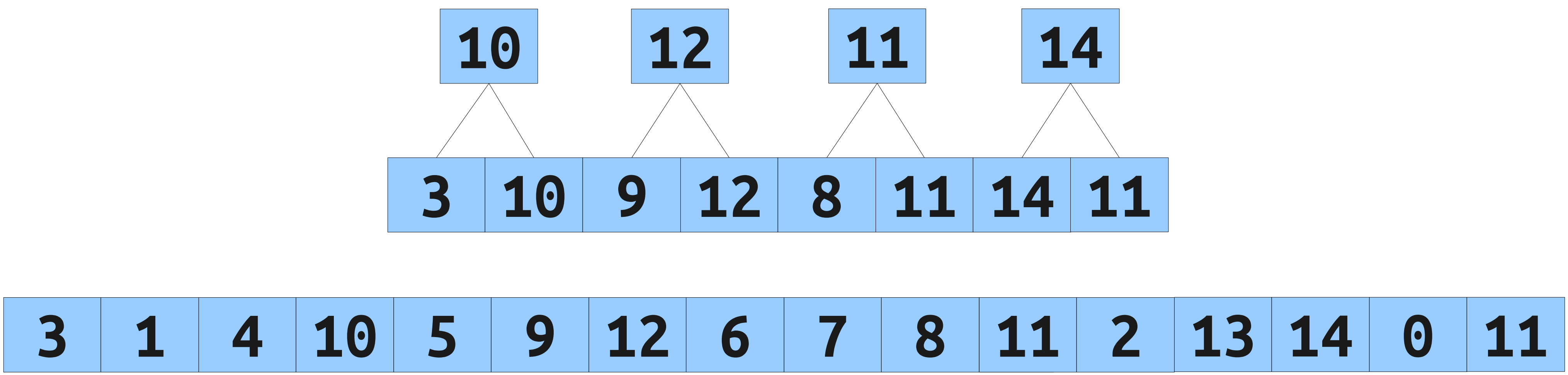
Finding the Maximum Value



Finding the Maximum Value

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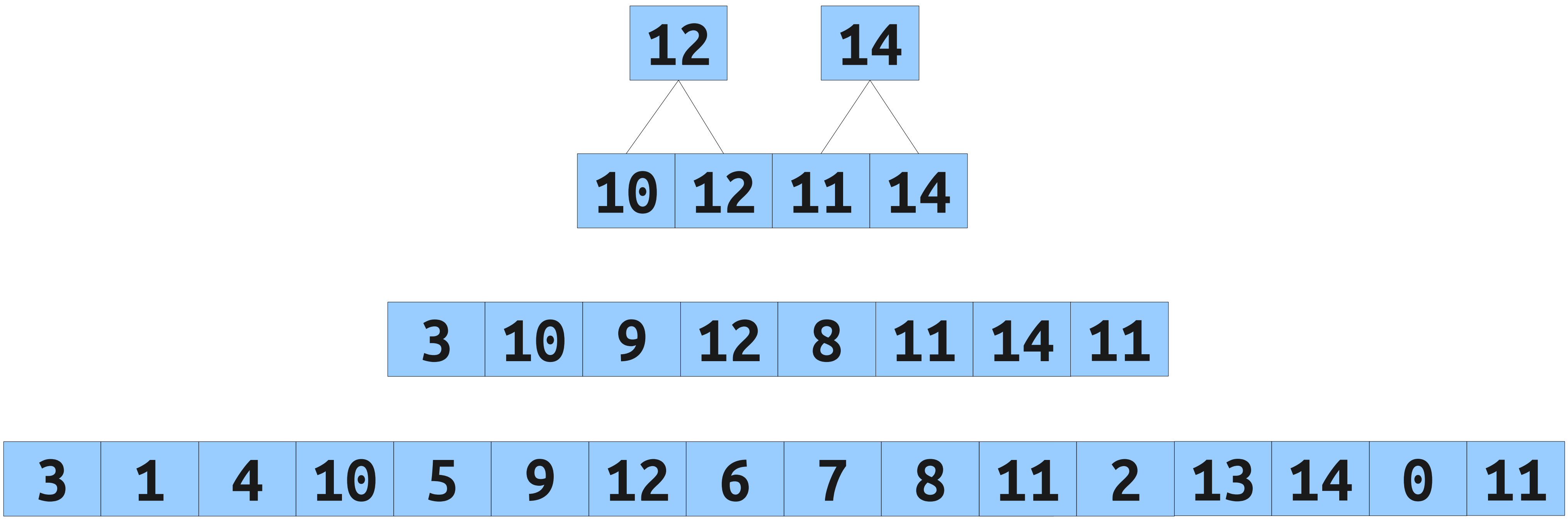
Finding the Maximum Value



Finding the Maximum Value

12	14

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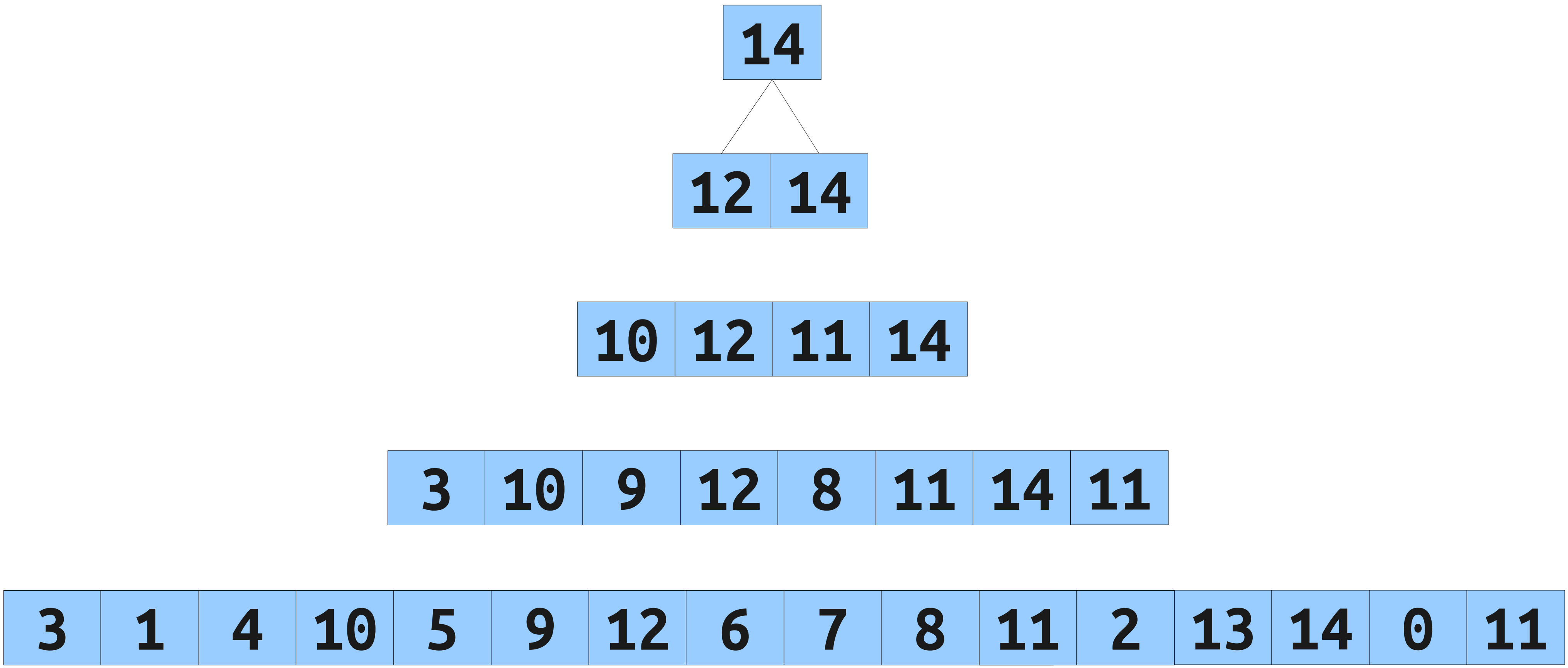

Finding the Maximum Value

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$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + \Theta(n)$$

Finding the Maximum Value

14

12	14

10	12	11	14

3	10	9	12	8	11	14	11

3	1	4	10	5	9	12	6	7	8	11	2	13	14	0	11

$T(1) \leq c$
$T(n) \leq T(\lceil n / 2 \rceil) + cn$

Finding the Maximum Value

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$$cn + cn / 2 + \ldots + c$$

$$T(1) \leq c$$
$$T(n) \leq T(n / 2) + cn$$

Finding the Maximum Value

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$T(1) \leq c$
$T(n) \leq T(n / 2) + cn$

$$cn + cn / 2 + \ldots + c$$
$$= cn\,(1 + \tfrac{1}{2} + \ldots + {}^{1}/_{n})$$

Finding the Maximum Value

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3	10	9	12	8	11	14	11

3	1	4	10	5	9	12	6	7	8	11	2	13	14	0	11

$$T(1) \leq c$$
$$T(n) \leq T(n / 2) + cn$$

$$\begin{aligned} & cn + cn / 2 + \dots + c \\ = \ & cn\,(1 + \tfrac{1}{2} + \dots + {}^{1}/_{n}) \\ \leq \ & cn\,(1 + \tfrac{1}{2} + \tfrac{1}{4} + \dots) \end{aligned}$$

Finding the Maximum Value

14

12	14

10	12	11	14

3	10	9	12	8	11	14	11

3	1	4	10	5	9	12	6	7	8	11	2	13	14	0	11

$$\begin{aligned} & cn + cn / 2 + \ldots + c \\ = \; & cn \left(1 + \tfrac{1}{2} + \ldots + {}^{1}/_{n}\right) \\ \leq \; & cn \left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots\right) \\ = \; & 2cn \end{aligned}$$

$$\begin{aligned} & T(1) \leq c \\ & T(n) \leq T(n / 2) + cn \end{aligned}$$

Finding the Maximum Value

14

12	14

10	12	11	14

3	10	9	12	8	11	14	11

3	1	4	10	5	9	12	6	7	8	11	2	13	14	0	11

$$\mathrm{T}(1) \le c$$
$$\mathrm{T}(n) \le \mathrm{T}(n / 2) + cn$$

$$\begin{aligned} & cn + cn / 2 + \ldots + c \\ = {} & cn \left(1 + \tfrac{1}{2} + \ldots + {}^{1}/_{n}\right) \\ \le {} & cn \left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots\right) \\ = {} & 2cn = \mathbf{O(}\boldsymbol{n}\mathbf{)} \end{aligned}$$

Three Recurrences

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Solves to $O(n \log n)$

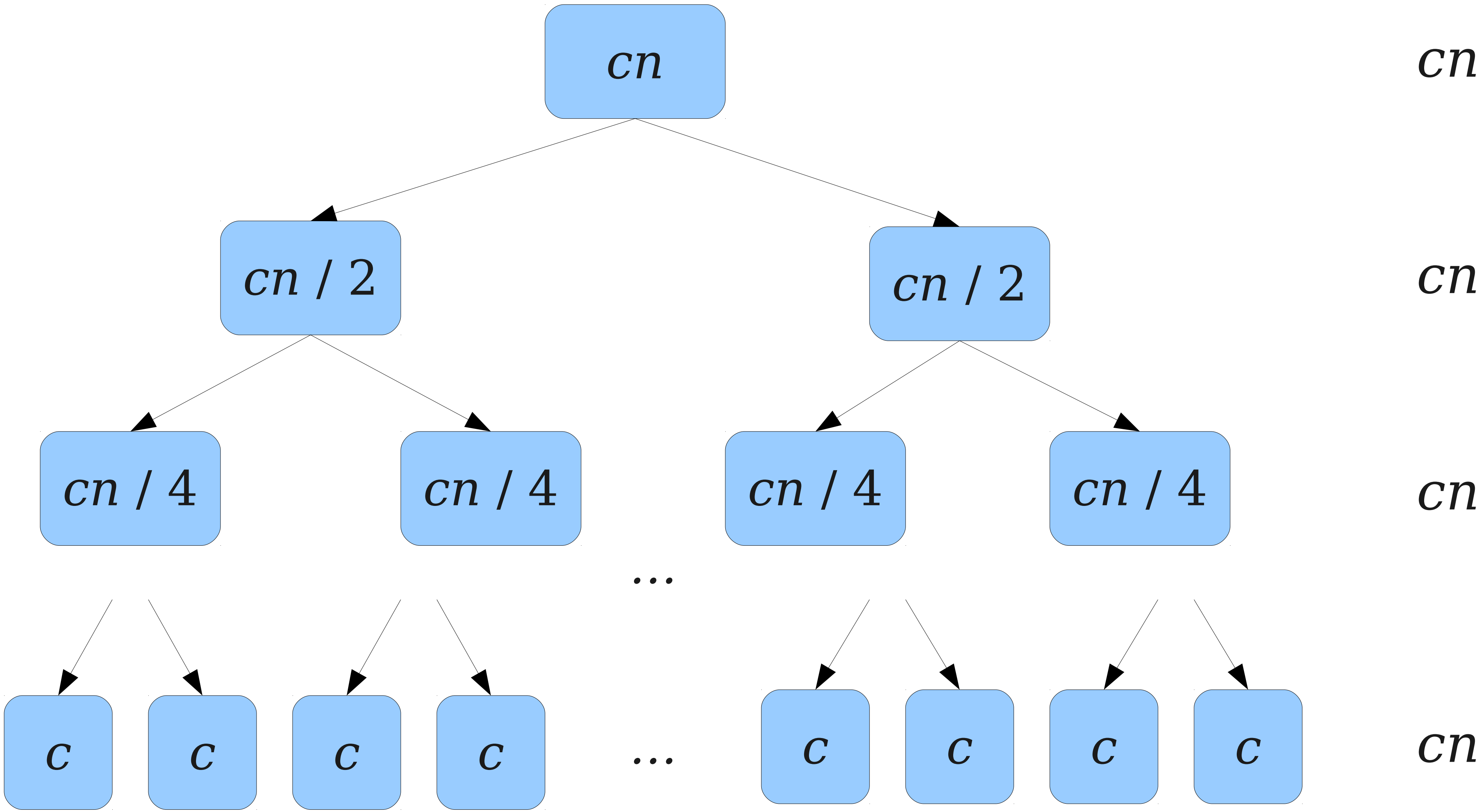
$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(1)$$

Solves to $O(n)$

$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + \Theta(n)$$

Solves to $O(n)$

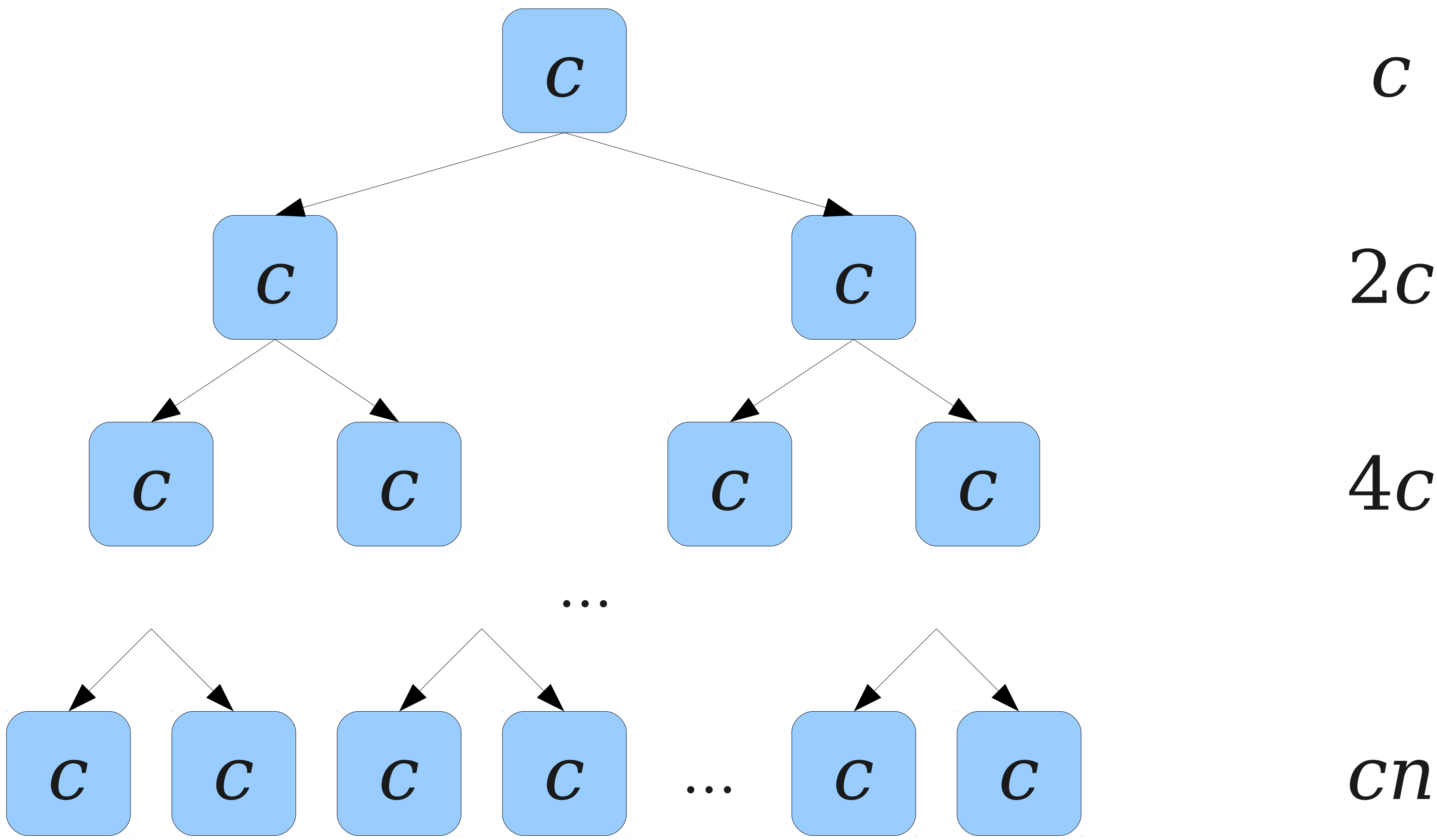
$T(1) \leq c$
$T(n) \leq 2T(n / 2) + cn$



$O(n \log n)$

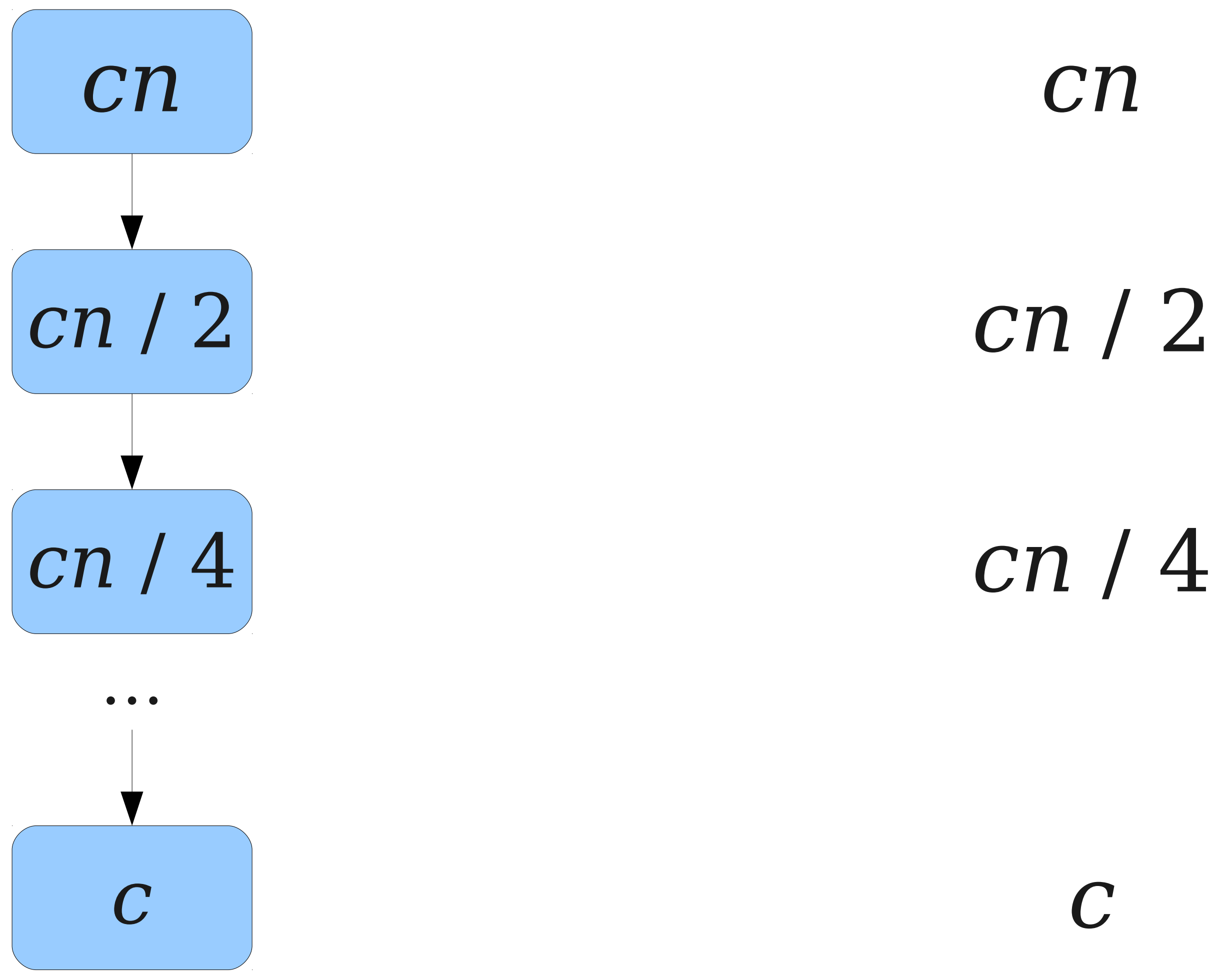
$T(1) \leq c$

$T(n) \leq 2T(n / 2) + c$



$$O(n)$$

$$T(1) \le c$$
$$T(n) \le T(n / 2) + cn$$



$$O(n)$$

Categorizing Recurrences

- The recurrences we have seen so far can be categorized into three groups:
  - **Topheavy recurrences**, where the majority of the runtime is dominating by the initial call.
    - Runtime is dominated by initial call.
  - **Balanced recurrences**, where each level in the tree does the same amount of work.
    - Runtime is determined by number of layers times the work per layer.
  - **Bottomheavy recurrences**, where the majority of the runtime is accounted for in the leaves.
    - Runtime is dominated by the work per leaf times the number of leaves.

The Master Theorem

- The **Master Theorem** (given on the next slide) is a theorem for asymptotically bounding recurrences of the type we've seen so far.
- Intuitively, categorizes recurrences into one of the three groups just mentioned, then determines the runtime based on that category.

The Master Theorem

Theorem: Let T(n) be defined as follows:

$$\boxed{\begin{array}{l} T(1) \leq \Theta(1) \\ T(n) \leq aT(\lceil n / b \rceil) + O(n^d) \end{array}}$$

Then

$$T(n) = \begin{cases} O\left(n^d\right) & \text{if } \log_b a < d \\ O\left(n^d \log n\right) & \text{if } \log_b a = d \\ O\left(n^{\log_b a}\right) & \text{if } \log_b a > d \end{cases}$$

Solving Existing Recurrences

- Consider the mergesort recurrence

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Solving Existing Recurrences

- Consider the mergesort recurrence

$$T(0) = \Theta(1)$$
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- What are a, b, and d?

Solving Existing Recurrences

- Consider the mergesort recurrence

$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(n) \leq 2T(\lceil n / 2 \rceil) + \Theta(n) \end{aligned}}$$

- What are a, b, and d? ***a* = 2, *b* = 2, *d* = 1**.

Solving Existing Recurrences

- Consider the mergesort recurrence

$$\boxed{\begin{array}{l} T(0) = \Theta(1) \\ T(1) = \Theta(1) \\ T(n) \leq 2T(\lceil n / 2 \rceil) + \Theta(n) \end{array}}$$

- What are a, b, and d? ***a* = 2, *b* = 2, *d* = 1**.
- What is $\log_b a$?

Solving Existing Recurrences

- Consider the mergesort recurrence

$$T(0) = \Theta(1)$$
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- What are a, b, and d? $\boldsymbol{a = 2, b = 2, d = 1}$.
- What is $\log_b a$? **1**

Solving Existing Recurrences

- Consider the mergesort recurrence

$$\boxed{\begin{array}{l} \mathrm{T}(0) = \Theta(1) \\ \mathrm{T}(1) = \Theta(1) \\ \mathrm{T}(n) \le 2\mathrm{T}(\lceil n / 2 \rceil) + \Theta(n) \end{array}}$$

- What are a, b, and d? $\boldsymbol{a = 2, b = 2, d = 1}$.
- What is $\log_b a$? **1**
- By the Master Theorem, $\mathrm{T}(n) =$ $\mathbf{O(}\boldsymbol{n}\ \mathbf{log}\ \boldsymbol{n}\mathbf{)}$.

Solving Existing Recurrences

- Consider the weakly unimodal maximum recurrence:

$$\boxed{\begin{aligned} &T(1) \leq c \\ &T(n) \leq 2T(n / 2) + c \end{aligned}}$$

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- Consider the weakly unimodal maximum recurrence:

$$\boxed{\begin{aligned} &\mathrm{T}(1) \le c \\ &\mathrm{T}(n) \le 2\mathrm{T}(\lceil n / 2 \rceil) + c \end{aligned}}$$

- What are a, b, d? ***a* = 2, *b* = 2, *d* = 0**

Solving Existing Recurrences

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Solving Existing Recurrences

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Solving Existing Recurrences

- Consider the weakly unimodal maximum recurrence:

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(\lceil n / 2 \rceil) + c \end{array}}$$

- What are a, b, d? $\boldsymbol{a = 2, b = 2, d = 0}$
- What is $\log_b a$? **1**
- By the Master Theorem, $T(n) = \mathbf{O(\boldsymbol{n})}$

Solving Existing Recurrences

- Consider the recurrence for the code to find the maximum value in an array:

$$\boxed{\begin{array}{l} \mathrm{T}(1) \le c \\ \mathrm{T}(n) \le \mathrm{T}(\lceil n / 2 \rceil) + cn \end{array}}$$

Solving Existing Recurrences

- Consider the recurrence for the code to find the maximum value in an array:

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- What are a, b, d?

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- What is $\log_b a$? **0**

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- Consider the recurrence for the code to find the maximum value in an array:

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- What are a, b, d? $\boldsymbol{a = 1, b = 2, d = 1}$
- What is $\log_b a$? $\mathbf{0}$
- By the Master Theorem, $T(n) = \boldsymbol{O(n)}$

Proving the Master Theorem

- We can prove the Master Theorem by writing out a generic proof using a recursion tree.
  - Draw out the tree.
  - Determine the work per level.
  - Sum across all levels.
- The three cases of the Master Theorem correspond to whether the recurrence is topheavy, balanced, or bottomheavy.

Simplifying the Recurrence

- The recurrence given by the Master Theorem is shown here:

$$\boxed{\begin{aligned} &\mathrm{T}(1) \le \Theta(1) \\ &\mathrm{T}(n) \le a\mathrm{T}(\lceil n / b \rceil) + \mathrm{O}(n^d) \end{aligned}}$$

Simplifying the Recurrence

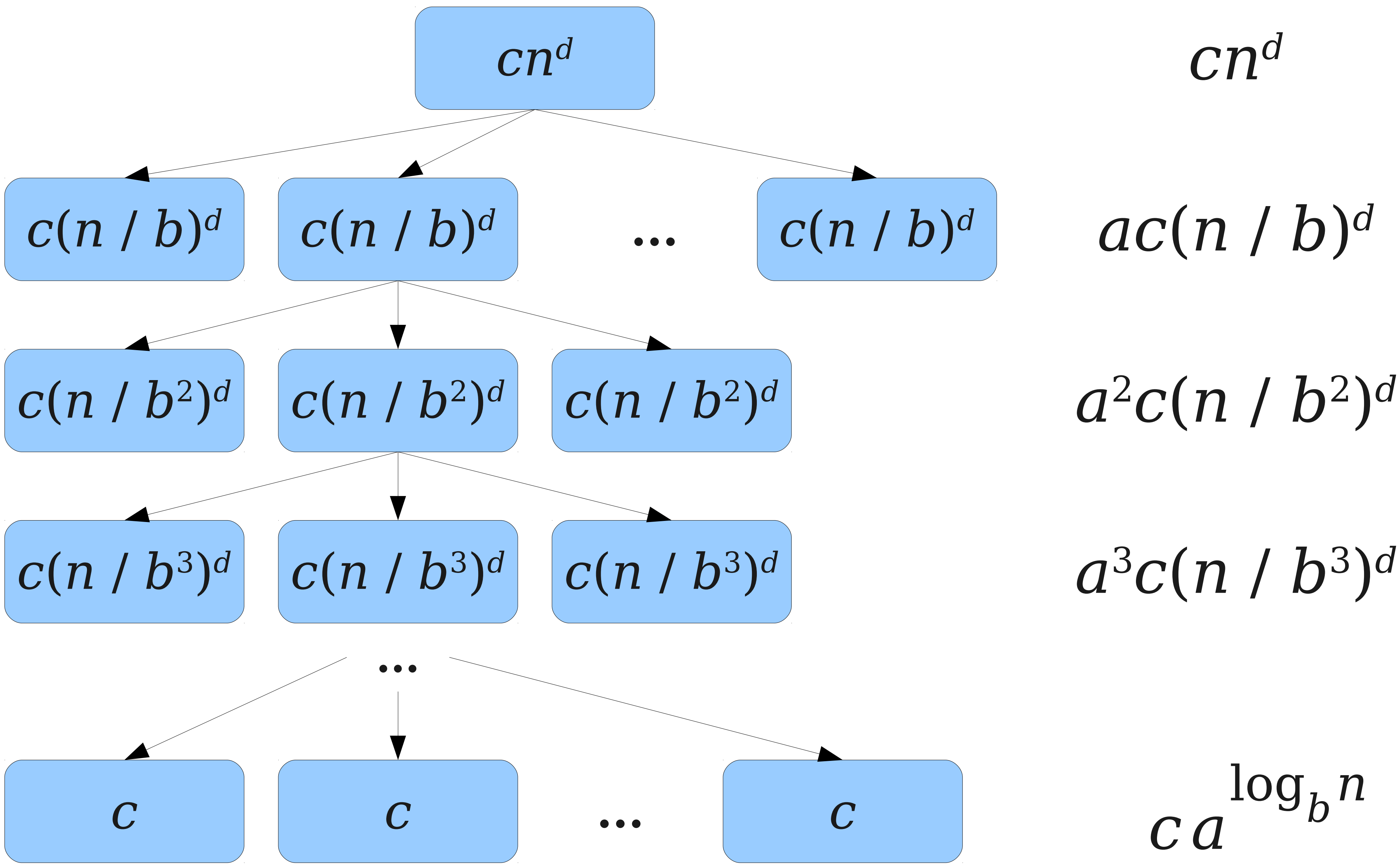
- The recurrence given by the Master Theorem is shown here:

$$\boxed{\begin{array}{l} T(1) \le \Theta(1) \\ T(n) \le a T(\lceil n / b \rceil) + O(n^d) \end{array}}$$

- We will apply our standard simplifications to this recurrence:
  - Assume inputs are powers of *b*.
  - Replace Θ and O with constant multiples.

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le a T(n / b) + c n^d \end{array}}$$

$$T(1) \le c$$
$$T(n) \le aT(n / b) + cn^d$$



Hairy Scary Math

- At internal level k of the tree, the work done is

$$a^k c(n / b^k)^d$$

Hairy Scary Math

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- Rearranging:

$$a^k c (n / b^k)^d = cn^d a^k / b^{dk}$$

Hairy Scary Math

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Hairy Scary Math

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- Rearranging:

$$\begin{aligned} a^k c (n / b^k)^d &= cn^d a^k / b^{dk} \\ &= cn^d (a / b^d)^k \end{aligned}$$

- Therefore:

$$\mathrm{T}(n) \leq c\, a^{\log_b n} + \sum_{k=0}^{\log_b n - 1} c\, n^d \left(\frac{a}{b^d}\right)^k$$

Hairy Scary Math

- At internal level k of the tree, the work done is

$$a^k c(n / b^k)^d$$

- Rearranging:

$$\begin{aligned} a^k\, c\, (n / b^k)^d &= cn^d\, a^k / b^{dk} \\ &= cn^d\, (a / b^d)^k \end{aligned}$$

- Therefore:

$$\begin{aligned} \mathrm{T}(n) &\leq c\,a^{\log_b n} + \sum_{k=0}^{\log_b n - 1} c\,n^d \left(\frac{a}{b^d}\right)^k \\ &= c\,a^{\log_b n} + c\,n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k \end{aligned}$$

Icky Tricky Math

- Let's see if we can simplify

$$\mathrm{T}(n) \quad \leq \quad c\,a^{\log_b n} + \sum_{k=0}^{c} n^d \log_b n - 1 \left(\frac{a}{b^d}\right)^k$$

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$$\mathrm{T}(n) \quad \leq \quad c\, a^{\log_b n} + \sum_{k=0}^{c} n^d \log_b n - 1 \left(\frac{a}{b^d}\right)^k$$

- Let's look at the first term. Note that

$$a^{\log_b n} \quad = \; \left(b^{\log_b a}\right)^{\log_b n}$$

Icky Tricky Math

- Let's see if we can simplify

$$T(n) \quad \leq \quad c\, a^{\log_b n} + \sum_{k=0}^{c} n^d \log_b n - 1 \left(\frac{a}{b^d}\right)^k$$

- Let's look at the first term. Note that

$$\begin{aligned} a^{\log_b n} \quad &= \left(b^{\log_b a}\right)^{\log_b n} \\ &= \quad b^{(\log_b a)(\log_b n)} \end{aligned}$$

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$$\mathrm{T}(n) \quad \le \quad c\,a^{\log_b n} + \sum_{k=0}^{c} n^d \log_b n - 1 \left(\frac{a}{b^d}\right)^k$$

- Let's look at the first term. Note that

$$\begin{aligned} a^{\log_b n} &= \left(b^{\log_b a}\right)^{\log_b n} \\ &= b^{(\log_b a)(\log_b n)} \\ &= \left(b^{\log_b n}\right)^{\log_b a} \\ &= n^{\log_b a} \end{aligned}$$

so $\quad \mathrm{T}(n) \le c\,n^{\log_b a} + c\,n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$

Frightening Enlightening Math

- All that's left to do now is to simplify

$$\mathrm{T}(n) \leq c\,n^{\log_b a} + c\,n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

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$$T(n) \leq c n^{\log_b a} + c n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

- **Case 1:** What if $a / b^d = 1$?

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- **Case 1:** What if $a / b^d = 1$?

$$a / b^d = 1$$
$$a = b^d$$

Frightening Enlightening Math

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$$\mathrm{T}(n) \leq c\,n^{\log_b a} + c\,n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

- **Case 1:** What if $a \,/\, b^d = 1$?

$$\boxed{\begin{gathered} a \,/\, b^d = 1 \\ a = b^d \\ \log_b a = d \end{gathered}}$$

Frightening Enlightening Math

- All that's left to do now is to simplify

$$T(n) \leq c n^{\log_b a} + c n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

- **Case 1:** What if $a / b^d = 1$? Then $\log_b a = d$

Frightening Enlightening Math

- All that's left to do now is to simplify

$$\mathrm{T}(n) \le cn^{\log_b a} + cn^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

- **Case 1:** What if $a / b^d = 1$? Then $\log_b a = d$, so

$$\mathrm{T}(n) \le cn^d + cn^d \sum_{k=0}^{\log_b n - 1} 1$$

Frightening Enlightening Math

- All that's left to do now is to simplify

$$T(n) \le cn^{\log_b a} + cn^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

- **Case 1:** What if $a / b^d = 1$? Then $\log_b a = d$, so

$$\begin{aligned} T(n) &\le cn^d + cn^d \sum_{k=0}^{\log_b n - 1} 1 \\ &= cn^d + cn^d \log_b n \end{aligned}$$

Frightening Enlightening Math

- All that's left to do now is to simplify

$$T(n) \le c n^{\log_b a} + c n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d} \right)^k$$

- **Case 1:** What if $a / b^d = 1$? Then $\log_b a = d$, so

$$\begin{aligned} T(n) &\le c n^d + c n^d \sum_{k=0}^{\log_b n - 1} 1 \\ &= c n^d + c n^d \log_b n \\ &= O(n^d \log n) \end{aligned}$$

Frightening Enlightening Math

- All that's left to do now is to simplify

$$\mathrm{T}(n) \leq cn^{\log_b a} + cn^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

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$$T(n) < cn^d + cn^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

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Frightening Enlightening Math

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$$T(n) \leq c\, n^{\log_b a} + c\, n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k$$

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$$\begin{aligned} T(n) &< c\, n^d + c\, n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k \\ &< c\, n^d + c\, n^d \sum_{k=0}^{\infty} \left(\frac{a}{b^d}\right)^k \\ &< c\, n^d \left(1 + \frac{1}{1 - a/b^d}\right) \\ &= O(n^d) \end{aligned}$$

Case 3: What if $a \,/\, b^d > 1$?

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$$\begin{aligned} \mathrm{T}(n) &\le c\,n^{\log_b a} + c\,n^d \sum_{k=0}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^k \\ &= c\,n^{\log_b a} + c\,n^d \frac{(a/b^d)^{\log_b n} - 1}{(a/b^d) - 1} \end{aligned}$$

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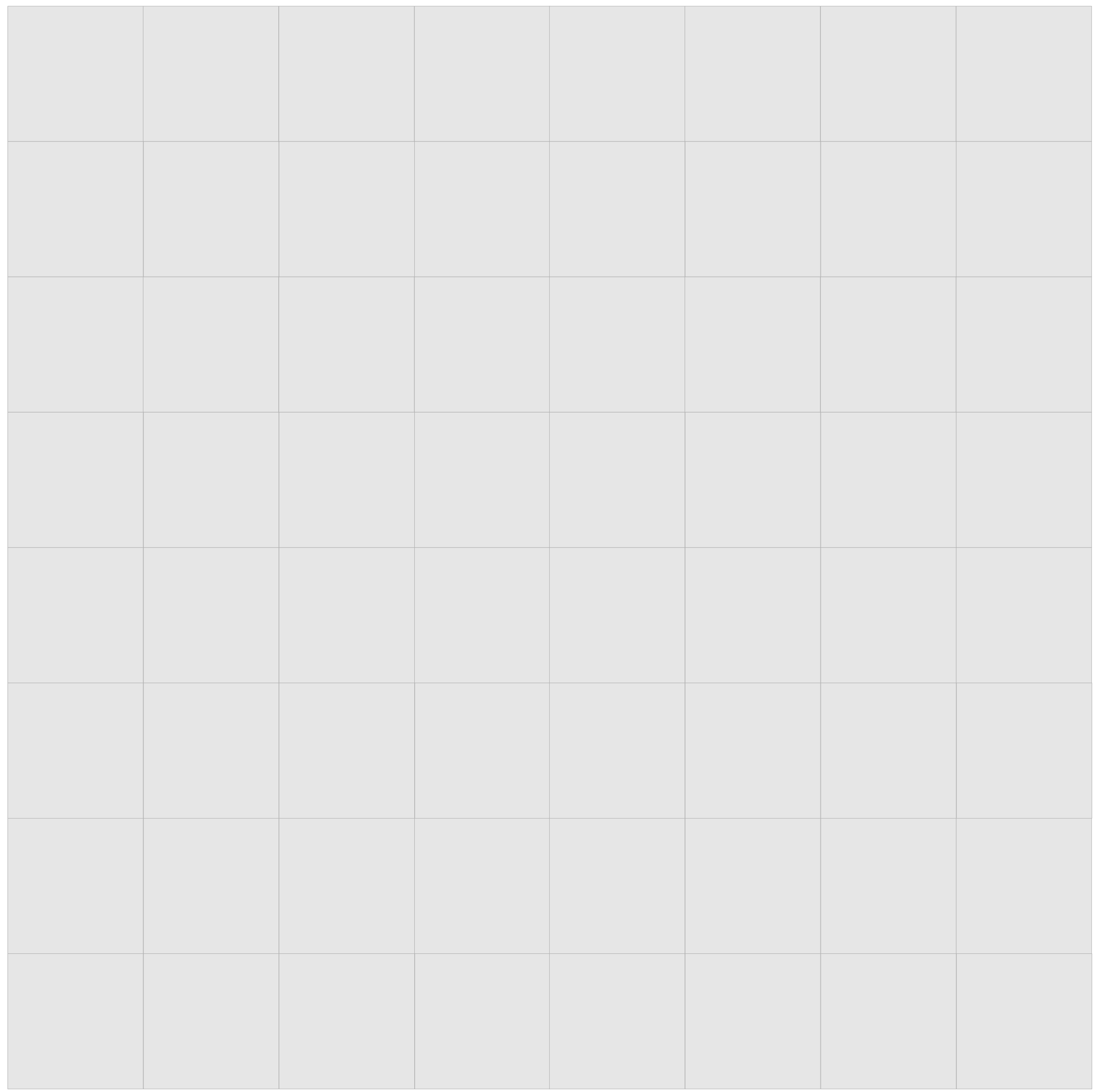
&= cn^{\log_b a} + cn^d (n^{\log_b a} / n^d) \Theta(1) \\
&= cn^{\log_b a} + cn^{\log_b a} \Theta(1) \\
&= \mathrm{O}(n^{\log_b a})
\end{aligned}
$$

Why the Master Theorem Matters

- The proof of the Master Theorem can be thought of as a single proof that works for all recurrences of the form handled by the theorem.
- From this point forward, we can just call back to the Master Theorem when applicable.
- Not all recurrences can be solved by the Master Theorem; more on that next time.

Applications of the Master Theorem:
A Sampler of Algorithms

Tiling with Triominoes



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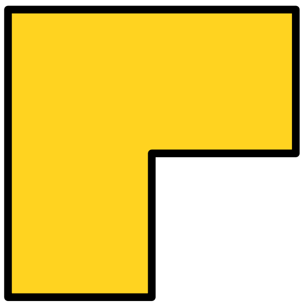

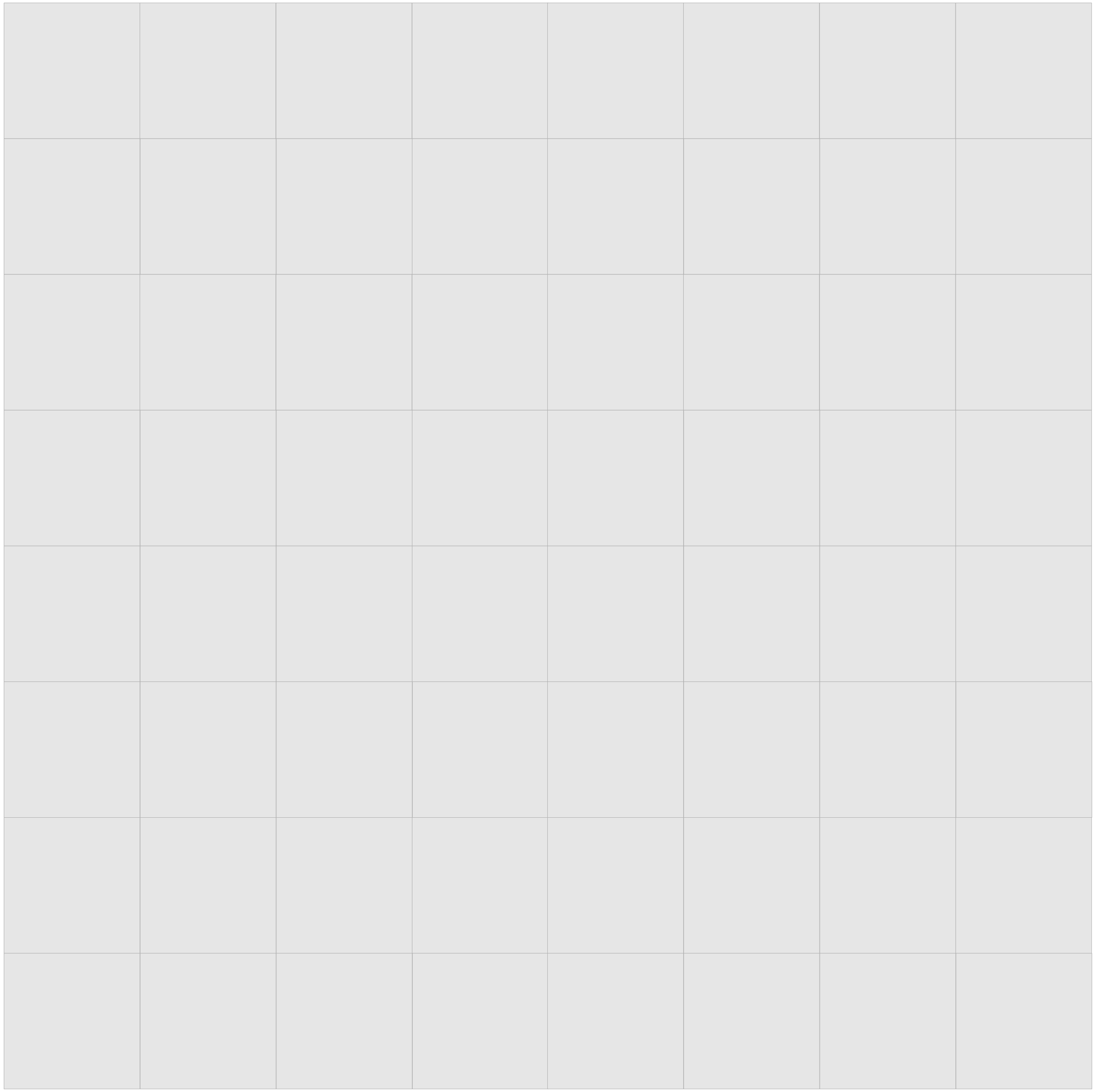

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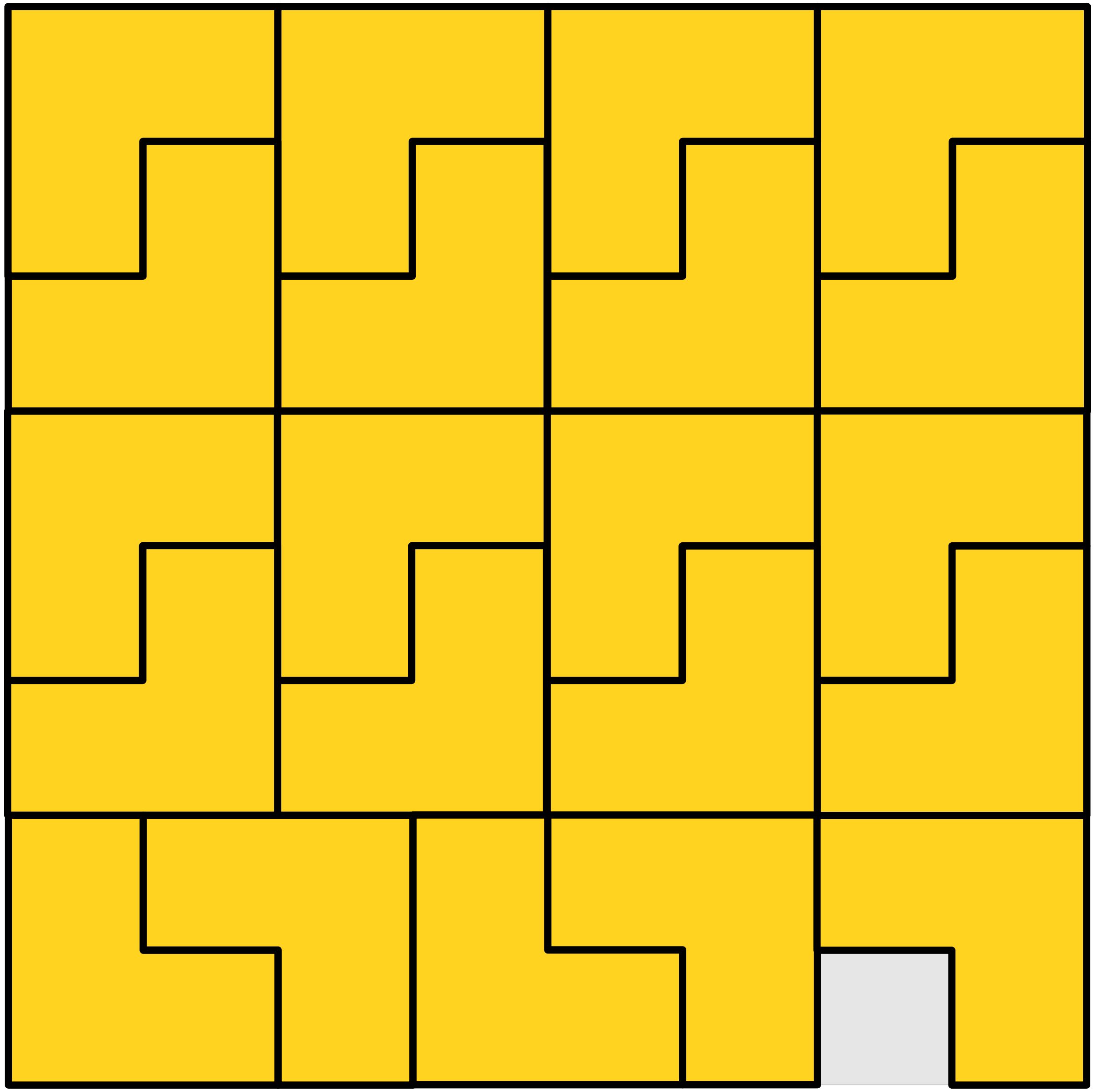

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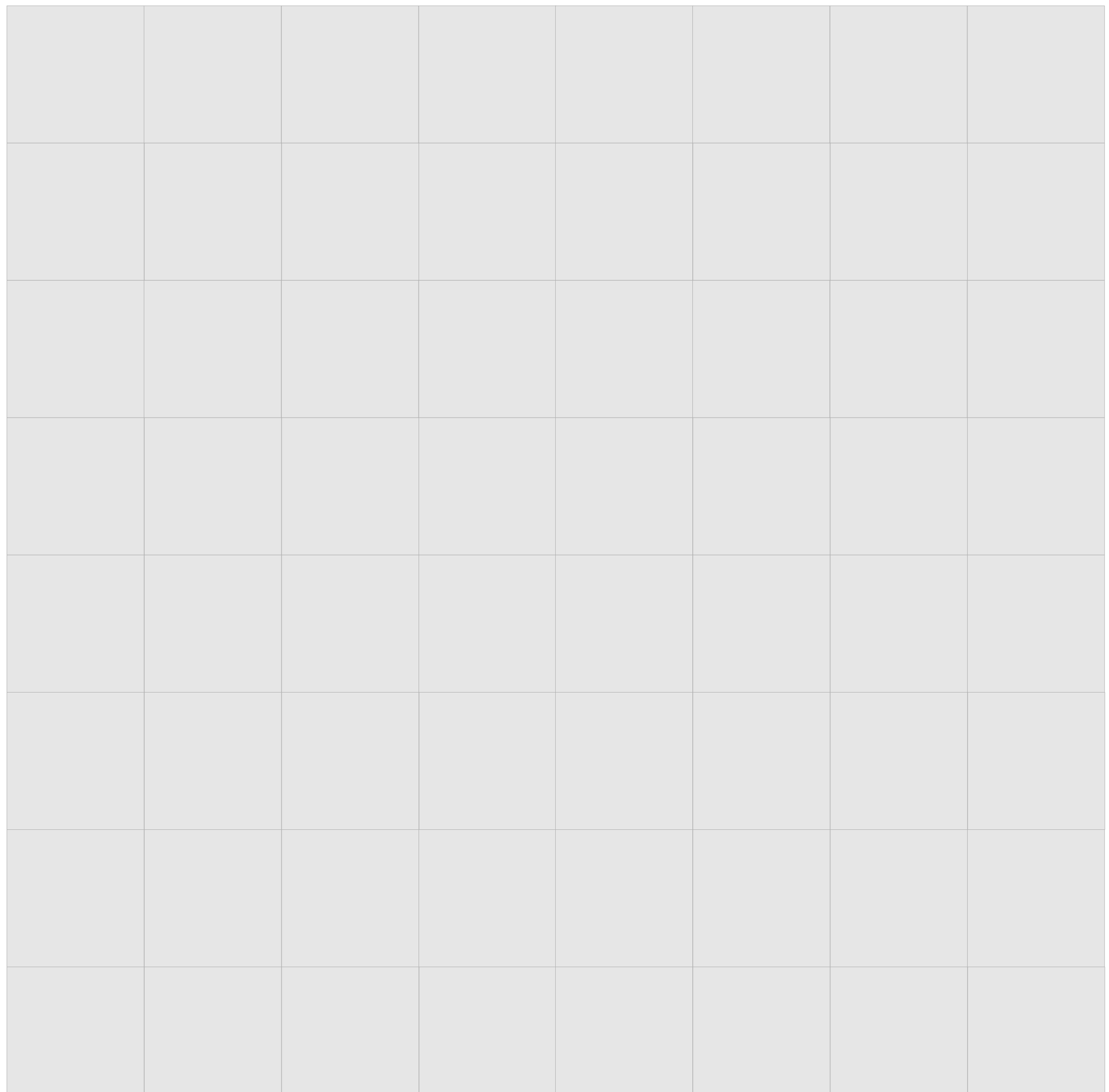

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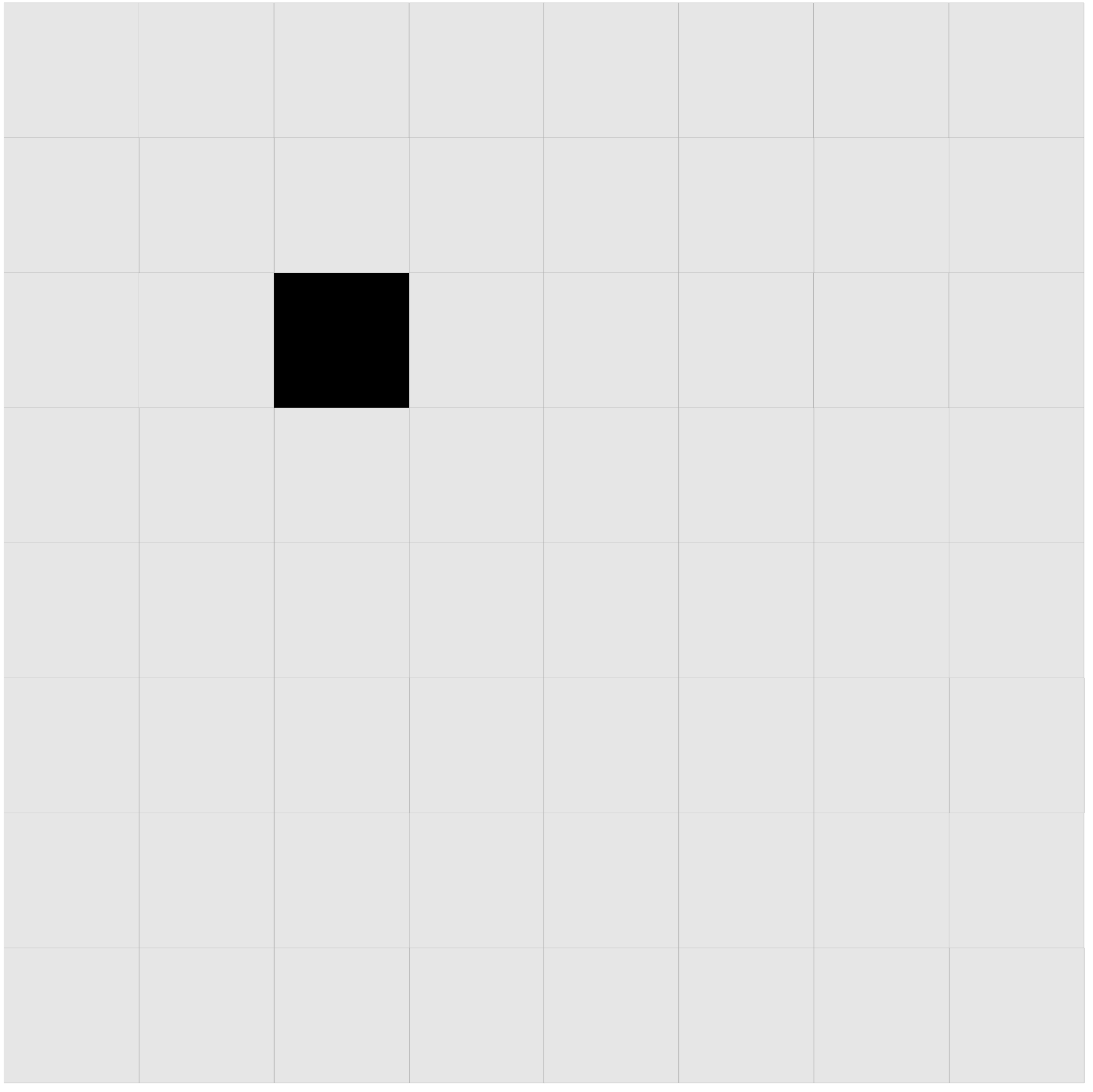

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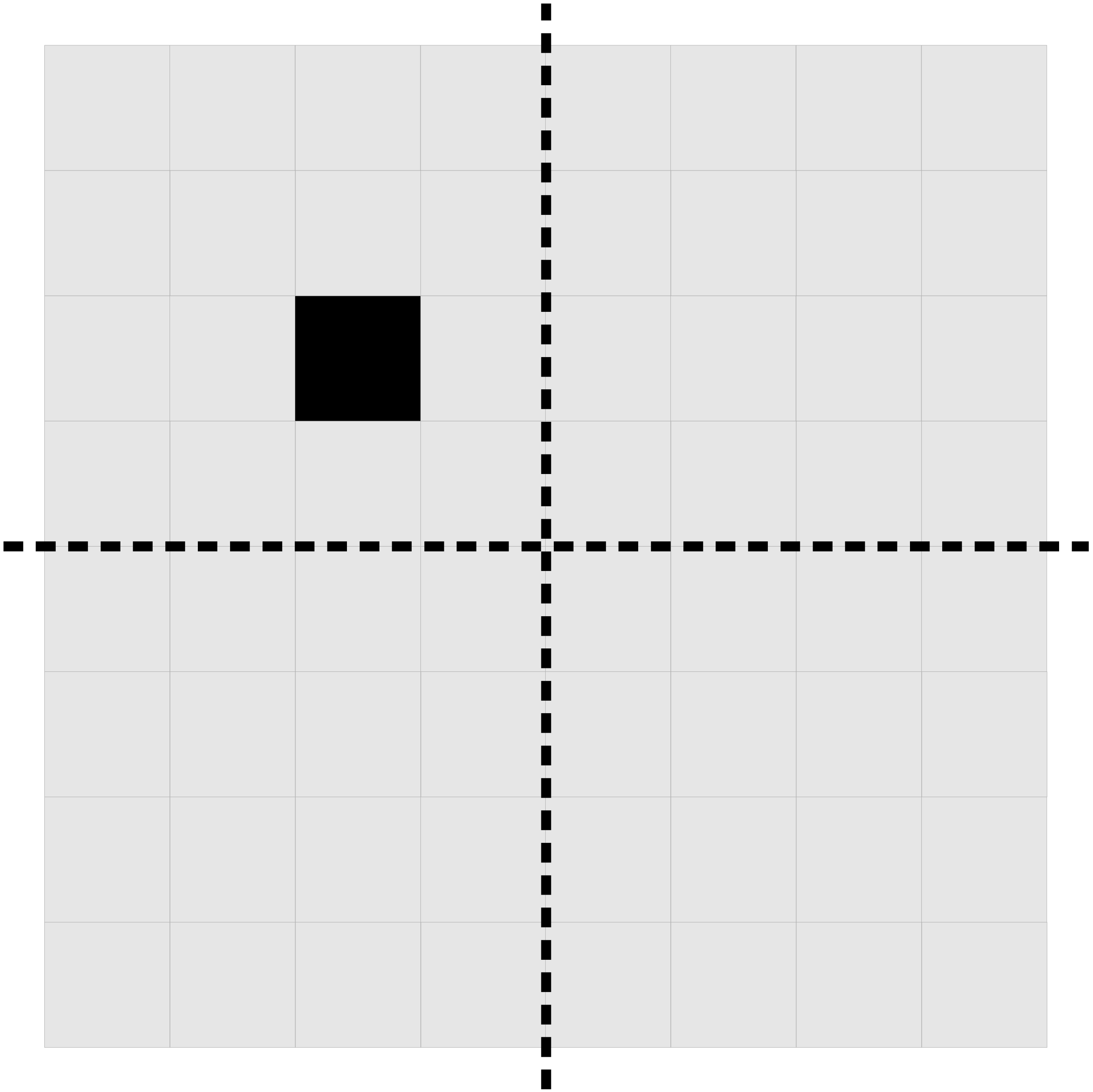

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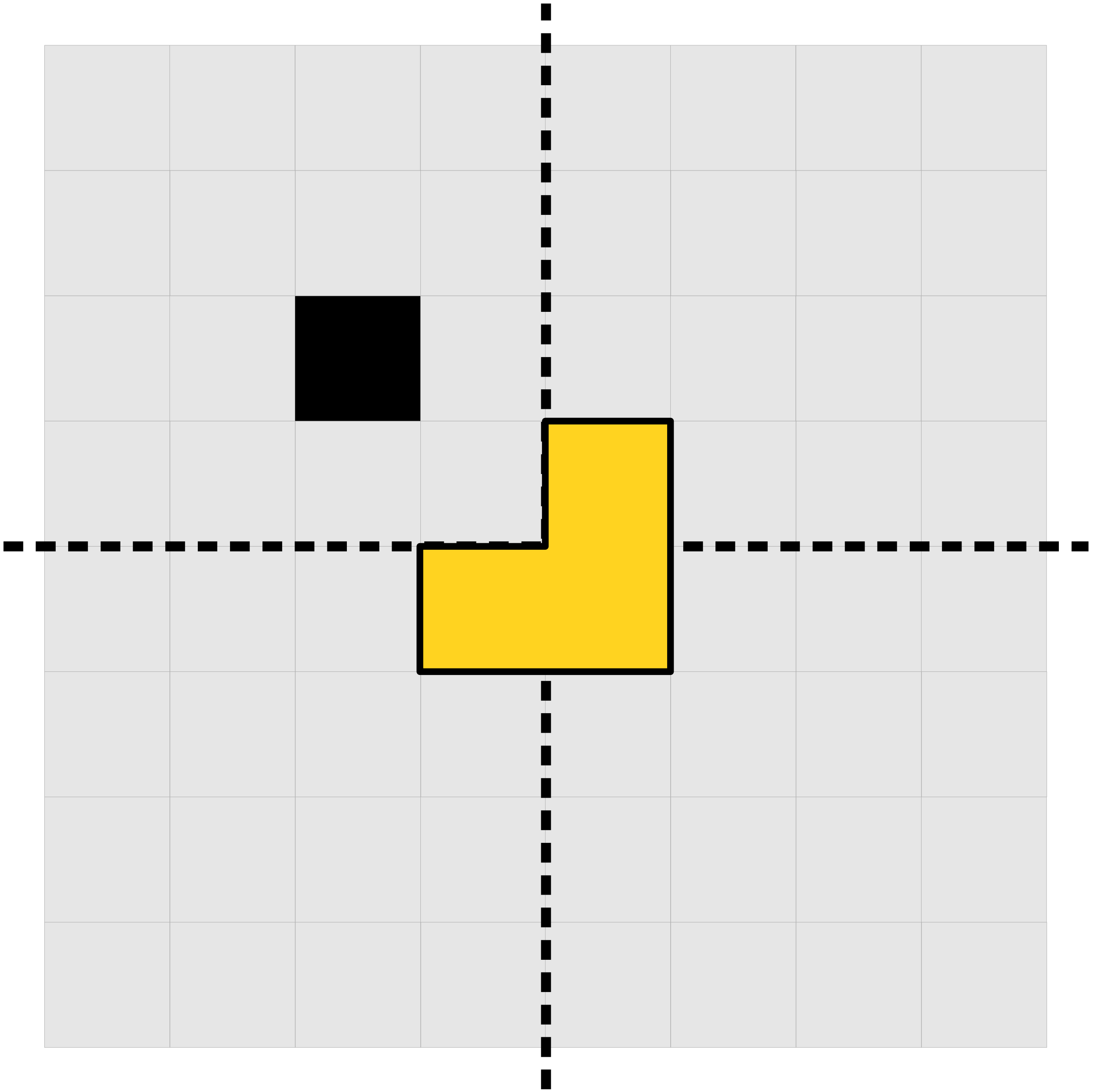

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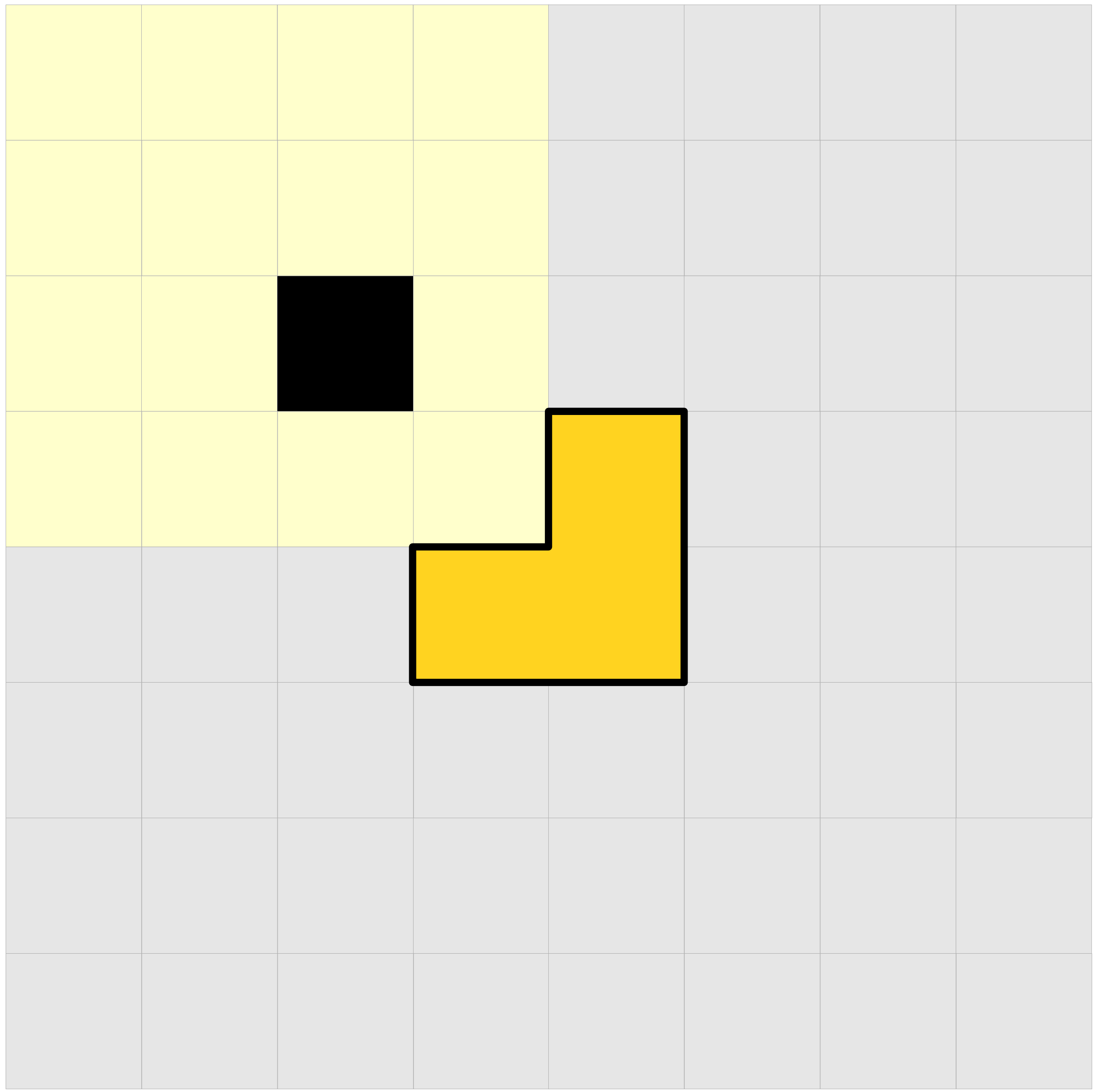

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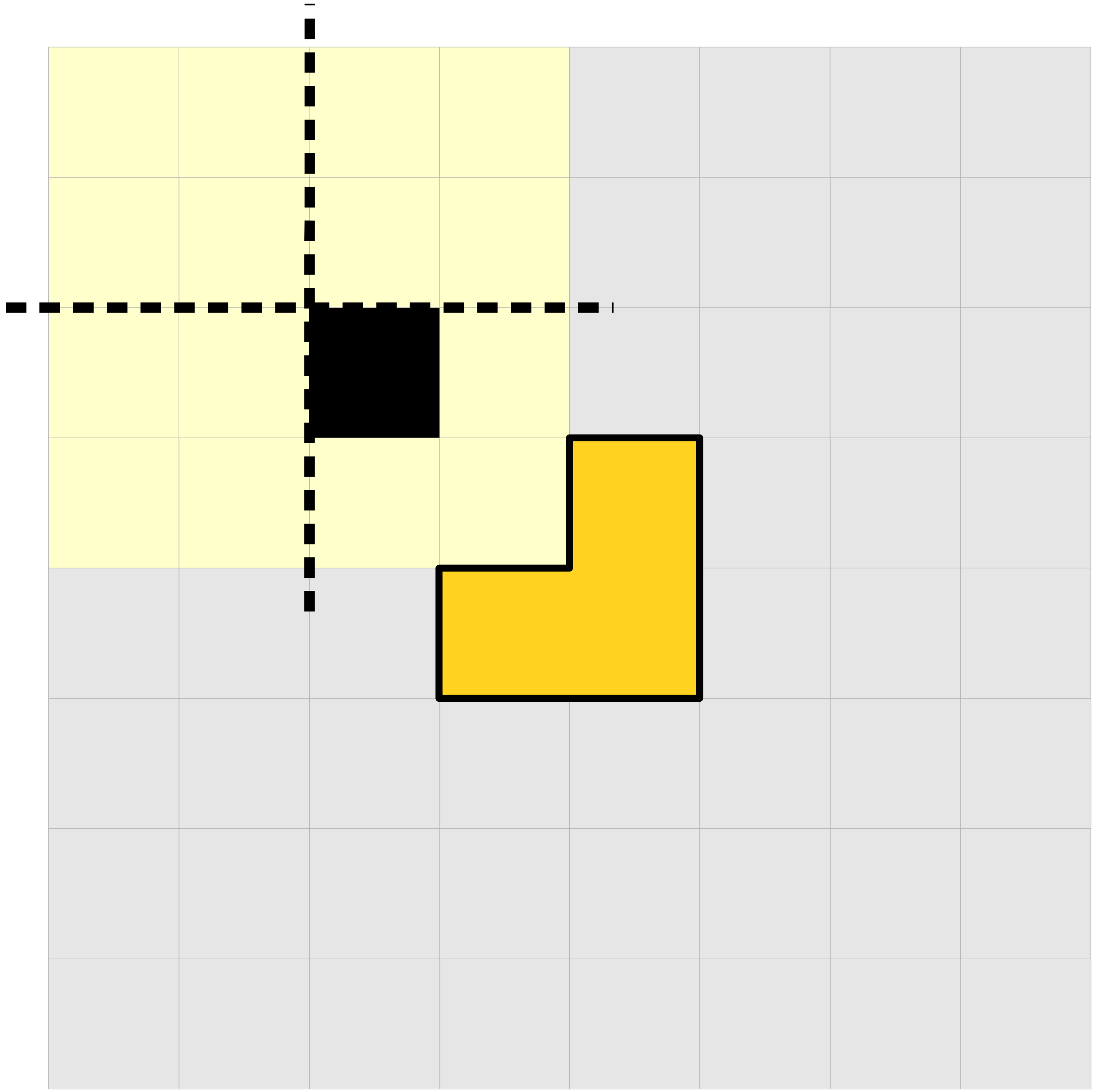

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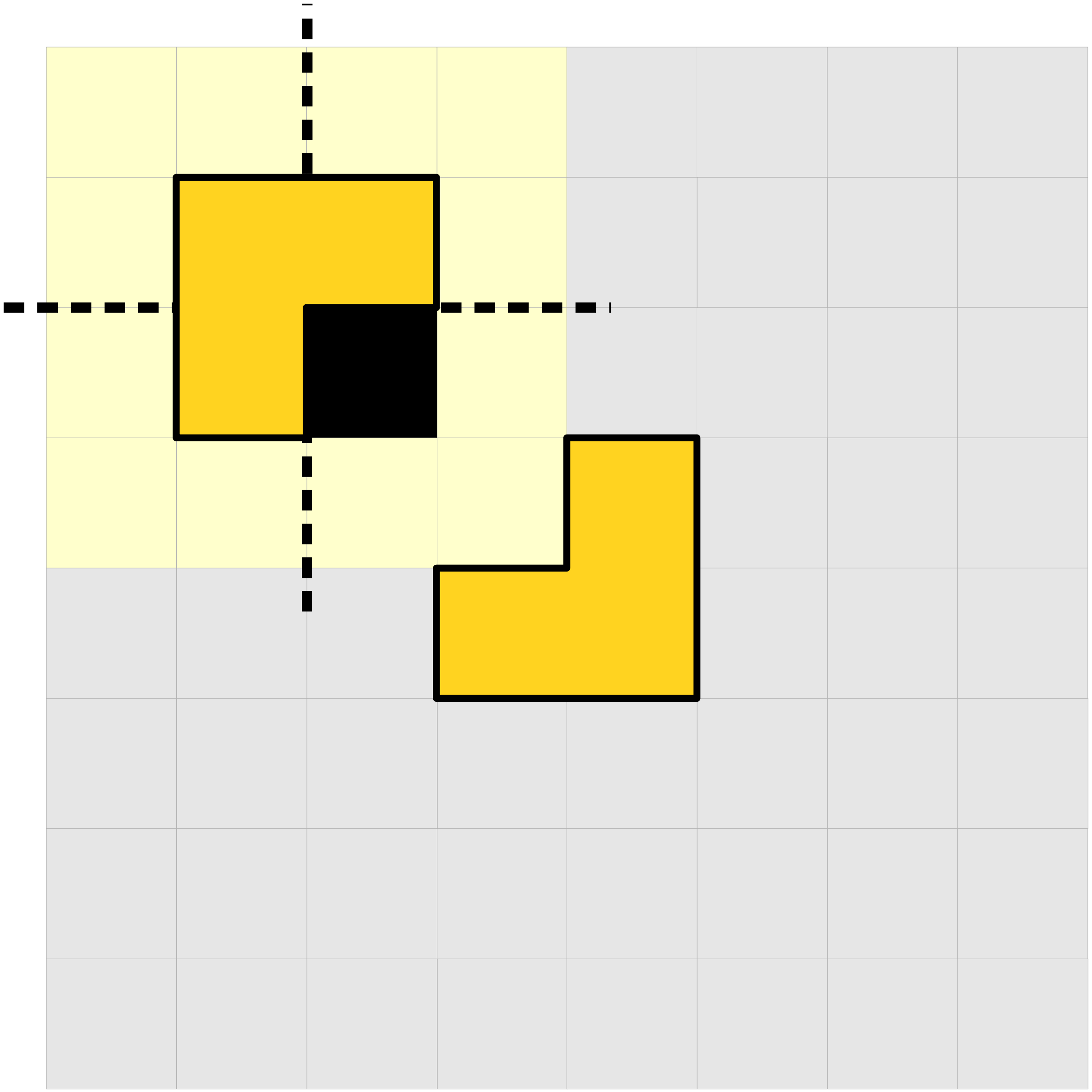

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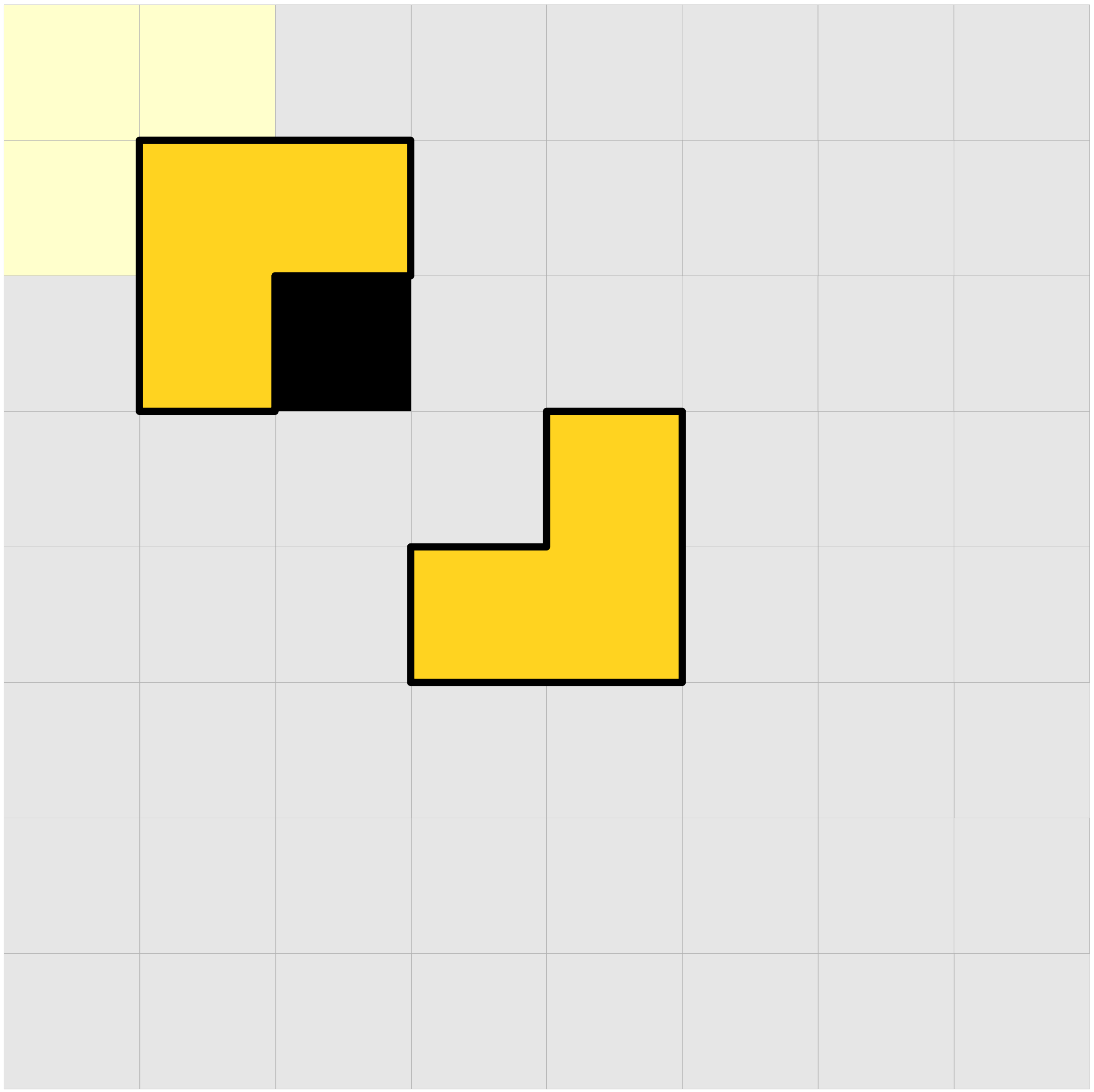

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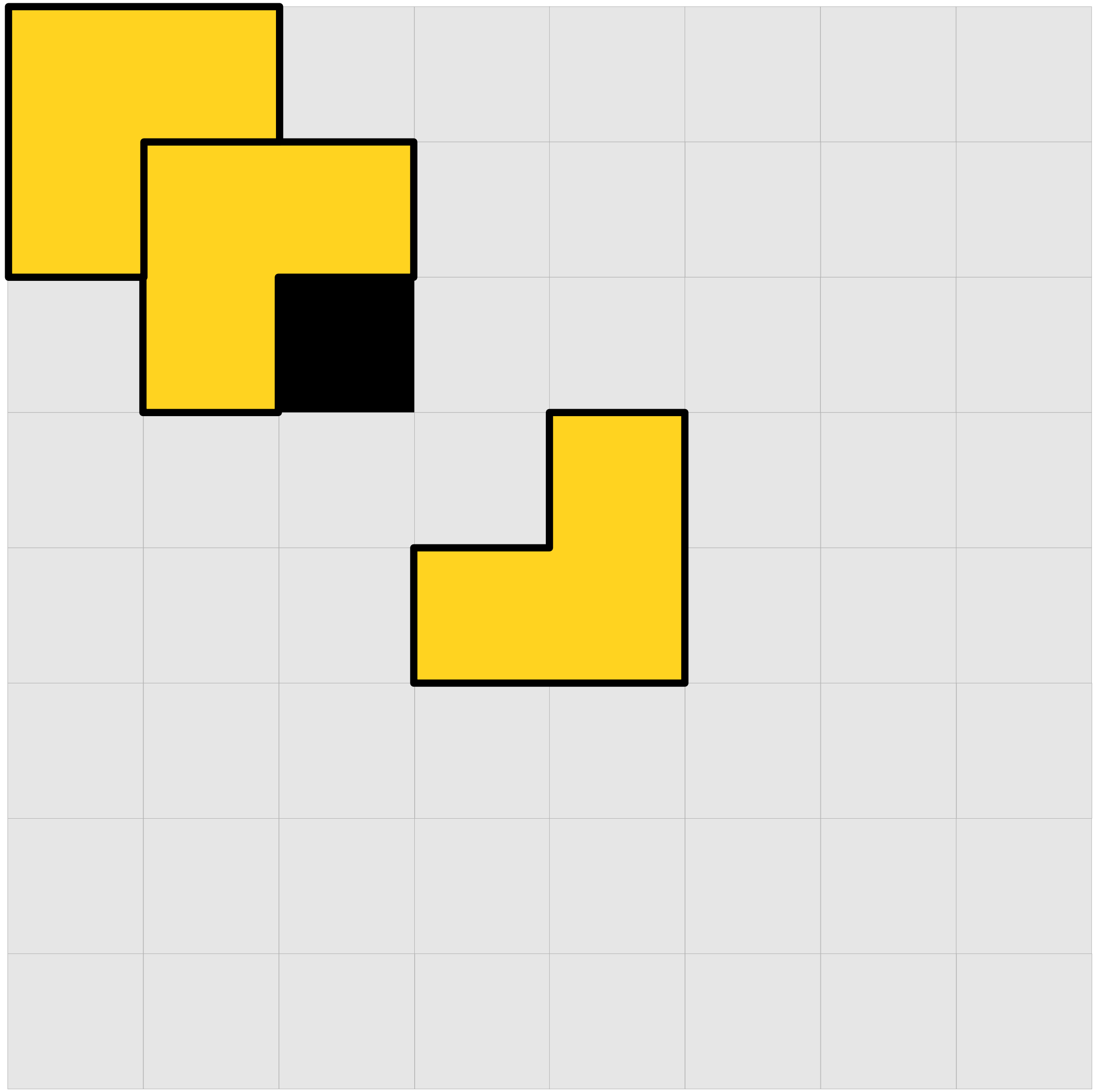

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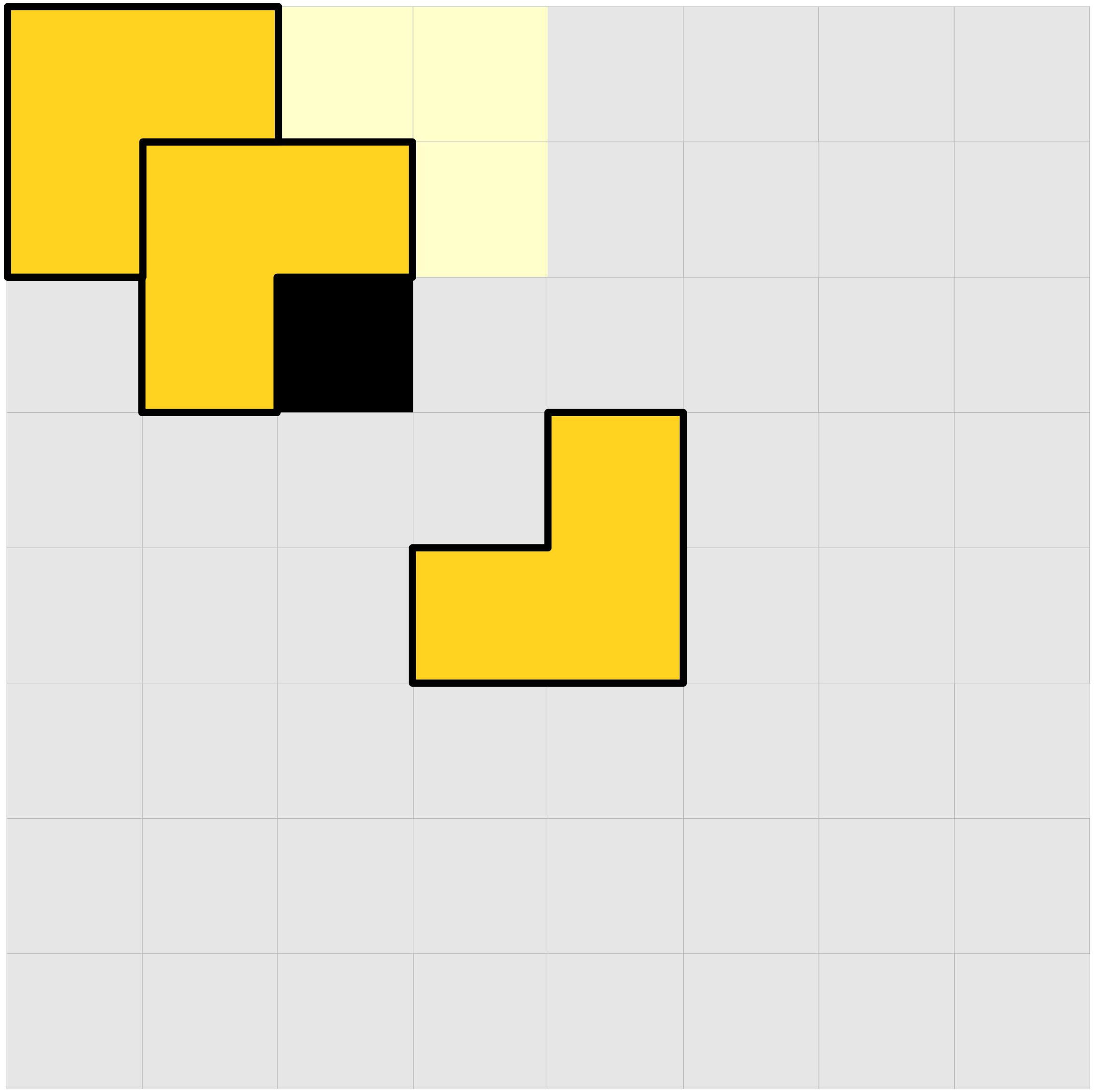

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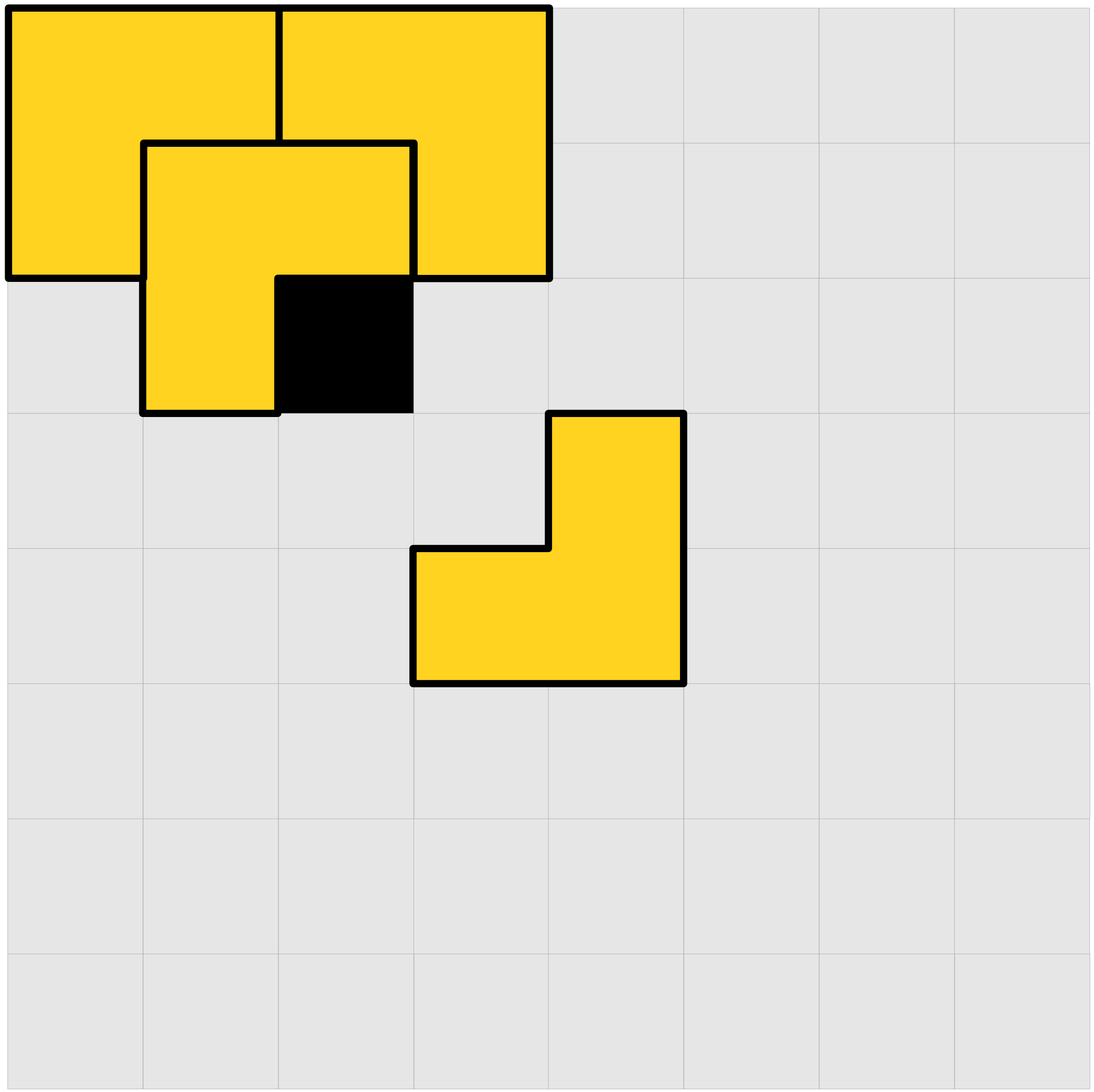

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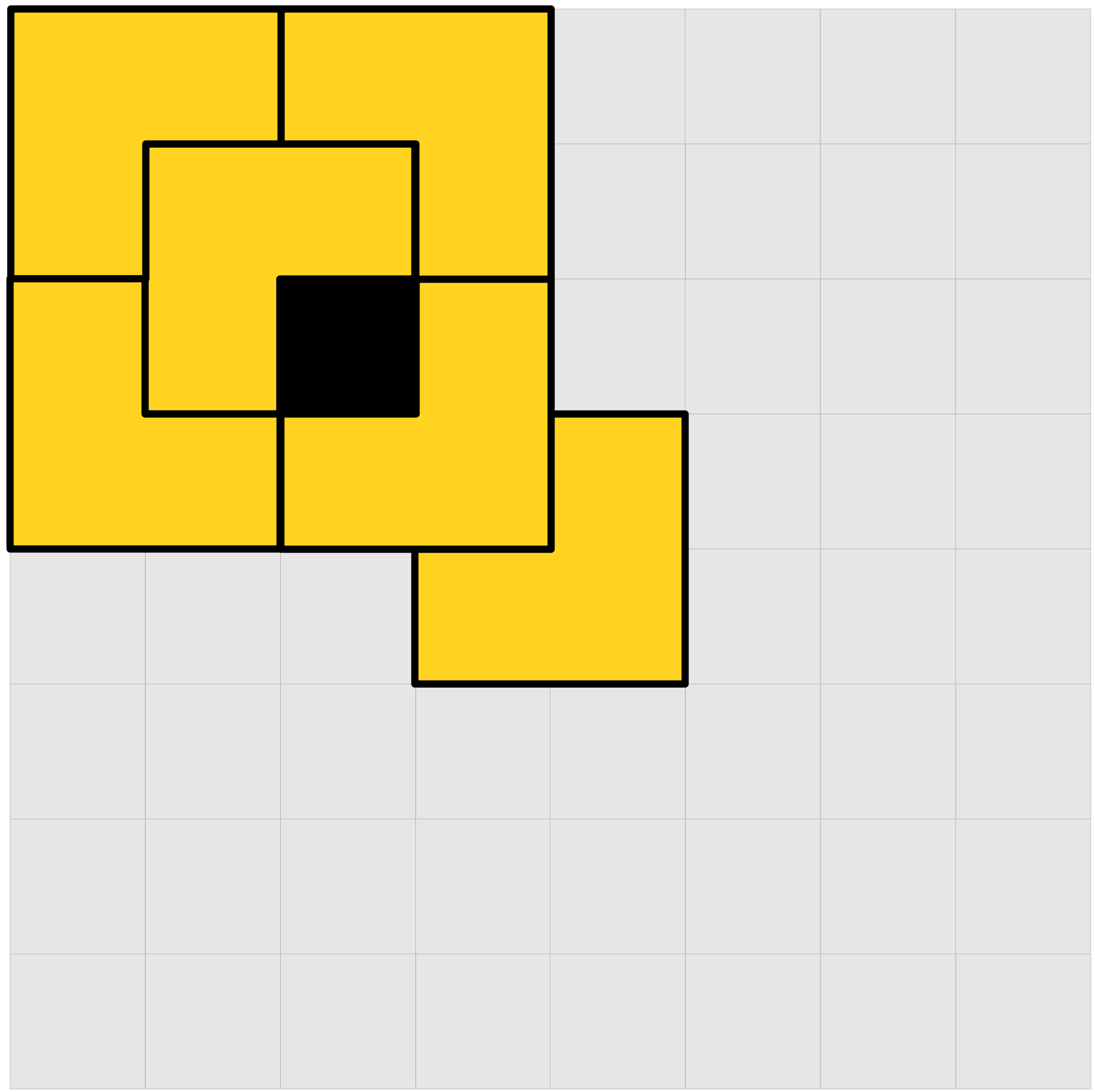

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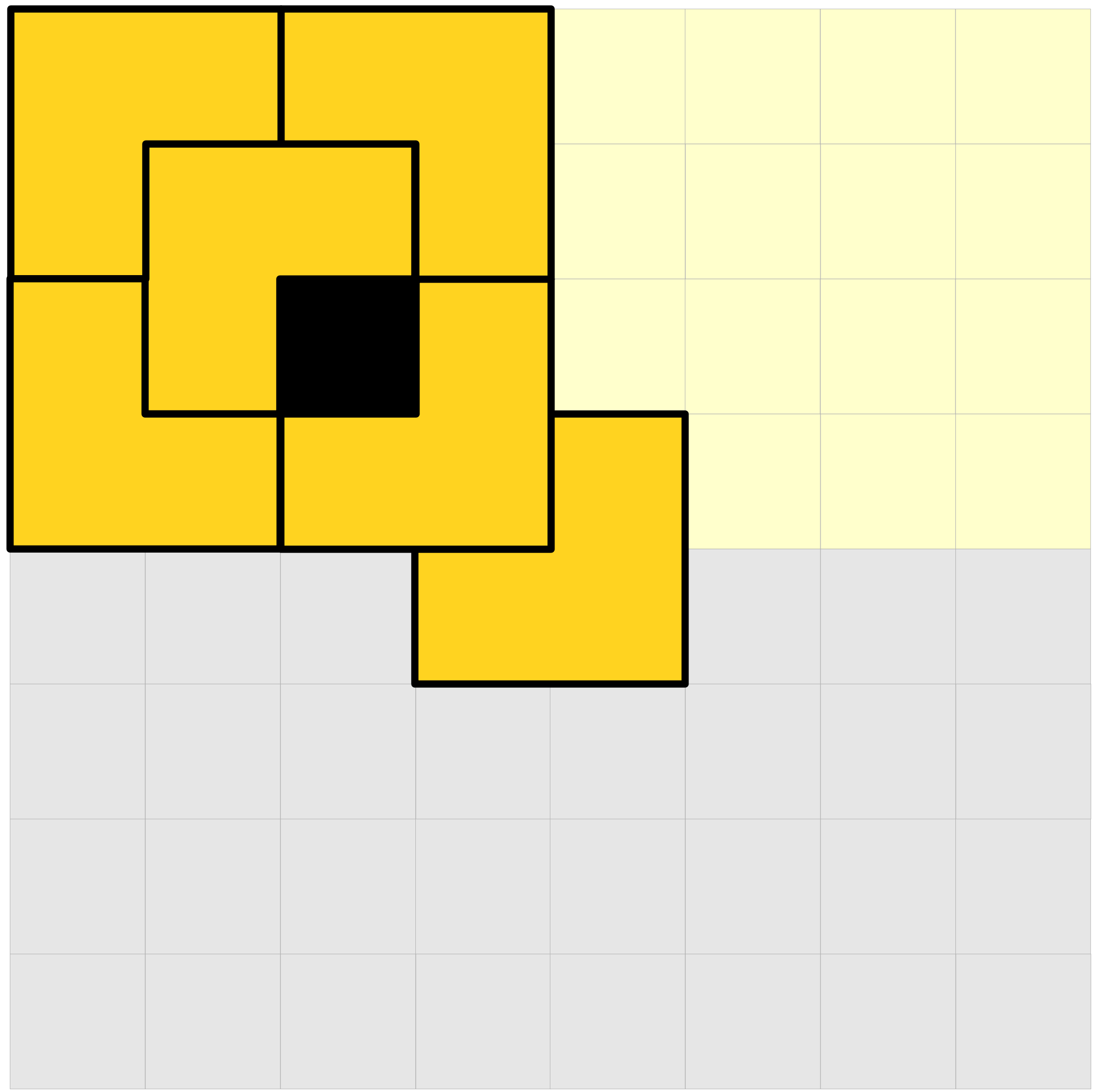

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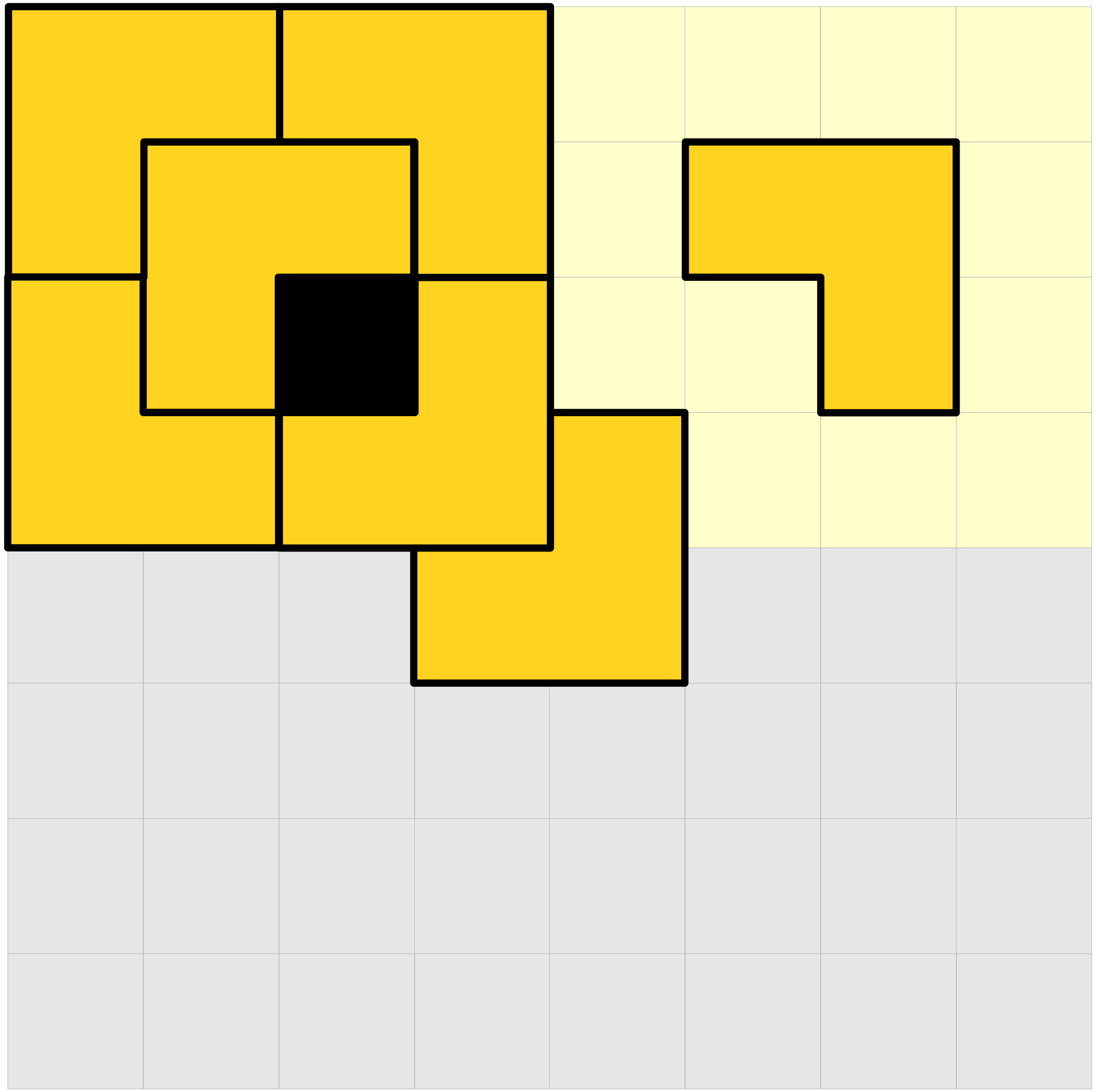

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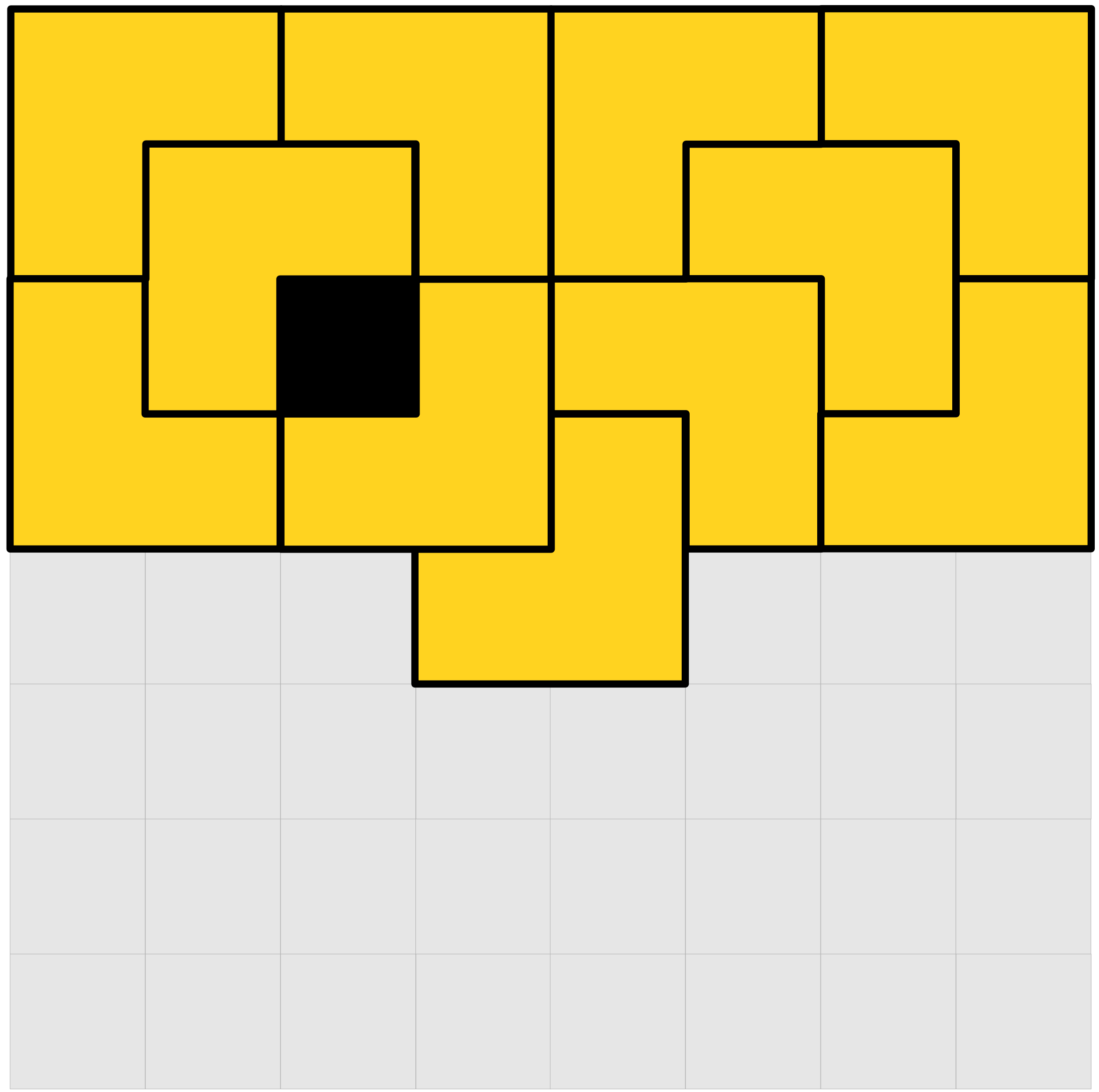

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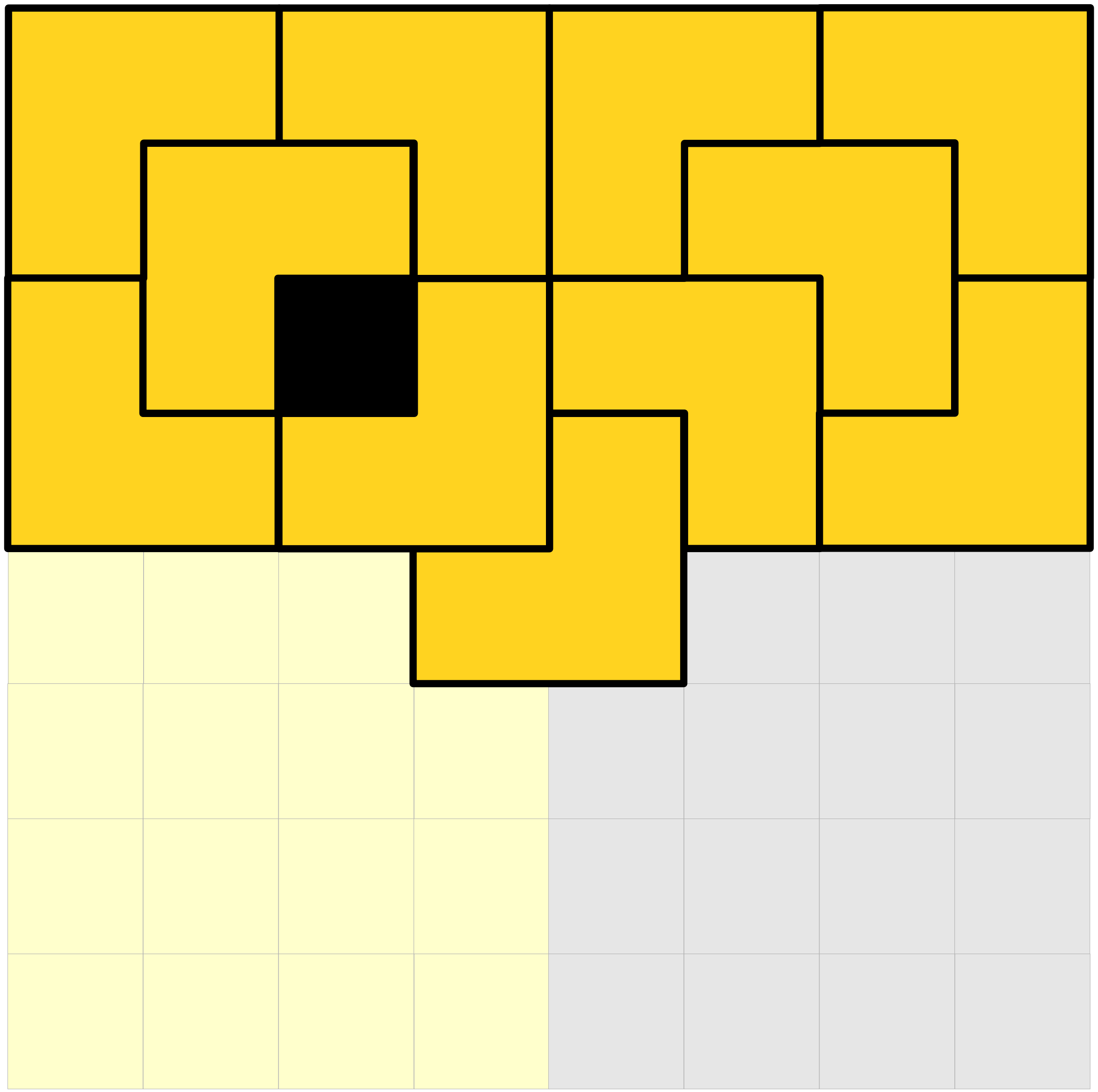

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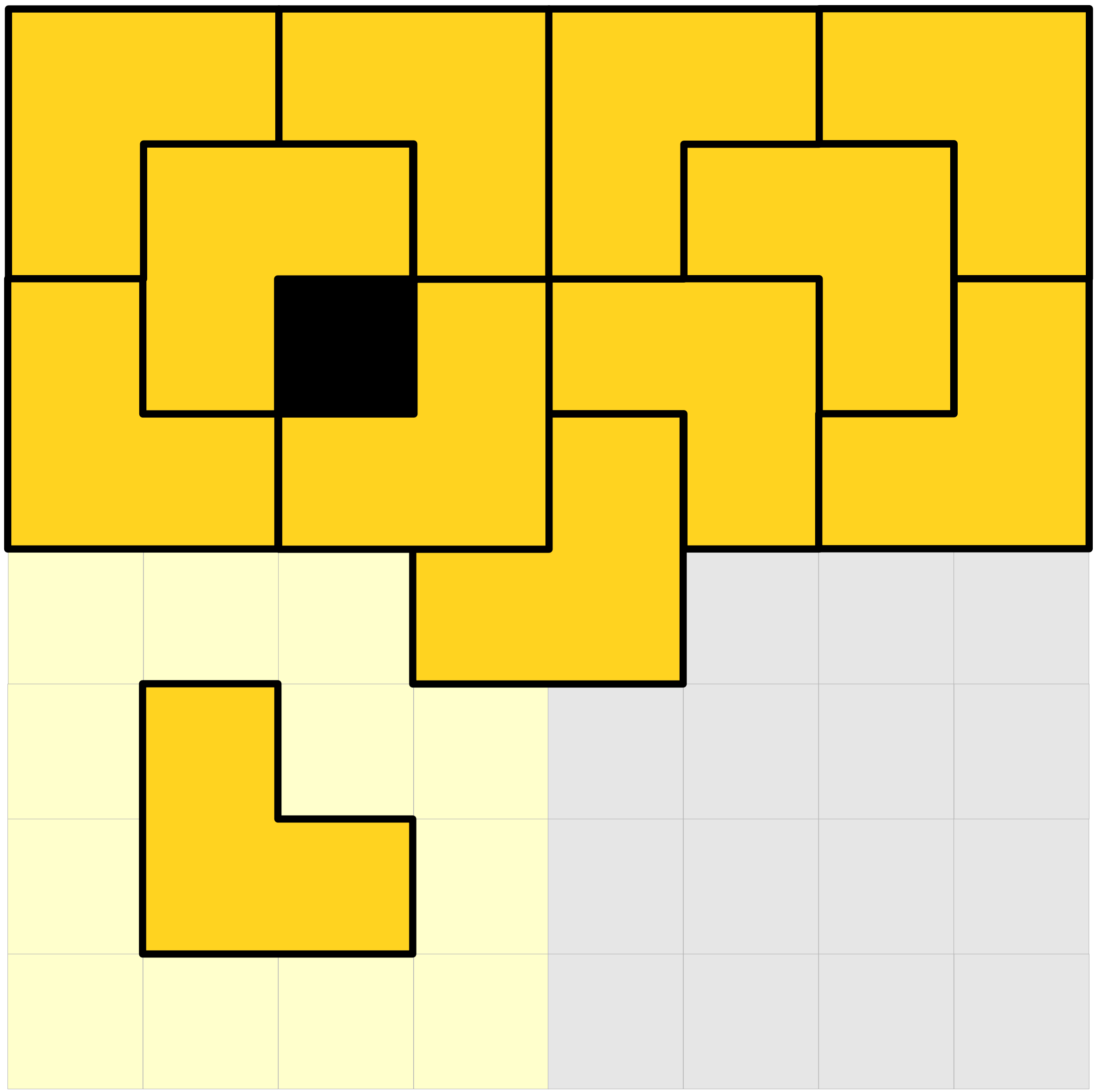

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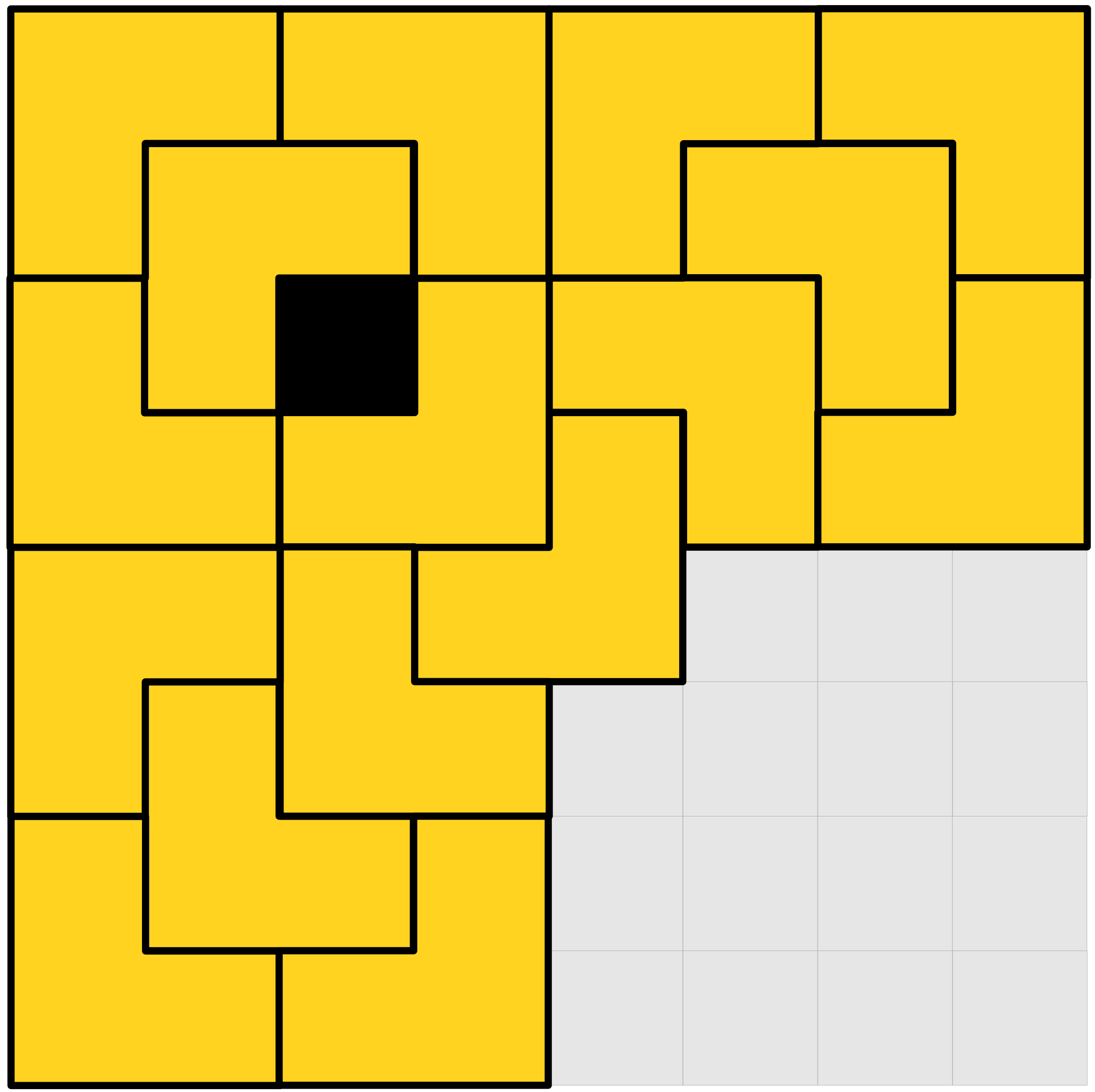

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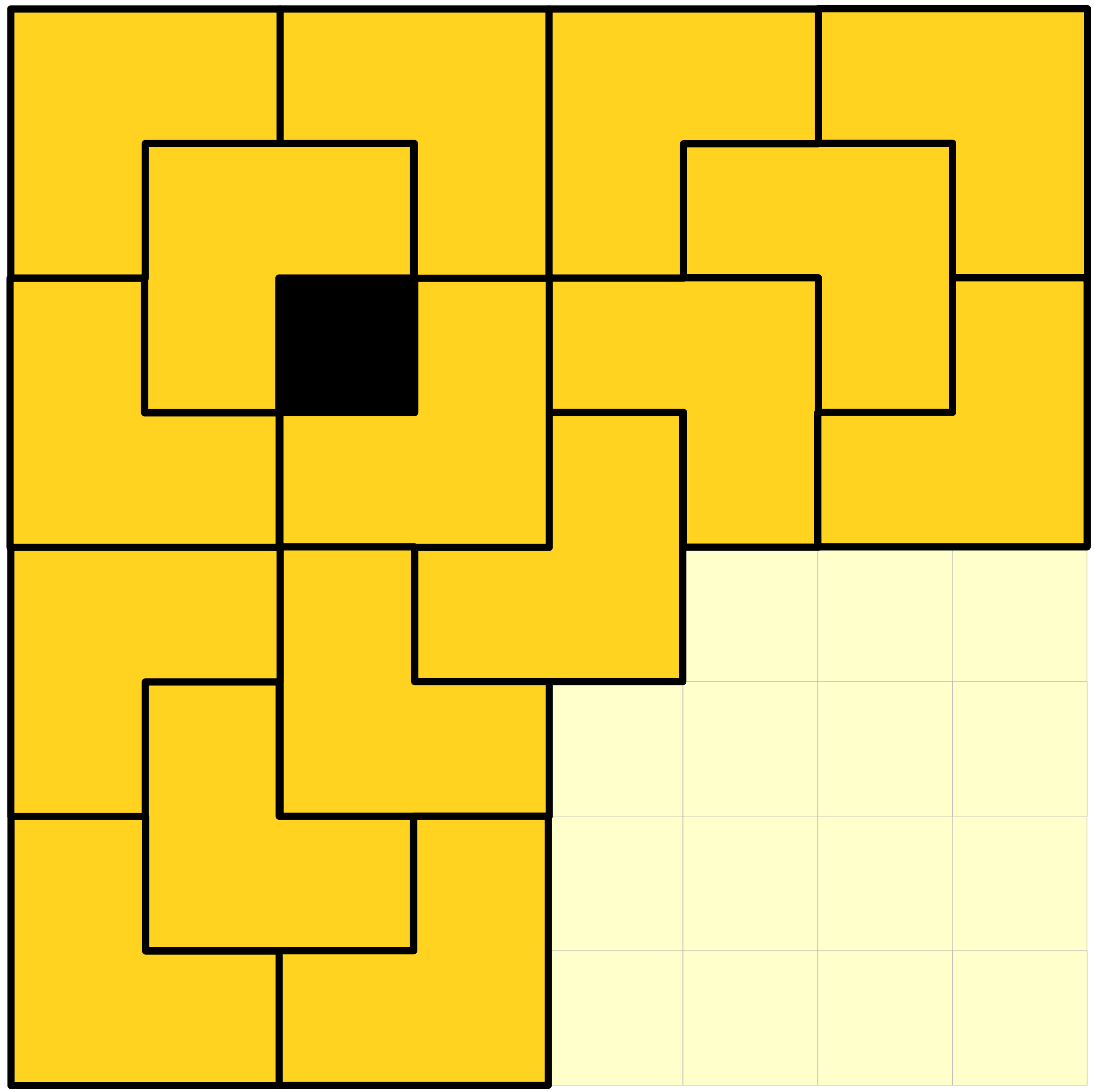

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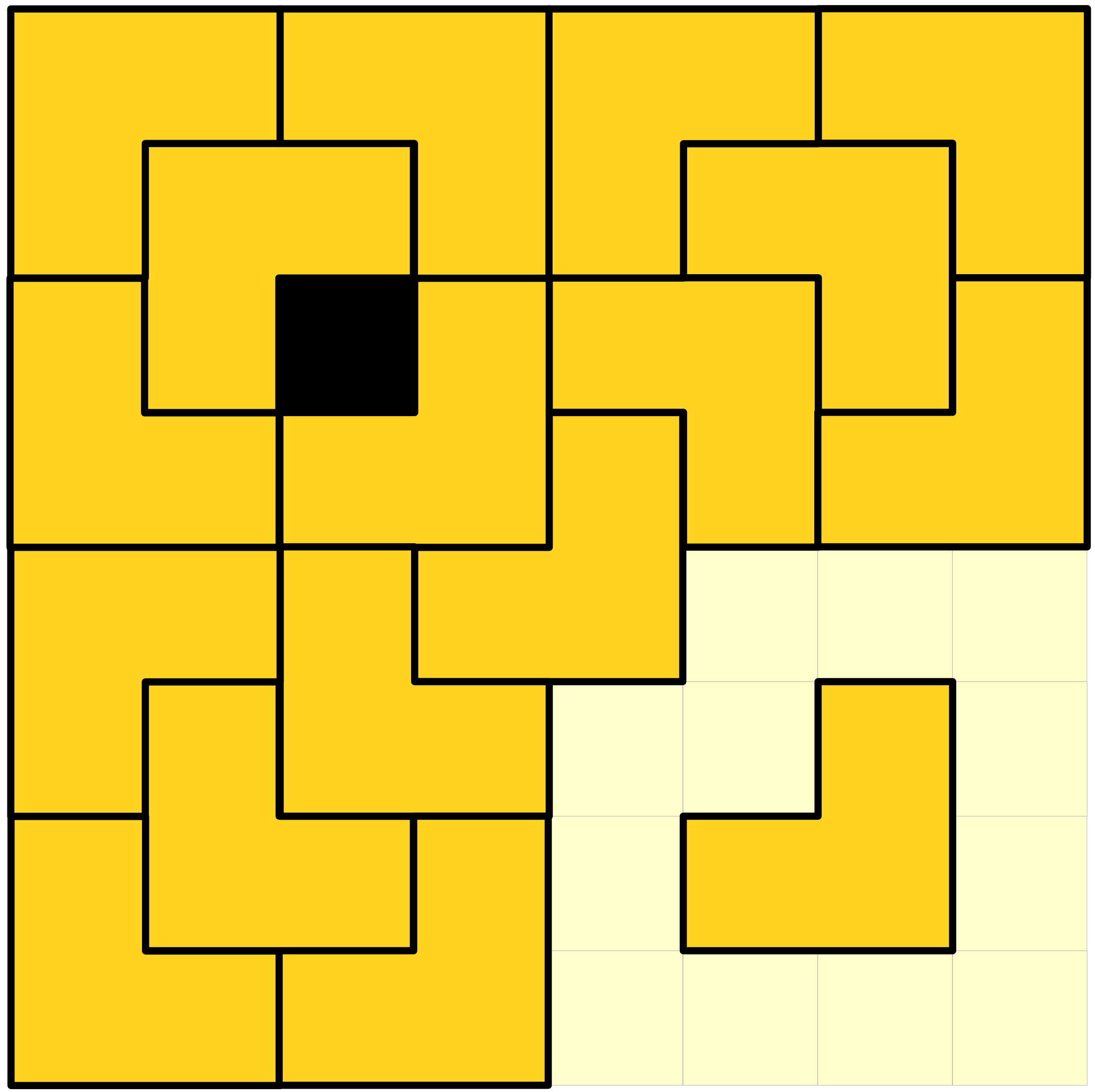

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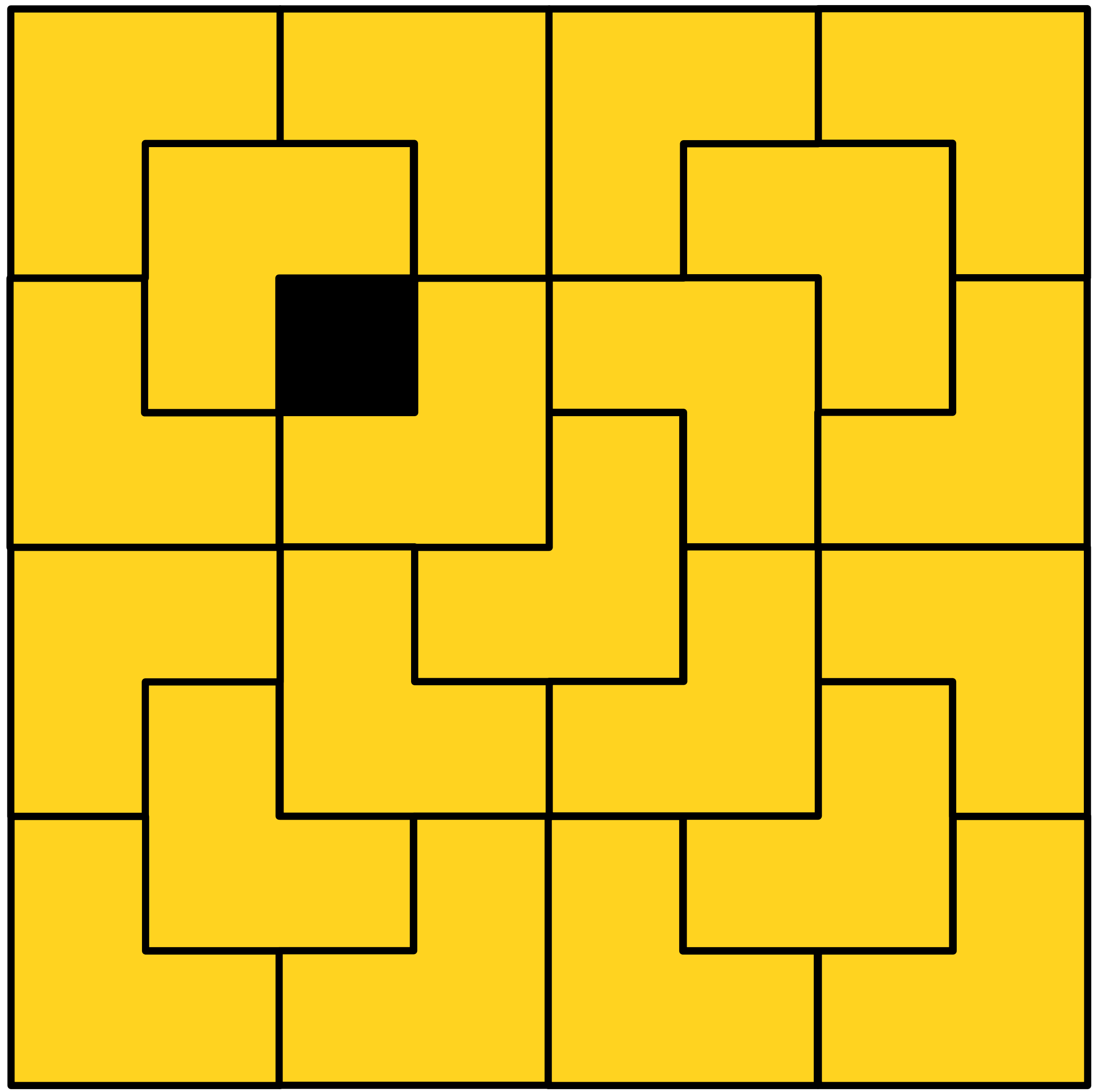

Tiling with Triominoes

- To tile a $2^k \times 2^k$ board missing a single square, do the following:
  - If the board has size 1×1, is has no uncovered squares (because one square is missing) and we're done.
  - Otherwise, place a triomino in the center to cover up one square from each quadrant that isn't missing a square, then recursively fill in the four smaller squares.

$$\boxed{\begin{aligned}&T(1) = \Theta(1)\\&T(n) = 4T(n / 2) + \Theta(1)\end{aligned}}$$

Solving the Recurrence

- We have the recurrence

$$\boxed{\begin{array}{l} T(1) = \Theta(1) \\ T(n) = 4T(n / 2) + \Theta(1) \end{array}}$$

- What are a, b, and d?
- What is $\log_b a$?
- What runtime do we get from the Master Theorem?
- Does that make sense?

Searching a Grid, Take Two

10	12	13	21	32	34	43	51
16	21	23	26	40	54	65	67
21	23	31	33	54	58	74	77
32	46	59	65	74	88	99	103
53	75	96	115	124	131	132	136
85	86	98	145	146	151	173	187

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21	**23**	**31**	**33**	**54**	**58**	**74**	**77**
32	**46**	**59**	**65**	**74**	**88**	**99**	**103**
53	**75**	**96**	**115**	**124**	**131**	**132**	**136**
85	**86**	**98**	**145**	**146**	**151**	**173**	**187**

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(Z) \leq 3T(\lceil Z / 4 \rceil) + \Theta(1)$$

Recursive Sorted Searching

- We now have the recurrence

$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(Z) \leq 3T(\lceil Z / 4 \rceil) + \Theta(1) \end{aligned}}$$

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- What are *a*, *b*, and *d*?

Recursive Sorted Searching

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- What are *a*, *b*, and *d*?
- What does this recurrence solve to?

Recursive Sorted Searching

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$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(Z) \leq 3T(\lceil Z / 4 \rceil) + \Theta(1) \end{aligned}}$$

- What are *a*, *b*, and *d*?
- What does this recurrence solve to?
- Since $T(Z) = O(Z^{\log_4 3})$, the runtime is **$O((mn)^{\log_4 3})$** $\approx O((mn)^{0.79})$

One More Example:

Integer Multiplication

Some Efficiency Claims

- Claim: The following can be done in $\Theta(1)$ time:
  - Multiplying two one-digit numbers.
  - Adding two one-digit numbers.
- Suppose that A and B have n digits each. Then these operations have the following costs:
  - Computing $A + B$: $\Theta(n)$
  - Computing $A - B$: $\Theta(n)$
  - Computing $A \cdot 10^k$: $O(n + k)$
  - Computing $A \bmod 10^k$: $O(n + k)$

Algorism Efficiency

- Recall: **Algorism** refers to place-value arithmetic.
- What is the cost of computing $A \cdot B$, where A and B are n-digit numbers?
  - Does $\Theta(n)$ rounds of the following:
    - Multiply each digit in A by a digit in B: $\Theta(n)$ time, including time to carry across columns.
    - Shift the resulting number $O(n)$ places: $O(n)$ time.
  - $\Theta(n)$ additions of $O(n)$-digit numbers: time $\Theta(n^2)$.
  - Overall runtime: $\boldsymbol{\Theta(n^2)}$.

A Quick History Lesson

Multiplying with Divide-and-Conquer

- Suppose that you want to multiply together two numbers X and Y, both of which are n digits long.
- Write

$$X = a \cdot 10^{\lfloor n/2 \rfloor} + b$$

$$Y = c \cdot 10^{\lfloor n/2 \rfloor} + d$$

  where $b, d < 10^{\lfloor n/2 \rfloor}$

- If $X = 13579$ and $Y = 24680$, what are a, b, c and d?

Multiplying with Divide-and-Conquer

- If $X = a \cdot 10^{\lfloor n/2 \rfloor} + b$ and $Y = c \cdot 10^{\lfloor n/2 \rfloor} + d$, then

Multiplying with Divide-and-Conquer

- If $X = a \cdot 10^{\lfloor n/2 \rfloor} + b$ and $Y = c \cdot 10^{\lfloor n/2 \rfloor} + d$, then

$$X \cdot Y = (a \cdot 10^{\lfloor n/2 \rfloor} + b) \cdot (c \cdot 10^{\lfloor n/2 \rfloor} + d)$$

Multiplying with Divide-and-Conquer

- If $X = a \cdot 10^{\lfloor n/2 \rfloor} + b$ and $Y = c \cdot 10^{\lfloor n/2 \rfloor} + d$, then

$$
\begin{aligned}
X \cdot Y &= (a \cdot 10^{\lfloor n/2 \rfloor} + b) \cdot (c \cdot 10^{\lfloor n/2 \rfloor} + d) \\
&= ac \cdot 10^{2\lfloor n/2 \rfloor} + ad \cdot 10^{\lfloor n/2 \rfloor} + bc \cdot 10^{\lfloor n/2 \rfloor} + bd
\end{aligned}
$$

Multiplying with Divide-and-Conquer

- If $X = a \cdot 10^{\lfloor n/2 \rfloor} + b$ and $Y = c \cdot 10^{\lfloor n/2 \rfloor} + d$, then

$$\begin{aligned} X \cdot Y &= (a \cdot 10^{\lfloor n/2 \rfloor} + b) \cdot (c \cdot 10^{\lfloor n/2 \rfloor} + d) \\ &= ac \cdot 10^{2\lfloor n/2 \rfloor} + ad \cdot 10^{\lfloor n/2 \rfloor} + bc \cdot 10^{\lfloor n/2 \rfloor} + bd \\ &= ac \cdot 10^{2\lfloor n/2 \rfloor} + (ad + bc) \cdot 10^{\lfloor n/2 \rfloor} + bd \end{aligned}$$

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- What is the cost of directly evaluating this expression?
  - Does 4 multiplications on numbers with $\lceil n / 2 \rceil$ digits.
  - Does three additions of numbers with $O(n)$ digits.
  - Does two multiplications by powers of ten, each of which takes $O(n)$ time.

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$$
\boxed{\begin{aligned}
&T(1) = \Theta(1) \\
&T(n) = 4T(\lceil n / 2 \rceil) + O(n)
\end{aligned}}
$$

Solving the Recurrence

- We now have the recurrence

$$\boxed{\begin{aligned} &T(1) = \Theta(1) \\ &T(n) = 4T(\lceil n / 2 \rceil) + O(n) \end{aligned}}$$

- What does the Master Theorem say?
- Runtime is $O(n^2)$. But that's no better than before...

Karatsuba's Observation

- Karatsuba arrived at this expression:

$$X \cdot Y = \boldsymbol{ac} \cdot 10^{2\lfloor n/2 \rfloor} + (\boldsymbol{ad + bc}) \cdot 10^{\lfloor n/2 \rfloor} + \boldsymbol{bd}$$

- Karatsuba's key question: Is it possible to compute $\boldsymbol{ac}$, $\boldsymbol{ad + bc}$, and $\boldsymbol{bd}$ without making four multiplications?

Karatsuba's Observation

- Consider these three products:

$$E = ac$$

$$F = bd$$

$$G = (a + b)(c + d) = ac + ad + bc + bd$$

- We can compute these values with two additions and three multiplications.
- Note that

$$ac = E$$

$$bd = F$$

$$ad + bc = G - E - F$$

Karatsuba's Algorithm

- Write $\boldsymbol{X = a \cdot 10^{\lfloor n/2 \rfloor} + b}$ and $\boldsymbol{Y = c \cdot 10^{\lfloor n/2 \rfloor} + d}$
- Recursively compute

$$\boldsymbol{E = ac \qquad F = bd \qquad G = (a + b)(c + d)}$$

- Then

$$\boldsymbol{X \cdot Y = E \cdot 10^{2\lfloor n/2 \rfloor} + (G - E - F) \cdot 10^{\lfloor n/2 \rfloor} + F}$$

- Does two multiplications by powers of ten (O(n) each), four additions (O(n) each), two subtractions (O(n) each), and three recursive multiplies on numbers with at most $\lceil n / 2 \rceil$ digits.

$$\boxed{\begin{array}{l} T(1) = \Theta(1) \\ T(n) = 3T(\lceil n / 2 \rceil) + O(n) \end{array}}$$

Karatsuba's Algorithm

- We now have the recurrence

$$T(1) = \Theta(1)$$
$$T(n) = 3T(\lceil n / 2 \rceil) + O(n)$$

- What does the Master Theorem tell us?
- Runtime is $O(n^{\log_2 3}) \approx$ **$O(n^{1.585})$**
- This is asymptotically better than the normal algorithm!
- **Standard algorism is not the optimal algorism algorithm!**

After Karatsuba

- Several other algorithms for multiplying numbers have arisen since Karatsuba's algorithm.
- **Toom-Cook** uses a similar set of techniques to multiply n-digit numbers in time $O(n^{\log_3 5})$.
- **Schönhage–Strassen** uses a completely different approach (based on the fast Fourier transform) to achieve $O(n \log n \log \log n)$ runtime.
- Recently (2008), **Fürer's algorithm** achieved runtime $n \log n \, 2^{O(\log^* n)}$, where $\log^* n$ is an *extremely* slowly-growing function.
- **Finding an optimal multiplication algorithm is still an open problem!**

Next Time

- The Selection Problem
- The Median of Medians Algorithm
- The Substitution Method