Greedy Algorithms
Part One
Announcements

• Problem Set Three due right now if using a late period.
• Solutions will be released at end of lecture.
Outline for Today

- **Greedy Algorithms**
  - Can myopic, shortsighted decisions lead to an optimal solution?
- **Lilypad Jumping**
  - Helping our amphibious friends home!
- **Activity Selection**
  - Planning your weekend!
Frog Jumping

- The frog begins at position 0 in the river. Its goal is to get to position $n$.
- There are lilypads at various positions. There is always a lilypad at position 0 and position $n$.
- The frog can jump at most $r$ units at a time.
- **Goal**: Find the path the frog should take to minimize jumps, assuming a solution exists.
Formalizing the Algorithm

- Let $J$ be an empty series of jumps.
- Let our current position $x = 0$.
- While $x < n$:
  - Find the furthest lilypad $l$ reachable from $x$ that is not after position $n$.
  - Add a jump to $J$ from $x$ to $l$'s location.
  - Set $x$ to $l$'s location.
- Return $J$. 
Greedy Algorithms

• A **greedy algorithm** is an algorithm that constructs an object $X$ one step at a time, at each step choosing the locally best option.

• In some cases, greedy algorithms construct the globally best object by repeatedly choosing the locally best option.
Greedy Advantages

- Greedy algorithms have several advantages over other algorithmic approaches:
  - **Simplicity**: Greedy algorithms are often easier to describe and code up than other algorithms.
  - **Efficiency**: Greedy algorithms can often be implemented more efficiently than other algorithms.
Greedy Challenges

- Greedy algorithms have several drawbacks:
  - **Hard to design**: Once you have found the right greedy approach, designing greedy algorithms can be easy. However, finding the right approach can be hard.
  - **Hard to verify**: Showing a greedy algorithm is correct often requires a nuanced argument.
Back to Frog Jumping

• We now have a simple greedy algorithm for routing the frog home: jump as far forward as possible at each step.

• We need to prove two properties:
  • The algorithm will find a legal series of jumps (i.e. it doesn't “get stuck”).
  • The algorithm finds an optimal series of jumps (i.e. there isn't a better path available).
Lemma 1: The greedy algorithm always finds a path from the start lilypad to the destination lilypad.

Proof: By contradiction; suppose it did not. Let the positions of the lilypads be $x_1 < x_2 < \ldots < x_m$. Since our algorithm didn't find a path, it must have stopped at some lilypad $x_k$ and not been able to jump to a future lilypad. In particular, this means it could not jump to lilypad $k + 1$, so $x_k + r < x_{k+1}$.

Since there is a path from lilypad 1 to the lilypad $m$, there must be some jump in that path that starts before lilypad $k + 1$ and ends at or after lilypad $k + 1$. This jump can't be made from lilypad $k$, so it must have been made from lilypad $s$ for some $s < k$. But then we have $x_s + r < x_k + r < x_{k+1}$, so this jump is illegal.

We have reached a contradiction, so our assumption was wrong and our algorithm always finds a path. ■
Proving Optimality

• How can we prove this algorithm finds an optimal series of jumps?

• **Key Proof Idea**: Consider an arbitrary optimal series of jumps \( J^* \), then show that our greedy algorithm produces a series of jumps no worse than \( J^* \).
  
  • We don't know what \( J^* \) is or that our algorithm is necessarily optimal. However, we can still use the existence of \( J^* \) in our proof.
Some Notation

- Let $J$ be the series of jumps produced by our algorithm and let $J^*$ be an optimal series of jumps.
  - Note that there might be multiple different optimal jump patterns.
- Let $|J|$ and $|J^*|$ denote the number of jumps in $J$ and $J^*$, respectively.
  - Note that $|J| \geq |J^*|$. (Why?)
The Key Lemma

- Let $p(i, J)$ denote the frog's position after taking the first $i$ jumps from jump series $J$.

- **Lemma**: For any $i$ in $0 \leq i \leq |J^*|$, we have $p(i, J) \geq p(i, J^*)$.
  
  - After taking $i$ jumps according to the greedy algorithm, the frog will be at least as far forward as if she took $i$ jumps according to the optimal solution.

- We can formalize this using induction.
**Lemma 2:** For all $0 \leq i \leq |J^*|$, we have $p(i, J) \geq p(i, J^*)$.

**Proof:** By induction. As a base case, if $i = 0$, then $p(0, J) = 0 \geq 0 = p(0, J^*)$ since the frog hasn't moved.

For the inductive step, assume that the claim holds for some $0 \leq i < |J^*|$. We will prove the claim holds for $i + 1$ by considering two cases:

**Case 1:** $p(i, J) \geq p(i + 1, J^*)$. Since each jump moves forward, we have $p(i + 1, J) \geq p(i, J)$, so we have $p(i + 1, J) \geq p(i + 1, J^*)$.

**Case 2:** $p(i, J) < p(i + 1, J^*)$. Each jump is of size at most $r$, so $p(i + 1, J^*) \leq p(i, J^*) + r$. By our IH, we know $p(i, J) \geq p(i, J^*)$, so $p(i + 1, J^*) \leq p(i, J) + r$. Therefore, the greedy algorithm can jump to position at least $p(i + 1, J^*)$. Therefore, $p(i + 1, J) \geq p(i + 1, J^*)$.

So $p(i + 1, J) \geq p(i + 1, J^*)$, completing the induction. ■
**Theorem:** Let $J$ be the series of jumps produced by the greedy algorithm and $J^*$ be any optimal series of jumps. Then $|J| = |J^*|$.  

**Proof:** Since $J^*$ is an optimal solution, we know that $|J^*| \leq |J|$. We will prove $|J^*| \geq |J|$. Suppose for contradiction that $|J^*| < |J|$. Let $k = |J^*|$. By Lemma 2, we have $p(k, J^*) \leq p(k, J)$. Because the frog arrives at position $n$ after $k$ jumps along series $J^*$, we know $n \leq p(k, J)$. Because the greedy algorithm never jumps past position $n$, we know $p(k, J) \leq n$, so $n = p(k, J)$. Since $|J^*| < |J|$, the greedy algorithm must have taken another jump after its $k$th jump, contradicting that the algorithm stops after reaching position $n$.

We have reached a contradiction, so our assumption was wrong and $|J^*| = |J|$, so the greedy algorithm produces an optimal solution. ■
The style of proof we just wrote is an example of a greedy stays ahead proof.

The general proof structure is the following:

- Find a series of measurements $M_1, M_2, \ldots, M_k$ you can apply to any solution.
- Show that the greedy algorithm's measures are at least as good as any solution's measures. (This usually involves induction.)
- Prove that because the greedy solution's measures are at least as good as any solution's measures, the greedy solution must be optimal. (This is usually a proof by contradiction.)
Another Problem: Activity Scheduling
Activity Scheduling

3 4 5 6 7 8 9 10 11 12 1

- Llama Hugging
- Skydiving
- Gardening
- Tree Climbing
- Evening Hike
- Salsa Dancing
- Fancy Dinner
- Bar Crawling
- Night Snorkeling
- Bonfire
- Navel Gazing
- Jazz Concert

Evening Hike
Activity Scheduling

- You are given a list of activities \((s_1, e_1), (s_2, e_2), \ldots, (s_n, e_n)\) denoted by their start and end times.
- All activities are equally attractive to you, and you want to maximize the number of activities you do.
- Goal: Choose the largest number of non-overlapping activities possible.
Thinking Greedily

- If we want to try solving this using a greedy approach, we should think about different ways of picking activities greedily.

- A few options:
  - **Be Impulsive:** Choose activities in ascending order of start times.
  - **Avoid Commitment:** Choose activities in ascending order of length.
  - **Finish Fast:** Choose activities in ascending order of end times.
Thinking Greedily

- Of the three options we saw, only the third one seems to work:
  
  Choose activities in ascending order of finishing times.

- More formally:
  
  - Sort the activities into ascending order by finishing time and add them to a set $U$.
  - While $U$ is not empty:
    - Choose any activity with the earliest finishing time.
    - Add that activity to $S$.
    - Remove from $U$ all activities that overlap $S$. 
Proving Legality

• **Lemma:** The schedule produced this way is a legal schedule.

• **Proof Idea:** Use induction to show that at each step, the set $U$ only contains activities that don't conflict with activities picked from $S$. 
To prove that the schedule $S$ produced by the algorithm is optimal, we will use another “greedy stays ahead” argument:

- Find some measures by which the algorithm is at least as good as any other solution.
- Show that those measures mean that the algorithm must produce an optimal solution.
Comparing Solutions

Muffin Collecting

Basket Weaving

Cupcake Baking

Pondering

Fancy Dinner

Meandering

Gallivanting

Wandering

Movies

Clubbing
Comparing Solutions

3 4 5 6 7 8 9 10 11 12 1

Muffin Collecting

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Movies

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Basket Weaving

Clubbing
Greedy Stays Ahead

- **Observation:** The $k$th activity chosen by the greedy algorithm finishes no later than the $k$th activity chosen in any legal schedule.

- We need to
  - Prove that this is actually true, and
  - Show that, if it's true, the algorithm is optimal.

- We'll do this out of order.
Some Notation

- Let $S$ be the schedule our algorithm produces and $S^*$ be any optimal schedule.
- Note that $|S| \leq |S^*|$.
- Let $f(i, S)$ denote the time that the $i$th activity finishes in schedule $S$.
- **Lemma**: For any $1 \leq i \leq |S|$, we have $f(i, S) \leq f(i, S^*)$. 
Theorem: The greedy algorithm for activity selection produces an optimal schedule.

Proof: Let $S$ be the schedule the algorithm produced and $S^*$ be any optimal schedule. Since $S^*$ is optimal, we have $|S| \leq |S^*|$. We will prove $|S| \geq |S^*|$. Assume for contradiction that $|S| < |S^*|$. Let $k = |S|$. By our lemma, we know $f(k, S) \leq f(k, S^*)$, so the $k$th activity in $S$ finishes no later than the $k$th activity in $S^*$. Since $|S| < |S^*|$, there is a $(k + 1)$st activity in $S^*$, and its start time must be after $f(k, S^*)$ and therefore after $f(k, S)$. Thus after the greedy algorithm added its $k$th activity to $S$, the $(k + 1)$st activity from $S^*$ would still belong to $U$. But the greedy algorithm ended after $k$ activities, so $U$ must have been empty. We have reached a contradiction, so our assumption must have been wrong. Thus the greedy algorithm must be optimal. ■
Lemma: If $S$ is a schedule produced by the greedy algorithm and $S^*$ is an optimal schedule, then for any $1 \leq i \leq |S|$, we have $f(i, S) \leq f(i, S^*)$.

Proof: By induction. For our base case, we prove $f(1, S) \leq f(1, S^*)$. The first activity the greedy algorithm selects must be an activity that ends no later than any other activity, so $f(1, S) \leq f(1, S^*)$.

For the inductive step, assume the claim holds for some $i$ in $1 \leq i < |S|$. Since $f(i, S) \leq f(i, S^*)$, the $i$th activity in $S$ finishes before the $i$th activity in $S^*$. Since the $(i+1)$st activity in $S^*$ must start after the $i$th activity in $S^*$ ends, the $(i + 1)$st activity in $S^*$ must start after the $i$th activity in $S$ ends. Therefore, the $(i+1)$st activity in $S^*$ must be in $U$ when the greedy algorithm selects its $(i+1)$st activity. Since the greedy algorithm selects the activity in $U$ with the lowest end time, we have $f(i + 1, S) \leq f(i + 1, S^*)$, completing the induction. $\blacksquare$
Summary

- Greedy algorithms aim for global optimality by iteratively making a locally optimal decision.

- To show correctness, typically need to show
  - The algorithm produces a legal answer, and
  - The algorithm produces an optimal answer.

- Often use “greedy stays ahead” to show optimality.
Next Time

- Minimum Spanning Trees
- Prim's Algorithm
- Exchange Arguments