CS 161: Homework 1

Due by October 7, 2016 at noon

Instructions: Please answer the following questions to the best of your ability. Provide full and rigorous proofs and include all relevant calculations. When writing proofs, please strive for clarity and brevity (in that order). Please see the course website for submission instructions and collaboration policy. Cite any sources that you use.

Question 1 (25 points)

1. What is the cardinality of the set of subsets of $\{1, 2, ..., n\}$?
2. Let X be a random variable denoting the cardinality of a randomly selected subset of $\{1, 2, ..., n\}$ (where each subset has equal probability of being selected). What is the expected value of X?
3. Let `rand(a,b)` return an integer uniformly at random from the range $[a, b]$. Each call to `rand` is independent of other calls to `rand`. Consider the following function to simulate rolling a number of dice:

```
roll(n,k):
  sum = 0
  for i = 1 to k:
    sum = sum + rand(1,n)
  return sum
```

   Calculate the expected value and variance of `roll(n,k)`.

Question 2 (25 points)

1. For each of the following functions, indicate whether $f = O(g)$, $f = \Omega(g)$, or both ($f = \Theta(g)$). Justify your answers.

(a)	$f(n) = 6n^2 + n\log n + 6$	$g(n) = n^2 + 2$
(b)	$f(n) = 7n\log\left(n^7\right)$	$g(n) = n\log n$
(c)	$f(n) = 3^n$	$g(n) = 3^{2n}$
(d)	$f(n) = n\log n$	$g(n) = (\log n)^2$
(e)	$f(n) = 100^n$	$g(n) = n!$
(f)	$f(n) = n^{\log\log n}$	$g(n) = (\log n)^{\log n}$

2. Show that $\sum_{i=1}^{n} \log i = \Theta(n\log n)$.
3. Show that $\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$.

Question 3 (25 points)

Suppose that G is a graph with $2n$ nodes and no triangles (cycles of length 3). G is a proper graph, *i.e.* it has no self-loops or multiple edges between the same pair of nodes. Prove that G has at most n^2 edges.

Question 4 (25 points)

In lecture, you saw how to solve the recurrence relation for MergeSort by drawing its recursion tree and adding up the total running time. Use this method to solve the following recurrences — that is, get a tight bound of the form $T(n) = \Theta(f(n))$ for the appropriate function f.

Show your work. If you wish, you may assume that n initially has the form $n = a^i$, for an appropriate constant a.

1. $T(n) = 3T(n/3) + 2n$
2. $T(n) = 4T(n/2) + \Theta(n)$
3. $T(n) = T(n-1) + c^n$, where $c > 0$ is a positive constant.

Note: In this question, do not use the "master theorem/method", which you will see in lecture soon.