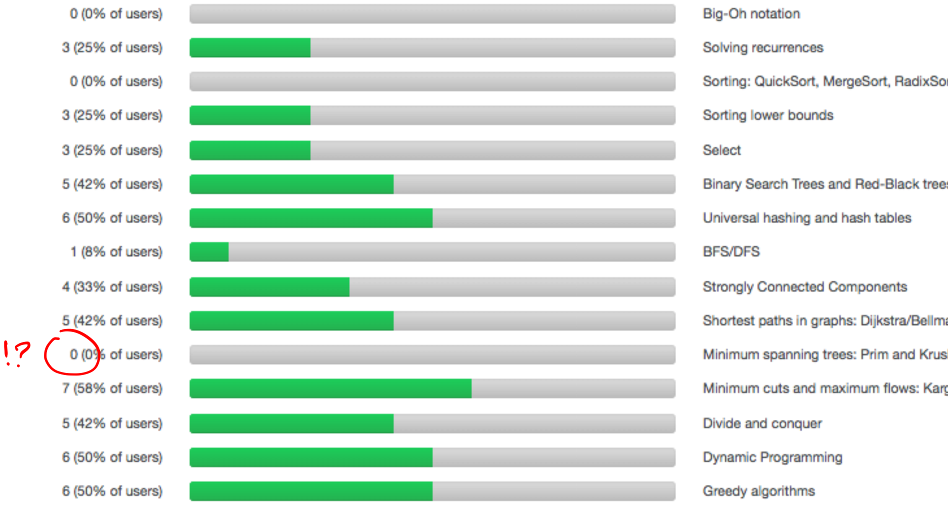
CS161 Review Session Practice Problems

WITH SOLUTION SKETCHES!

12/6/2017

Poll results

As of 1am this morning...



Agenda

- I have a bunch of practice problems.
- Y'all vote on topics and we'll do them.
- I can also answer particular questions about the material.
- Topics I have problems for:
  - Grab-bag (multiple choice, etc)
  - Hashing
  - Red-Black Trees
  - Ford-Fulkerson
  - Dynamic Programming
  - Greedy algorithms
  - Divide and conquer
  - Randomized algs

We ended up talking about these in class!

Multiple choice warmup!

For each of the following quantities, **identify all of the options** that correctly describe the quantity.

(a) The function $f(n)$, where $f(n) = n\log(n)$. (A), (C)

(b) $T(n)$ given by $T(n) = T(n/4) + \Theta(n^2)$ with $T(n) = 1$ for all $n \leq 8$.
(A)(B)(C)

(c) $T(n)$ which is the running time of the following algorithm:

```
mysteryAlg( n ):
    if n < 3:
        return 1
    return mysteryAlg( n/2 ) + mysteryAlg( (n/2) + 1 )
```

where above all division is integer division (so a/b means $\lfloor a/b \rfloor$).

(A), (C), (D)

$(A)\ O(n^2)$ $(B)\ \Theta(n^2)$ $(C)\ \Omega(n)$ $(D)\ O(n)$ $(E)\ O(\log^2(n))$.

Prove or give a counter-example

Let $G = (V, E)$ be an undirected weighted graph, and let T be a minimum spanning tree in G. Decide whether the following statements **must be true** or **may be false**, and prove it!

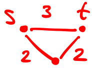
(a) For any pair of distinct vertices $s, t \in V$, there is a unique path from s to t in T.

True **False**

Otherwise there would be a cycle!

(b) For any pair of distinct vertices $s, t \in V$, the cost of a path between s and t in T is minimal among all paths from s to t in G.

True **False**



Hashing warm-up

Let $\mathcal{U}$ be a universe of size m, where m is a prime, and consider the following two hash families which hash $\mathcal{U}$ into n buckets, where n is much smaller than m.

- First, consider $\mathcal{H}_1$, which is the set of all functions from $\mathcal{U}$ to $\{1, \ldots, n\}$:

$$\mathcal{H}_1 = \{h \mid h : \mathcal{U} \to \{1, \ldots, n\}\}$$

- Second, let $p = m$ (so p is prime since we assumed m to be prime), and choose $\mathcal{H}_2$ to be

$$\mathcal{H}_2 = \{h_{a,b} \mid a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\}\},$$

  where $h_{a,b}(x) = (ax + b \mod p) \mod n$.

- You want to implement a hash table using one of these two families. Why would you choose $\mathcal{H}_2$ over $\mathcal{H}_1$? **Choose the best answer.**

(A) $\mathcal{H}_1$ isn't a universal hash family.

(B) Storing an element of $\mathcal{H}_1$ takes a lot of space.

(C) Storing all of $\mathcal{H}_1$ takes a lot of space.

Shortest Paths

- When might you prefer breadth-first search to Dijkstra's algorithm?

  If the graph is unweighted

- When might you prefer Floyd-Warshall to Bellman-Ford?

  If you want shortest paths between all pairs of vertices.

- When might you prefer Bellman-Ford to Dijkstra's algorithm?

  If there are negative edge wts

Suppose that $b_1, \ldots, b_n$ are n distinct integers in a **uniformly random order**. Consider the following algorithm:

```
findMax(b_1,...,b_n):
    currentMax = -Infinity
    for i = 1,...,n:
        if b_i > currentMax:
            currentMax = b_i
    return currentMax
```

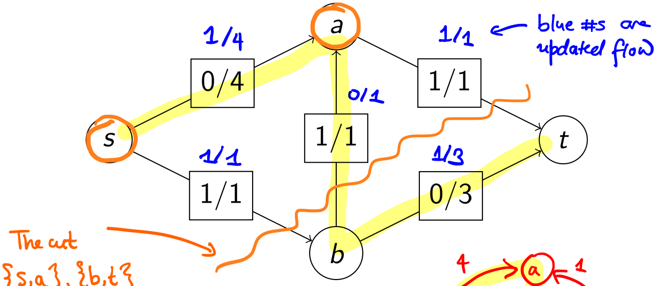
$\mathbb{E}\{\text{\# times currentMax updated}\}$
$= \mathbb{E}\left\{ \sum_{i=1}^{n} \mathbb{1}\{ b_i > b_1, \ldots, b_{i-1} \} \right\}$
$= \sum_{i=1}^{n} \mathbb{P}\{ b_i > b_1, \ldots, b_{i-1} \}$
$= \sum_{i=1}^{n} 1/i$
$= \Theta(\log(n))$

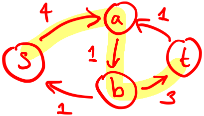
since $b_1, \ldots, b_i$ are uniform — the probability that any one is largest is $\frac{1}{i}$.

What is the expected number of times that currentMax is updated? (Asymptotic notation is fine).

$\Theta(\log(n))$

Consider the following flow on a graph. The notation x/y means that an edge has flow x out of capacity y.





- Draw the residual graph for this flow.
- Find an augmenting path in the residual graph and use it to increase the flow. The path highlighted above results in the flow marked above.
- Find a minimum cut and prove (not by exhaustion) that it is a minimum cut. The cut {s,a},{b,t} has value 2, which is minimal since $2 \leq$ max flow $=$ min cut ≤ 2
  b/c the updated flow above has value 2
  This cut has value 2

Dynamic Programming!

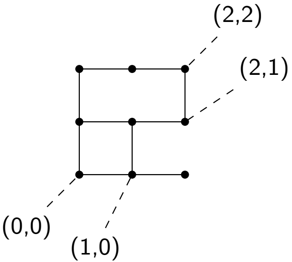
- Suppose that roads in a city are laid out in an $n \times n$ grid, but some of the roads are obstructed.
- For example, for $n = 3$, the city may look like this:

Define $M[i,j]$ = #paths from $(0,0)$ to (i,j).

$$M[i,j] = \mathbb{1}\{(i-1,j) \bullet\!\!-\!\!\bullet (i,j)\} \cdot M[i-1,j] + \mathbb{1}\{(i,j-1) \bullet\!\!-\!\!\bullet (i,j)\} \cdot M[i,j-1]$$

Alg:

- Initialize $M[i,j] = 0 \;\; \forall\, i,j \in \{0, \dots, n-1\}$
- $M[0,0] \leftarrow 1$
- for $i = 0, \_\_, n-1$:
  - for $j = 0, \_\_, n-1$:
    - if $i > 0$ and there is a road $(i-1,j) \bullet\!\!-\!\!\bullet (i,j)$:
      - $M[i,j] \mathrel{+}= M[i-1,j]$
    - if $j > 0$ and there is a road $(i,j) \bullet\!\!-\!\!\bullet (i,j-1)$:
      - $M[i,j] \mathrel{+}= M[i,j-1]$
- Return $M[n-1,n-1]$.



where we have only drawn the roads that are not blocked. You want to count the number of ways to get from $(0,0)$ to $(n-1, n-1)$, using paths that only go up and to the right. In the example above, the number of paths is 3.

- Design a DP algorithm to solve this problem.

Divide and Conquer!

- Given an array A of length n, we say that an array B is a *circular shift* of A if there is an integer k between 1 and n (inclusive) so that

$$B = A[k:n] + A[1:k],$$

  where $+$ denotes concatenation.
- For example, if $A = [2,5,6,8,9]$, then $B = [6,8,9,2,5]$ is a circular shift of A (with $k = 2$). The sorted array A itself is also a circular shift of A (with $k = 1$).
- Design a $O(\log(n))$-time algorithm that takes as input an array B which is a circular shift of a sorted array which contains distinct positive integers, and returns the value of the largest element in B. For example, give B as above, your algorithm should return 9.

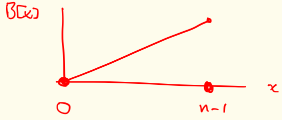
Solution on next page.

SOLUTION for DIVIDE + CONQUER

```
def findMax(B):
    n ← len(B)
    if B[0] ≤ B[n-1]:   \\case 1
        return B[n-1]
    mid = ⌊n/2⌋ + 1
    if B[mid] > B[0]:   \\case 2
        return findMax( B[mid:n] )
    if B[mid] < B[0]:   \\case 3
        return findMax( B[:mid+1] )
```

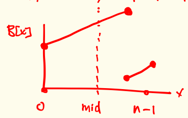
Idea:

- In CASE 1, the situation looks like



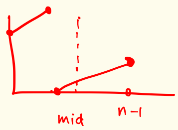
So we return B[n-1]

- In CASE 2, it looks like



so the max is on the right side and we recurse on B[mid:]

In CASE 3, it looks like



so the max is on the left side and we recurse on B[:mid+1]

Greedy Algorithms!

There are n final exams on Dec. 13 at Stanford; exam i is scheduled to begin at time a_i and end at time b_i. Two exams which overlap cannot be administered in the same classroom; two exams i and j are defined to be *overlapping* if $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ (including if $b_i = a_j$, so one starts exactly at the time that the other ends). Design an algorithm which solves the following problem.

- **Input:** Arrays A and B of length n so that $A[i] = a_i$ and $B[i] = b_i$.
- **Output:** The smallest number of classrooms necessary to schedule all of the exams, and an optimal assignment of exams to classrooms.
- **Running time:** $O(n \log(n) + nk)$, where k is the minimum number of classrooms needed.
- **For example:** Suppose there are three exams, with start and finish times as given below:

i	1	2	3
a_i	12pm	4pm	2pm
b_i	3pm	6pm	5pm

SOLUTION on next page

Then the exams can be scheduled in two rooms; Exam 1 and Exam 2 can be scheduled in Room 1 and Exam 3 can be scheduled in Room 2.

Solution to Scheduling Problem

```
def scheduleRooms(A, B):      // IDEA. Sort exams by start time.
                              // Greedily put exams into any room
                              // that can accomodate them.
                              // If there is no such room, start a new room.
    n ← len(A)
    C = [ (A[i], i) for i = 0, ..., n-1 ]
    sort C        // increasing order by start time.
    rooms = [ ]   // list of rooms
    endTimes = [ ]
    for i = 0, ___, n-1:
        for r = 0, ___, len(rooms) - 1:
            if C[i][1] > endTimes[r]:
                rooms[r].append( C[i][1] )
                endTimes[r] = B[C[i][1]]
                break
        Else: // did not break
            rooms.append( C[i][1] )
            endTimes.append( [ B[C[i][1]] ] )
    Return rooms.
```

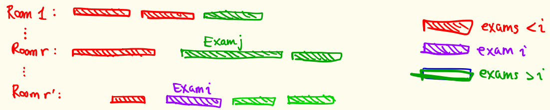
Correctness by induction

Inductive hyp: After adding the i^{th} exam, there is an optimal schedule that extends the current solution.

Base Case: After adding 0 exams, there is an optimal sol. extending this. :)

Inductive Step: Suppose the inductive hyp holds for $i-1$, and let S be the optimal schedule that extends it.

If S puts exam i where we would put it (say, room r) then we are done, so suppose that S puts exam i in room r'!

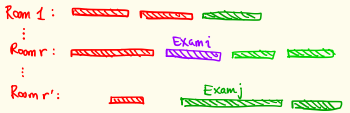


time

Let $j > i$ be the next exam scheduled in Room r.

Then $a_j \geq a_i$, since a_i had the smallest start time of all exams not yet picked.

So consider the schedule S' where we swap the rest of room r' w/ the rest of room r:



This is still a valid schedule, and uses the same # rooms as S, so it's also optimal. And it puts exam i in room r, so we're done.

CONCLUSION: At the end of the alg., there's still an optimal solution extending the current one so the current one is optimal.

Universal Hash Families

- Definition: A hash family $\mathcal{H}$ (mapping $\mathcal{U}$ into n buckets) is **2-universal** if for all $x \neq y \in \mathcal{U}$ and for all $a, b \in \{1, \ldots, n\}$,

$$\mathbb{P}((h(x), h(y)) = (a, b)) = \frac{1}{n^2}.$$

(a) Show that if $\mathcal{H}$ is 2-universal, then it is universal.

(b) Show that the converse is not true. That is, there is a universal family that's not 2-universal.

(a) Suppose that $\mathcal{H}$ is 2-universal. Then $\forall\, x \neq y \in \mathcal{U}$,

$$\mathbb{P}_{h \in \mathcal{H}}\{h(x) = h(y)\} = \sum_{t \in \{1,\ldots,n\}} \mathbb{P}\{(h(x), h(y)) = (t,t)\}$$
$$= \sum_{t \in \{1,\ldots,n\}} 1/n^2$$
$$= 1/n.$$

So by definition $\mathcal{H}$ is universal.

(b) Consider: $n=2$, $\mathcal{U} = \{x, y\}$, $\mathcal{H} = \{h_1, h_2\}$, where:

	x	y
h_1	0	0
h_2	1	0

Then $\mathbb{P}_{h \in \mathcal{H}}\{h(x) = h(y)\} = 1/2$.

But $\mathbb{P}_{h \in \mathcal{H}}\{(h(x), h(y)) = (0,0)\} = 1/2$, not $1/4$.

More universal hash families

Say that $\mathcal{H}$ is a universal hash family, containing functions $h : \mathcal{U} \to \{1, \ldots, n\}$. Consider the following game.

- You choose $h \in \mathcal{H}$ uniformly at random and keep it secret.
- A bad guy chooses $x \in \mathcal{U}$, and asks you for $h(x)$. (You give it to them).
- The bad guy chooses $y \in \mathcal{U} \setminus \{x\}$, and tries to get $h(y) = h(x)$.
- If $h(x) = h(y)$, the bad guy wins. Otherwise, you win.

One of the following two is true.

1. There is a universal hash family $\mathcal{H}$ so that the bad guy wins with probability 1.
2. For any universal hash family $\mathcal{H}$, the probability that the bad guy wins is at most $1/n$.

To see this, consider the hash family $\mathcal{H} = \{h_1, h_2\}$ w/ $\mathcal{U} = \{x, y, z\}$, $n=2$, given by

	x	y	z
h_1	0	0	1
h_2	1	0	1

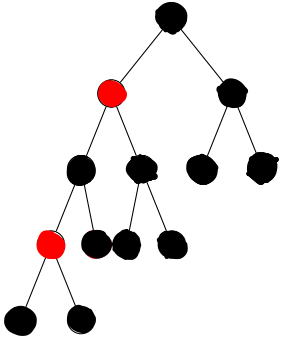
This is universal since
$\mathbb{P}\{h(x) = h(y)\} = 1/2$
$\mathbb{P}\{h(x) = h(z)\} = 1/2$
$\mathbb{P}\{h(y) = h(z)\} = 0 < 1/2$.

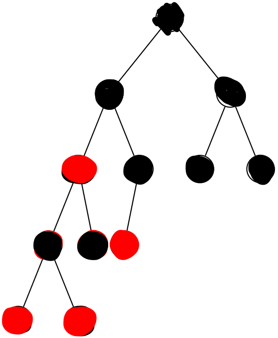
Bad guy's algorithm:
If $h(x) = 1$, choose z
If $h(x) = 0$, choose y

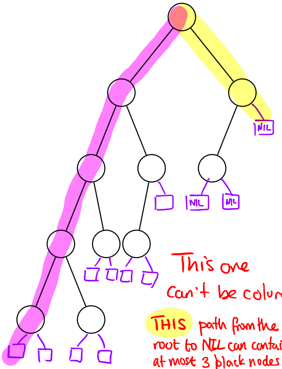
Which is true and why?

Red-Black Trees

Which of the following can be colored as a red-black tree? Either give a coloring or explain why not.







This one can't be colored.

THIS path from the root to NIL can contain at most 3 black nodes (incuding NIL).

But THIS one must have at least 4: The root, NIL, and then at least two internal ones.