

Lecture 12

More Bellman-Ford, Floyd-Warshall,
and Dynamic Programming!

WiCS Speed Mentoring Night!

WiCS Speed Mentoring Night

network with grad students + learn about their research!

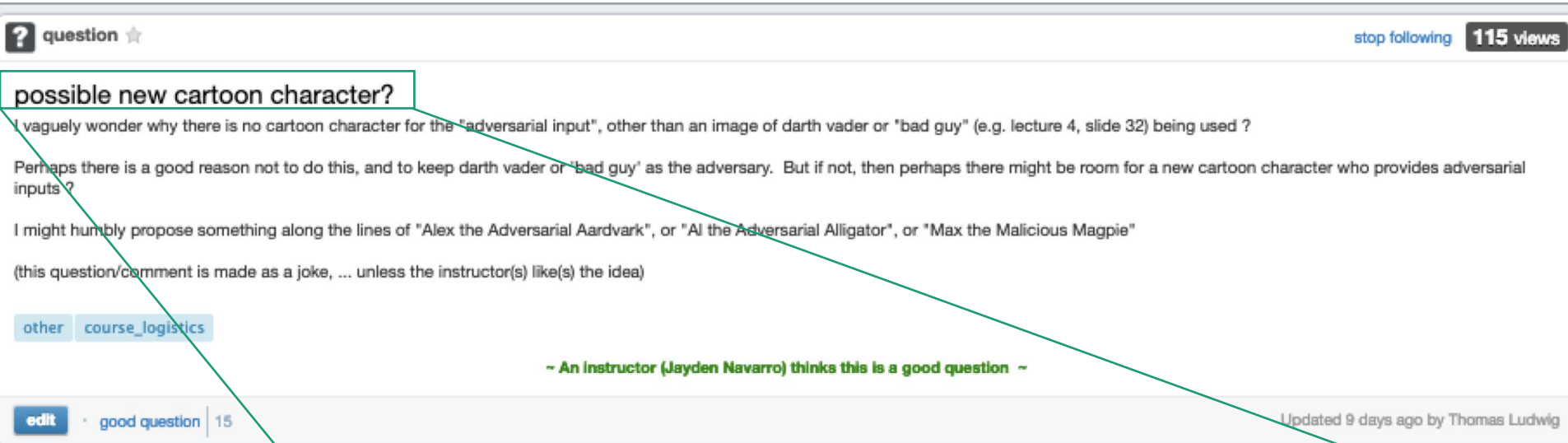
who Grad students sharing their research!
date + time FEB 26 @ 6-7 PM
location Gates 219



Announcements

- HW5 due Friday
- The midterm was hard!
 - But I think you guys did great!
 - Grades and solutions released soon...

Important feedback from Piazza

A screenshot of a Piazza question interface. The question title is "possible new cartoon character?". The content discusses the lack of a cartoon character for "adversarial input" and proposes "Alex the Adversarial Aardvark", "Al the Adversarial Alligator", or "Max the Malicious Magpie". The question is marked as a "good question" by an instructor and has 15 answers. A green box highlights the question title, and a green line connects it to a larger green box below containing the same text. The interface includes a "question" header with a star icon, a "stop following" button, and a "115 views" badge. The question title is in a green box. The content is in a light gray box. The question is marked as a "good question" by an instructor. The question has 15 answers. The question is updated 9 days ago by Thomas Ludwig.

? question ☆ stop following 115 views

possible new cartoon character?

I vaguely wonder why there is no cartoon character for the "adversarial input", other than an image of darth vader or "bad guy" (e.g. lecture 4, slide 32) being used ?

Perhaps there is a good reason not to do this, and to keep darth vader or "bad guy" as the adversary. But if not, then perhaps there might be room for a new cartoon character who provides adversarial inputs ?

I might humbly propose something along the lines of "Alex the Adversarial Aardvark", or "Al the Adversarial Alligator", or "Max the Malicious Magpie"

(this question/comment is made as a joke, ... unless the instructor(s) like(s) the idea)

other course_logistics

~ An instructor (Jayden Navarro) thinks this is a good question ~

edit · good question 15 Updated 9 days ago by Thomas Ludwig

possible new cartoon character?

Anakin the Adversarial Aardvark

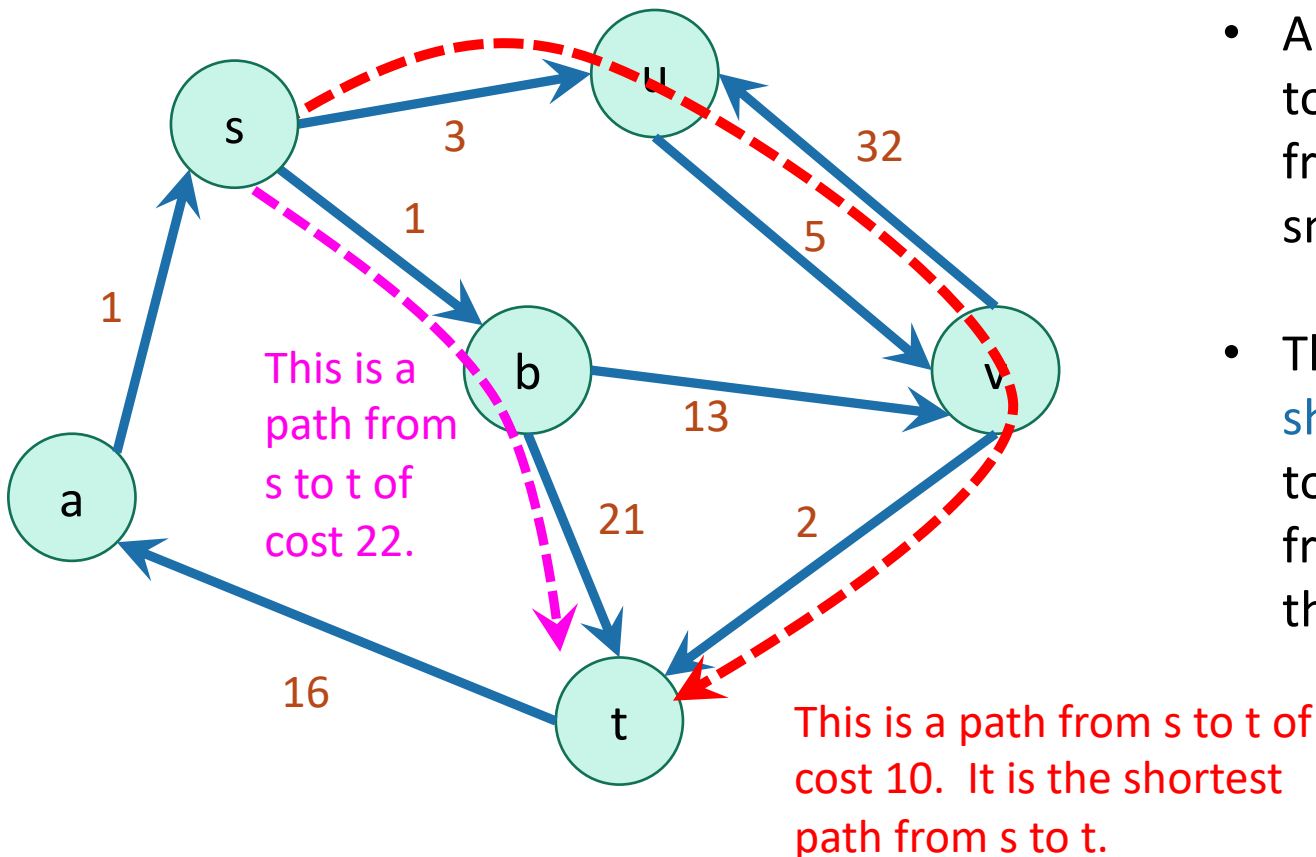


Today

- Bellman-Ford continued
- Bellman-Ford is a special case of ***Dynamic Programming!***
- What is dynamic programming?
 - Warm-up example: Fibonacci numbers
- Another example:
 - Floyd-Warshall Algorithm

Recall

- A weighted directed graph:

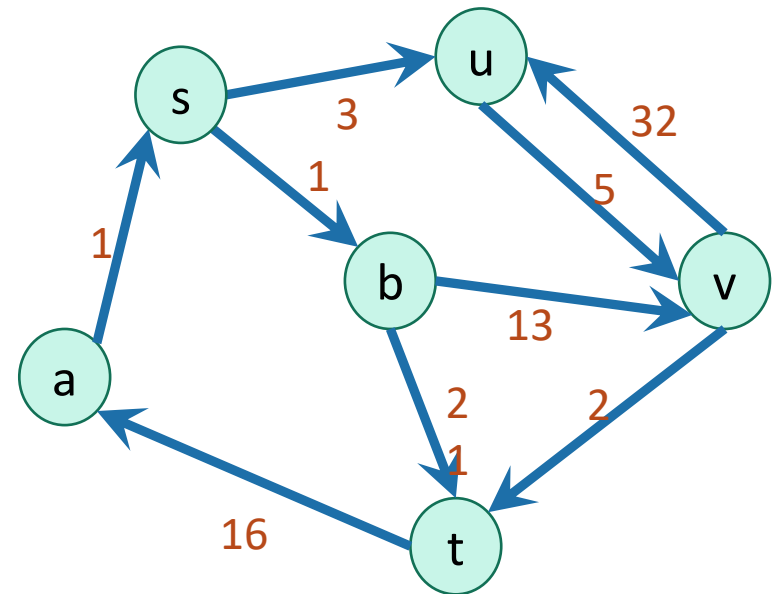


- Weights on edges represent **costs**.
- The **cost of a path** is the sum of the weights along that path.
- A **shortest path** from s to t is a directed path from s to t with the smallest cost.
- The **single-source shortest path problem** is to find the shortest path from s to v for all v in the graph.

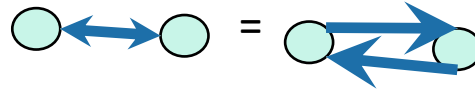
Last time

- Dijkstra's algorithm!
- Bellman-Ford algorithm!
 - Both solve single-source shortest path in weighted graphs.

We sped through the Bellman-Ford algorithm so let's do that now in a bit more detail.



Bellman-Ford



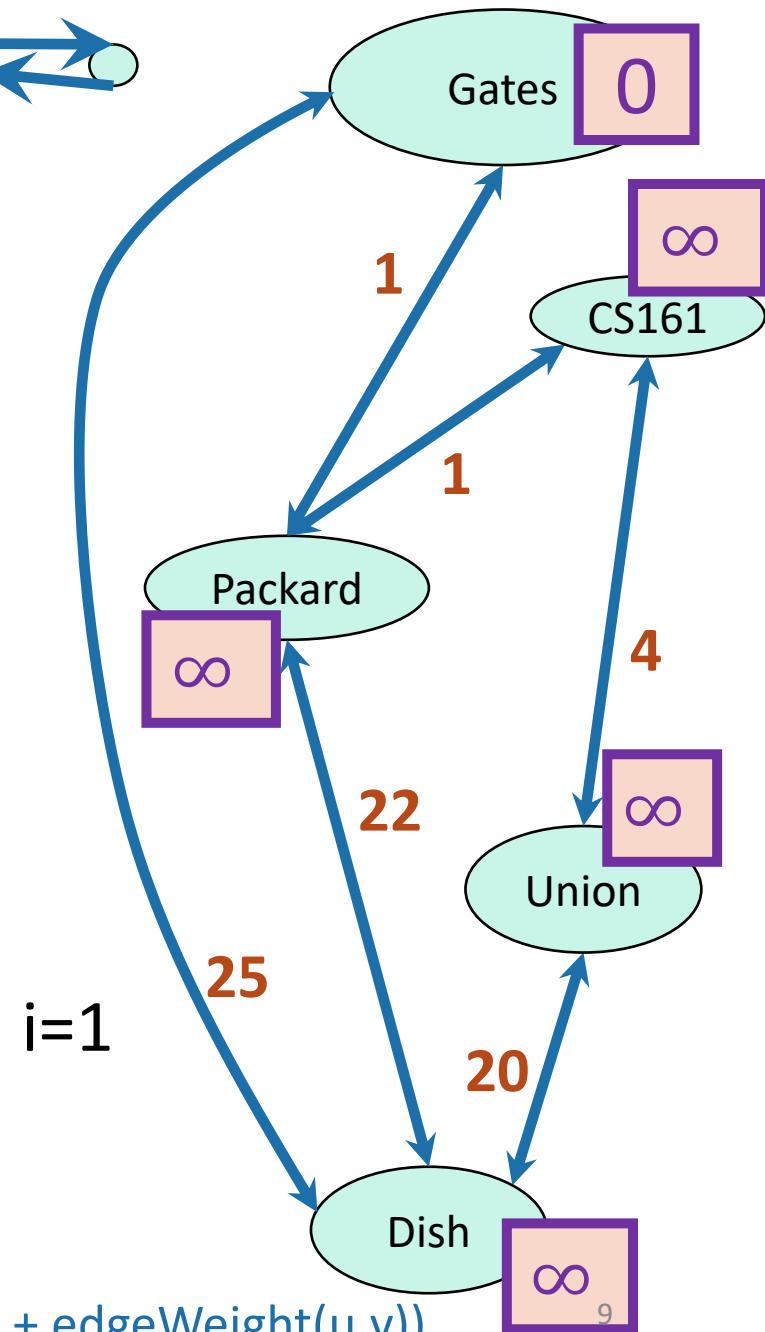
How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$					
$d^{(2)}$					
$d^{(3)}$					
$d^{(4)}$					

WHOOOPS! The pseudocode was messed up during lecture! It's still a bit messed up since $d^{(i+1)}$ might not get initialized before it's used – just assume that everything is initialized to Infinity. (We'll re-write the pseudocode later in the lecture so it won't matter too much.)

- For $i=0, \dots, n-2$:
 - For u in V :
 - For v in $u.\text{neighbors}$:

- $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

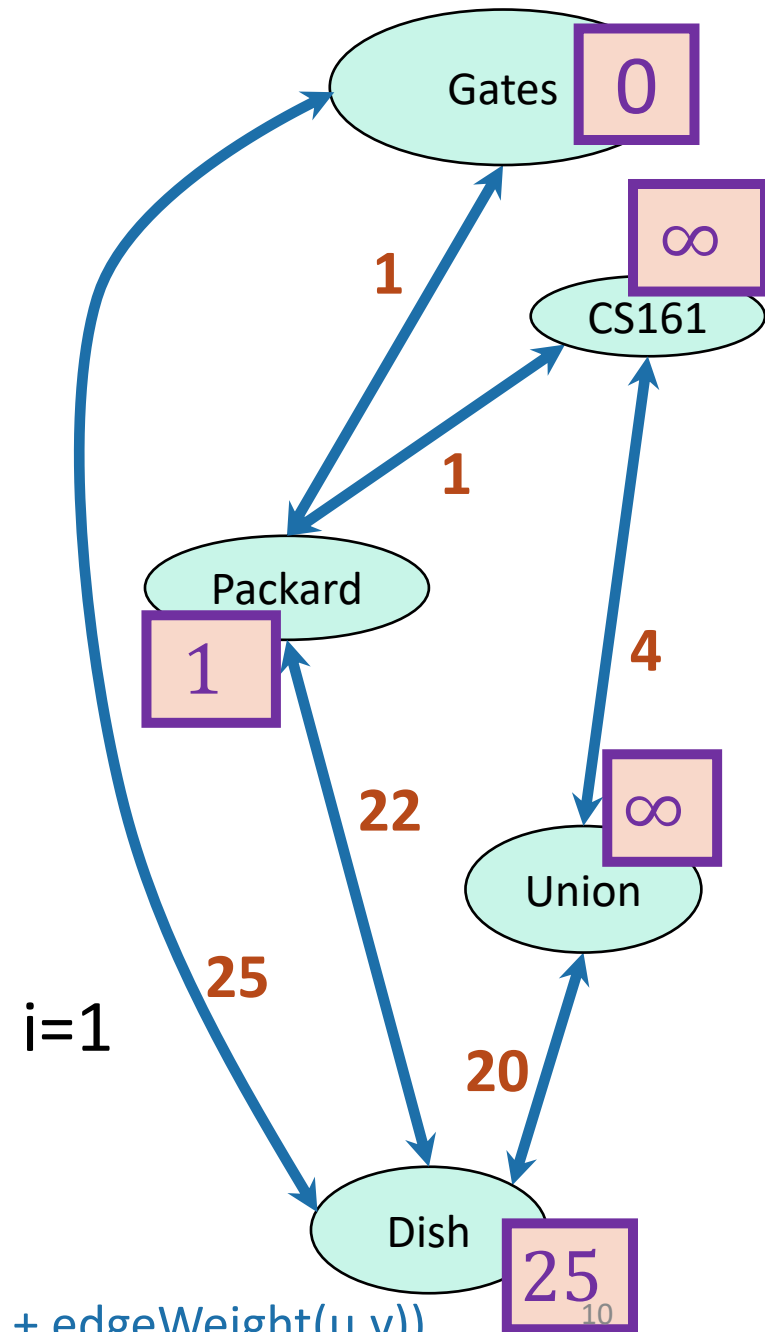


Bellman-Ford

How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	1	∞	∞	25
$d^{(2)}$					
$d^{(3)}$					
$d^{(4)}$					

- For $i=0, \dots, n-2$:
 - For u in V :
 - For v in $u.\text{neighbors}$:
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

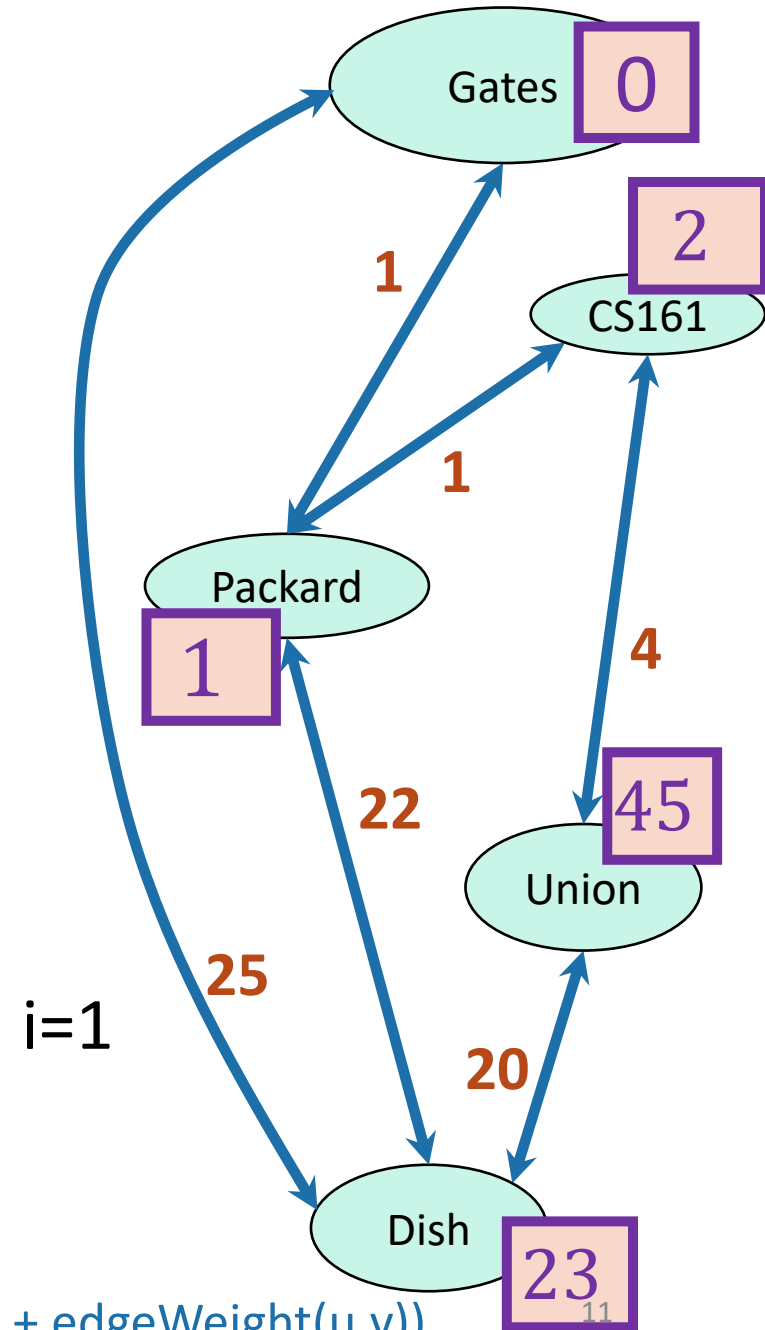


Bellman-Ford

How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	1	∞	∞	25
$d^{(2)}$	0	1	2	45	23
$d^{(3)}$					
$d^{(4)}$					

- For $i=0, \dots, n-2$:
 - For u in V :
 - For v in $u.\text{neighbors}$:
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

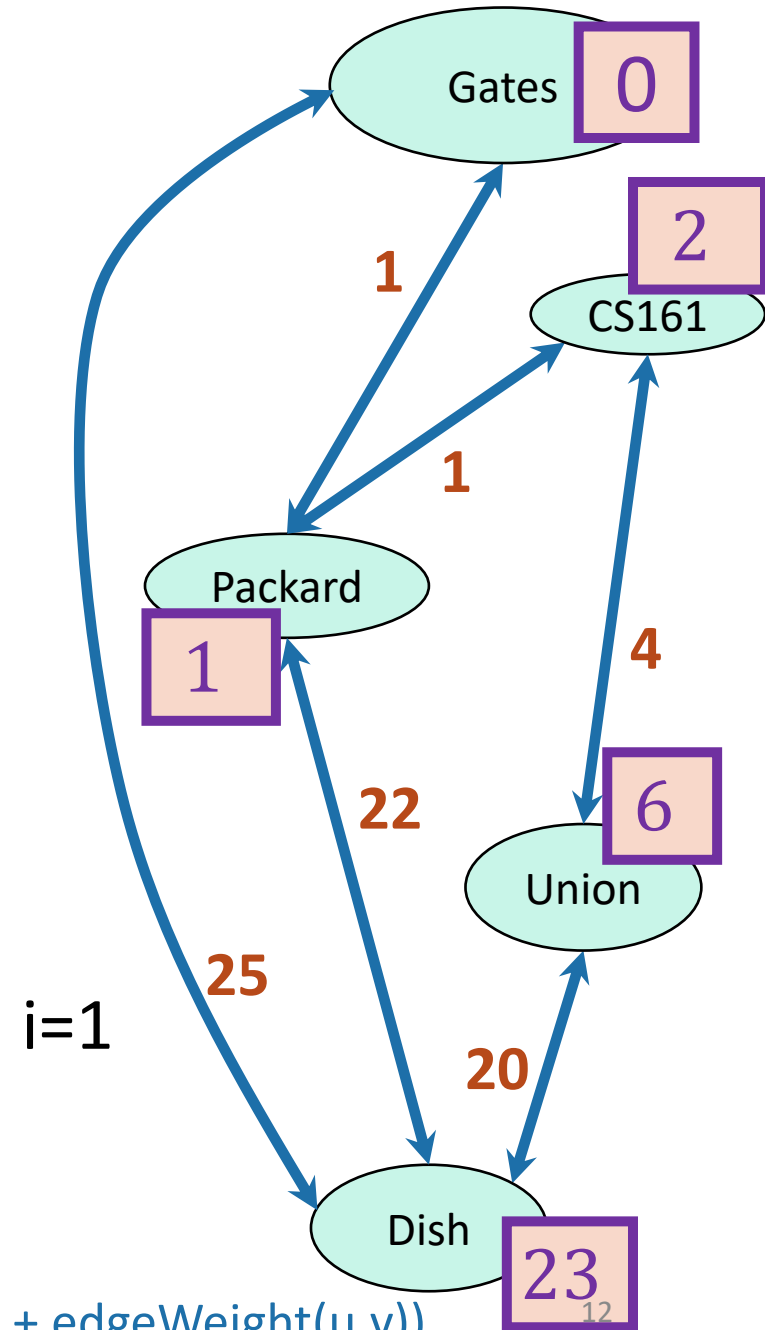


Bellman-Ford

How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	1	∞	∞	25
$d^{(2)}$	0	1	2	45	23
$d^{(3)}$	0	1	2	6	23
$d^{(4)}$					

- For $i=0, \dots, n-2$:
 - For u in V :
 - For v in $u.\text{neighbors}$:
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$



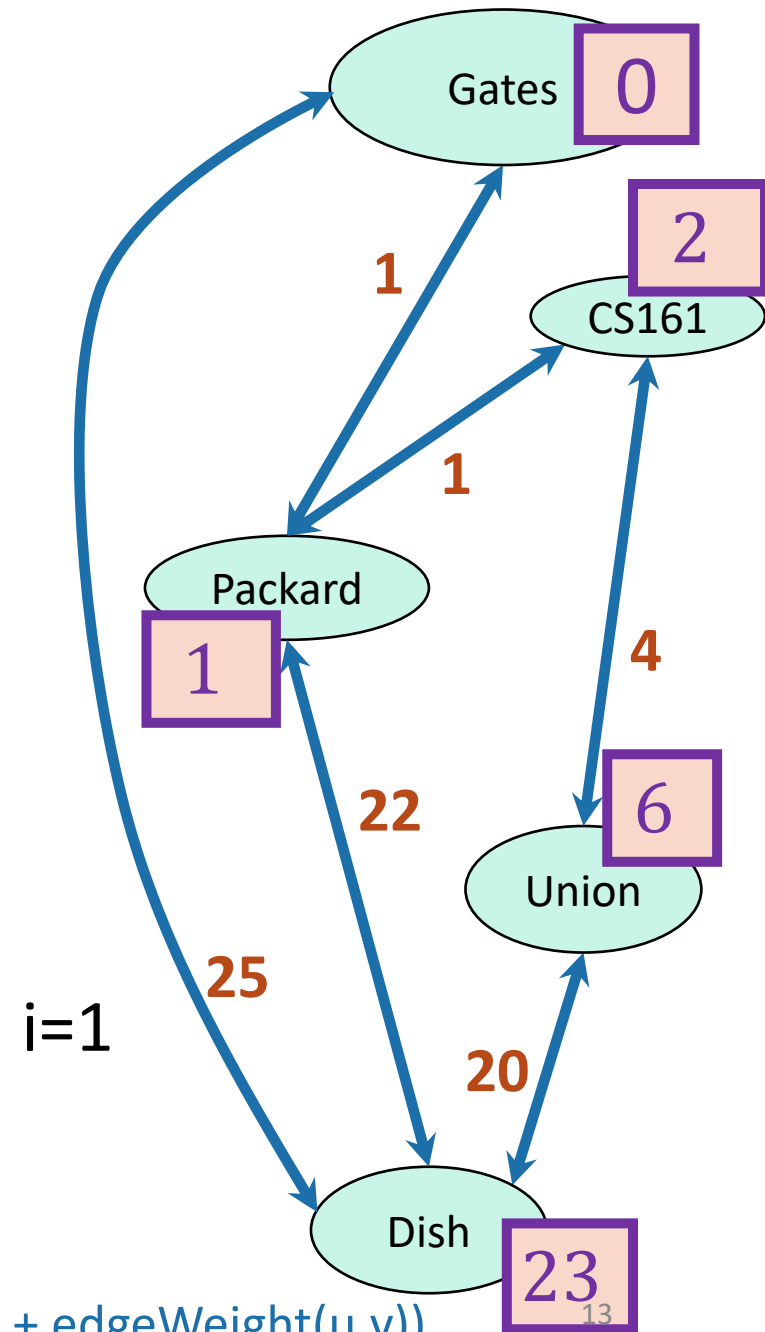
Bellman-Ford

How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	1	∞	∞	25
$d^{(2)}$	0	1	2	45	23
$d^{(3)}$	0	1	2	6	23
$d^{(4)}$	0	1	2	6	23

These are the final distances!

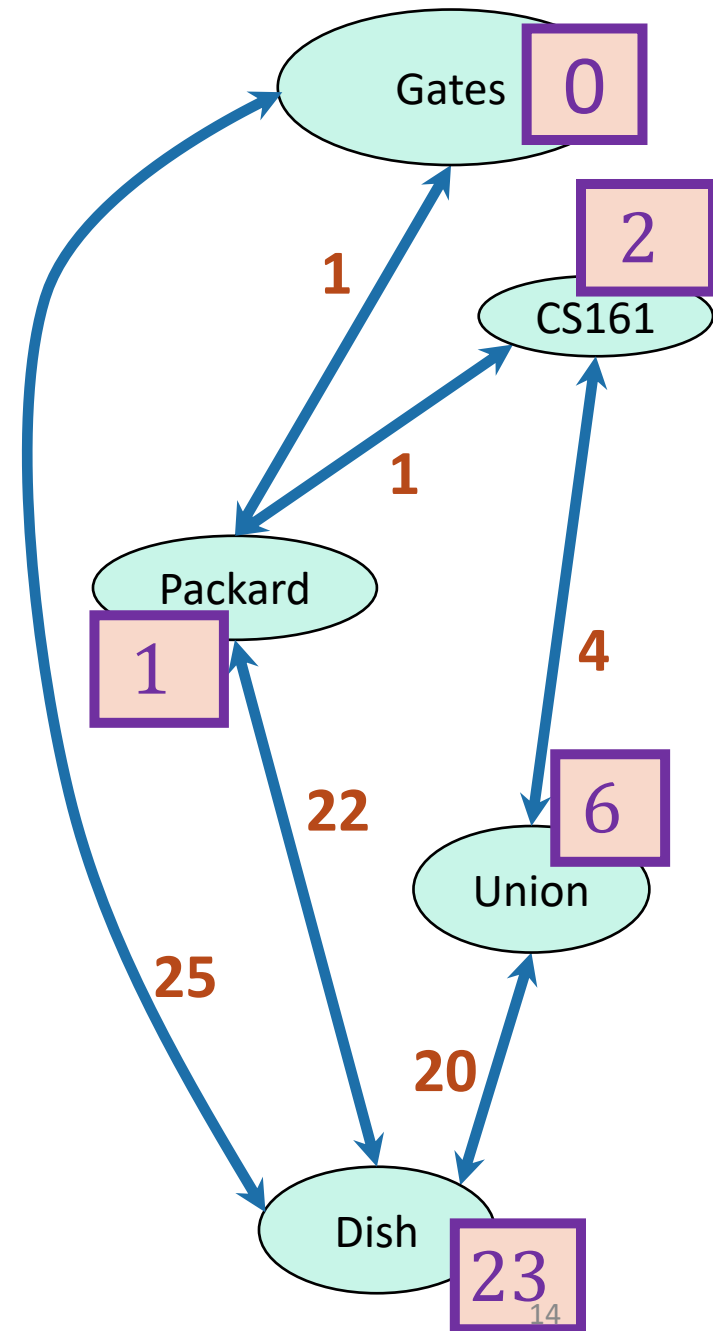
- For $i=0, \dots, n-2$:
 - For u in V :
 - For v in $u.\text{neighbors}$:
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i+1)}[v], d^{(i)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$



Interpretation of $d^{(i)}$

$d^{(i)}[v]$ is equal to the cost of the shortest path between s and v with at most i edges.

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	1	∞	∞	25
$d^{(2)}$	0	1	2	45	23
$d^{(3)}$	0	1	2	6	23
$d^{(4)}$	0	1	2	6	23



Why does Bellman-Ford work?

- Inductive hypothesis:
 - $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v **with at most i edges**.
- Conclusion:
 - $d^{(n-1)}[v]$ is equal to the cost of the shortest path between s and v **with at most $n-1$ edges**.

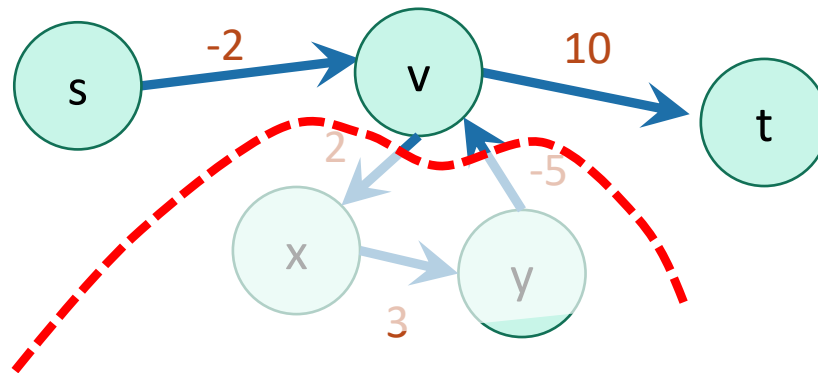
Do the base case and
inductive step!



Aside: simple paths

Assume there is no negative cycle.

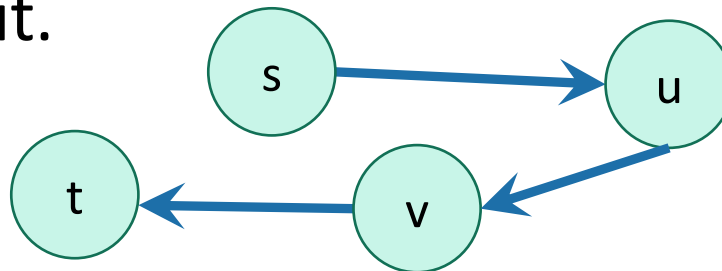
- Then there is a shortest path from s to t , and moreover there is a **simple** shortest path.



This cycle isn't helping.
Just get rid of it.

- A **simple path** in a graph with n vertices has at most $n-1$ edges in it.

Can't add another edge
without making a cycle!



"Simple" means
that the path has
no cycles in it.



- So there is a shortest path with at most $n-1$ edges

Why does it work?

- Inductive hypothesis:
 - $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v **with at most i edges**.
- Conclusion:
 - $d^{(n-1)}[v]$ is equal to the cost of the shortest path between s and v **with at most $n-1$ edges**.
 - **If there are no negative cycles**, $d^{(n-1)}[v]$ is equal to the cost of the shortest path.

Notice that negative edge **weights** are fine.
Just not negative cycles.

Bellman-Ford* algorithm

Bellman-Ford*(G,s):

- Initialize arrays $d^{(0)}, \dots, d^{(n-1)}$ of length n to be all ∞
- $d^{(0)}[s] = 0$
- **For** $i=0, \dots, n-2$:
 - **For** u in V :
 - **For** v in $u.outNeighbors$:
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + w(u,v))$
- Now, $dist(s,v) = d^{(n-1)}[v]$ for all v in V .
 - (Assuming G has no negative cycles)

Here, Dijkstra picked a special vertex u – Bellman-Ford will just look at all the vertices u .

*Slightly different than some versions of Bellman-Ford...but this way is pedagogically convenient for today's lecture.

We can simplify the pseudocode a bit

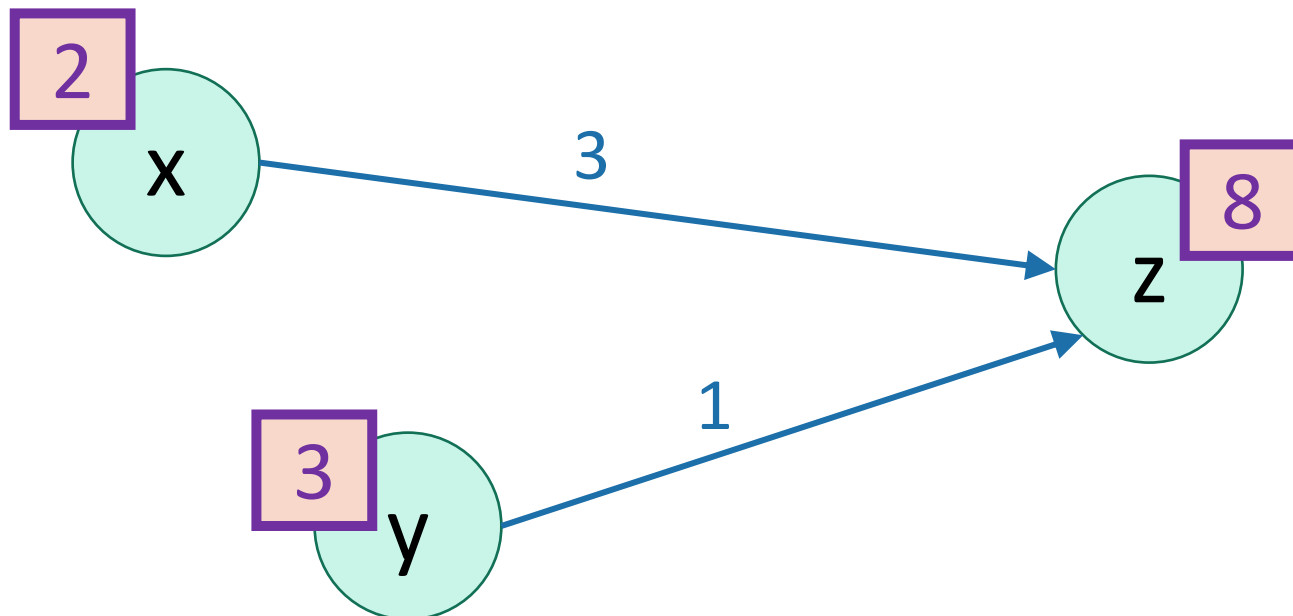
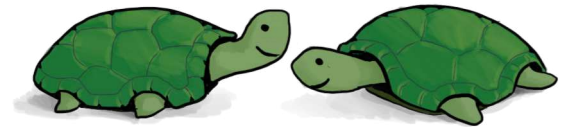
- This will be useful later...

One step of Bellman-Ford

What will happen to z if we run these for-loops?

- **For** u in V:
 - **For** v in u.outNeighbors:

- $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + w(u,v))$

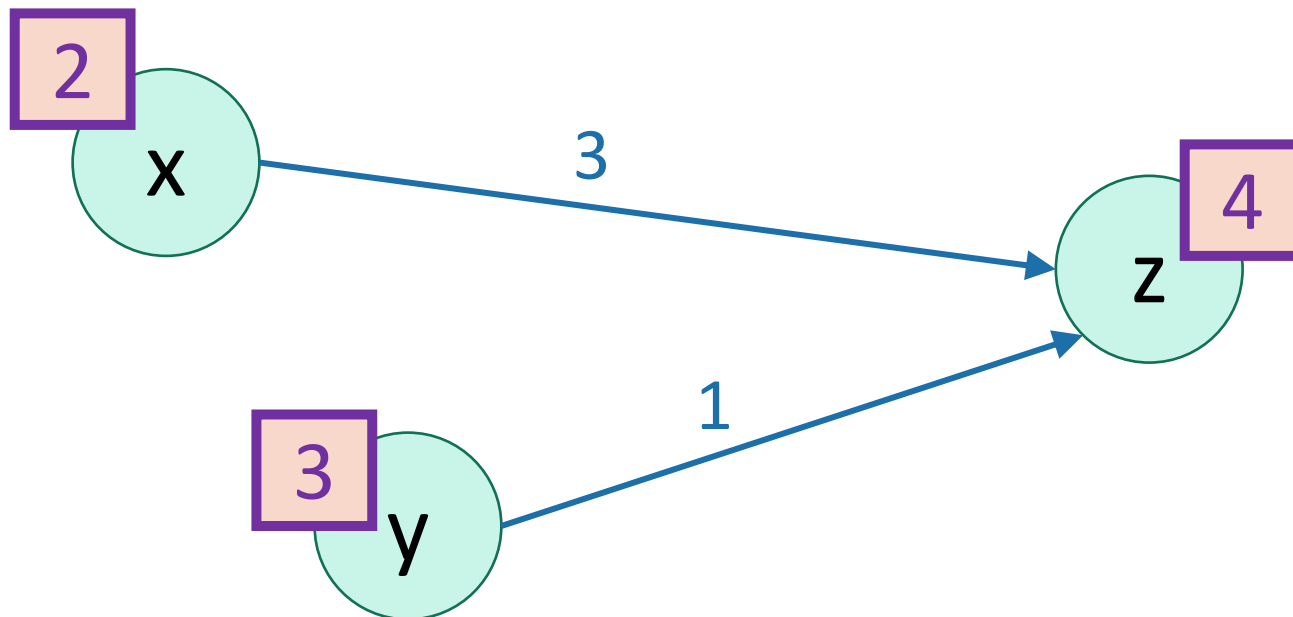
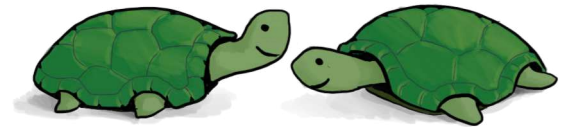


One step of Bellman-Ford

What will happen to z if we run these for loops?

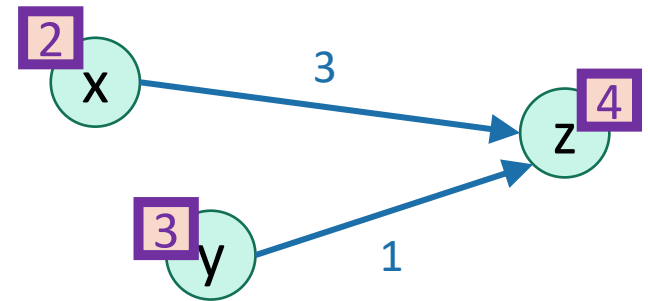
- **For** u in V:
 - **For** v in u.outNeighbors:

- $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + w(u,v))$



One step of Bellman-Ford

- **For** u in V :
 - **For** v in $u.outNeighbors$:
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + w(u,v))$
- Each vertex z finds the in-neighbor u so that $d^{(i)}[u] + w(u,z)$ is smallest and goes with that.
- (Unless z chooses not to update).
- So we can equivalently write:
 - **For** z in V :
 - $d^{(i+1)}[z] \leftarrow \min(d^{(i)}[z], \min_{u \text{ in } z.inNbrs} \{d^{(i)}[u] + w(u,z)\})$



Bellman-Ford* algorithm

Bellman-Ford*(G,s):

- Initialize arrays $d^{(0)}, \dots, d^{(n-1)}$ of length n
- $d^{(0)}[v] = \infty$ for all v in V
- $d^{(0)}[s] = 0$
- **For** $i=0, \dots, n-2$:
 - **For** v in V :
 - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v] , \min_{u \text{ in } v.\text{inNbrs}} \{d^{(i)}[u] + w(u,v)\})$
- Now, $\text{dist}(s,v) = d^{(n-1)}[v]$ for all v in V .

*Slightly different than some versions of Bellman-Ford...but this way is pedagogically convenient for today's lecture.

Note on implementation

- Don't actually keep all n arrays around.
- Just keep two at a time: “last round” and “this round”

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	1	∞	∞	25
$d^{(2)}$	0	1	2	45	23
$d^{(3)}$	0	1	2	6	23
$d^{(4)}$	0	1	2	6	23

Only need these two in order to compute $d^{(4)}$

Bellman-Ford take-aways

- Running time is $O(mn)$
 - For each of n rounds, update m edges.
- Works fine with negative edges.
- Does not work with negative cycles.
 - But it can detect negative cycles!
 - (We waved our hands at this last time)

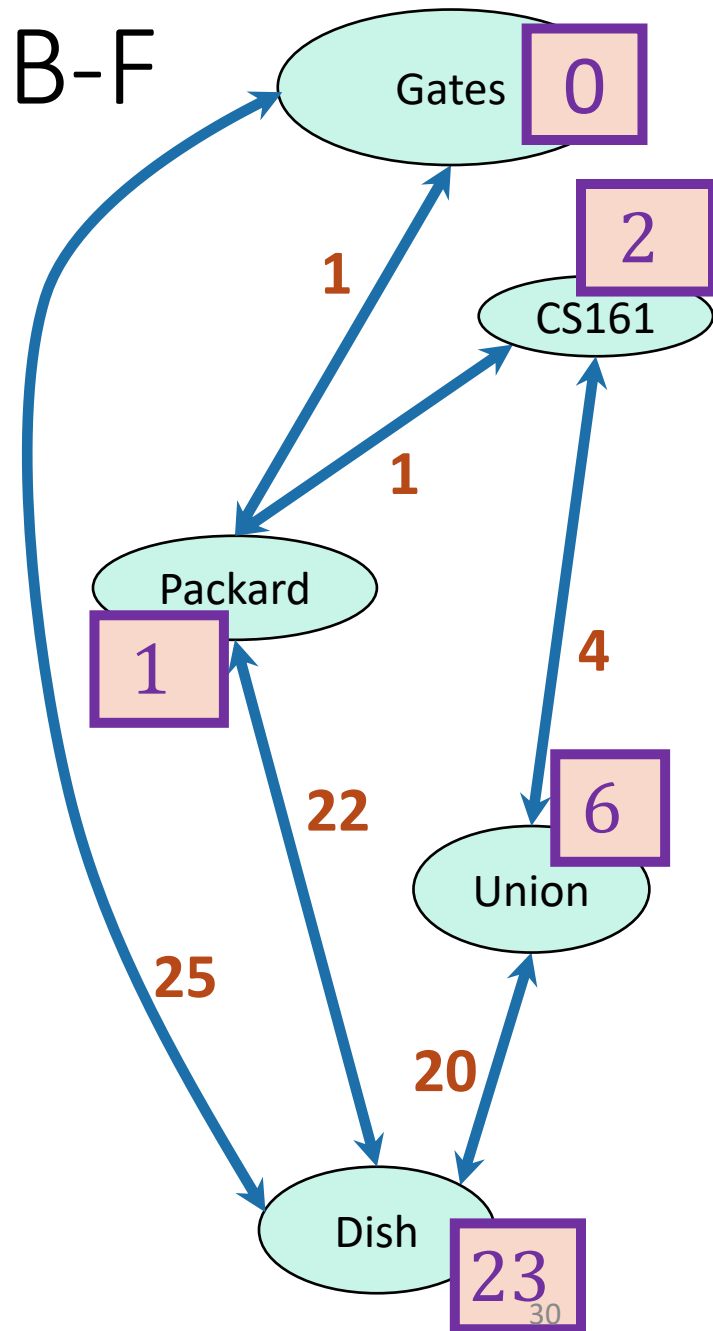
Go through the slides from last time, or CLRS, and understand how to modify Bellman-Ford to handle negative cycles!



Important thing about B-F for the rest of this lecture

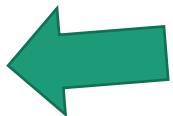
$d^{(i)}[v]$ is equal to the cost of the
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Bellman-Ford is an example of...
Dynamic Programming!

Today:

- Example of Dynamic programming: 
 - Fibonacci numbers.
 - (And Bellman-Ford)
- What is dynamic programming, exactly?
 - And why is it called “dynamic programming”?
- Another example: Floyd-Warshall algorithm
 - An “all-pairs” shortest path algorithm

Pre-Lecture exercise:

How not to compute Fibonacci Numbers

- Definition:
 - $F(n) = F(n-1) + F(n-2)$, with $F(0) = F(1) = 1$.
 - The first several are:
 - 1
 - 1
 - 2
 - 3
 - 5
 - 8
 - 13, 21, 34, 55, 89, 144,...
- Question:
 - Given n , what is $F(n)$?

Candidate algorithm

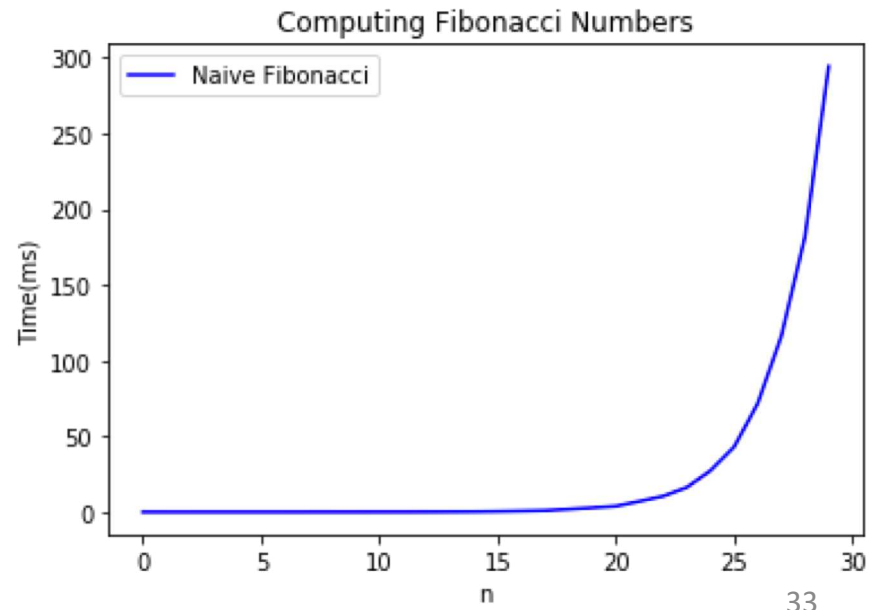
See CLRS Problem 4-4 for a walkthrough of how fast the Fibonacci numbers grow!



```
def Fibonacci(n):  
    if n == 0 or n == 1:  
        return 1  
    return Fibonacci(n-1) + Fibonacci(n-2)
```

Running time?

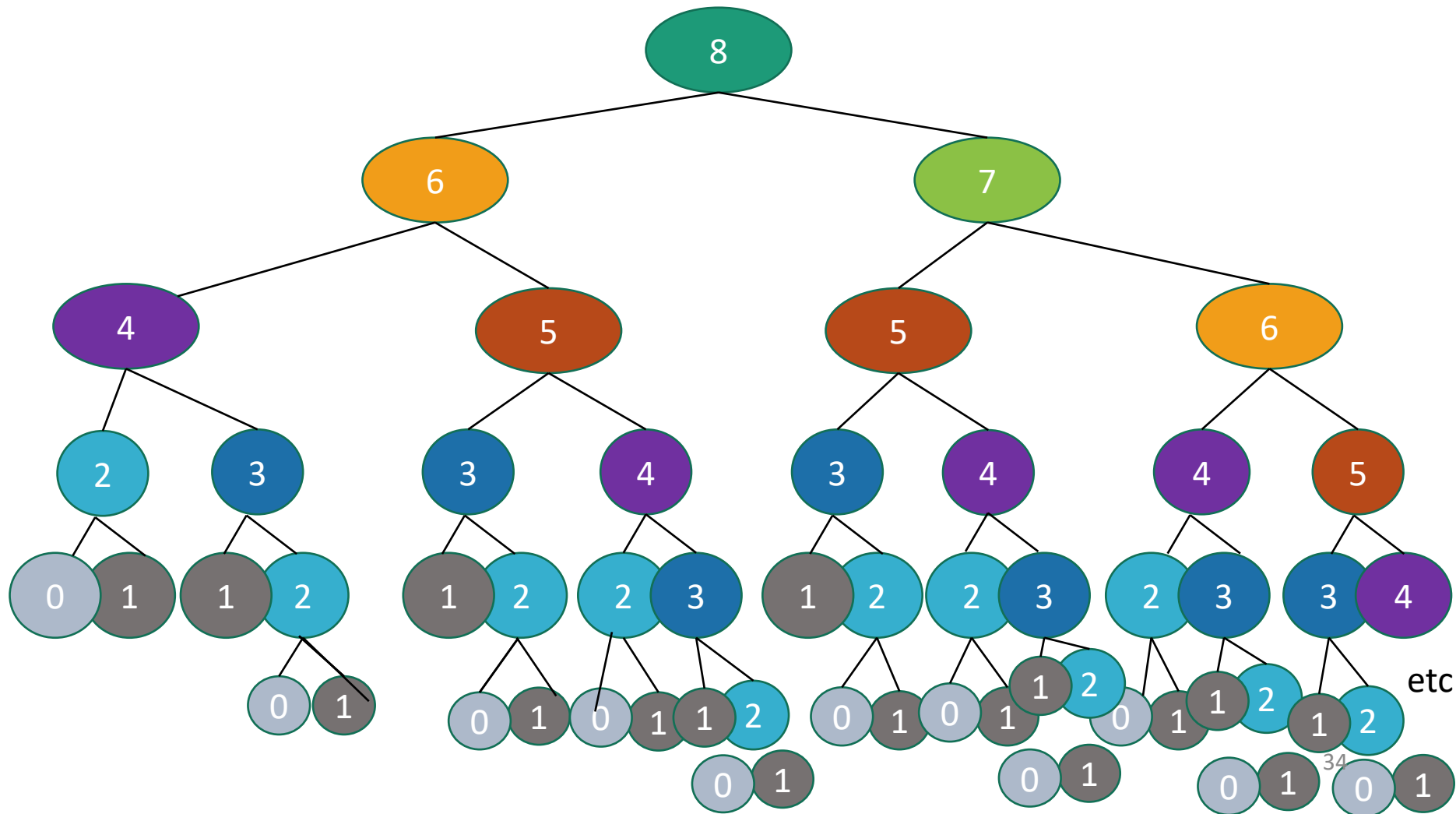
- $T(n) = T(n-1) + T(n-2) + O(1)$
- $T(n) \geq T(n-1) + T(n-2)$ for $n \geq 2$
- So $T(n)$ grows *at least* as fast as the Fibonacci numbers themselves...
- Fun fact, that's like ϕ^n where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
- aka, **EXPONENTIALLY QUICKLY** 😞



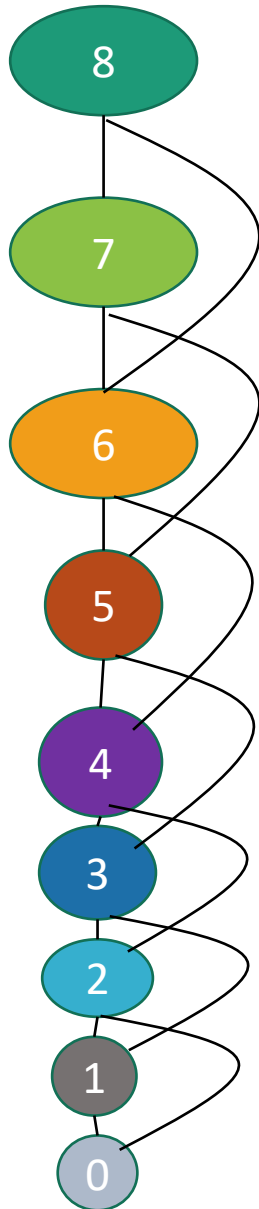
See IPython notebook for lecture 12

What's going on?
Consider $\text{Fib}(8)$

**That's a lot of
repeated
computation!**

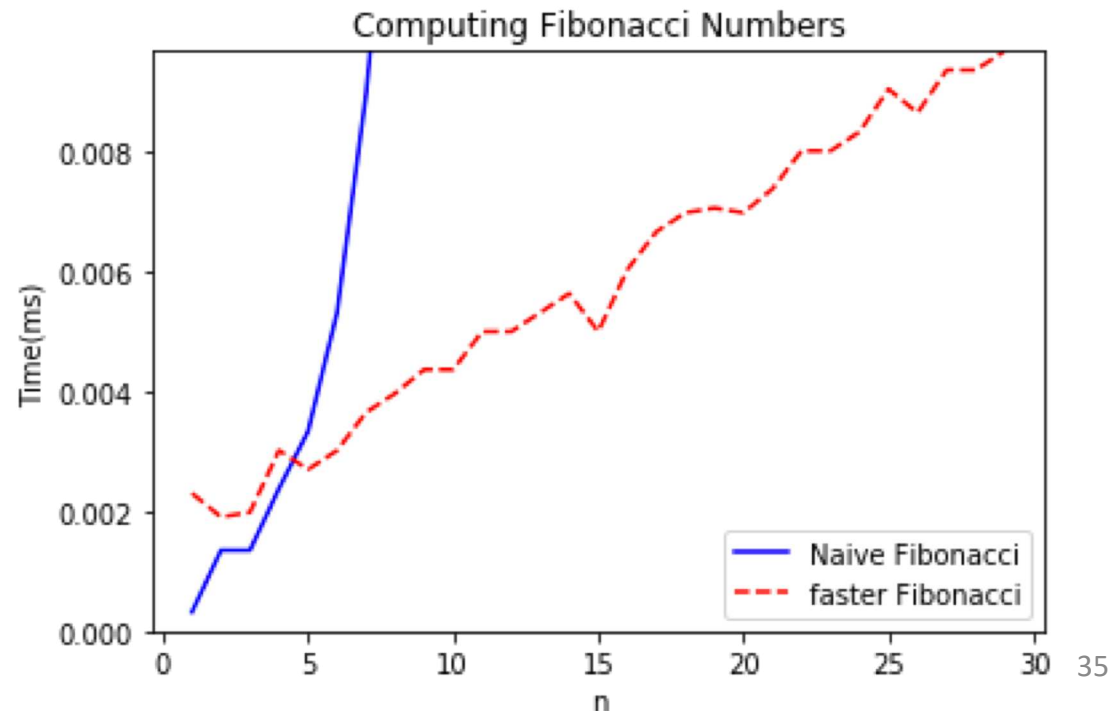


Maybe this would be better:



```
def fasterFibonacci(n):  
    • F = [1, 1, None, None, ..., None ]  
      • \\ F has length n + 1  
    • for i = 2, ..., n:  
      • F[i] = F[i-1] + F[i-2]  
    • return F[n]
```

Much better running time!



This was an example of...

*Dynamic
programming!*

What is *dynamic programming*?

- It is an algorithm design paradigm
 - like divide-and-conquer is an algorithm design paradigm.
- Usually it is for solving **optimization problems**
 - eg, *shortest* path
 - (Fibonacci numbers aren't an optimization problem, but they are a good example...)

Elements of dynamic programming

1. Optimal sub-structure:

- Big problems break up into sub-problems.
 - Fibonacci: $F(i)$ for $i \leq n$
 - Bellman-Ford: Shortest paths with at most i edges for $i \leq n$
- The solution to a problem can be expressed in terms of solutions to smaller sub-problems.
 - Fibonacci:

$$F(i+1) = F(i) + F(i-1)$$

- Bellman-Ford:

$$d^{(i+1)}[v] \leftarrow \min\{d^{(i)}[v], \min_u \{d^{(i)}[u] + \text{weight}(u,v)\}\}$$

Shortest path with at most i edges from s to v

Shortest path with at most i edges from s to u .

Elements of dynamic programming

2. Overlapping sub-problems:

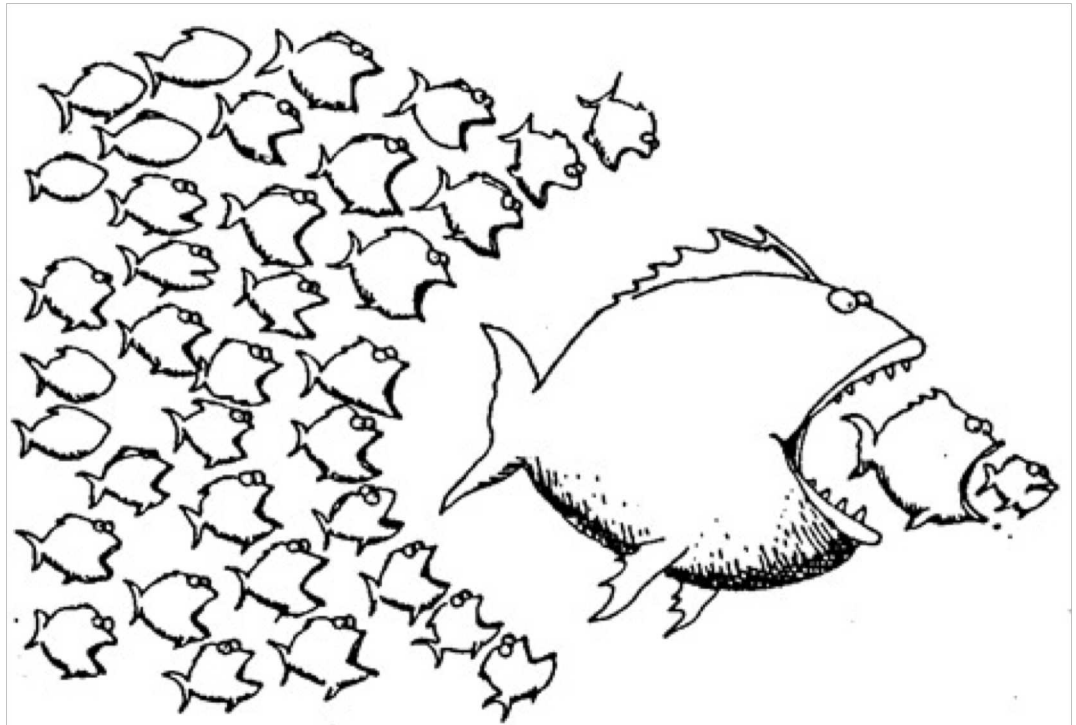
- The sub-problems overlap.
 - Fibonacci:
 - Both $F[i+1]$ and $F[i+2]$ directly use $F[i]$.
 - And lots of different $F[i+x]$ indirectly use $F[i]$.
 - Bellman-Ford:
 - Many different entries of $d^{(i+1)}$ will directly use $d^{(i)}[v]$.
 - And lots of different entries of $d^{(i+x)}$ will indirectly use $d^{(i)}[v]$.
- This means that we can save time by solving a sub-problem just once and storing the answer.

Elements of dynamic programming

- Optimal substructure.
 - Optimal solutions to sub-problems can be used to find the optimal solution of the original problem.
- Overlapping subproblems.
 - The subproblems show up again and again
- Using these properties, we can design a **dynamic programming** algorithm:
 - Keep a table of solutions to the smaller problems.
 - Use the solutions in the table to solve bigger problems.
 - At the end we can use information we collected along the way to find the solution to the whole thing.

Two ways to think about and/or implement DP algorithms

- Top down
- Bottom up



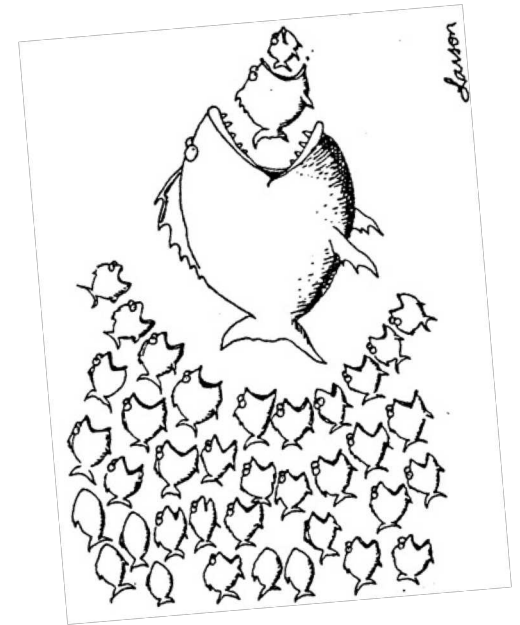
This picture isn't hugely relevant but I like it.

Larson

Bottom up approach

what we just saw.

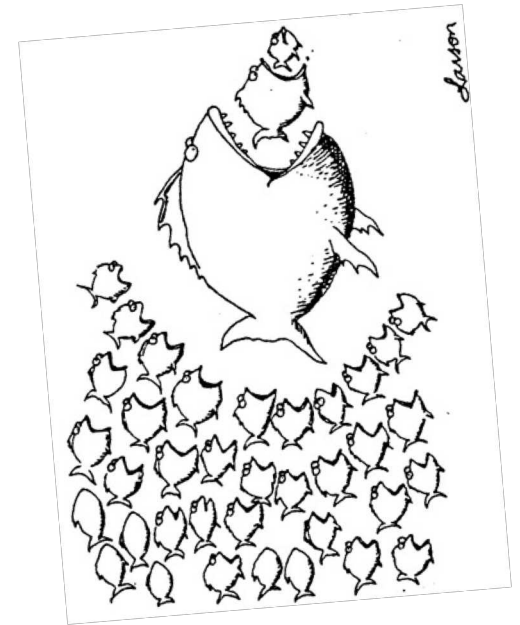
- For Fibonacci:
- Solve the small problems first
 - fill in $F[0], F[1]$
- Then bigger problems
 - fill in $F[2]$
- ...
- Then bigger problems
 - fill in $F[n-1]$
- Then finally solve the real problem.
 - fill in $F[n]$



Bottom up approach

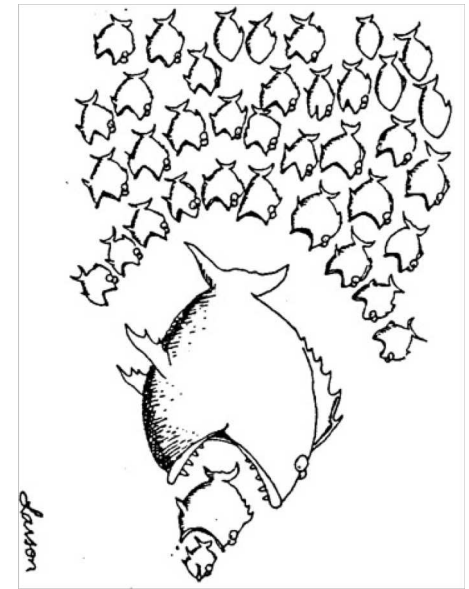
what we just saw.

- For Bellman-Ford:
- Solve the small problems first
 - fill in $d^{(0)}$
- Then bigger problems
 - fill in $d^{(1)}$
- ...
- Then bigger problems
 - fill in $d^{(n-2)}$
- Then finally solve the real problem.
 - fill in $d^{(n-1)}$



Top down approach

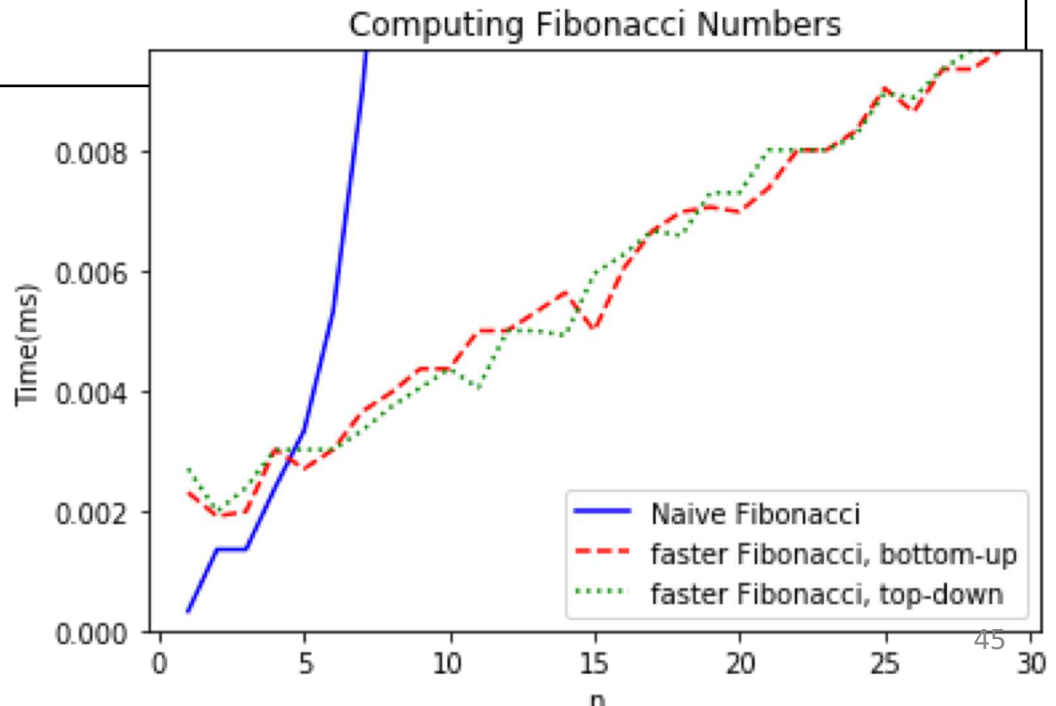
- Think of it like a recursive algorithm.
- To solve the big problem:
 - Recurse to solve smaller problems
 - Those recurse to solve smaller problems
 - etc..
- The difference from divide and conquer:
 - Keep track of what small problems you've already solved to prevent re-solving the same problem twice.
 - Aka, “**memo-ization**”



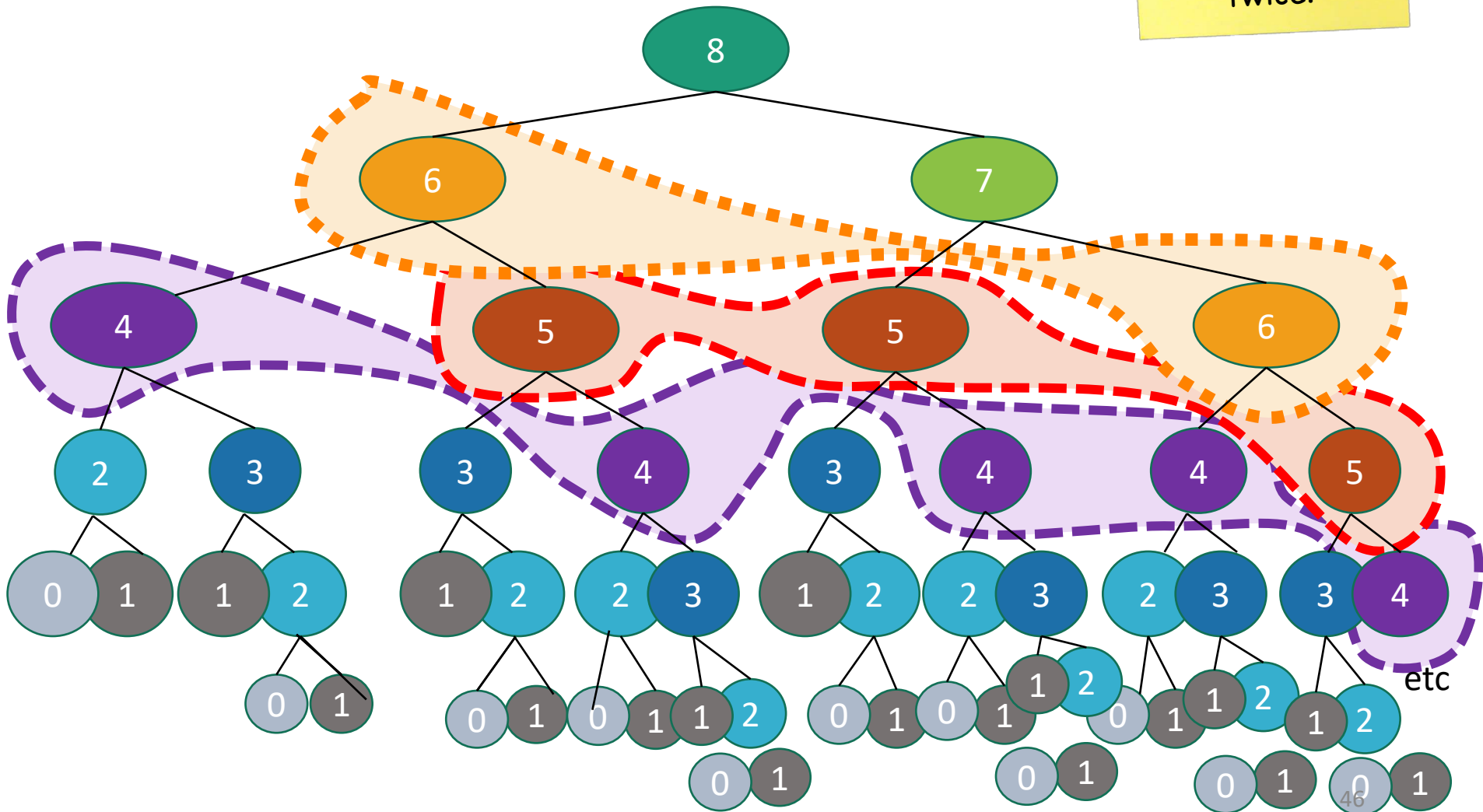
Example of top-down Fibonacci

- define a global list `F = [1,1,None, None, ..., None]`
- **def** `Fibonacci(n):`
 - **if** `F[n] != None:`
 - **return** `F[n]`
 - **else:**
 - `F[n] = Fibonacci(n-1) + Fibonacci(n-2)`
 - **return** `F[n]`

Memo-ization:
Keeps track (in F)
of the stuff you've
already done.



Collapse
repeated nodes
and don't do
the same work
twice!



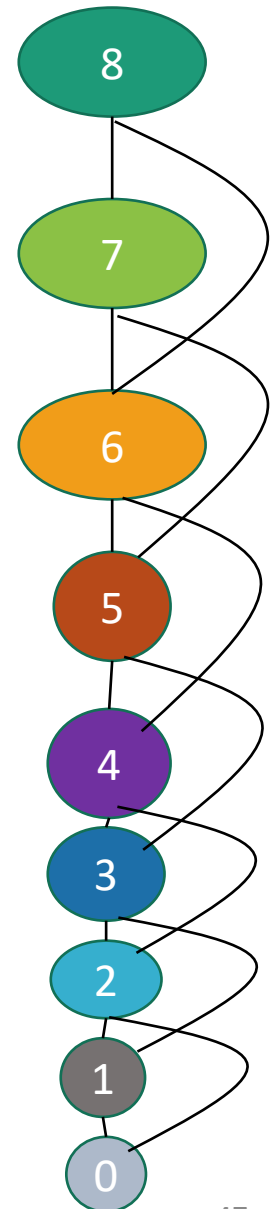
Memo-ization Visualization

ctd

Collapse
repeated nodes
and don't do the
same work
twice!

But otherwise
treat it like the
same old
recursive
algorithm.

```
• define a global list F = [1,1,None, None, ..., None]
• def Fibonacci(n):
    • if F[n] != None:
        • return F[n]
    • else:
        • F[n] = Fibonacci(n-1) + Fibonacci(n-2)
    • return F[n]
```



What have we learned?

- ***Dynamic programming:***

- Paradigm in algorithm design.
- Uses **optimal substructure**
- Uses **overlapping subproblems**
- Can be implemented **bottom-up** or **top-down**.
- It's a fancy name for a pretty common-sense idea:



Why “*dynamic programming*” ?

- **Programming** refers to finding the optimal “program.”
 - as in, a shortest route is a *plan* aka a *program*.
- **Dynamic** refers to the fact that it’s multi-stage.
- But also it’s just a fancy-sounding name.



Manipulating computer code in an action movie?

Why “*dynamic programming*” ?

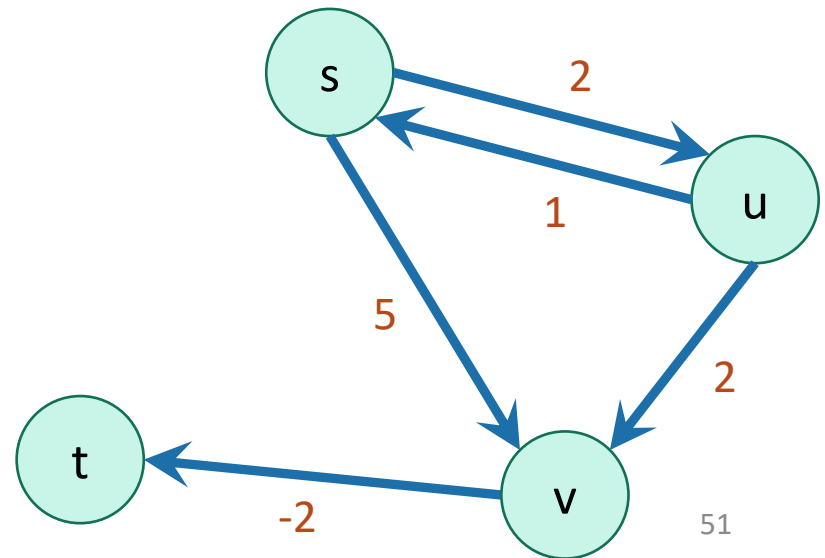
- Richard Bellman invented the name in the 1950's.
- At the time, he was working for the RAND Corporation, which was basically working for the Air Force, and government projects needed flashy names to get funded.
- From Bellman's autobiography:
 - “It's impossible to use the word, dynamic, in the pejorative sense...I thought dynamic programming was a good name. It was something not even a Congressman could object to.”

Floyd-Warshall Algorithm

Another example of DP

- This is an algorithm for **All-Pairs Shortest Paths (APSP)**
 - That is, I want to know the shortest path from u to v for **ALL pairs** u, v of vertices in the graph.
 - Not just from a special single source s .

Source	Destination				
	s	u	v	t	
	s	0	2	4	2
	u	1	0	2	0
	v	∞	∞	0	-2
	t	∞	∞	∞	0



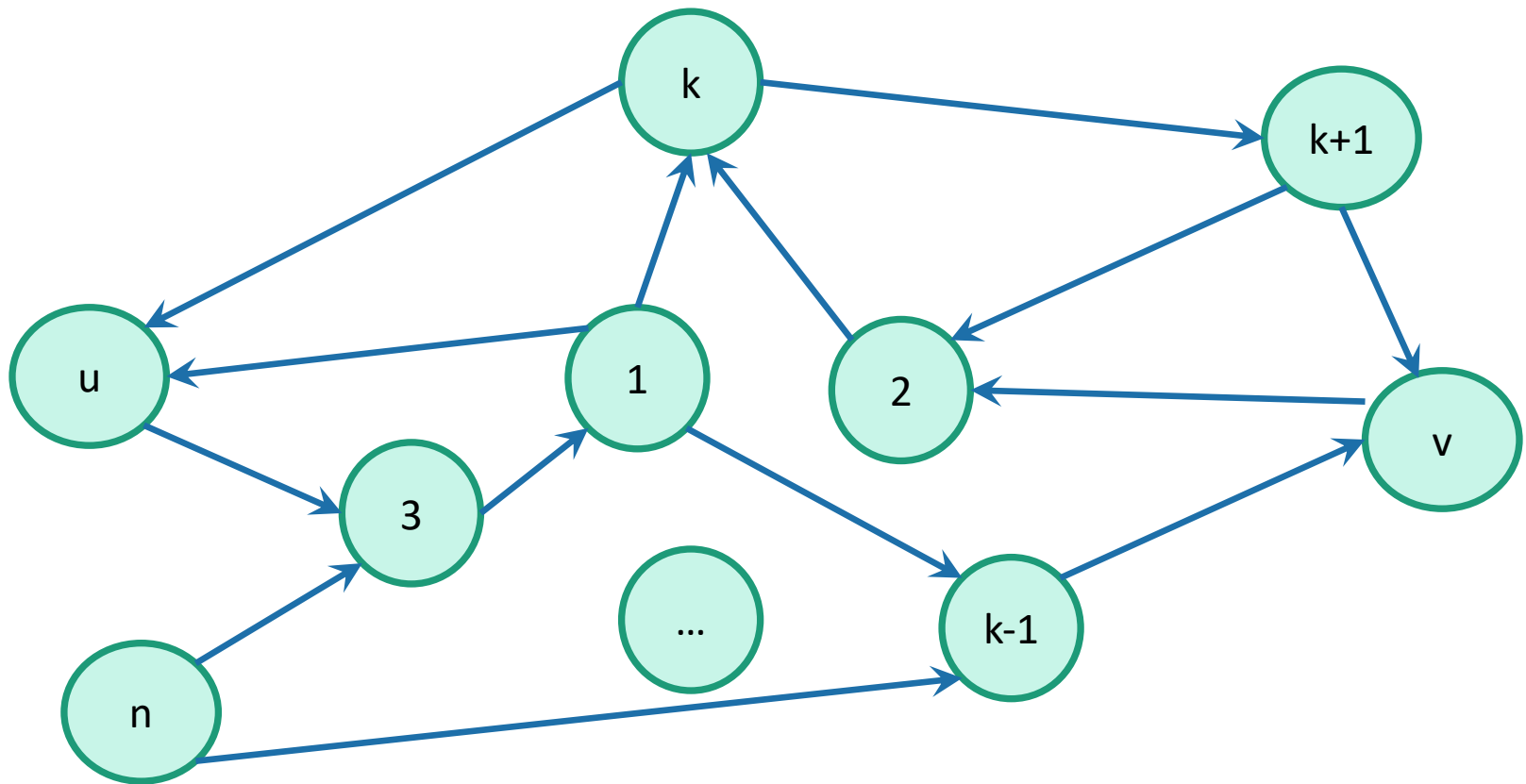
Floyd-Warshall Algorithm

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 - That is, I want to know the shortest path from u to v for **ALL pairs** u, v of vertices in the graph.
 - Not just from a special single source s .
- Naïve solution (if we want to handle negative edge weights):
 - For all s in G :
 - Run Bellman-Ford on G starting at s .
 - Time $O(n \cdot nm) = O(n^2m)$,
 - may be as bad as n^4 if $m=n^2$

Can we do better?

Optimal substructure



Optimal substructure

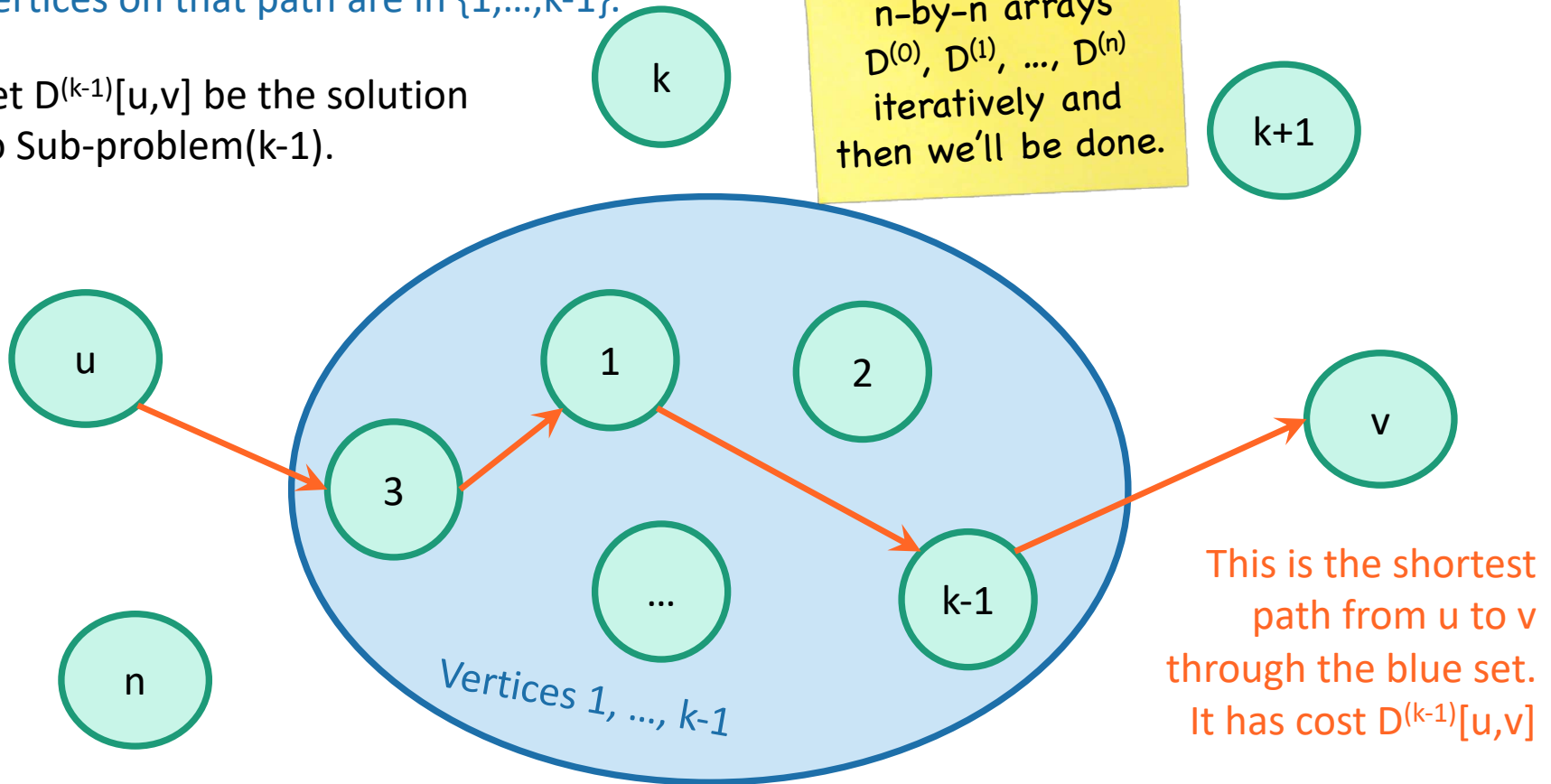
Label the vertices $1, 2, \dots, n$
(We omit some edges in the picture below – meant to be a cartoon, not an example).

Sub-problem($k-1$):

For all pairs, u, v , find the cost of the shortest path from u to v , so that all the internal vertices on that path are in $\{1, \dots, k-1\}$.

Let $D^{(k-1)}[u, v]$ be the solution to Sub-problem($k-1$).

Our DP algorithm will fill in the n -by- n arrays $D^{(0)}, D^{(1)}, \dots, D^{(n)}$ iteratively and then we'll be done.



Optimal substructure

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(We omit some edges in the picture below – meant to be a cartoon, not an example).

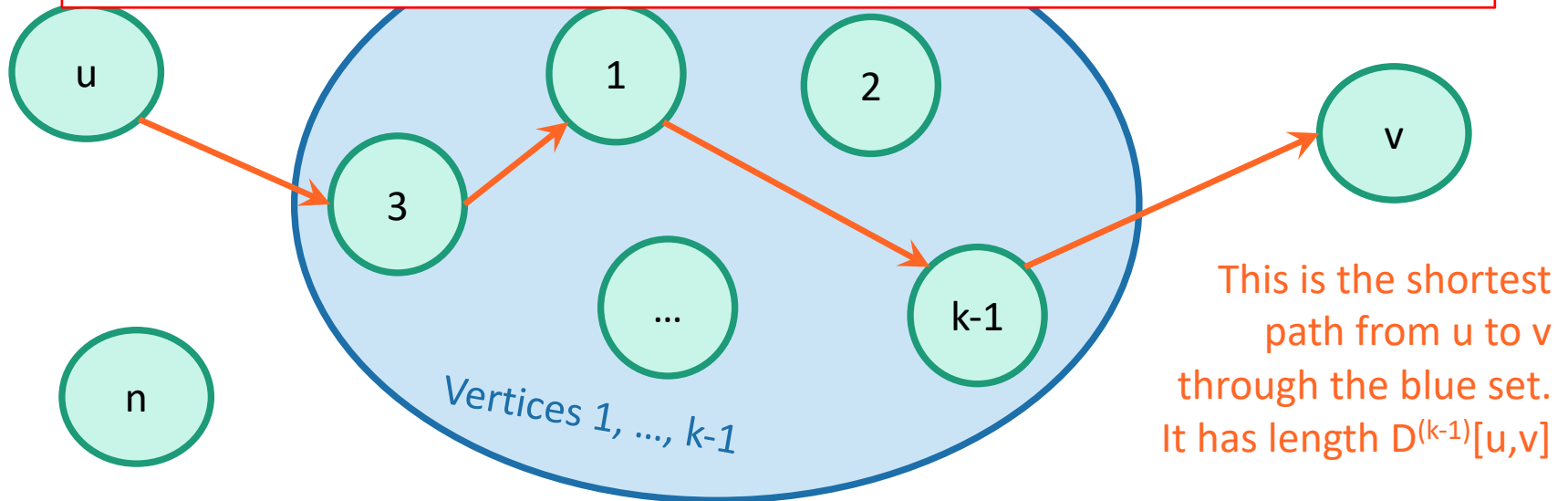
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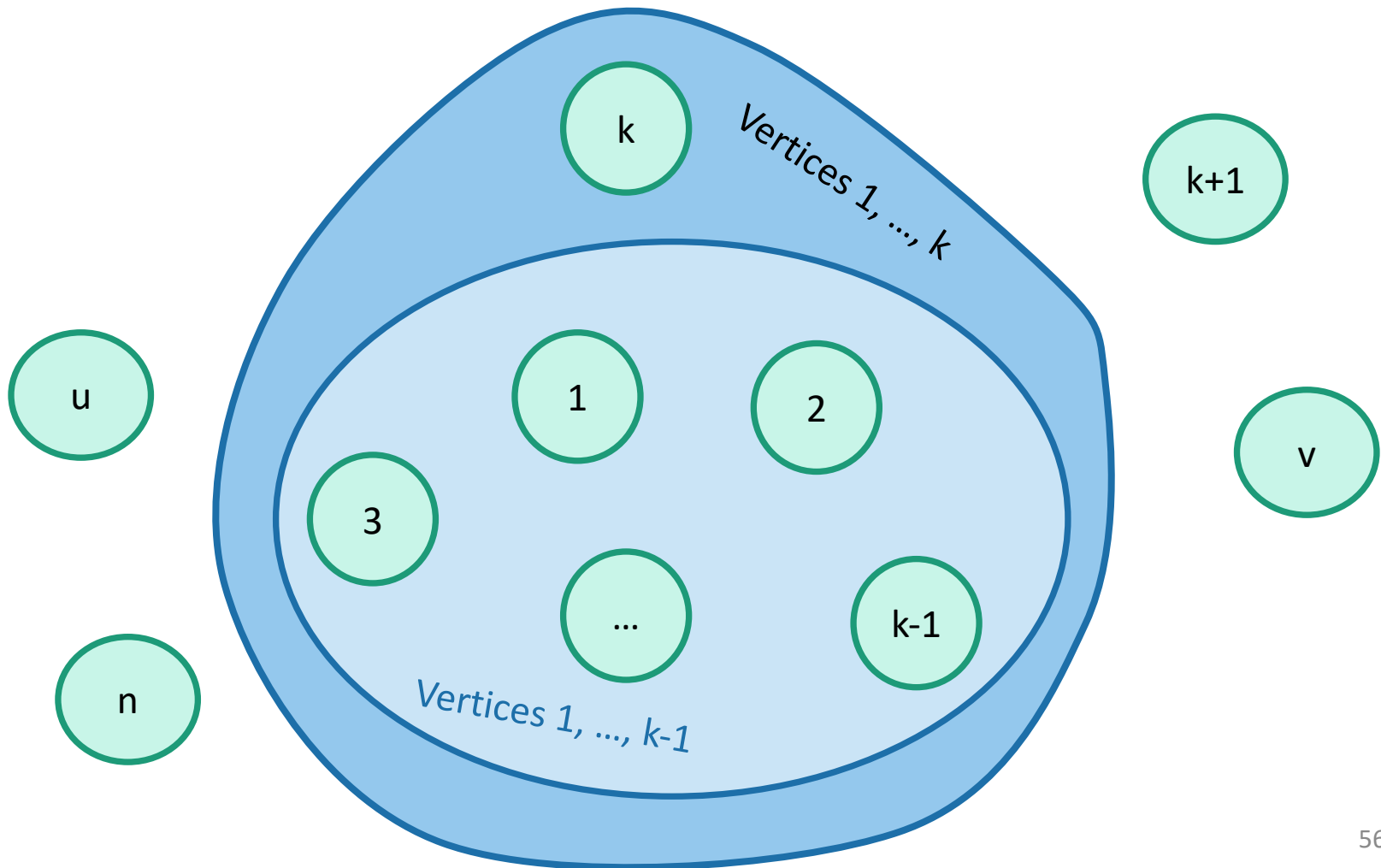
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Question: How can we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?



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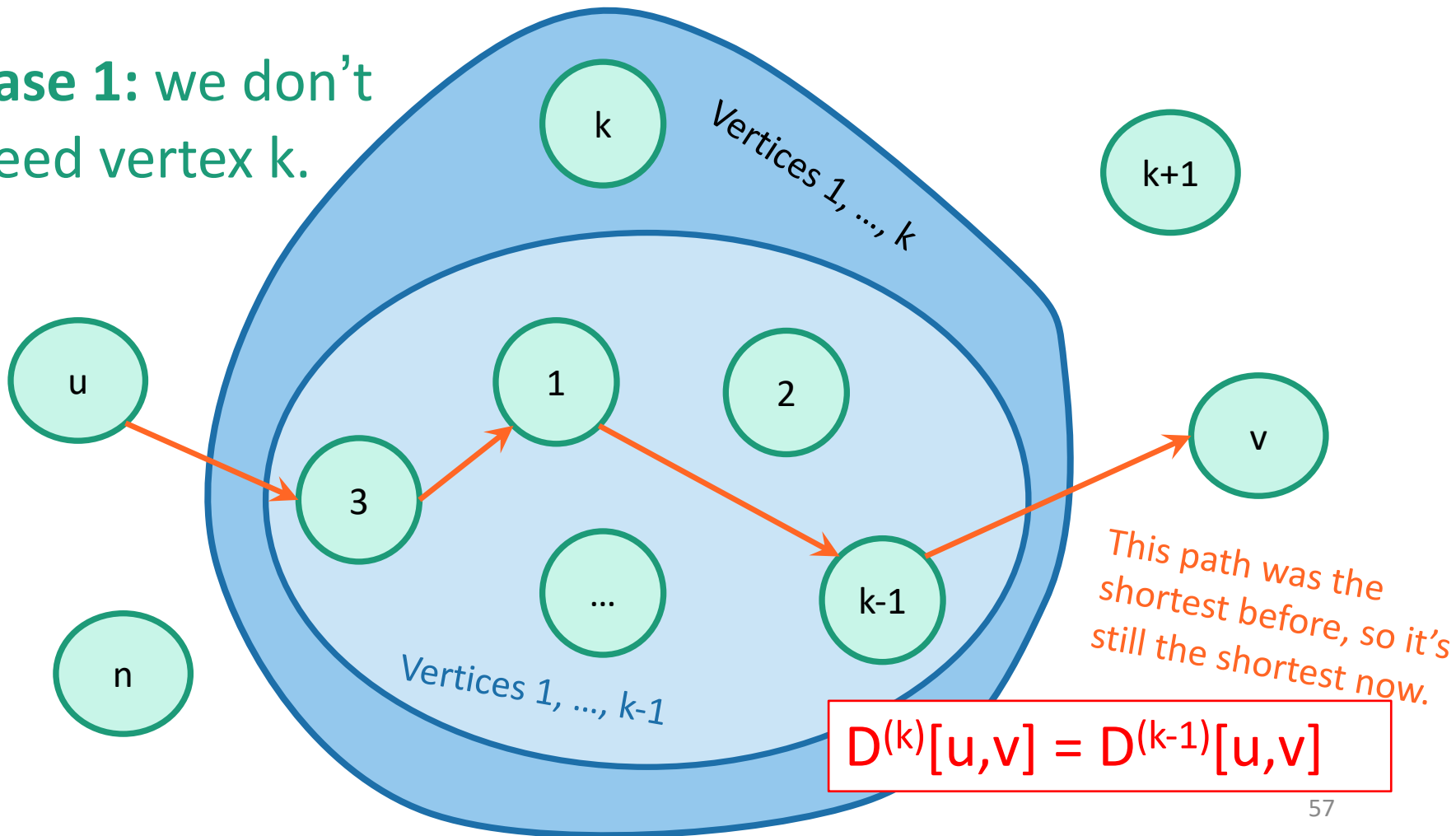
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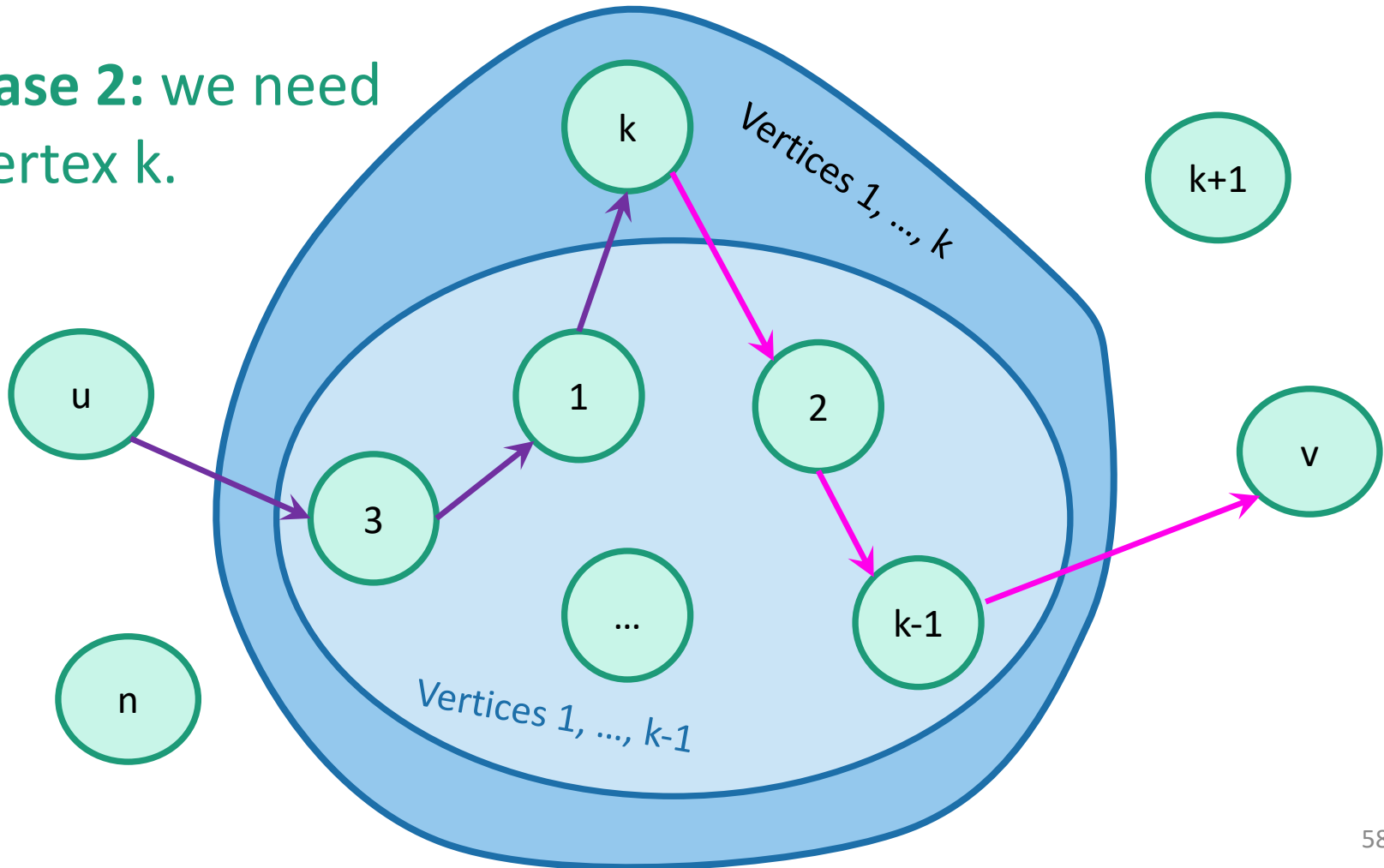
Case 1: we don't need vertex k .



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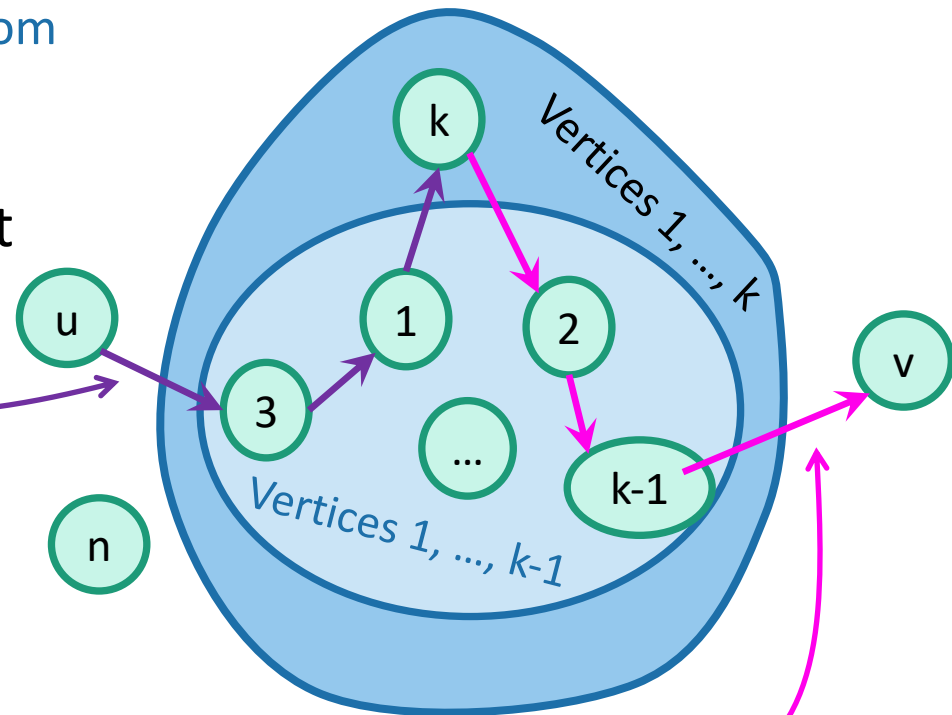
Case 2: we need vertex k .



Case 2 continued

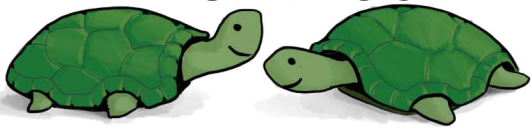
- Suppose there are **no negative cycles**.
 - Then WLOG the shortest path from u to v through $\{1, \dots, k\}$ is **simple**.
- If that path passes through k , it must look like this: 
- This path is the shortest path from u to k through $\{1, \dots, k-1\}$.
 - sub-paths of shortest paths are shortest paths
- Similarly for this path.

Case 2: we need vertex k .

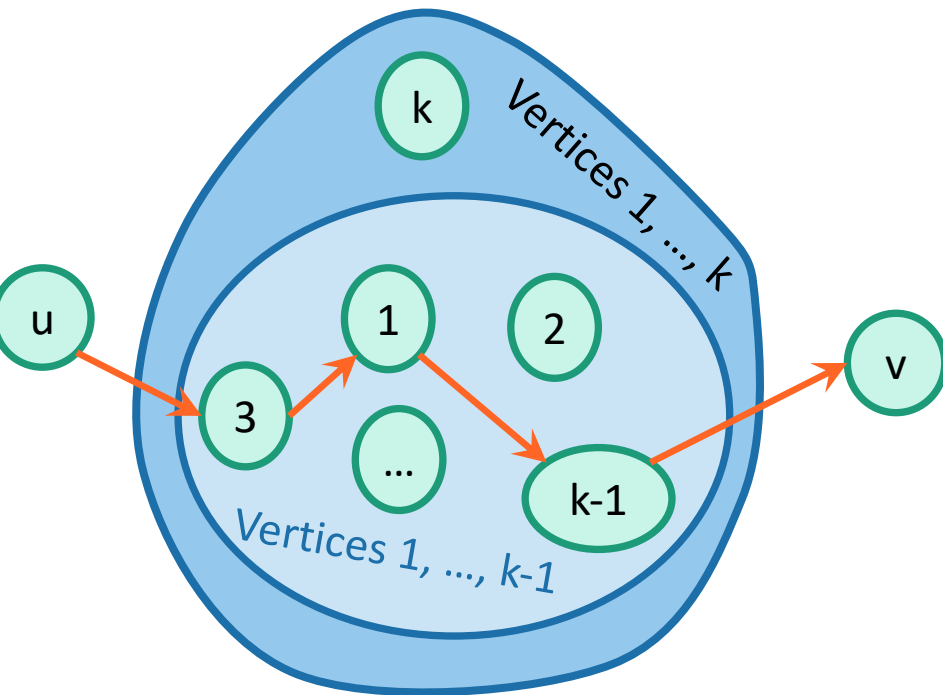


$$D^{(k)}[u, v] = D^{(k-1)}[u, k] + D^{(k-1)}[k, v]$$

How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

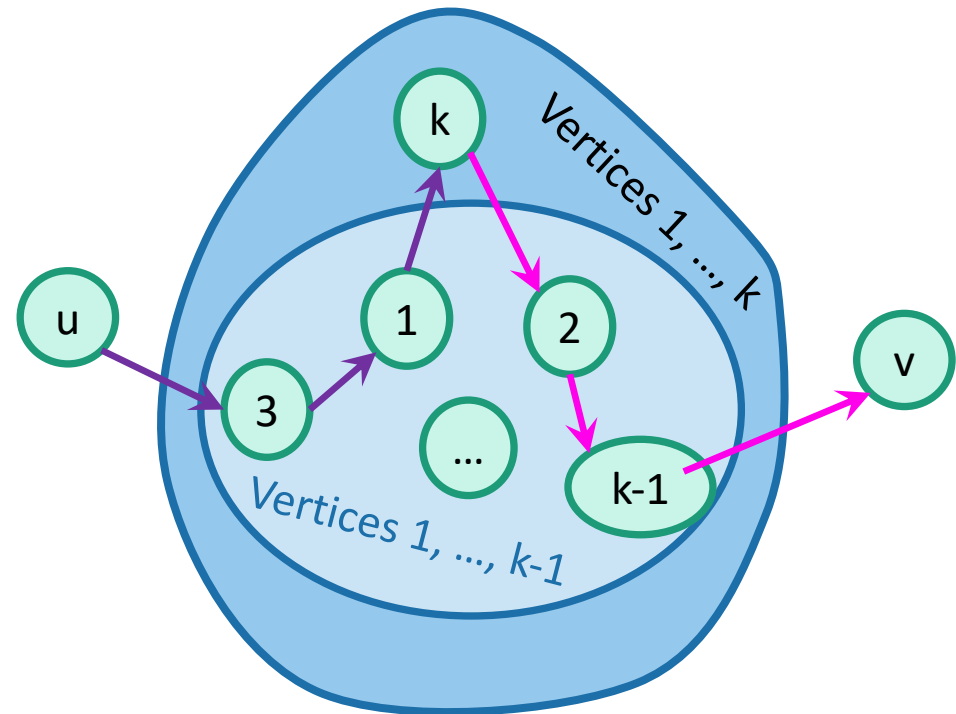


Case 1: we don't need vertex k .



$$D^{(k)}[u,v] = D^{(k-1)}[u,v]$$

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$$D^{(k)}[u,v] = D^{(k-1)}[u,k] + D^{(k-1)}[k,v]$$

How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

- $D^{(k)}[u,v] = \min\{ D^{(k-1)}[u,v], D^{(k-1)}[u,k] + D^{(k-1)}[k,v] \}$

Case 1: Cost of
shortest path
through $\{1, \dots, k-1\}$

Case 2: Cost of shortest path
from u to k and then from k to v
through $\{1, \dots, k-1\}$

- Optimal substructure:
 - We can solve the big problem using solutions to smaller problems.
- Overlapping sub-problems:
 - $D^{(k-1)}[k,v]$ can be used to help compute $D^{(k)}[u,v]$ for lots of different u 's.

How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

- $D^{(k)}[u,v] = \min\{ D^{(k-1)}[u,v], D^{(k-1)}[u,k] + D^{(k-1)}[k,v] \}$

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- Using our *Dynamic programming* paradigm, this immediately gives us an algorithm!



Floyd-Warshall algorithm

- Initialize n-by-n arrays $D^{(k)}$ for $k = 0, \dots, n$

- $D^{(k)}[u,u] = 0$ for all u , for all k
- $D^{(k)}[u,v] = \infty$ for all $u \neq v$, for all k
- $D^{(0)}[u,v] = \text{weight}(u,v)$ for all (u,v) in E .

The base case checks out: the only path through zero other vertices are edges directly from u to v .

- **For** $k = 1, \dots, n$:

- **For** pairs u,v in V^2 :

- $D^{(k)}[u,v] = \min\{ D^{(k-1)}[u,v], D^{(k-1)}[u,k] + D^{(k-1)}[k,v] \}$

- **Return** $D^{(n)}$

This is a bottom-up *Dynamic programming* algorithm.

We've basically just shown

- Theorem:

If there are **no negative cycles** in a weighted directed graph G , then the Floyd-Warshall algorithm, running on G , returns a matrix $D^{(n)}$ so that:

$$D^{(n)}[u,v] = \text{distance between } u \text{ and } v \text{ in } G.$$

Work out the
details of a proof!



- Running time: $O(n^3)$

- Better than running Bellman-Ford n times!

- Storage:

- Need to store **two** n -by- n arrays, and the original graph.

As with Bellman-Ford, we don't really need to store all n of the $D^{(k)}$.

What if there *are* negative cycles?

- Just like Bellman-Ford, Floyd-Warshall can detect negative cycles:
 - Negative cycle $\Leftrightarrow \exists v$ s.t. there is a path from v to v that goes through all n vertices that has cost < 0 .
 - Negative cycle $\Leftrightarrow \exists v$ s.t. $D^{(n)}[v,v] < 0$.
- Algorithm:
 - Run Floyd-Warshall as before.
 - If there is some v so that $D^{(n)}[v,v] < 0$:
 - **return negative cycle.**

What have we learned?

- The Floyd-Warshall algorithm is another example of *dynamic programming*.
- It computes All Pairs Shortest Paths in a directed weighted graph in time $O(n^3)$.

Recap

- Two shortest-path algorithms:
 - Bellman-Ford for single-source shortest path
 - Floyd-Warshall for all-pairs shortest path
- ***Dynamic programming!***
 - This is a fancy name for:
 - Break up an optimization problem into smaller problems
 - The optimal solutions to the sub-problems should be sub-solutions to the original problem.
 - Build the optimal solution iteratively by filling in a table of sub-solutions.
 - Take advantage of overlapping sub-problems!

Next time

- More examples of *dynamic programming*!

We will stop bullets with our
action-packed coding skills,
and also maybe find longest
common subsequences.

