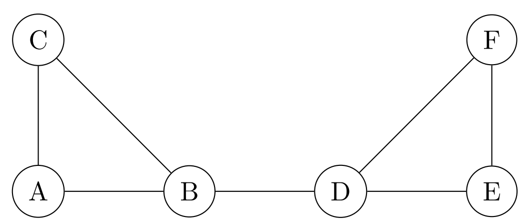
Some practice problems about Karger's algorithm

March 17, 2019

Note: While Karger's algorithm is fair game for the final, since we didn't have a proper homework assignment on it, it will not be weighted very heavily.

1. Consider the following graph:



   (a) What is the global mimimum cut of this graph?

   (b) What is the probability that Karger's algorithm chooses an edge of the minimum cut with its first choice?

   (c) What is the probability that one run of Karger's algorithm returns a minimum cut on this graph? How does it compare to the bound of $1/\binom{n}{2}$ that we saw in class? (You can either try to compute the probability exactly or else implement it and run it a bunch of times – note that you definitely wouldn't be asked this on an exam, but it's a great way to understand how Karger's algorithm works!)

2. Let n be some number. Suppose you have a magic button that, when pressed, outputs "Hooray!" with probability $1/n^3$. Otherwise it outputs "Hmmm..." Prove that if you push the button $T = n^3 \ln(100)$ times, that you will see at least one "Hooray" with probability at least 99/100.

SOLUTIONS:

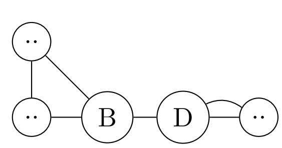
1. (a) The minimum cut is $\{A, B, C\}, \{D, E, F\}$.

   (b) The probability that Karger's algorithm chooses the edge $\{B, D\}$ which crosses the cut on the first round is 1/7.

   (c) I get

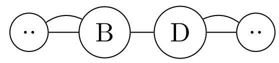
$$\frac{6}{7} \times \left(\left(\frac{2}{6} \times \frac{3}{4} \times \frac{2}{3} \right) + \left(\frac{3}{6} \times \frac{4}{5} \times \frac{2}{3} \right) \right) \approx 0.37.$$

   The way I did this was the following logic. In the first round, the probability that I don't choose the edge $\{B, D\}$ is 6/7. If I don't, then either way I collapse one of the triangles on either side, so up to symmetry I get something like this:



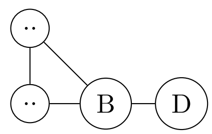
Then there are two choices which won't make the algorithm fail. With probability 3/6, I choose one of the edges in the remaining triangle, and with probability 2/6 I choose one of the edges in the double-edge. (And with probability 1/6 I choose the edge $\{B, D\}$ and fail).

If the first thing happened, I end up with (up to symmetries):



After that, the next edge I take has probability 4/5 of not being $\{B, D\}$, and after that the next edge has probability 2/3. Thus, in this case, the probability of success (conditioned on the event that the first edge chosen wasn't $\{B, D\}$) is $(3/6) \times (4/5) \times (2/3))$, which appears in the answer above.

On the other hand, if the second thing happened, I end up with (up to symmetries):



The the probability that I don't choose $\{B, D\}$ on the next round is 3/4, and in the next round I must collapse the triangle to a double edge. After that, the probability that I don't choose $\{B, D\}$ is 2/3. Thus, in this case, the probability of success (conditioned on the event that the first edge chosen wasn't $\{B, D\}$) is $(2/6) \times (3/4) \times (2/3))$, which appears in the answer above.

Now, $1/\binom{n}{2} = 1/\binom{6}{2} = 0.0666$, so it seems that the bound we proved in class was pessimistic for this graph.

2. Just like in class, we have

$$\begin{aligned} P\{\text{ no Hoorays }\} &= (1 - 1/n^3)^T \\ &\leq e^{-T/n^3} \\ &= e^{-n^3 \ln(100)/n^3} \\ &= e^{-\ln(100)} \\ &= \frac{1}{100}. \end{aligned}$$

Thus the probability that we see at least one Hooray is at least 99/100.