Lecture 17

What we’ve done and what’s to come

Advertisement: AI for Social Good

- TLDR:

  Apply for CS21: AI For Social Good at bit.ly/AIForGoodApp.

- CS21SI: AI for Social Good is a 2-unit student-taught class in which students learn about and apply cutting-edge artificial intelligence techniques to real-world social good spaces, such as healthcare, government, education, and the environment.
- The course alternates between lectures on machine learning theory and discussions with invited speakers, who will challenge students to apply techniques in their social good domains. You can learn more about the class at http://web.stanford.edu/class/cs21si/
- Apply by 11:59pm on Sunday, March 17 at bit.ly/AIForGoodApp.

Announcements

- Final exam:
  - 8:30am – 11:30am, Thursday 3/21.
  - Location:
    - Last name A-LIA: Cubberley Auditorium
    - Last name LIB-Z: Hewlett 200
  - You may bring two double-sided sheets of notes
  - Format: similar to the midterm

- See announcements from Monday 3/11 for more info, as well as study tips!

More announcements

- Course feedback open!
  - Fill it out on axess!
  - Your feedback is super-important!
  - It will help us make the course better!

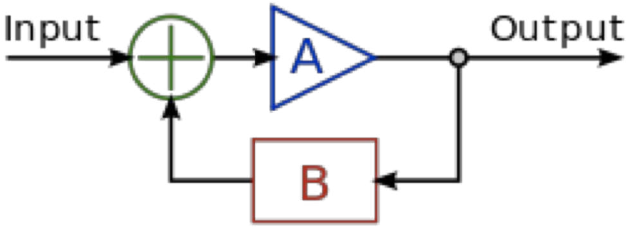
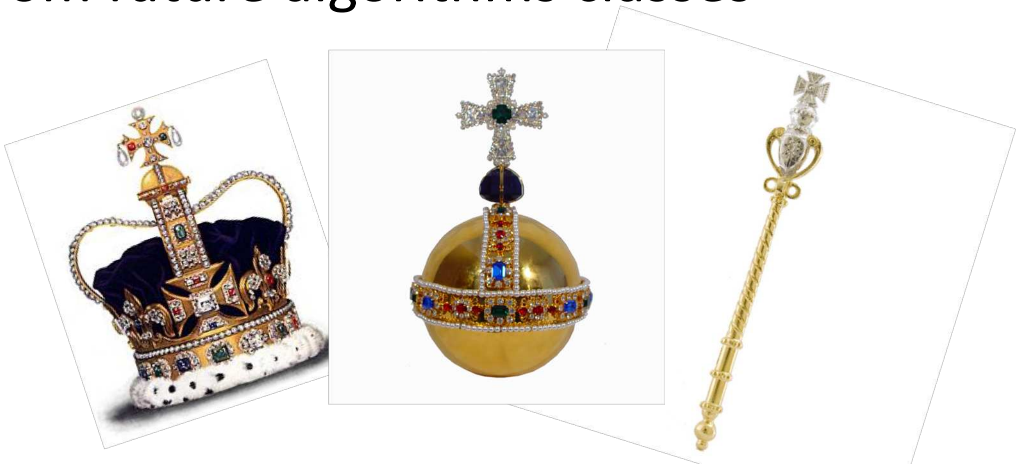


Figure 1: Feedback

Today

- What just happened?
  - A whirlwind tour of CS161

- What's next?
  - A few gems from future algorithms classes



It’s been a fun ride…

Sorting and friends!

Data structures: BSTs and Hashing!

Graphs!

Greedy algorithms!

Dynamic Programming!

Scheduling and etc.

Minimum Cuts

Divide-and-conquer and recurrence relations

LCS, Knapsack(s)

MSTs: Prim and Kruskal

BFS, DFS, SCCs

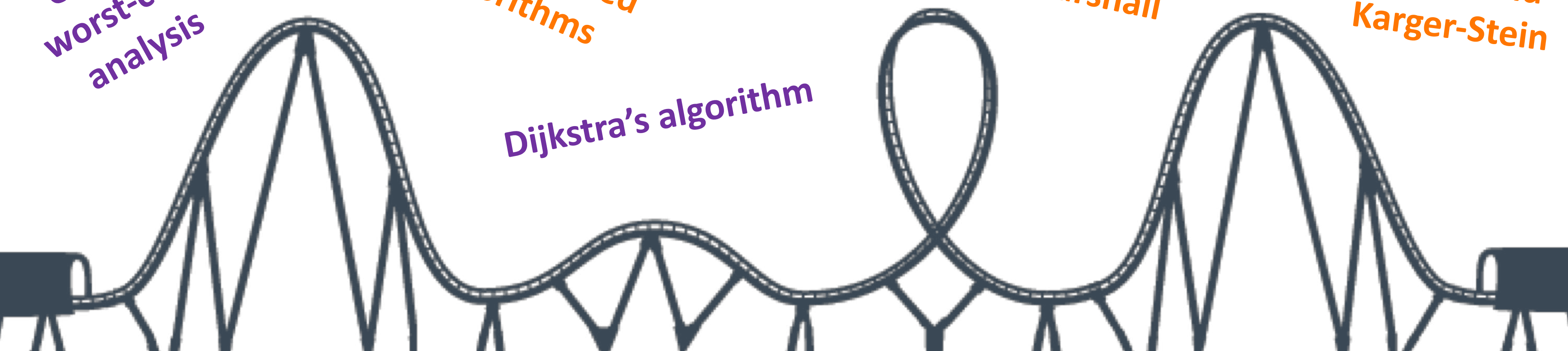
O() and worst-case analysis

Randomized algorithms

Bellman-Ford, Floyd-Warshall

Karger and Karger-Stein

Dijkstra’s algorithm



What have we learned?

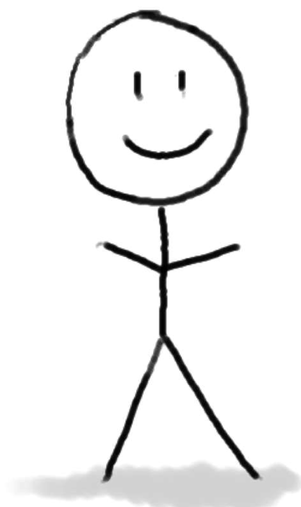
16 lectures in 12 slides.

General approach
to algorithm design and analysis

Does it work?

Is it fast?

Can I do better?



Algorithm designer

To answer these questions we need both **rigor** and **intuition**:

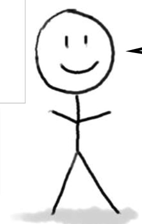
Plucky the Pedantic Penguin

Detail-oriented
Precise
Rigorous

Lucky the Lackadaisical Lemur

Big-picture
Intuitive
Hand-wavey

We needed more details



Does it work?
Is it fast?
Can I do better?

What
does that
mean??

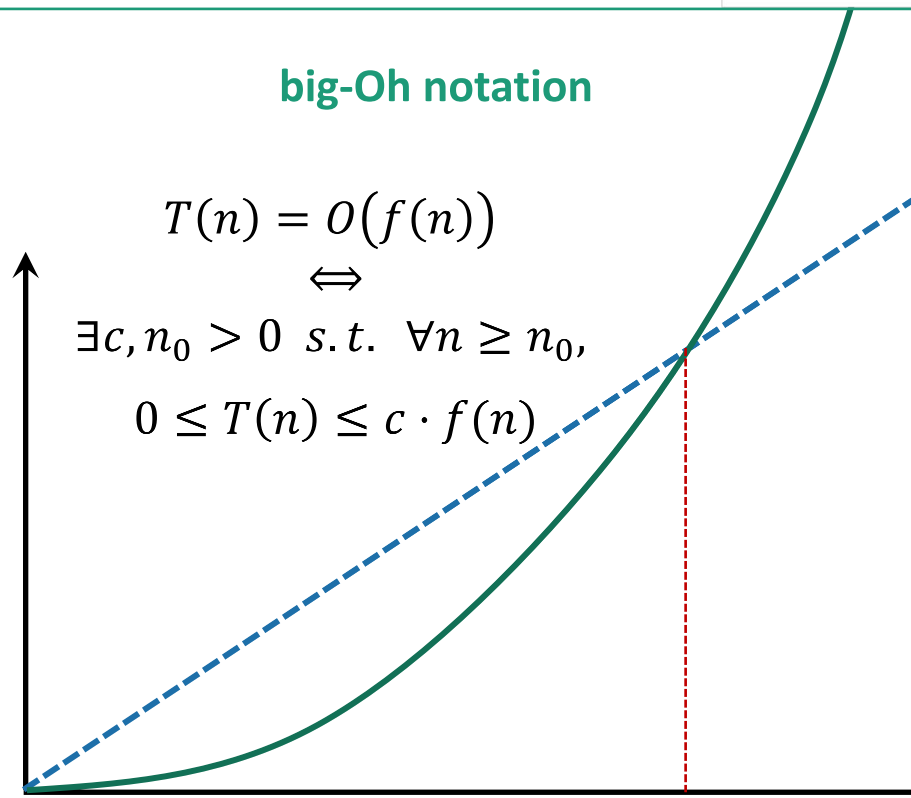
Worst-case analysis

Here is an input!

Anakin the Adversarial Aardvark

big-Oh notation

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$



Algorithm design paradigm: divide and conquer

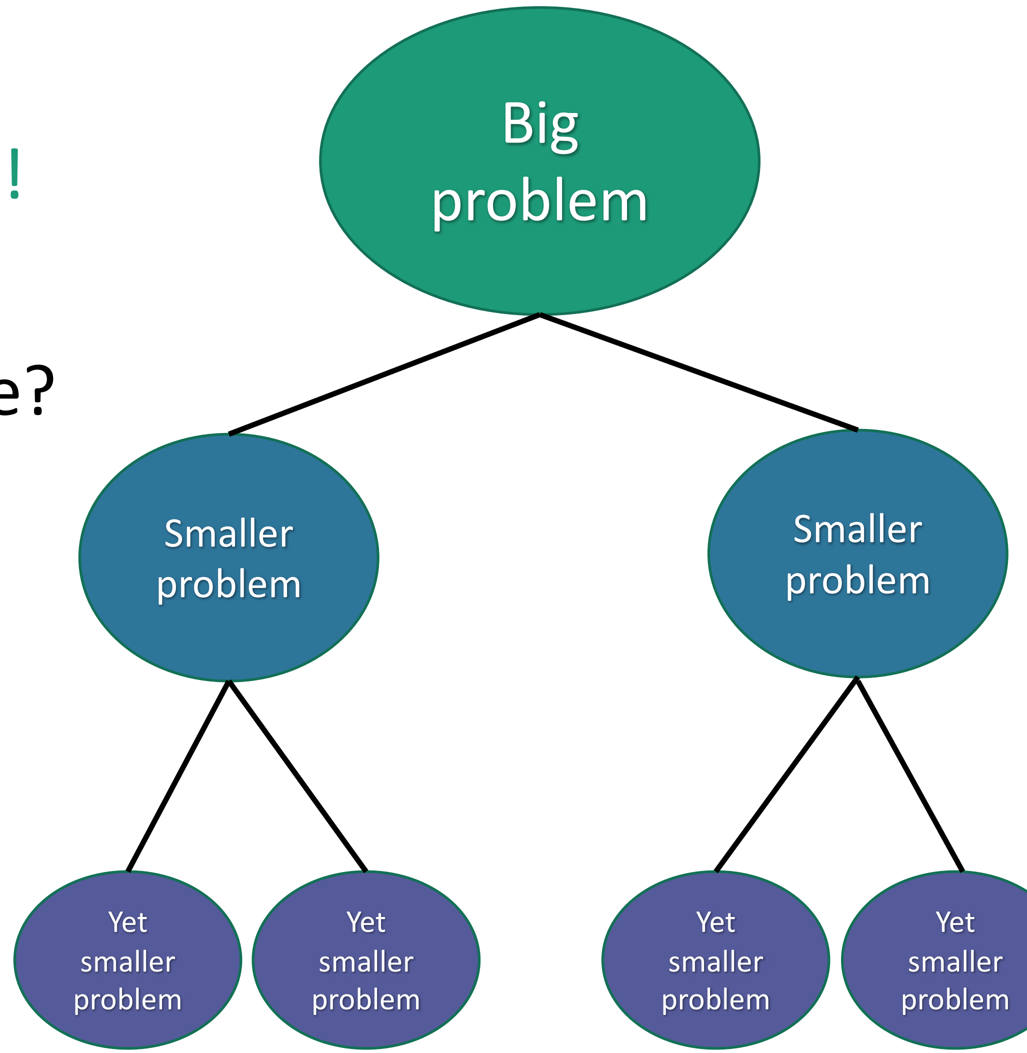
- Like MergeSort!
- Or Karatsuba's algorithm!
- Or SELECT!
- How do we analyze these?

By careful analysis!

Plucky the Pedantic Penguin

Useful shortcut, the **master method** is.

Jedi master Yoda



While we're on the topic of sorting

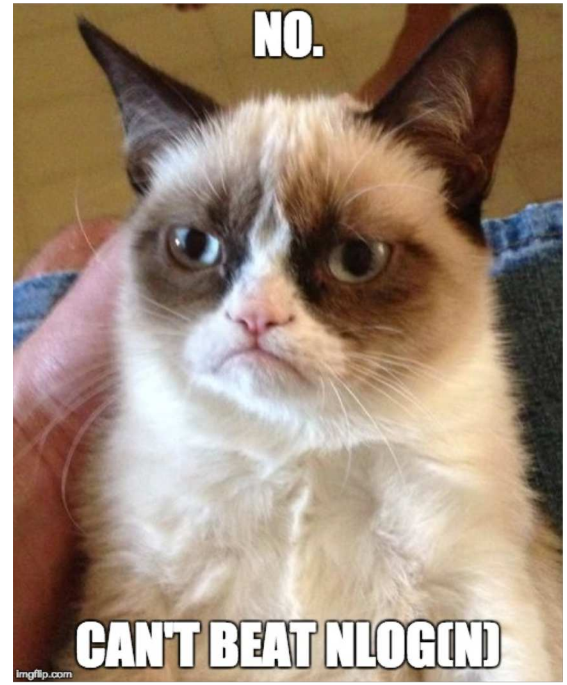
Why not use randomness?

- We analyzed **QuickSort!**
- Still worst-case input, but we use randomness after the input is chosen.
- Always correct, usually fast.
  - This is a Las Vegas algorithm

All this sorting is making me wonder…

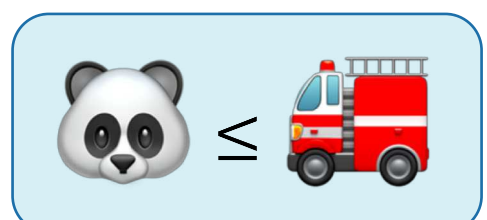
Can we do better?

- Depends on who you ask:



- **RadixSort** takes time O(n) if the objects are, for example, small integers!

- Can't do better in a **comparison-based** model.



beyond sorted arrays/linked lists:

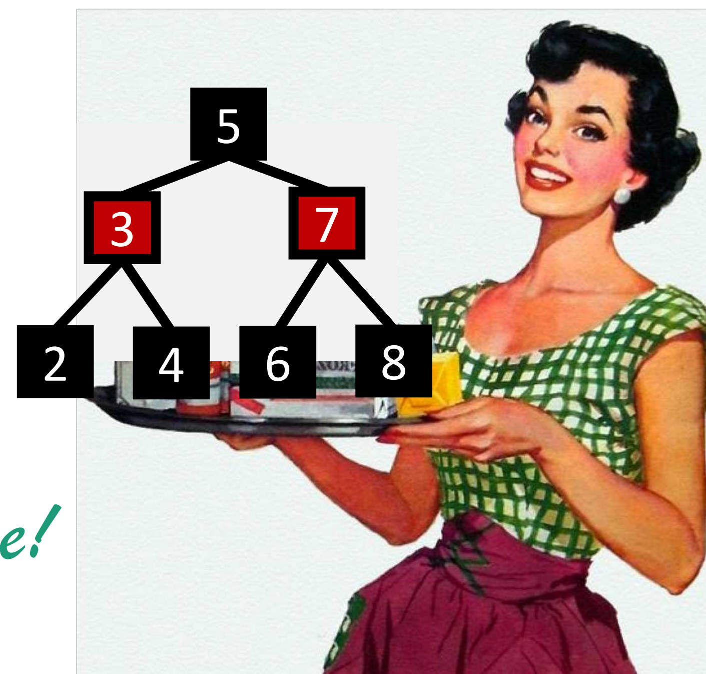
Binary Search Trees!

- Useful data structure!
- Especially the self-balancing ones!

Red-Black tree!

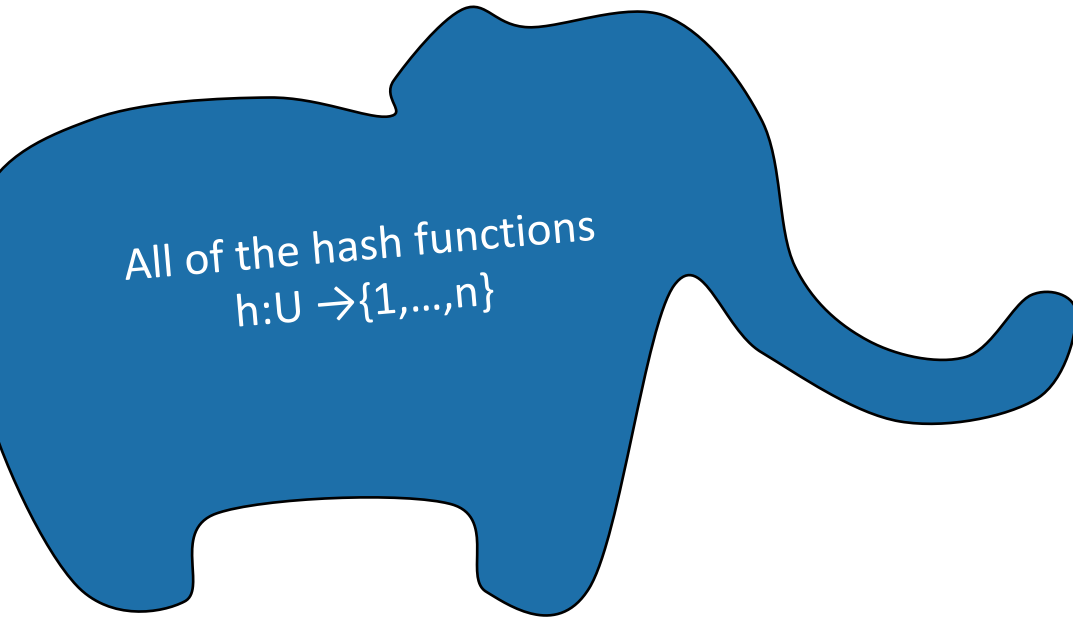
Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

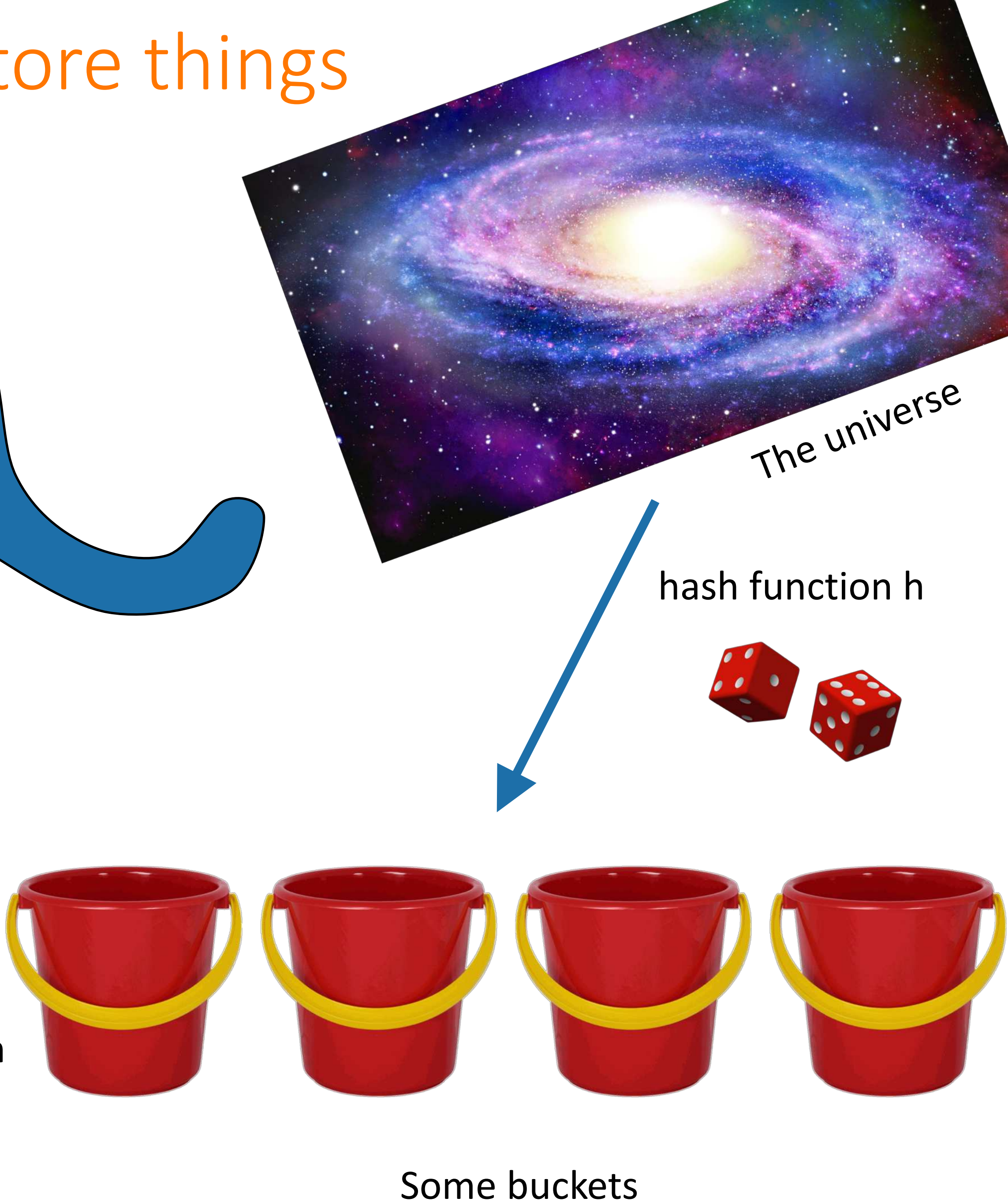
It's just good sense!



Another way to store things

Hash tables!



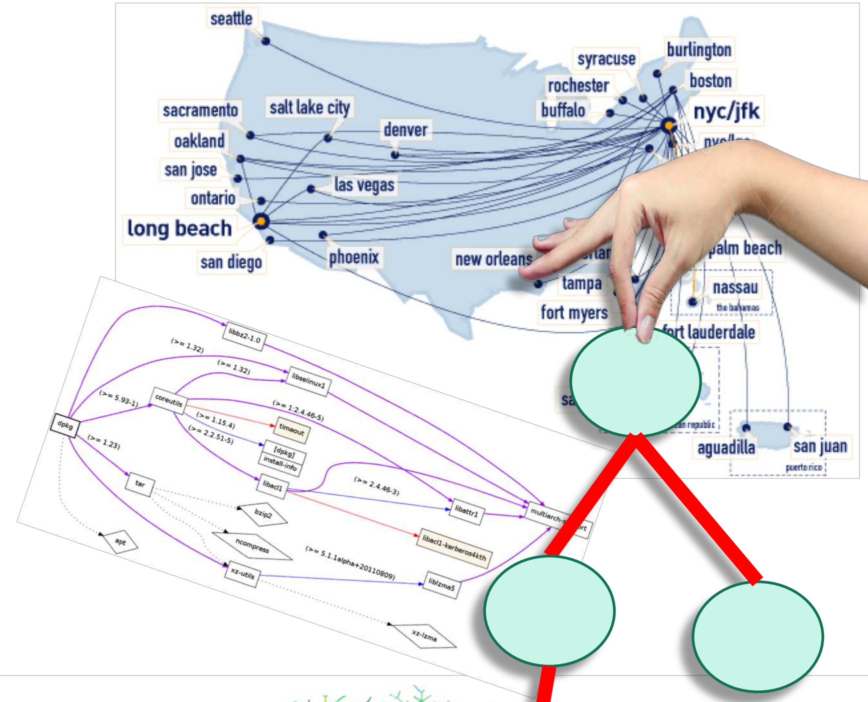


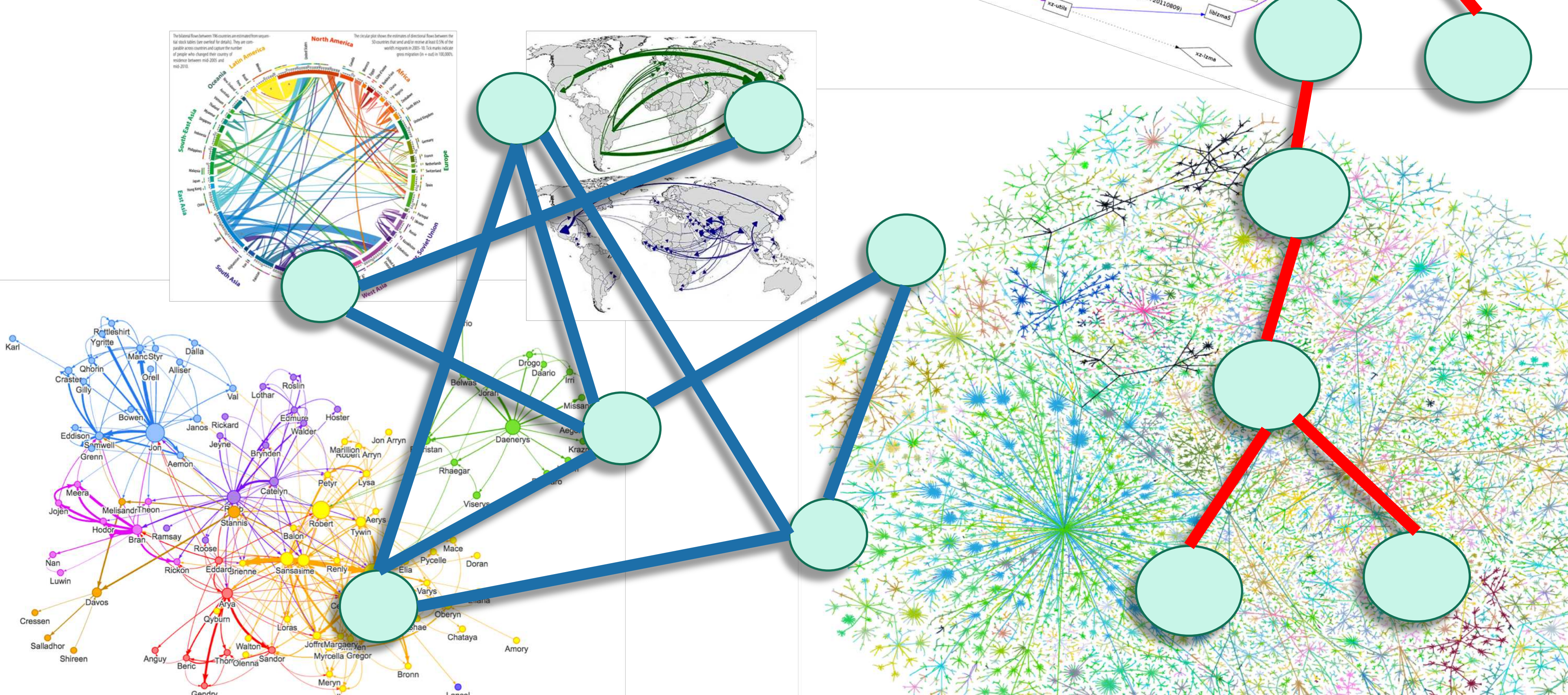
Choose h randomly from a universal hash family.

It's better if the hash family is small!
Then it takes less space to store h.

OMG GRAPHS

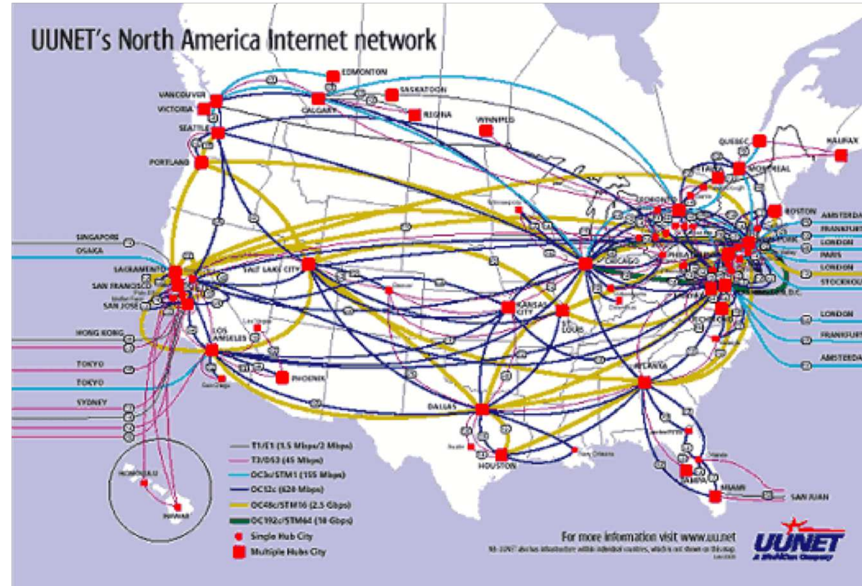
- BFS, DFS, and applications!
- SCCs, Topological sorting, …

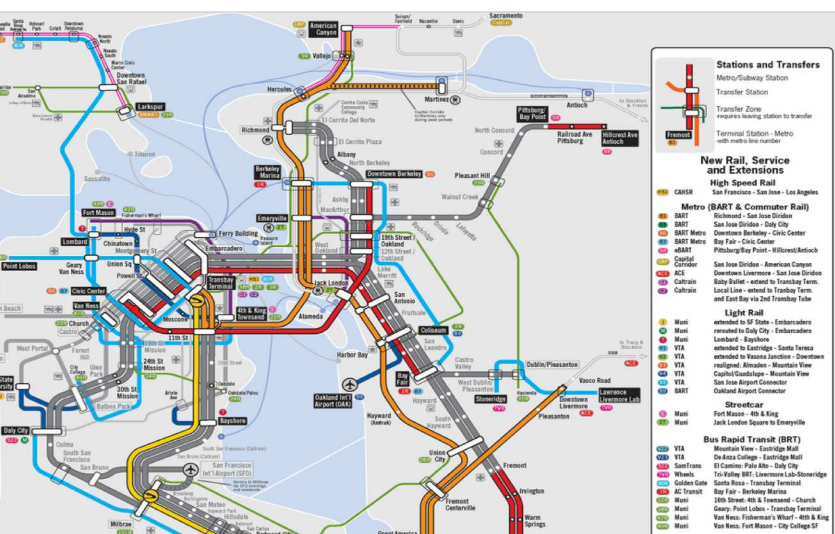




A fundamental graph problem:
shortest paths

- Eg, transit planning, packet routing, …
- Dijkstra!
- Bellman-Ford!
- Floyd-Warshall!

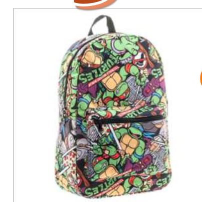





```
DN0a22a0e3:~ mary$ traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 10.34.160.2 (10.34.160.2)  38.168 ms  31.272 ms  28.841 ms
 2  [AS0] cwa-vrtr.sunet (10.21.196.28)  33.769 ms  28.245 ms  24.373 ms
 3  [AS32] 171.66.2.229 (171.66.2.229)  24.468 ms  20.115 ms  23.223 ms
 4  [AS32] hpr-svl-rtr-vlan8.sunet (171.64.255.235)  24.644 ms  24.962 ms  17.453 ms
 5  [AS2152] hpr-svl-hpr2--stan-ge.cenic.net (137.164.27.161)  22.129 ms  4.902 ms  3.642 ms
 6  [AS2152] hpr-lax-hpr3--svl-hpr3-100ge.cenic.net (137.164.25.73)  12.125 ms  43.361 ms  32.3
 7  [AS2152] hpr-i2--lax-hpr2-r&e.cenic.net (137.164.26.201)  40.174 ms  38.399 ms  34.499 ms
 8  [AS0] et-4-0-0.4079.sdn-sw.lasv.net.internet2.edu (162.252.70.28)  46.573 ms  23.926 ms  17
 9  [AS0] et-5-1-0.4079.rtsw.salt.net.internet2.edu (162.252.70.31)  30.424 ms  25.770 ms  23.1
10  [AS0] et-4-0-0.4079.sdn-sw.denv.net.internet2.edu (162.252.70.8)  47.454 ms  57.273 ms  73.
11  [AS0] et-4-1-0.4079.rtsw.kans.net.internet2.edu (162.252.70.11)  70.825 ms  67.809 ms  62.1
12  [AS0] et-4-1-0.4070.rtsw.chic.net.internet2.edu (198.71.47.206)  77.937 ms  57.421 ms  63.6
13  [AS0] et-0-1-0.4079.sdn-sw.ashb.net.internet2.edu (162.252.70.60)  77.682 ms  71.993 ms  73
14  [AS0] et-4-1-0.4079.rtsw.wash.net.internet2.edu (162.252.70.65)  71.565 ms  74.988 ms  71.0
15  [AS21320] internet2-gw.mx1.lon.uk.geant.net (62.40.124.44)  154.926 ms  145.606 ms  145.872
16  [AS21320] ae0.mx1.lon2.uk.geant.net (62.40.98.79)  146.565 ms  146.604 ms  146.801 ms
17  [AS21320] ae0.mx1.par.fr.geant.net (62.40.98.77)  153.289 ms  184.995 ms  152.682 ms
18  [AS21320] ae2.mx1.gen.ch.geant.net (62.40.98.153)  160.283 ms  160.104 ms  164.147 ms
19  [AS21320] swice1-100ge-0-3-0-1.switch.ch (62.40.124.22)  162.068 ms  160.595 ms  163.095 ms
20  [AS559] swizh1-100ge-0-1-0-1.switch.ch (130.59.36.94)  165.824 ms  164.216 ms  163.983 ms
21  [AS559] swiez3-100ge-0-1-0-4.switch.ch (130.59.38.109)  164.269 ms  164.370 ms  163.929 ms
22  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  164.082 ms  170.645 ms  165.372
23  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  164.773 ms  165.193 ms  172.158 ms
```

Bellman-Ford and Floyd-Warshall were examples of…

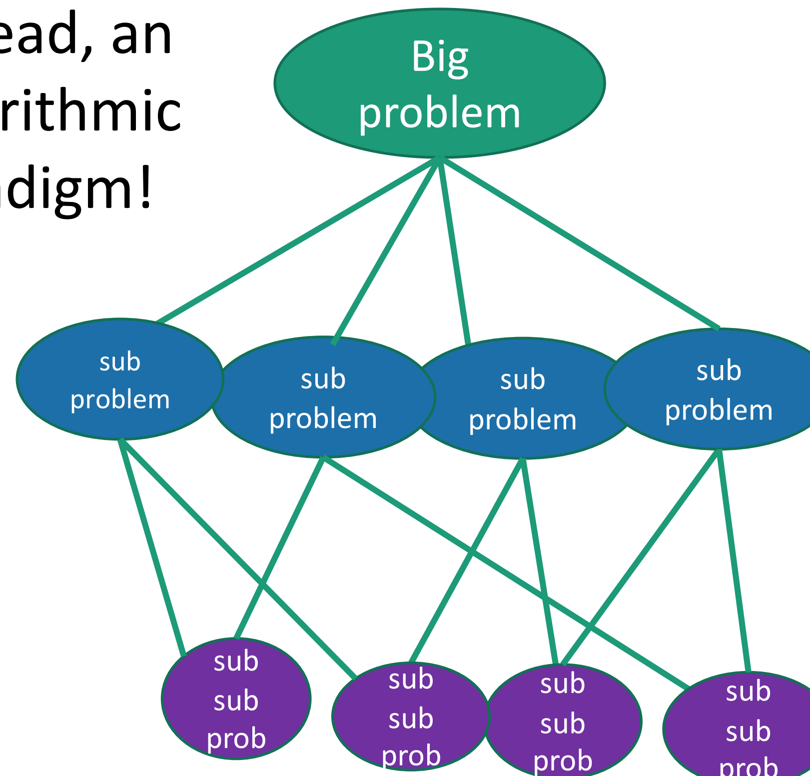
Dynamic Programming!



We saw many other examples, including Longest Common Subsequence and Knapsack Problems.

- Not programming in an action movie.

Instead, an algorithmic paradigm!



- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Steps 3-5:** Use dynamic programming: fill in a table to find the answer!

Sometimes we can take even better advantage of optimal substructure…with

Greedy algorithms

- Make a series of choices, and commit!

- Intuitively we want to show that our greedy choices never rule out success.
- Rigorously, we usually analyzed these by induction.
- Examples!
    - Activity Selection
    - Job Scheduling
    - Huffman Coding
    - **Minimum Spanning Trees**

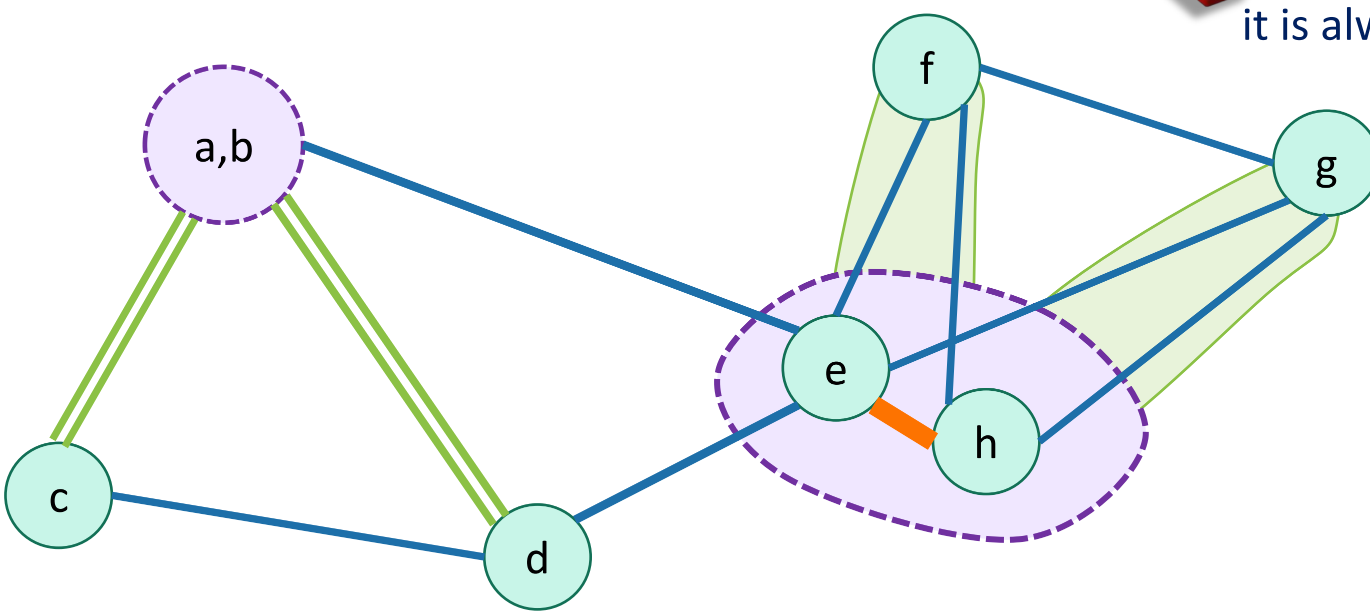
Prim's algorithm: greedily grow a tree

Kruskal's algorithm: greedily grow a forest

Minimum cuts

- Karger's algorithm!
- Karger-Stein algorithm!

Karger's algorithm is a **Monte-Carlo algorithm**: it is always fast but might be wrong.



And now we're here

What have we learned?

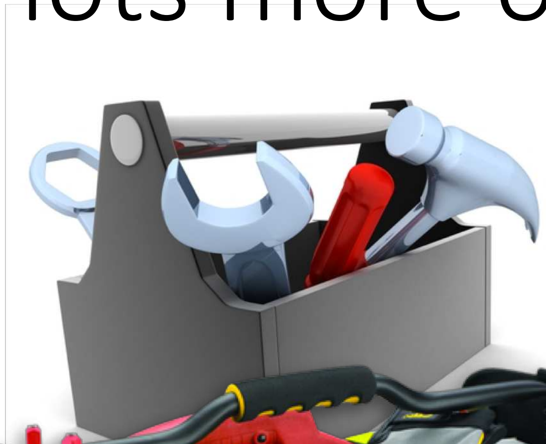
- A few algorithm design paradigms:
  - Divide and conquer, Dynamic Programming, Greedy
- A few analysis tools:
  - Worst-case analysis, asymptotic analysis, recurrence relations, probability tricks, proofs by induction
- A few common objects:
  - Graphs, arrays, trees, hash functions
- A LOT of examples!

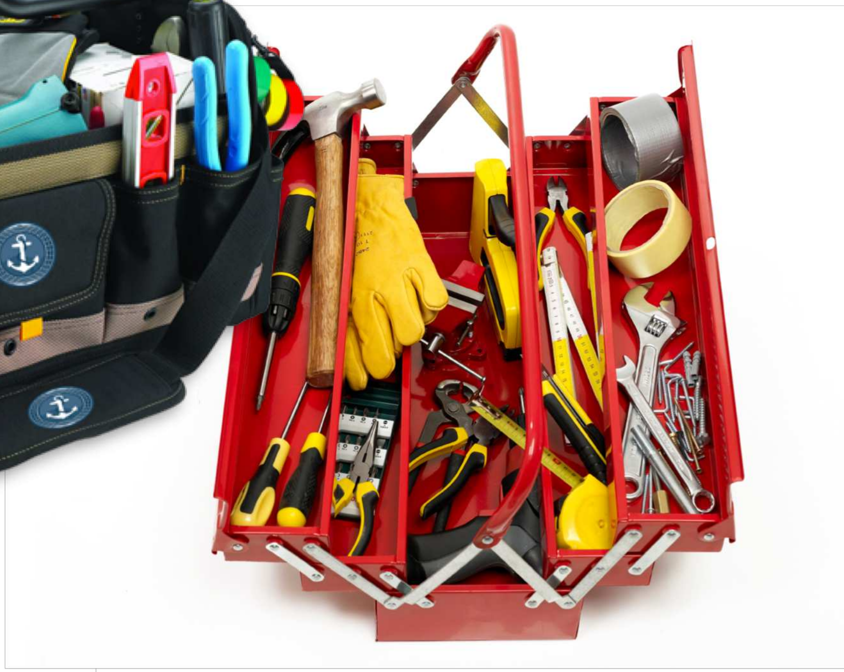
What have we learned?

We've filled out a toolbox

- Tons of examples give us intuition about what algorithmic techniques might work when.
- The technical skills make sure our intuition works out.

But there's lots more out there





- What's next???

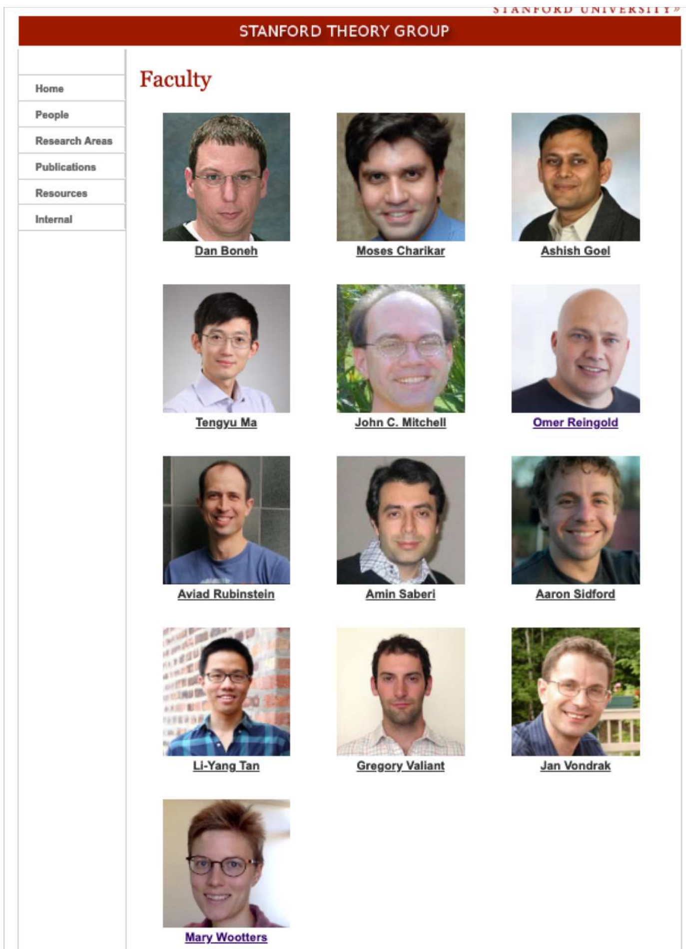
A taste of what's to come

- CS154 – Introduction to Complexity
- CS166 – Data Structures ← This Spring!!!
- CS167 – Readings in Algorithms
- CS168 – The Modern Algorithmic Toolbox ← This Spring!!!
- MS&E 212 – Combinatorial Optimization
- CS250 – Error Correcting Codes
- CS254 – Computational Complexity
- CS255 – Introduction to Cryptography
- CS261 – Optimization and Algorithmic Paradigms
- CS264 – Beyond Worst-Case Analysis
- CS265 – Randomized Algorithms
- CS269Q – Quantum Computing ← This Spring!!!
- CS269O – Introduction to Optimization Theory ← This Spring!!!
- EE364A/B – Convex Optimization I and II

...and many many more
upper-level topics courses!

findSomeTheoryCourses():

- go to theory.stanford.edu
- Click on "People"
- Look at what we're teaching!



Today

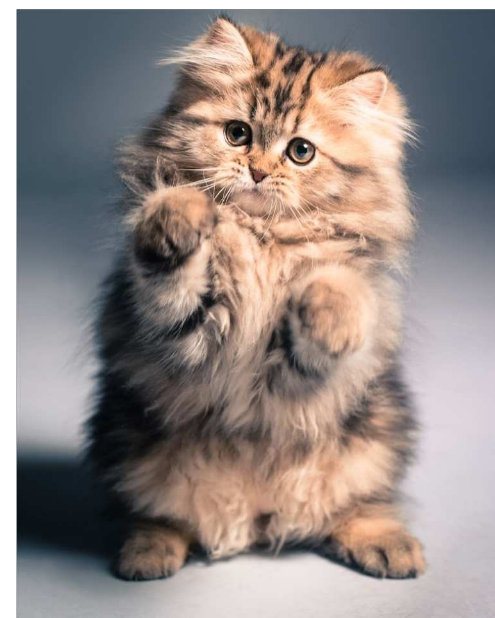
A few gems

- Linear programming

- Random projections

- Low-degree polynomials

This will be pretty fluffy, without much detail – take more CS theory classes for more detail!



NOTHING AFTER THIS POINT WILL BE ON THE FINAL EXAM

Linear Programming

- This is a fancy name for optimizing a linear function subject to linear constraints.
- For example:

Maximize

$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

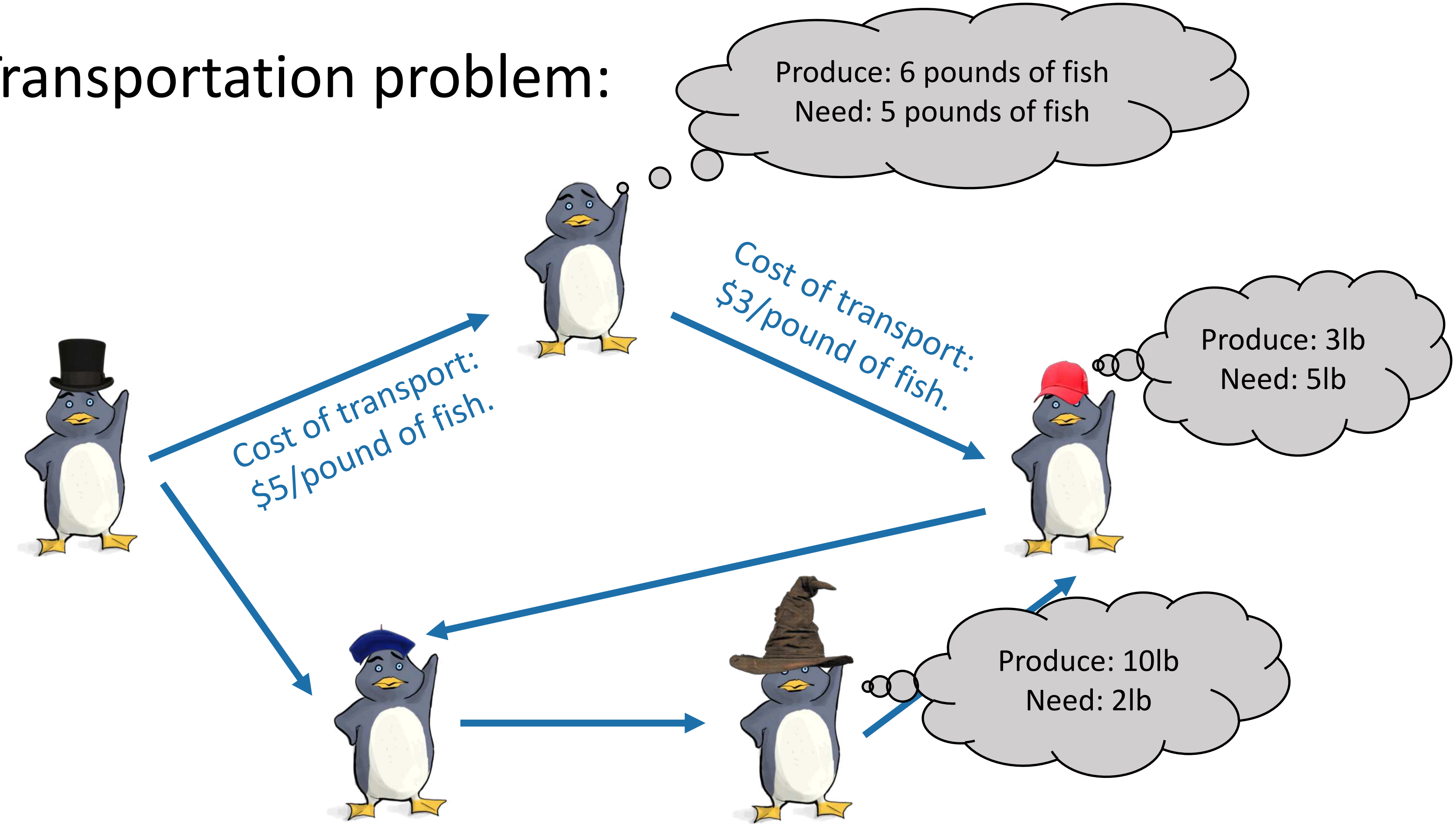
- It turns out the be an extremely general problem.

Goal: minimize transportation cost subject to meeting everyone's needs.

Example

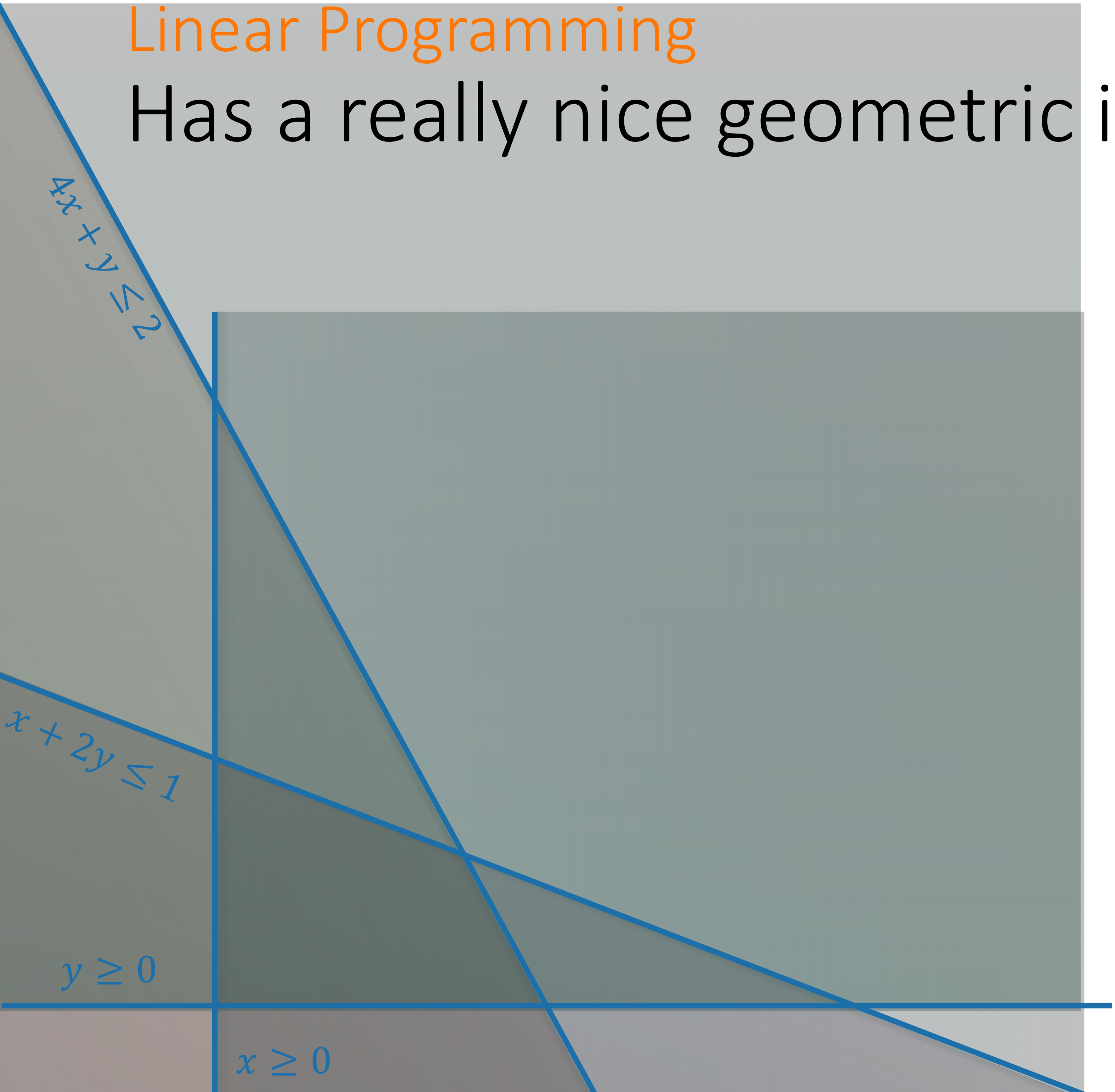
min $\sum_{edges}$ cost on that edge s.t. Supply/demand constraints are met.

- Transportation problem:



Linear Programming

Has a really nice geometric intuition



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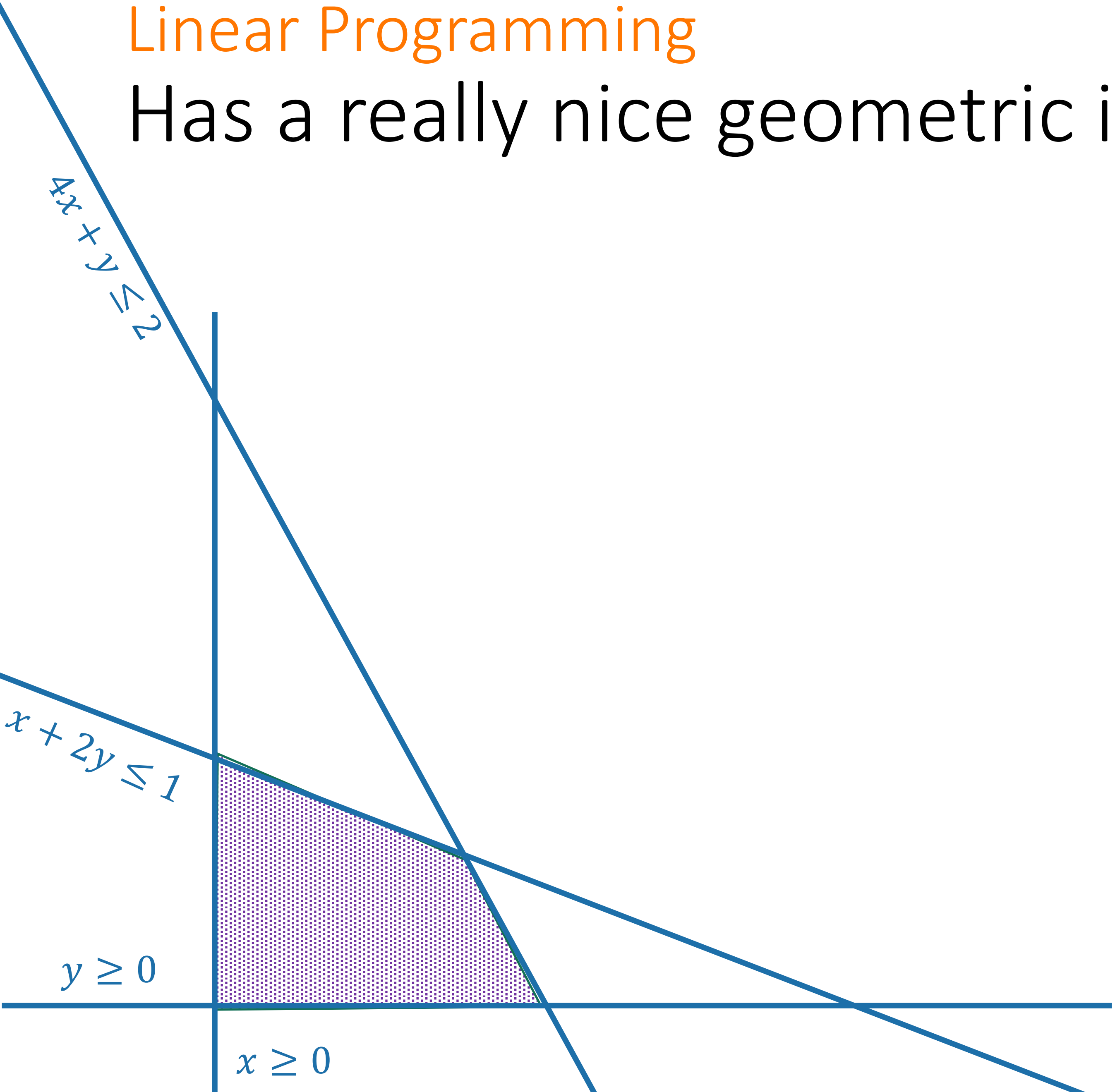

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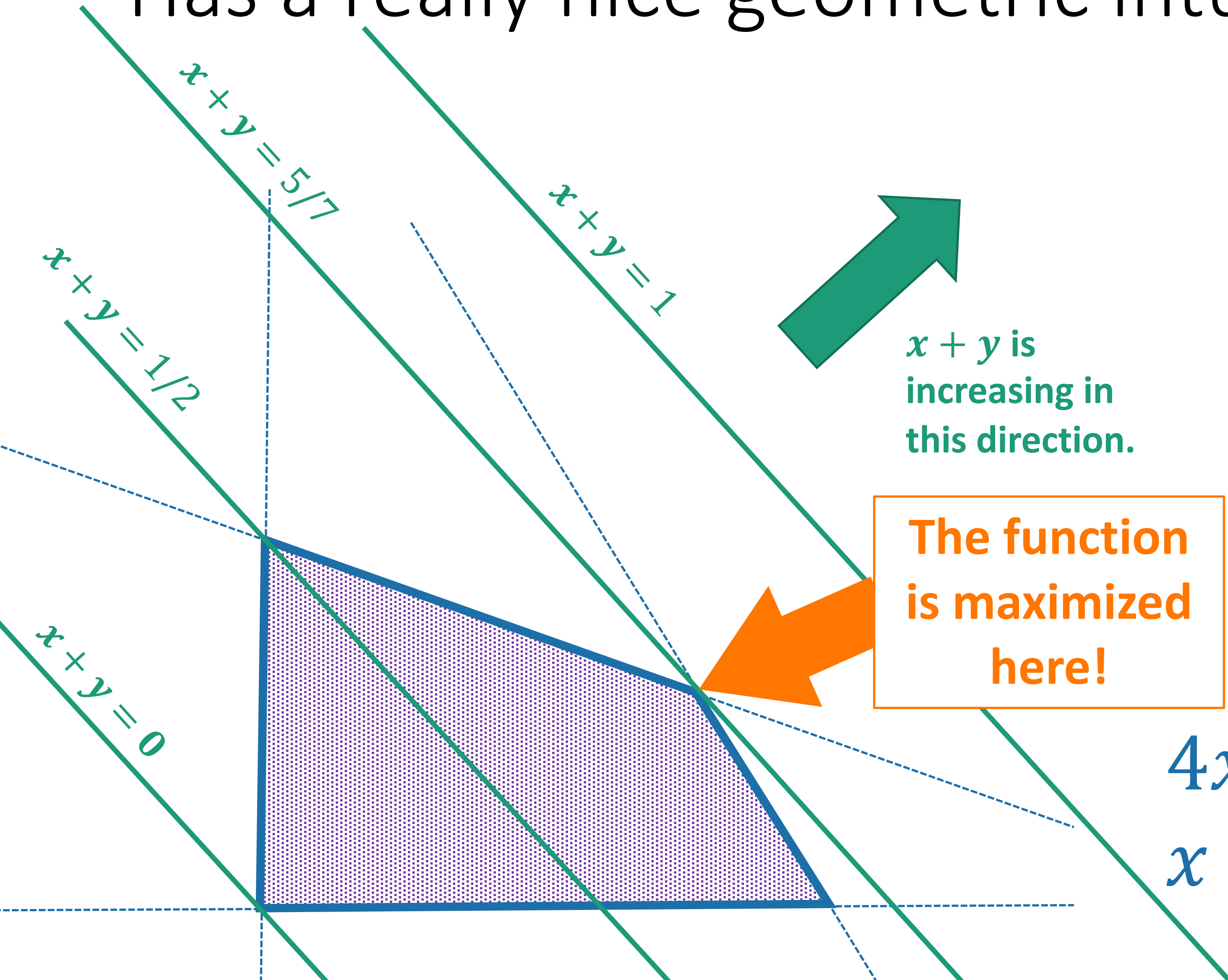


Maximize

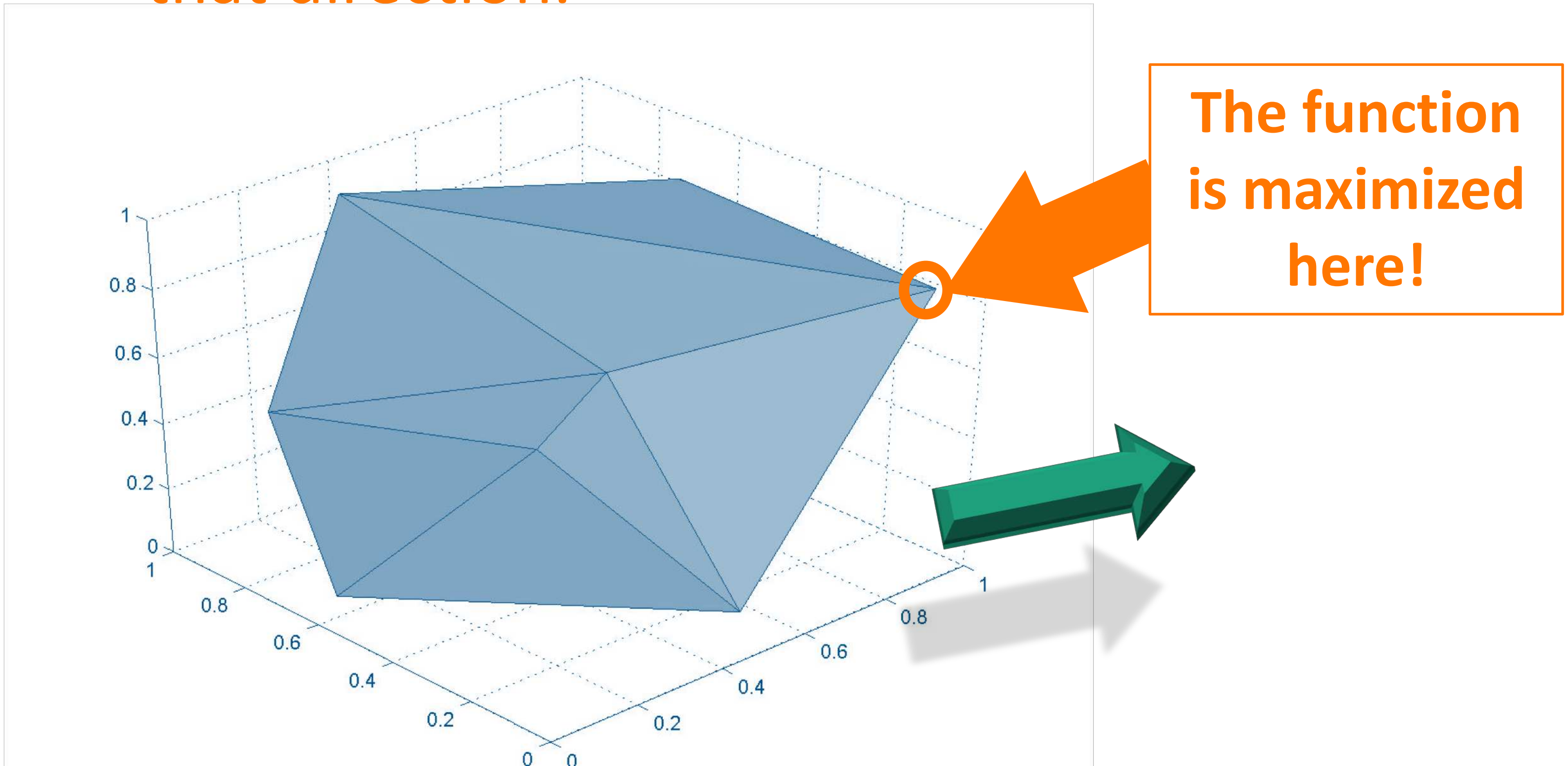
$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

In general

- The constraints define a **polytope**
- The function defines a **direction**
- We just want to find the vertex that is furthest in that direction.



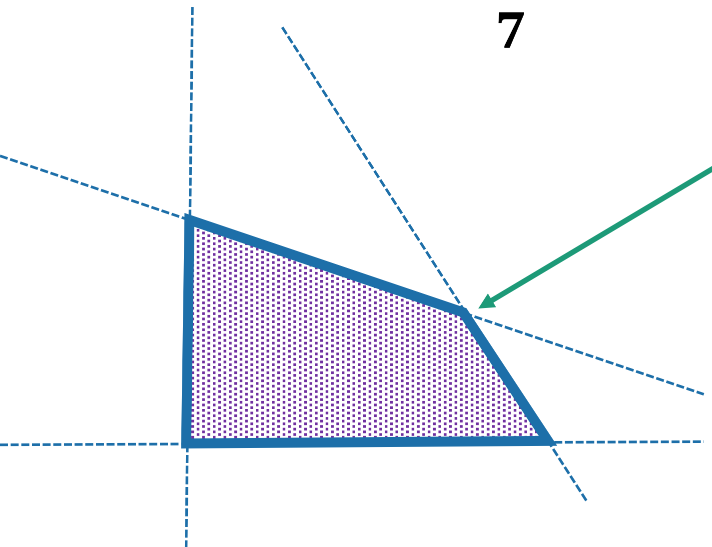
Duality

How do we know we have an optimal solution?

I claim that the optimum is 5/7.

Proof: say x and y satisfy the constraints.

- $\boldsymbol{x + y = \frac{1}{7}(4x + y) + \frac{3}{7}(x + 2y)}$
- $\boldsymbol{\leq \frac{1}{7} \cdot 2 + \frac{3}{7} \cdot 1}$
- $\boldsymbol{= \frac{5}{7}}$



You can check this point has value 5/7. So it must be optimal!

Maximize

$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

cute, but

How did you come up with 1/7, 3/7?

I claim that the optimum is 5/7.

Proof: say x and y satisfy the constraints.

- $x + y \leq \boxed{}(4x + y) + \boxed{}(x + 2y)$
- $\leq \boxed{} \cdot 2 + \boxed{}\, 1$
- $= \boxed{}$

- I want to choose things to put here
- So that I minimize this
- Subject to these things

Maximize

$x + y$

subject to

$x \geq 0$

$y \geq 0$

$4x + y \leq 2$

$x + 2y \leq 1$

That’s a linear program!

- How did I find those special values 1/7, 3/7?
- I solved some linear program.
- It’s called the dual program.

Minimize the upper bound you get,
subject to the proof working.

Maximize stuff
subject to stuff

Primal

The optimal values are
the same!

Minimize other stuff
subject to other stuff

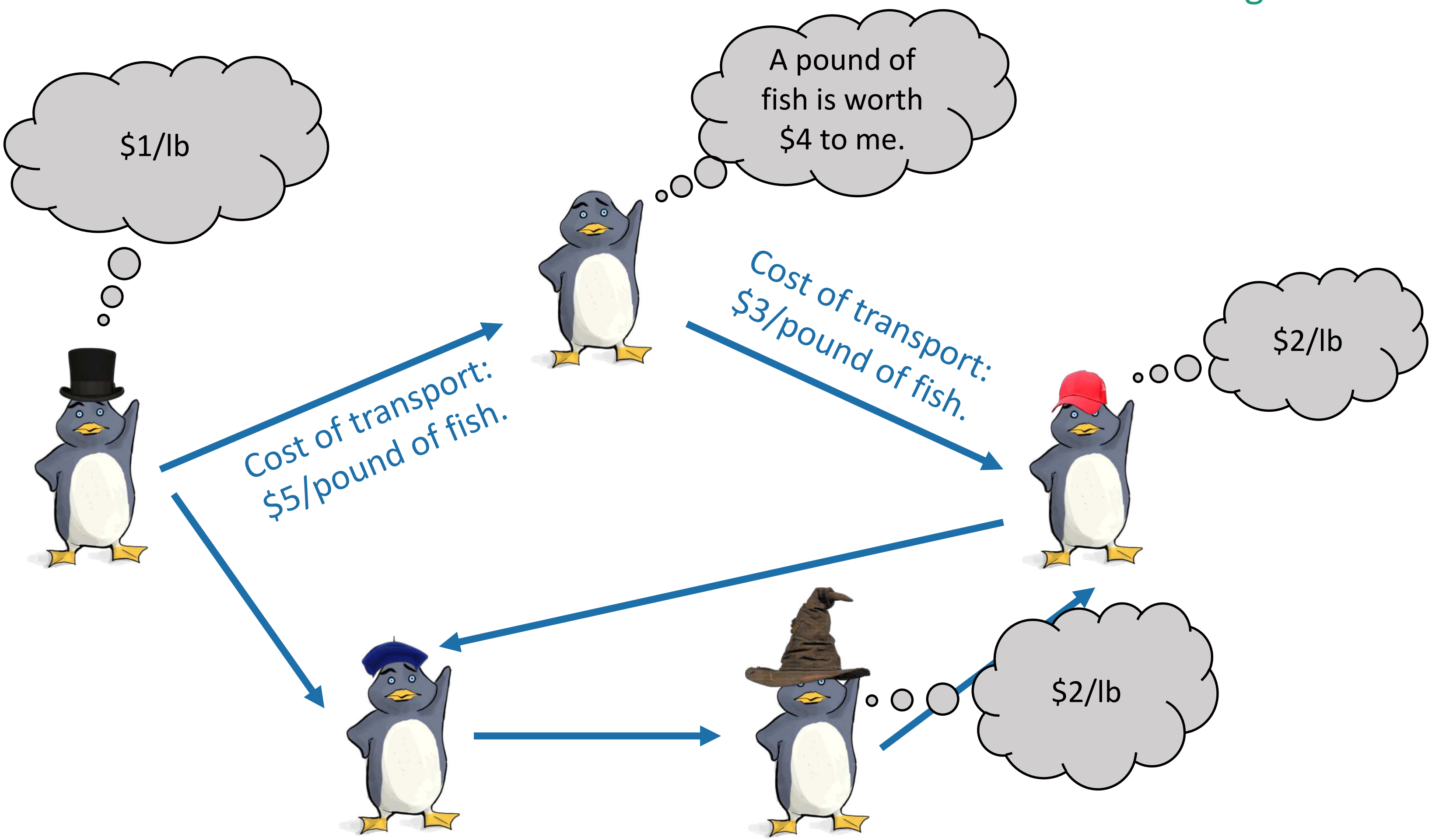
Dual

dual to transportation:
price-setting

Goal: maximize total prices so that there are no arbitrage opportunities.

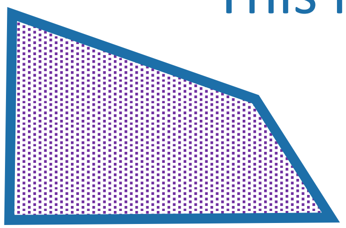
max $\sum_{v \in V}$ price at v s.t.

Cannot gain by buying from one penguin, transporting, and selling to another.



LPs and Duality are really powerful

- This general phenomenon shows up all over the place
  - Transportation and price-setting is a special case.
- Duality helps us reason about an optimization problem
  - The dual provides a **certificate** that we've solved the primal.
- We can solve LPs quickly!
  - For example, by intelligently bouncing around the vertices of the feasible region.
  - This is an **extremely powerful algorithmic primitive**.



Today

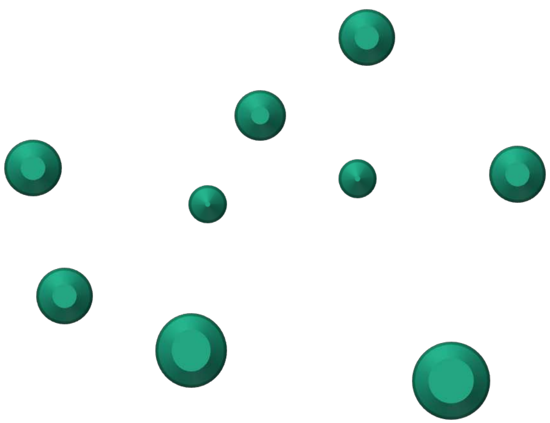
A few gems

- Linear programming

- Random projections
- Low-degree polynomials

One of my favorite tricks

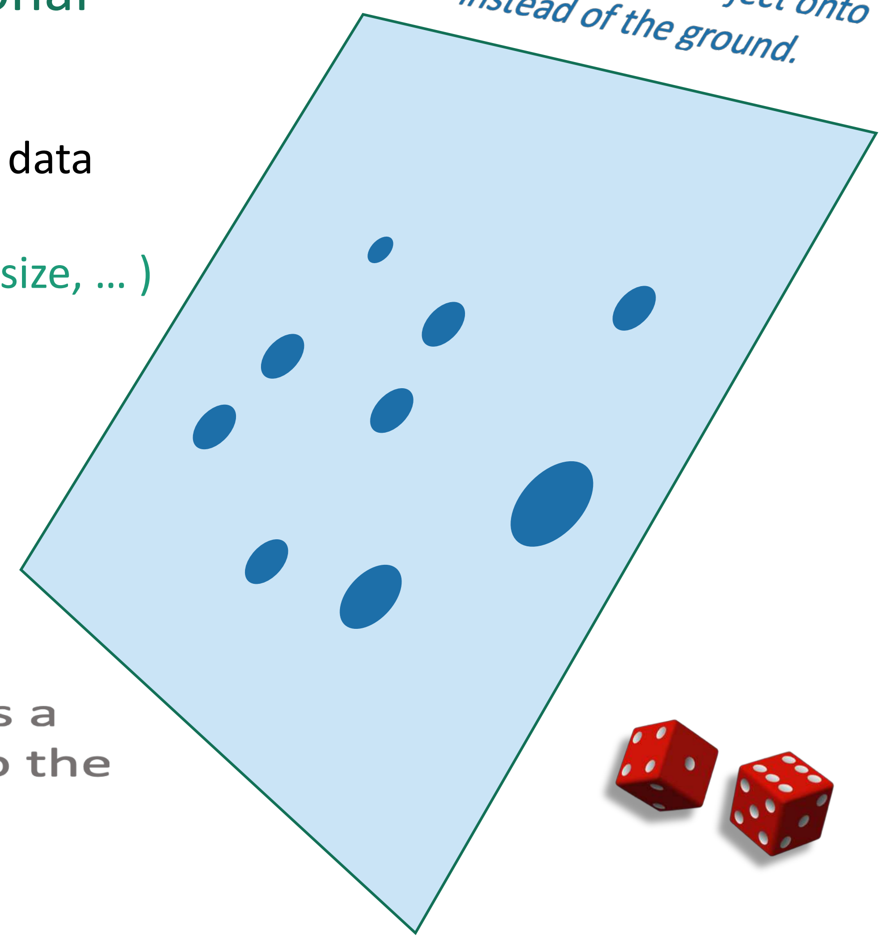
Take a random projection and hope for the best.

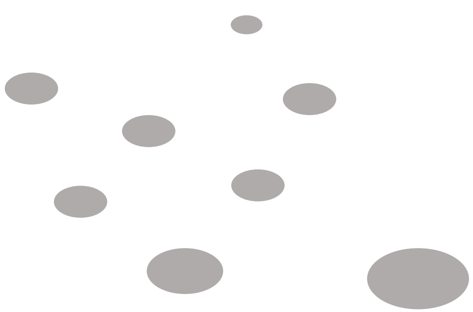


High-dimensional set of points

For example, each data point is a vector
(age, height, shoe size, …)

Choose a random subspace to project onto instead of the ground.





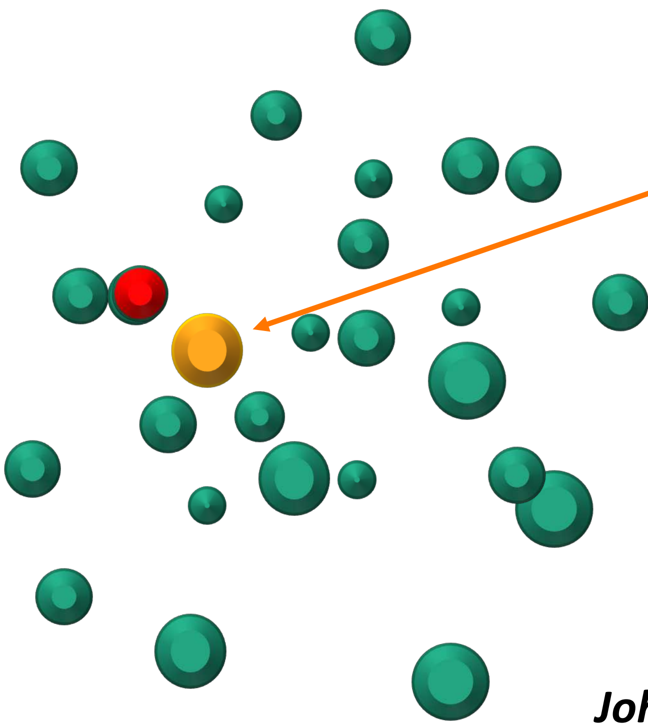
Their shadow is a projection onto the ground.

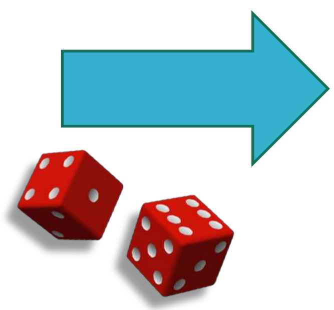
Why would we do this?

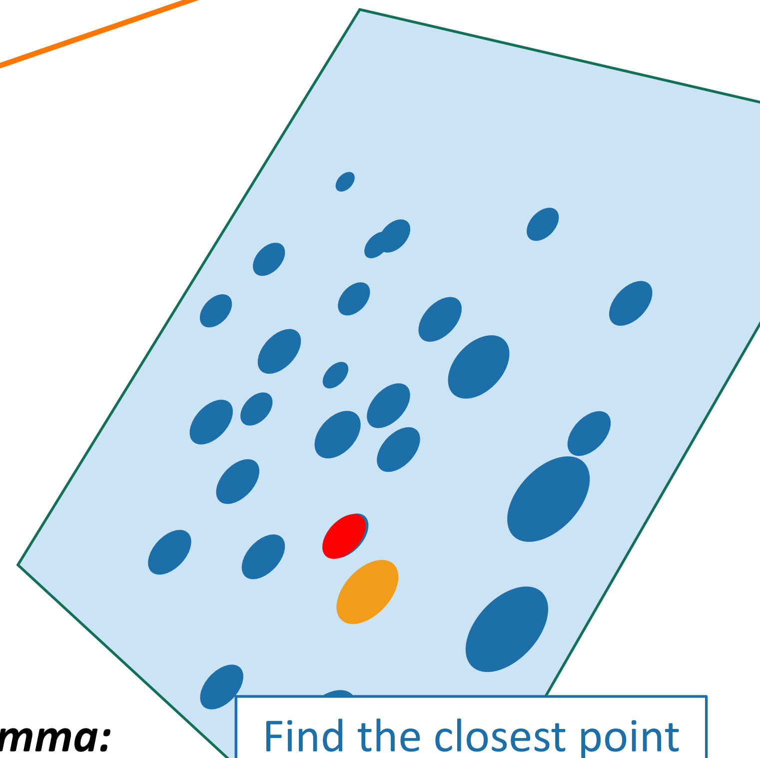
- High dimensional data takes a long time to process.
- Low dimensional data can be processed quickly.
- **“THEOREM”:** Random projections approximately preserve properties of data that you care about.

Example: nearest neighbors

- I want to find which point is closest to **this one**.







That takes a really long time in high dimensions.

Johnson-Lindenstrauss Lemma: *Euclidean distance is approximately preserved by random projections.*

Find the closest point down here, you're probably pretty correct.

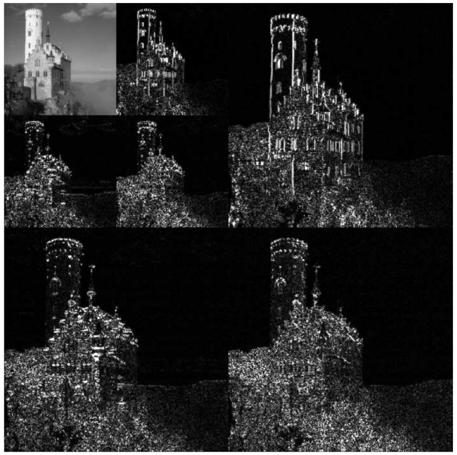
Another example:
Compressed Sensing

- Start with a sparse vector
    - Mostly zero or close to zero

(**5** , 0 , 0 , 0 ,0 , 0.01 , 0.01 , **5.8** , **32** , **14** , 0 , 0 , 0 , **12** , 0 , 0 , **5** , 0 , .03)

- For example:

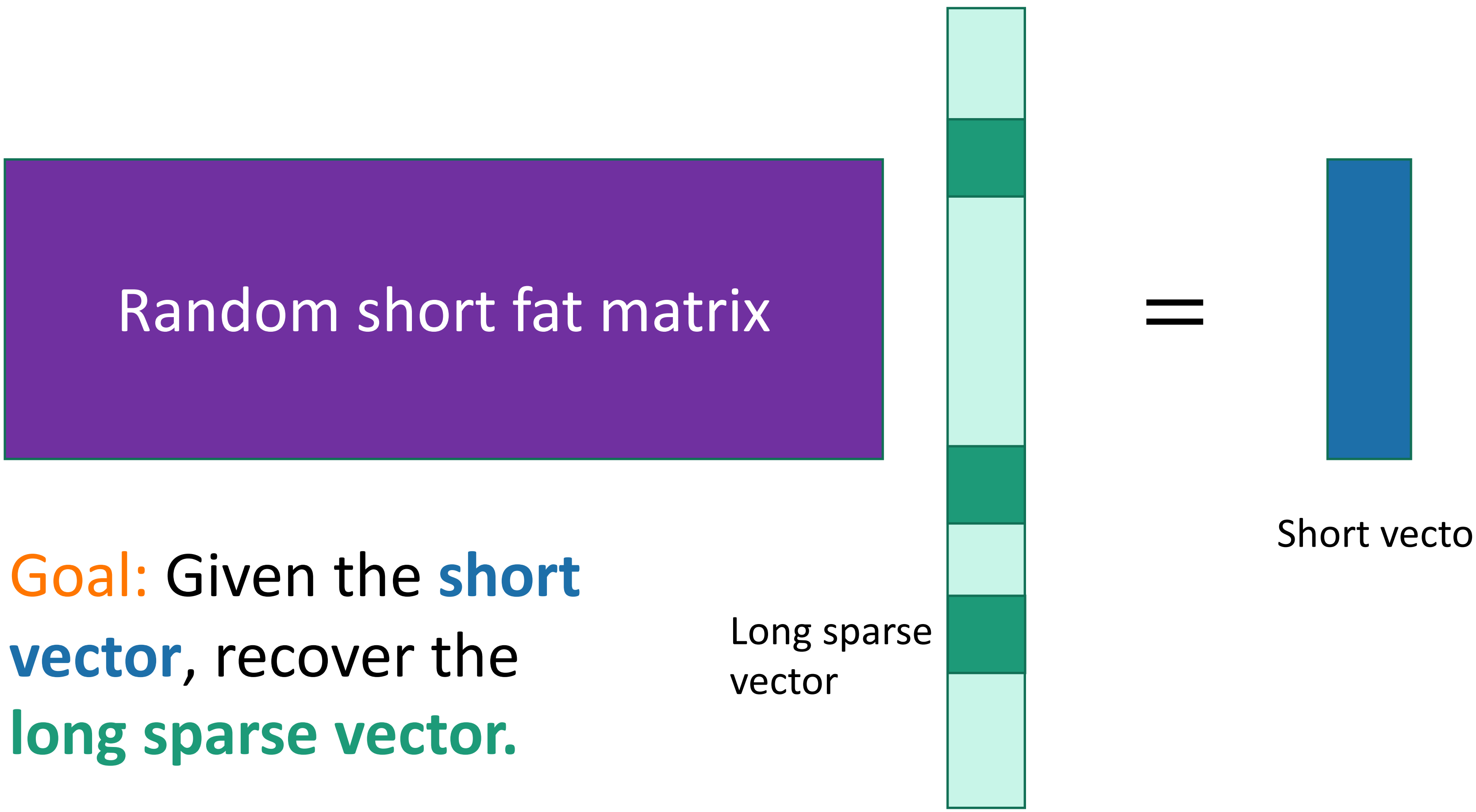
This image is sparse



This image is sparse after I take a wavelet transform.

Compressed sensing continued

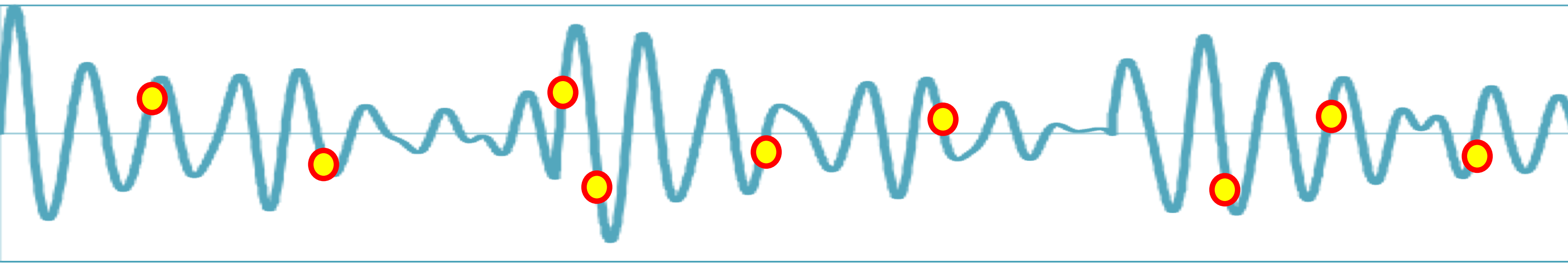
- Take a random projection of that sparse vector:



Goal: Given the **short vector**, recover the **long sparse vector.**

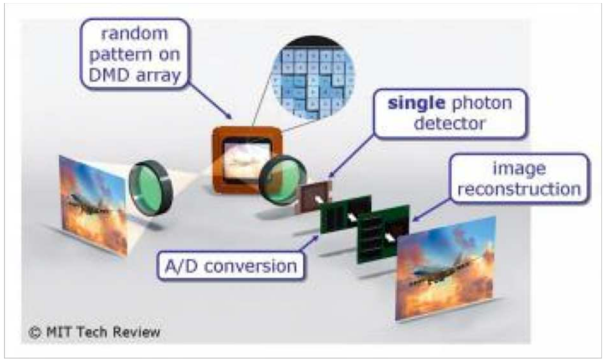
Why would I want to do that?

- Image compression and signal processing
- Especially when you **never have space to store the whole sparse vector to begin with**.



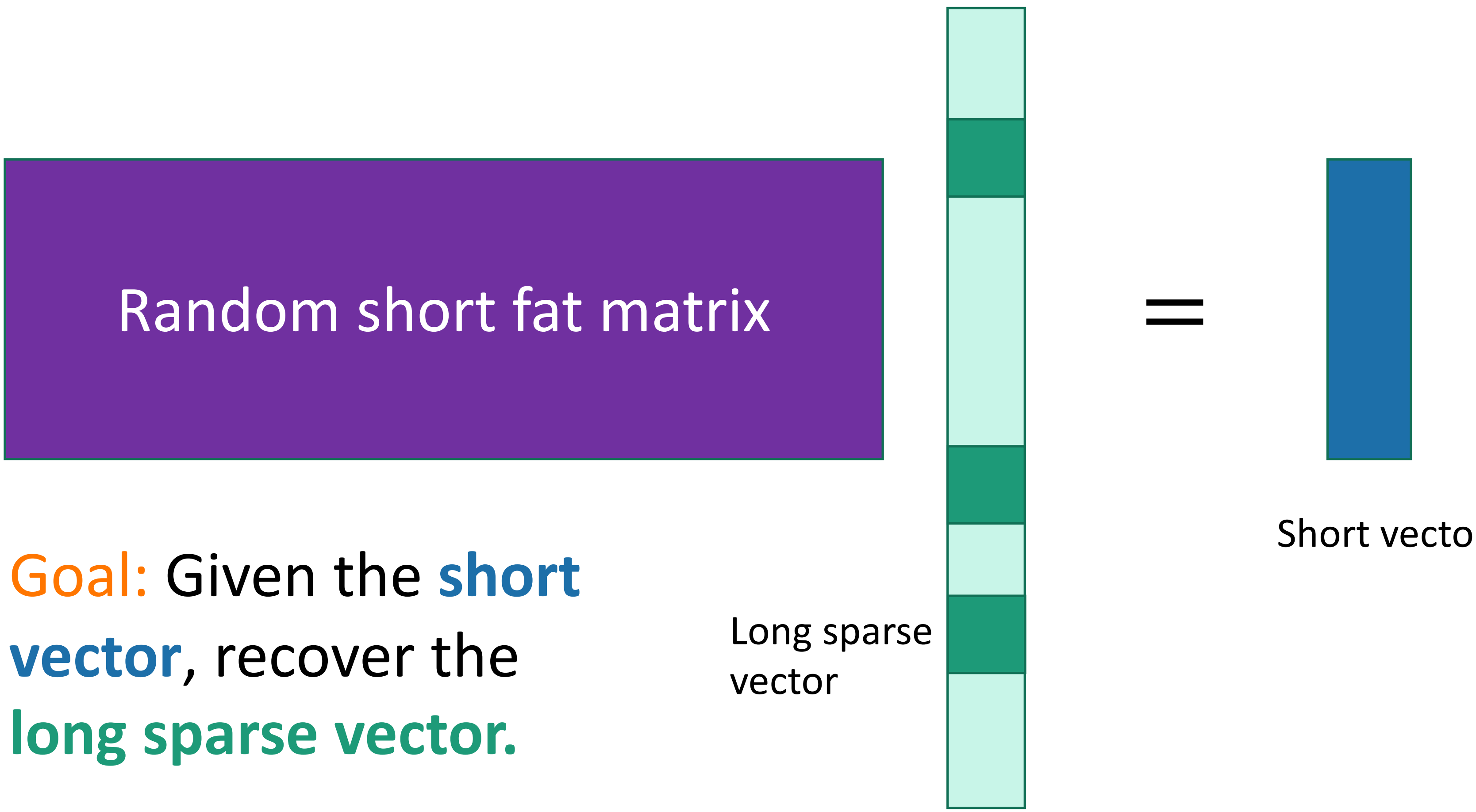
Randomly sampling (in the time domain) a signal that is sparse in the Fourier domain.

Random measurements in an fMRI means you spend less time inside an fMRI



A "single pixel camera" is a thing.

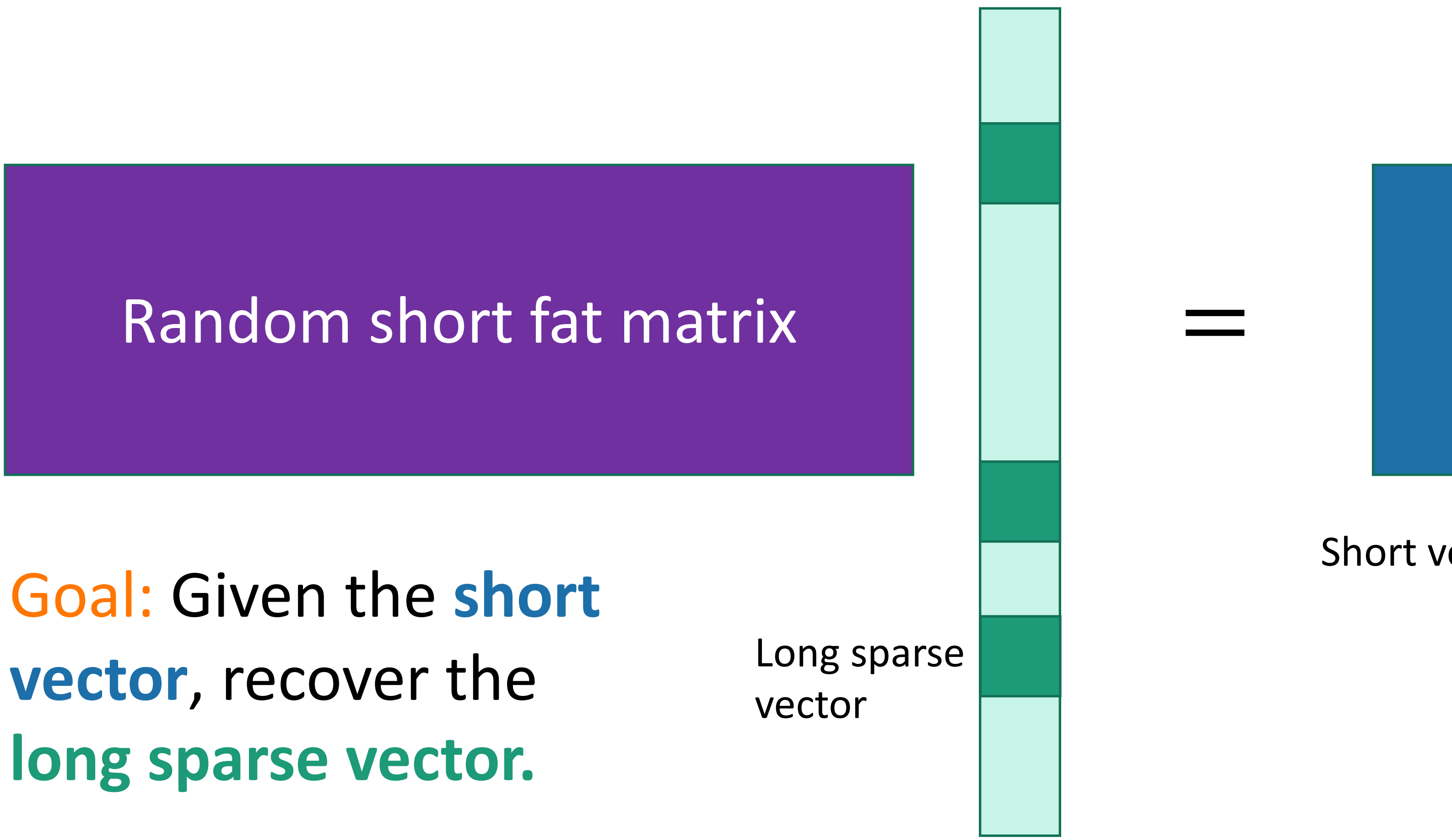
All examples of this:



Goal: Given the **short vector**, recover the **long sparse vector.**

But why should this be possible?

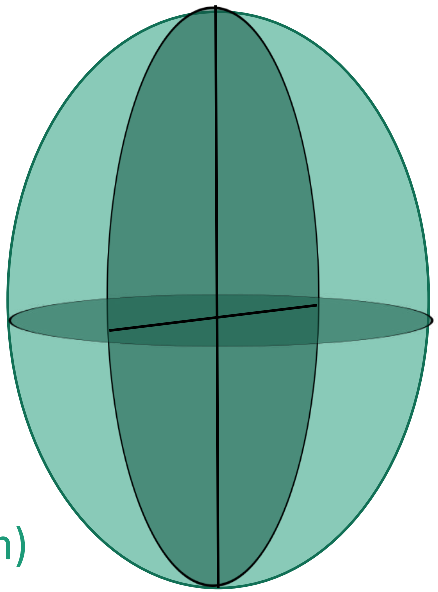
- There are tons of long vectors that map to the short vector!



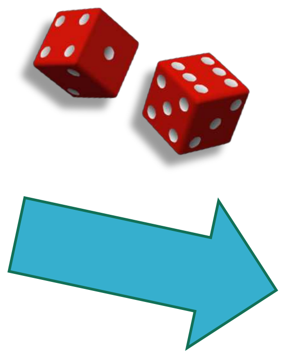
Goal: Given the **short vector**, recover the **long sparse vector.**

Back to the geometry

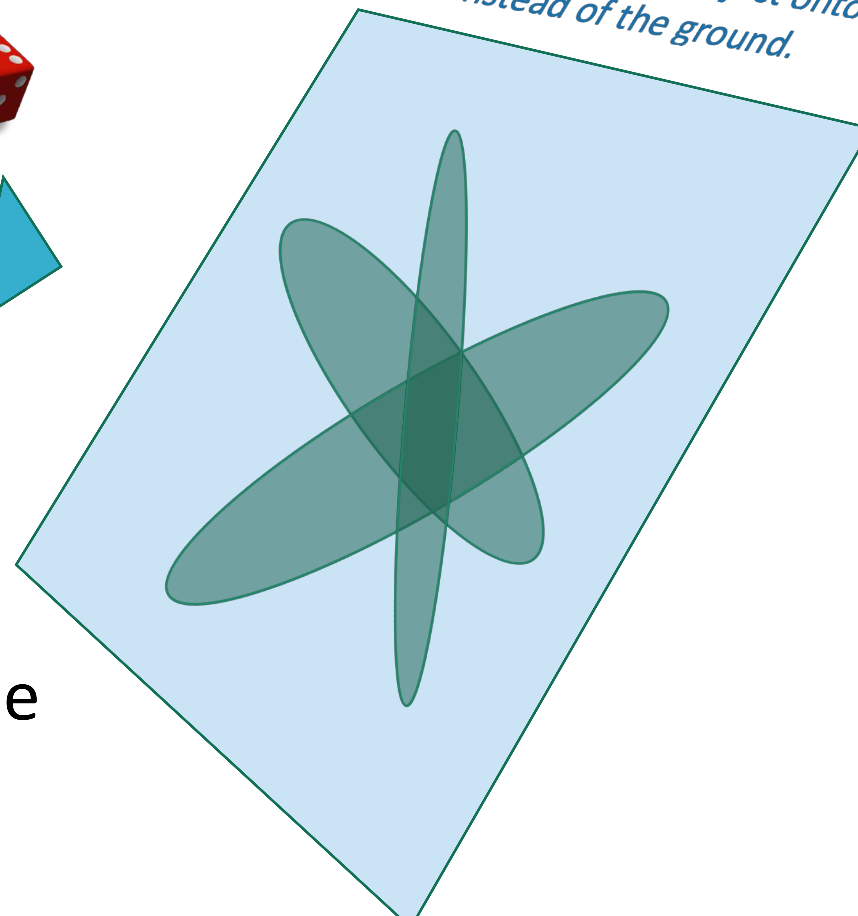
All of the sparse vectors



(Infinitely many of them)



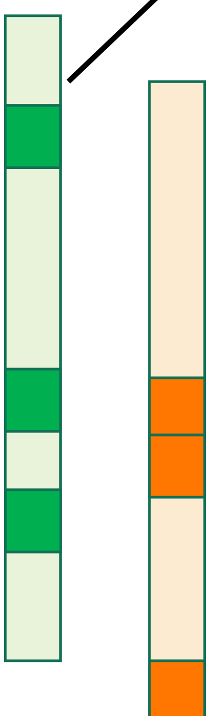
Choose a random subspace to project onto instead of the ground.

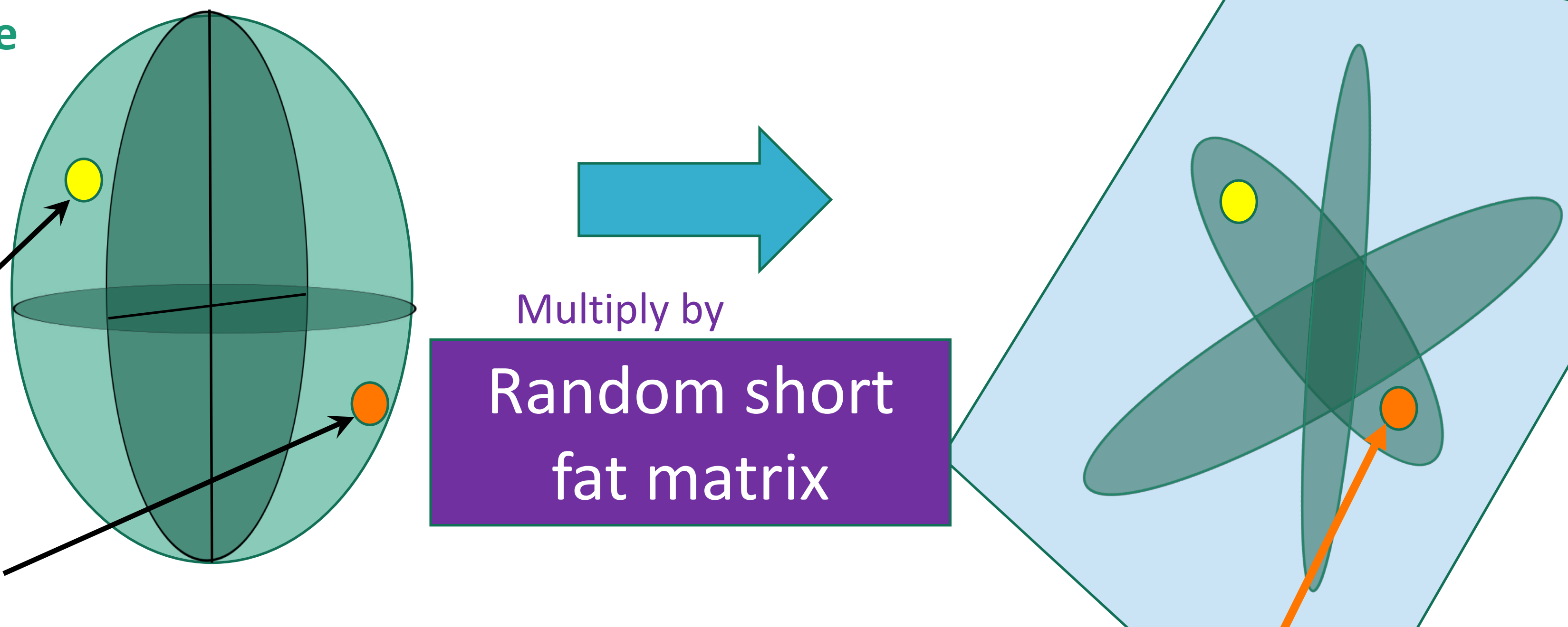


Theorem:

random projections preserve the geometry of sparse vectors too.

If we don’t care about algorithms, that’s more than enough.





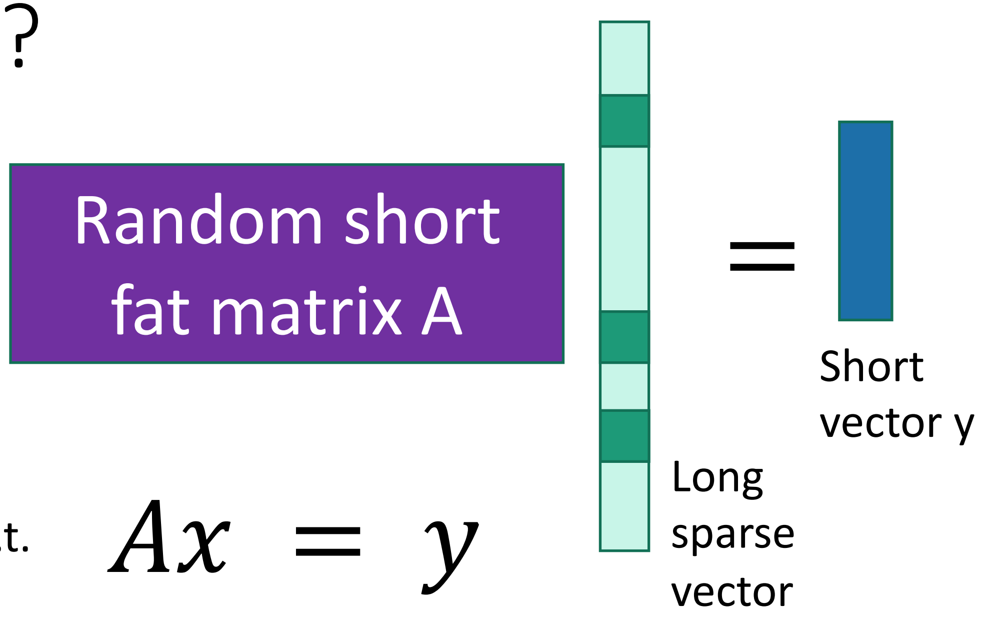
There may be tons of vectors that map to this point, but only one of them is sparse!

This means that, in theory, we can invert that arrow.

How do we do this efficiently??

Goal: Given the **short vector**, recover the **long sparse vector.**

An efficient algorithm?



What we'd like to do is:

Minimize number of nonzero entries in x s.t. $Ax = y$

This isn't a nice function

Problem: I don't know how to do that efficiently!

Instead:

Minimize $\sum_i |x_i|$ s.t. $Ax = y$

- It turns out that because the geometry of sparse vectors is preserved, this optimization problem **gives the same answer**.
- We can use **linear programming** to solve this quickly!

Today

A few gems

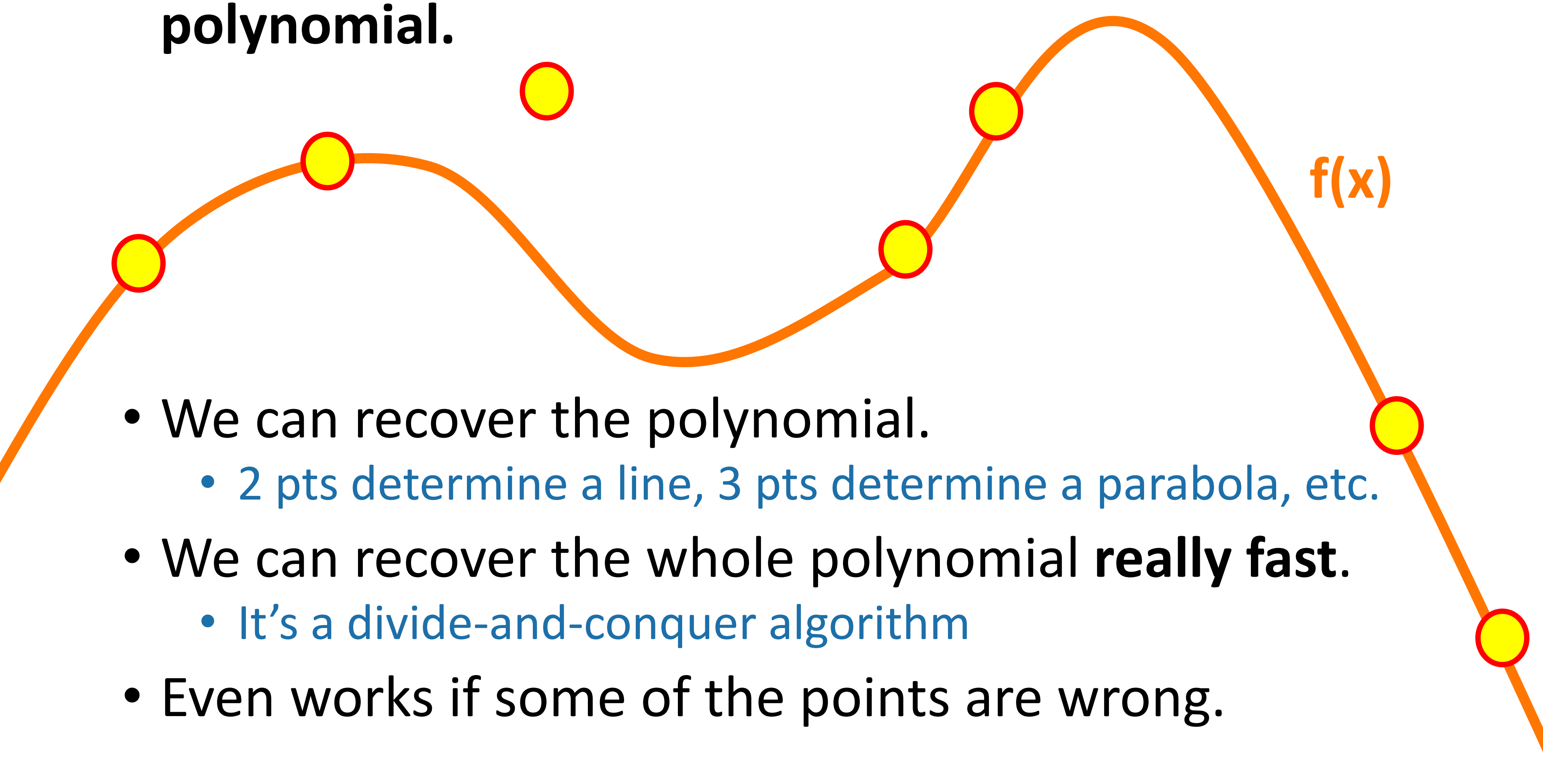
- Linear programming

- Random projections

- Low-degree polynomials

Another of my favorite tricks
Polynomial interpolation

- Say we have a few evaluation points of a **low-degree polynomial.**



- We can recover the polynomial.
  - 2 pts determine a line, 3 pts determine a parabola, etc.
- We can recover the whole polynomial **really fast**.
  - It's a divide-and-conquer algorithm
- Even works if some of the points are wrong.

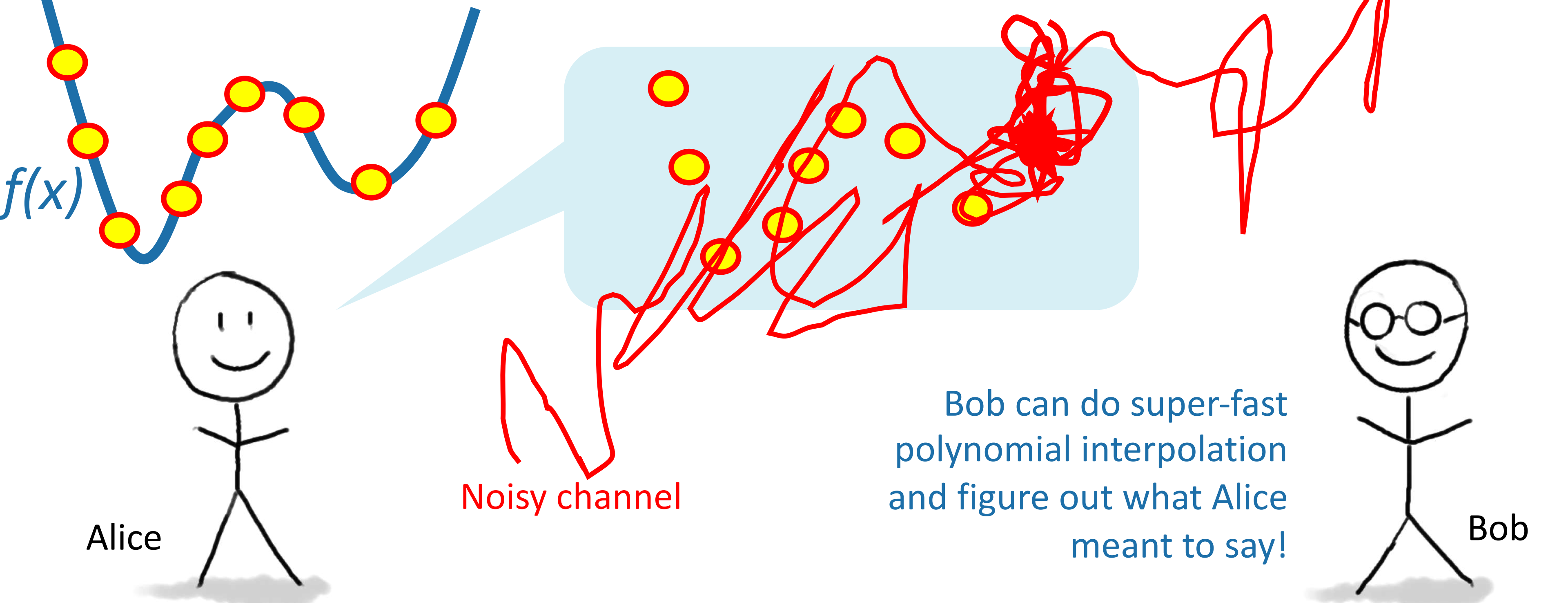
One application:

Communication and Storage

- Alice wants to send a message to Bob

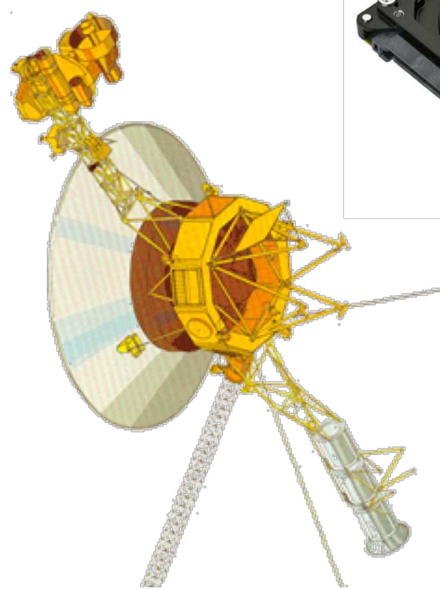
"Hi, Bob!"

$$f(x) = \boldsymbol{H} + \boldsymbol{I} \cdot x + \boldsymbol{B} \cdot x^2 + \boldsymbol{O} \cdot x^3 + \boldsymbol{B} \cdot x^4$$



This is actually used in practice

- It’s called “Reed-Solomon Encoding”



Another application:

Designing “random” projections that are better than random

The matrix that treats the big long vector as Alice’s message polynomial and evaluates it REAL FAST at random points.

- This is still ”random enough” to make the LP solution work.
- It is much more efficient to manipulate and store!

Today

A few gems

- Linear programming
- Random projections
- Low-degree polynomials

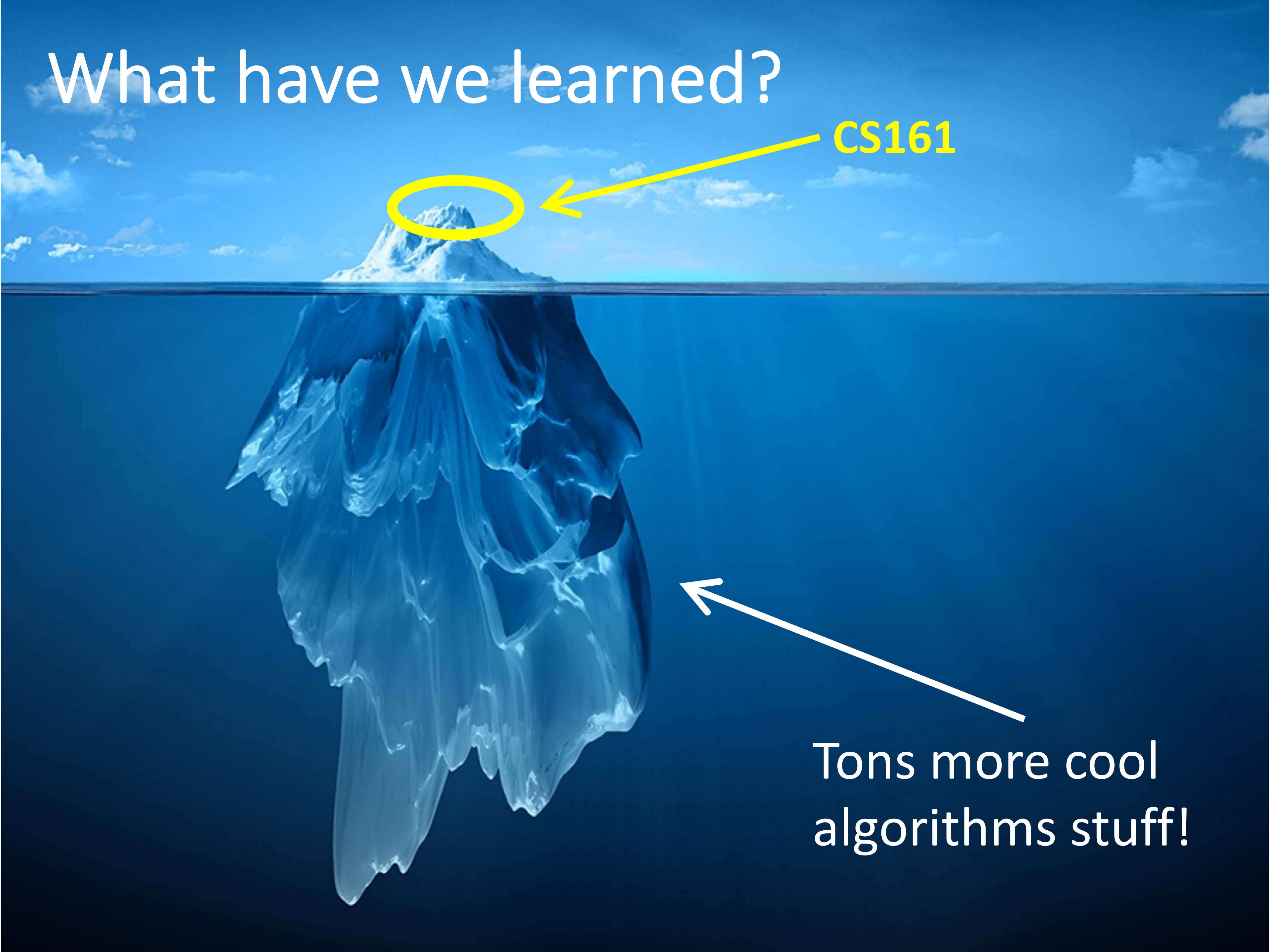
To learn more:

CS168, CS261, …

CS168, CS261, CS264, CS265, …

CS168, CS250, …

What have we learned?



To see more…

- Take more classes!
- Come hang out with the theory group!
  - Theory lunch, Thursdays at noon
  - Theory seminar, usually Fridays at 3pm
  - Join the theory-seminar mailing list for updates

theory.stanford.edu

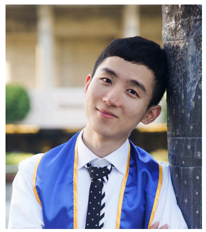
Stanford theory group:
We are very friendly.

A few final messages…

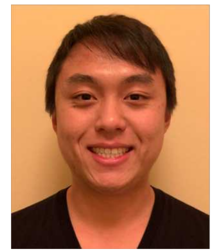
1. Thanks to the TAs!!!

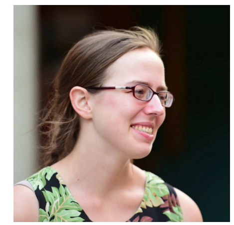
tell them you appreciate them!


Susanna


Jiaxi


Noa

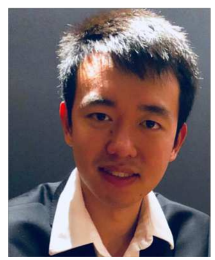

Nick


Reyna


Duligur


Aaron


Ingerid


Haojun


Megha

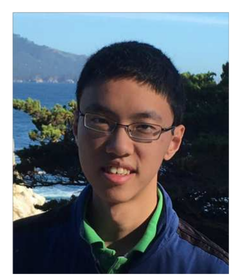

Richard

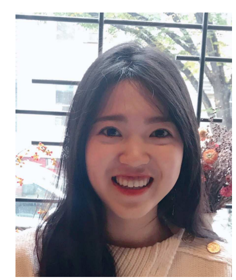

Dana


Jayden


Shashwat


Maxime


Brian


Anna

2.

THANKS

to you!!!!!!