Lecture 4

Median and Selection

Announcements!

- HW1 due Friday.
- HW2 posted Friday.

- I'm going to try to either take a short break around 11:20. If you need to leave at 11:20, please wait for that break so it's not disruptive.
  - (And if I forget, raise your hand at 11:20 and remind me to take that break).

Sections!

- Thursday (x2) and Friday
  - Check website for schedule.
- In general, think of section as reviewing that week's material so you'll be ready to go when HW is released on Friday.
  - This week a bit different; will cover both Weeks 1 and 2 material.

Piazza Heroes!

- Top student answerers:

Name, Email	number of answers
Jabari Hastings	26
Ashish Paliwa	23
Trenton Chang	10
Pranav Jain	8
Avery Wang	7
Richard Lin	7
Adam Leon	2
Andrew Han	2
Brahm Capoor	2
Esther Cherin Kim	2

Last Time:
Solving Recurrence Relations

- A **recurrence relation** expresses $T(n)$ in terms of $T(\text{less than } n)$
- For example, $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$
- Two methods of solution:
    1. Master Theorem (aka, generalized “tree method”)
    2. Substitution method (aka, guess and check)

The Master Theorem

- Suppose $a \geq 1, b > 1,$ and d are constants (that don't depend on n).
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Three parameters:

a : number of subproblems

b : factor by which input size shrinks

d : need to do n^d work to create all the subproblems and combine their solutions.

A powerful theorem it is…

Jedi master Yoda

The Substitution Method

- Step 1: Guess what the answer is.
- Step 2: Prove by induction that your guess is correct.
- Step 3: Profit.

The plan for today

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.

A fun recurrence relation

- $T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$ for $n > 10$.
- Base case: $T(n) = 1 \text{ when } 1 \leq n \leq 10$

The Substitution Method

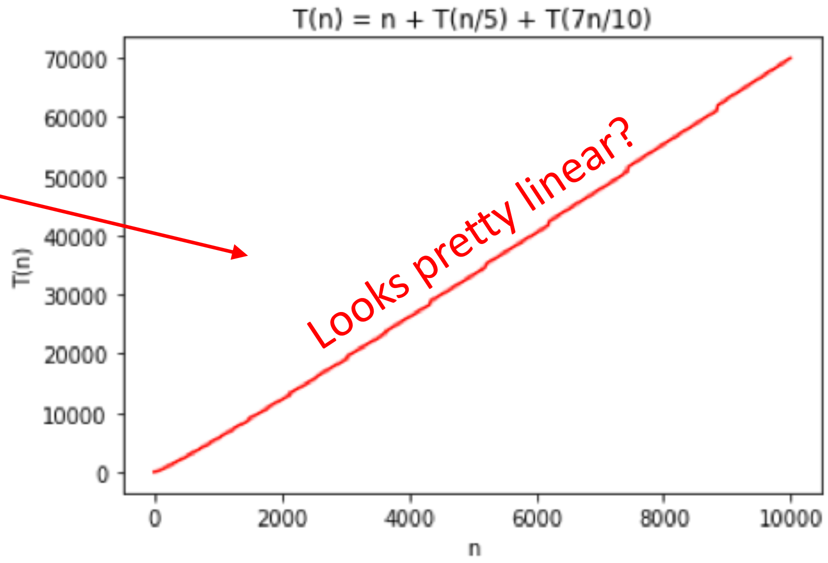
- Step 1: Guess what the answer is.
- Step 2: Prove by induction that your guess is correct.
- Step 3: Profit.

Step 1: guess the answer

$$T(n) \le T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n \text{ for } n > 10.$$

Base case: $T(n) = \ 1 \text{ when } 1 \le n \le 10$

- Trying to work backwards gets gross fast…
- We can also just try it out.
  - (see IPython Notebook)
- Let's guess O(n) and try to prove it.



Step 2: prove our guess is right

$$T(n) \le T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n \text{ for } n > 10.$$

Base case: $T(n) = 1$ when $1 \le n \le 10$

C is some constant we'll have to fill in later!

- Inductive Hypothesis: $T(j) \le \boldsymbol{C}j$ for all $1 \le j \le n$.
- Base case: $1 = T(j) \le \boldsymbol{C}j$ for all $1 \le j \le 10$
- Inductive step:
  - Assume that the IH holds for n=k-1.
  - $$\begin{aligned} T(k) &\le k + T\left(\frac{k}{5}\right) + T\left(\frac{7k}{10}\right) \\ &\le k + \boldsymbol{C} \cdot \left(\frac{k}{5}\right) + \boldsymbol{C} \cdot \left(\frac{7k}{10}\right) \\ &= k + \frac{\boldsymbol{C}}{5}k + \frac{7\boldsymbol{C}}{10}k \\ &\le \boldsymbol{C}k \text{ ??} \end{aligned}$$
  - (aka, want to show that IH holds for k=n).

Whatever we choose C to be, it should have C≥1

Let's solve for C and make this true!
C = 10 works.
(on board)

- Conclusion:
  - There is some $\boldsymbol{C}$ so that for all $n \ge 1$, $T(n) \le \boldsymbol{C}n$
  - Aka, T(n) = O(n). (Technically we also need $0 \le T(n)$ here…)

$T(n) \leq n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$ for $n > 10$.

Base case: $T(n) = 1$ when $1 \leq n \leq 10$

Step 3: Profit

(Aka, pretend we knew this all along).

(Assume that $T(n) \geq 0$ for all n. Then,)

Theorem: $T(n) = O(n)$

Proof:

- Inductive Hypothesis: $T(j) \leq \mathbf{10}j$ for all $1 \leq j \leq n$.
- Base case: $1 = T(j) \leq \mathbf{10}j$ for all $1 \leq j \leq 10$
- Inductive step:
  - Assume the IH holds for n=k-1.
  - $T(k) \leq k + T\left(\frac{k}{5}\right) + T\left(\frac{7k}{10}\right)$
    $\leq k + \mathbf{10} \cdot \left(\frac{k}{5}\right) + \mathbf{10} \cdot \left(\frac{7k}{10}\right)$
    $= k + 2k + 7k = \mathbf{10}k$
  - Thus IH holds for n=k.
- Conclusion:
  - For all $n \geq 1, T(n) \leq \mathbf{10}n$
  - (Also $0 \leq T(n)$ for all $n \geq 1$ since we assumed so.)
  - Aka, T(n) = O(n), using the definition with $n_0 = 1, c = 10$.

Plucky added the stuff about $T(n) \geq 0$ after lecture because this is part of the definition of O() and we were ignoring it…

$$T(n) \leq n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \text{ for } n > 10.$$

Base case: $T(n) = 1$ when $1 \leq n \leq 10$

Step 3: Profit

(Aka, pretend we knew this all along).

(Assume that $T(n) \geq 0$ for all n. Then,)

Theorem: $T(n) = O(n)$

Proof:

- Inductive Hypothesis: $T(n) \leq \mathbf{10}n$.
- Base case: $1 = T(n) \leq \mathbf{10}n$ for all $1 \leq \mathrm{n} \leq 10$
- Inductive step:
  - Assume the IH holds for all $1 \leq n \leq k-1$.
  - $T(k) \leq k + T\left(\frac{k}{5}\right) + T\left(\frac{7k}{10}\right)$
    $\leq k + \mathbf{10} \cdot \left(\frac{k}{5}\right) + \mathbf{10} \cdot \left(\frac{7k}{10}\right)$
    $= k + 2k + 7k = \mathbf{10}k$
  - Thus IH holds for n=k too.
- Conclusion:
  - For all $n \geq 1, T(n) \leq \mathbf{10}n$
  - (Also $0 \leq T(n)$ for all $n \geq 1$ since we assumed so.)
  - Aka, T(n) = O(n), using the definition with $n_0 = 1, c = 10$.

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What have we learned?

- The substitution method can work when the master theorem doesn't.
  - For example with different-sized sub-problems.

- Step 1: generate a guess
  - Throw the kitchen sink at it.
- Step 2: try to prove that your guess is correct
  - You may have to leave some constants unspecified till the end – then see what they need to be for the proof to work!!
- Step 3: profit
  - Pretend you didn't do Steps 1 and 2 and write down a nice proof.

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.

The k-SELECT problem

from your pre-lecture exercise

For today, assume all arrays have distinct elements.

A is an array of size n, k is in {1,…,n}

- SELECT(A, k):
  - Return the k'th smallest element of A.

7	4	3	8	1	5	9	14

- SELECT(A, 1) = 1
- SELECT(A, 2) = 3
- SELECT(A, 3) = 4
- SELECT(A, 8) = 14

- SELECT(A, 1) = MIN(A)
- SELECT(A, n/2) = MEDIAN(A)
- SELECT(A, n) = MAX(A)

Being sloppy about floors and ceilings!

Note that the definition of Select is 1-indexed…

On your pre-lecture exercise…

An O(nlog(n))-time algorithm

- SELECT(A, k):
  - A = MergeSort(A)
  - **return** A[k-1]

It's k-1 and not k since my pseudocode is 0-indexed and the problem is 1-indexed…

- Running time is O(n log(n)).
- So that's the benchmark….

Can we do better?

We're hoping to get O(n)

Show that you can't do better than O(n).

Goal: An O(n)-time algorithm

- On your pre-lecture exercise: SELECT(A, 1).
  - (aka, MIN(A))
- MIN(A):
  - ret = ∞
  - **For** i=0, ..., n-1:
    - If A[i] < ret:
      - ret = A[i]
  - **Return** ret

This stuff is O(1)

This loop runs O(n) times

- Time O(n). Yay!

Also on your pre-lecture exercise

How about SELECT(A,2)?

- SELECT2(A):
  - ret = ∞
  - minSoFar = ∞
  - **For** i=0, .., n-1:
    - **If** A[i] < ret and A[i] < minSoFar:
      - ret = minSoFar
      - minSoFar = A[i]
    - **Else** if A[i] < ret and A[i] >= minSoFar:
      - ret = A[i]
  - **Return** ret

(The actual algorithm here is not very important because this won't end up being a very good idea…)

Still O(n)

SO FAR SO GOOD.

SELECT(A, n/2) aka MEDIAN(A)?

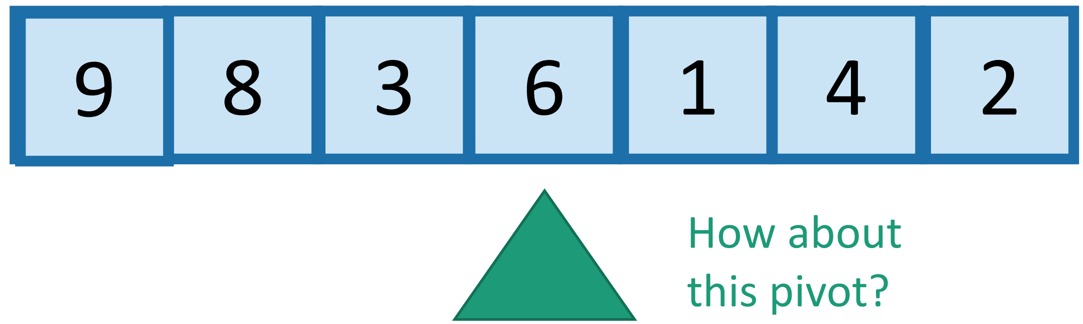
- MEDIAN(A):
  - ret = ∞
  - minSoFar = ∞
  - secondMinSoFar = ∞
  - thirdMinSoFar = ∞
  - fourthMinSoFar = ∞
  - ….
- This is not a good idea for large k (like n/2 or n).
- Basically this is just going to turn into something like INSERTIONSORT…and that was $O(n^2)$.

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.

Idea: divide and conquer!

Say we want to find SELECT(A, k)



How about this pivot?

First, pick a "pivot." We'll see how to do this later.

Next, partition the array into "bigger than 6" or "less than 6"

This PARTITION step takes time O(n). (Notice that we don't sort each half).

L = array with things smaller than A[pivot]

R = array with things larger than A[pivot]

Idea: divide and conquer!

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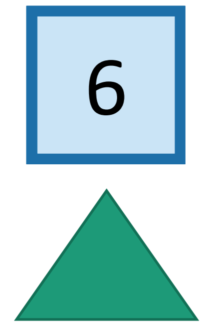

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This PARTITION step takes time O(n). (Notice that we don't sort each half).

3	1	4	2

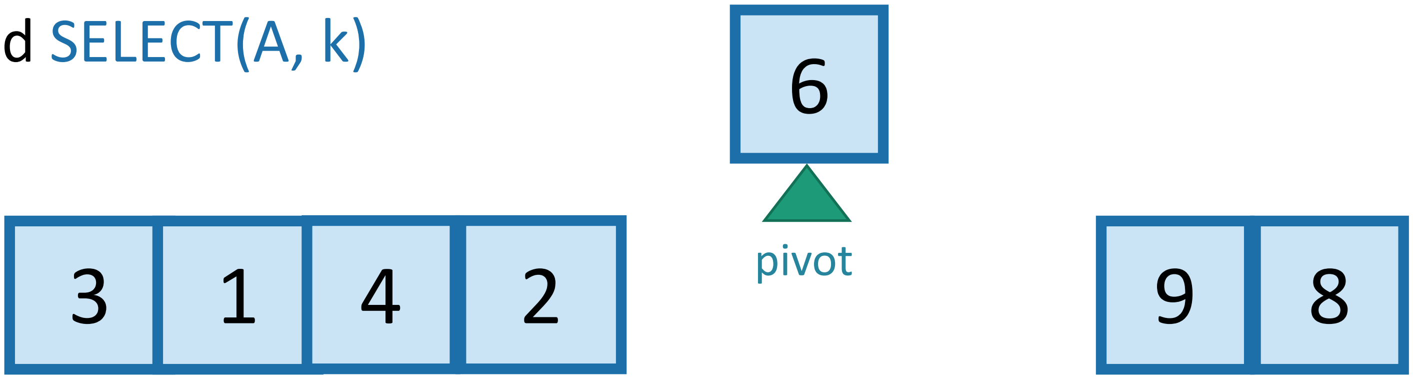
L = array with things smaller than A[pivot]

9	8

R = array with things larger than A[pivot]

Idea continued…

Say we want to find SELECT(A, k)



L = array with things smaller than A[pivot]

R = array with things larger than A[pivot]

- If k = 5 = len(L) + 1:
  - We should return A[pivot]
- If k < 5:
  - We should return SELECT(L, k)
- If k > 5:
  - We should return SELECT(R, k – 5)

This suggests a recursive algorithm

(still need to figure out how to pick the pivot…)

Pseudocode

- **getPivot**(A) returns some pivot for us.
  - How?? We'll see later…
- **Partition**(A,p) splits up A into L, A[p], R.
  - See Lecture 4 IPython notebook for code

- **Select**(A,k):
  - **If** len(A) <= 50:
    - A = **MergeSort**(A)
    - **Return** A[k-1]
  - p = **getPivot**(A)
  - L, pivotVal, R = **Partition**(A,p)
  - **if** len(L) == k-1:
    - return pivotVal
  - **Else if** len(L) > k-1:
    - return **Select**(L, k)
  - **Else if** len(L) < k-1:
    - return **Select**(R, k – len(L) – 1)

Base Case: If the len(A) = O(1), then any sorting algorithm runs in time O(1).

Case 1: We got lucky and found exactly the k'th smallest value!

Case 2: The k'th smallest value is in the first part of the list

Case 3: The k'th smallest value is in the second part of the list

Let’s make sure it works

- [IPython Notebook for Lecture 4]

Now we should be convinced

- No matter what procedure we use for **`getPivot`**(A), **`Select`**(A,k) returns a correct answer.

Formally prove the correctness of **`Select`**! (Hint: Induction!)

Siggi the Studious Stork

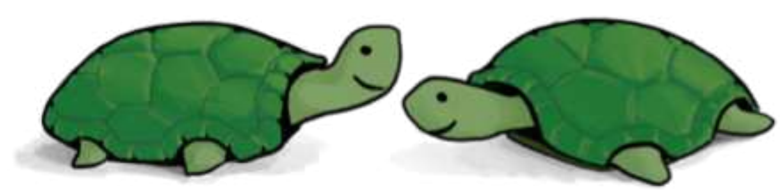
What is the running time?

Assuming we pick the pivot in time O(n)…

- $T(n) = \begin{cases} T(\mathbf{len(L)}) + O(n) & \mathbf{len(L)} > k - 1 \\ T(\mathbf{len(R)}) + O(n) & \mathbf{len(L)} < k - 1 \\ O(n) & \mathbf{len(L)} = k - 1 \end{cases}$

- What are **len(L)** and **len(R)**?
- That depends on how we pick the pivot…

What would be a "good" pivot?
What would be a "bad" pivot?



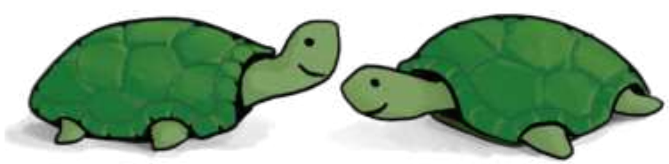
Think-Pair-Share Terrapins

The best way would be to always pick the pivot so that len(L) = k-1. But say we don't have control over k, just over how we pick the pivot.

The ideal pivot

- We split the input exactly in half:
  - len(L) = len(R) = (n-1)/2

What happens in that case?



In case it's helpful…

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

The ideal pivot

- We split the input exactly in half:
  - len(L) = len(R) = (n-1)/2
- Let's pretend that's the case and use the **Master Theorem**!
  - $T(n) \leq T\left(\frac{n}{2}\right) + O(n)$
  - So a = 1, b = 2, d = 1
  - $T(n) \leq O\left(n^d\right) = O(n)$

Apply here, the Master Theorem does NOT. Making unsubstantiated assumptions about problem sizes, we are.

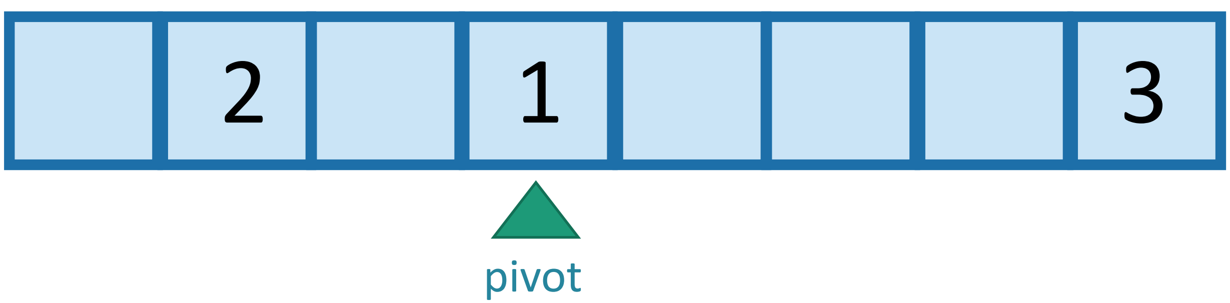
Jedi master Yoda

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

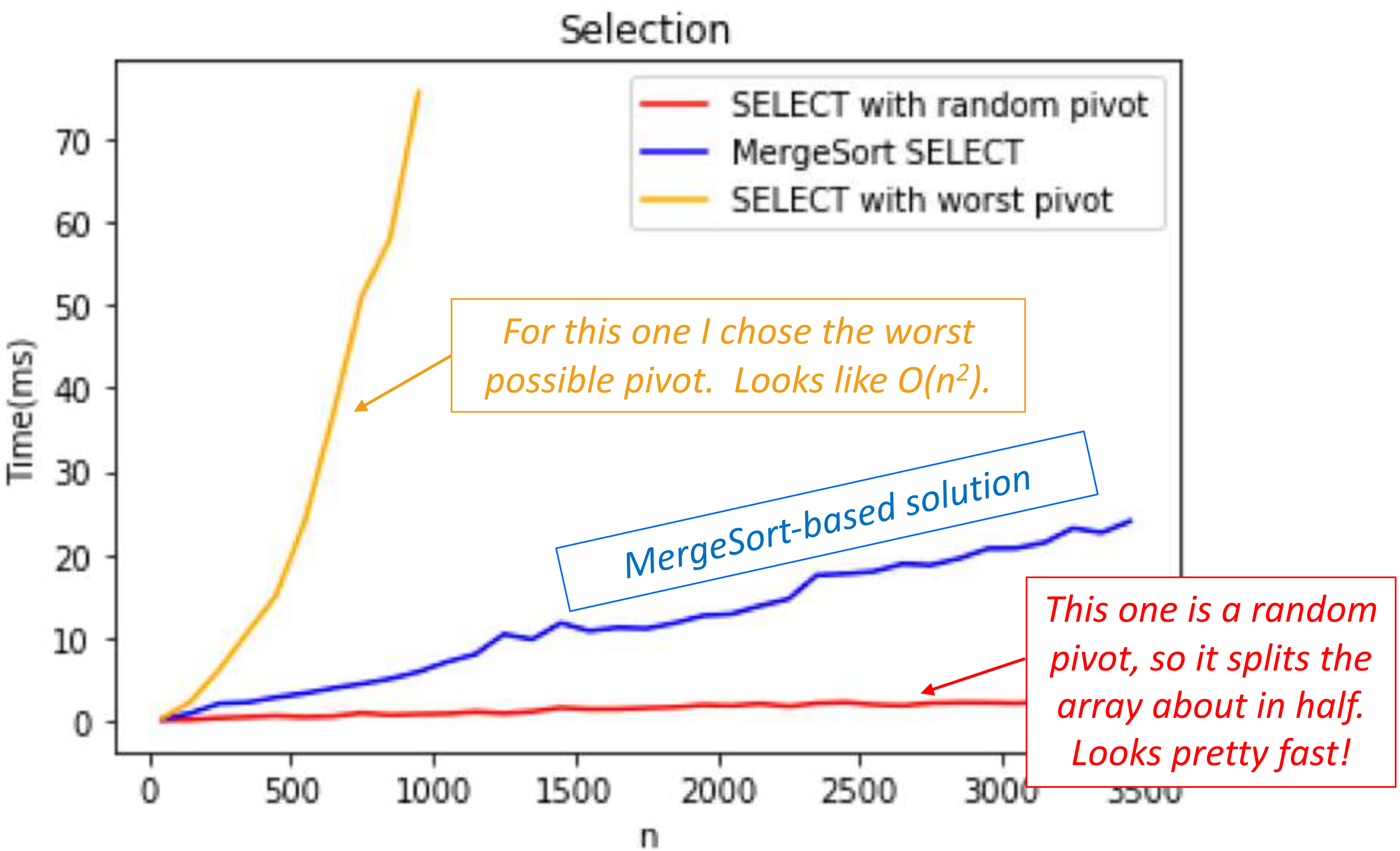
$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

The worst pivot

- Say our choice of pivot doesn't depend on A.
- A bad guy who **knows what pivots we will choose** gets to come up with A.



The distinction matters!



See Lecture 4 IPython notebook for code that generated this picture.

How do we pick a good pivot?

- Randomly?
  - That works well if there's no bad guy.
  - But if there is a bad guy who gets to see our pivot choices, that's just as bad as the worst-case pivot.

Aside:

- In practice, there is often no bad guy. In that case, just pick a random pivot and it works really well!
- (More on this next week)

How do we pick a good pivot?

- For today, let's assume there's this bad guy.
- Reasons:
  - This gives us a very strong guarantee
  - We'll get to see a really clever algorithm.
    - Necessarily it will look at A to pick the pivot.
  - We'll get to use the substitution method.

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.

Approach

- First, we’ll figure out what the ideal pivot would be.
  - But we won’t be able to get it.
- Then, we’ll figure out what a **pretty good** pivot would be.
  - But we still won’t know how to get it.
- Finally, we will see how to get our pretty good pivot!
  - And then we will celebrate.

How do we pick our ideal pivot?

- We'd like to live in the ideal world.

- Pick the pivot to divide the input in half.
- Aka, pick the median!
- Aka, pick SELECT(A, n/2)!

How about a good enough pivot?

- We'd like to approximate the ideal world.

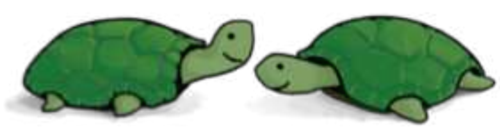
- Pick the pivot to divide the input about in half!
- Maybe this is easier!

A good enough pivot

We still don't know that we can get such a pivot, but at least it gives us a goal and a direction to pursue!

Lucky the lackadaisical lemur

- We split the input not quite in half:
  - 3n/10 < len(L) < 7n/10
  - 3n/10 < len(R) < 7n/10
- If we could do that (let's say, in time O(n)), the **Master Theorem** would say:
  - $T(n) \leq T\left(\frac{7n}{10}\right) + O(n)$



Think-Pair-Share Terrapins!

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

A good enough pivot

We still don't know that we can get such a pivot, but at least it gives us a goal!

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- If we could do that (let's say, in time O(n)), the **Master Theorem** would say:
  - $T(n) \leq T\left(\frac{7n}{10}\right) + O(n)$
  - So a = 1, b = 10/7, d = 1
  - $T(n) \leq O\left(n^d\right) = O(n)$

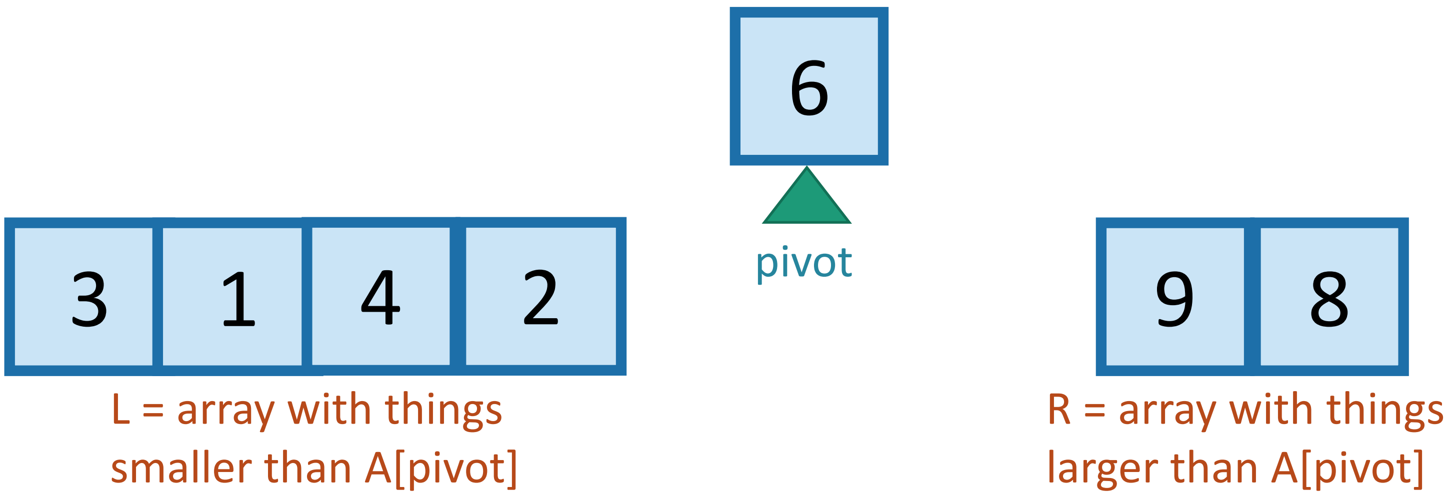
STILL GOOD!

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Goal

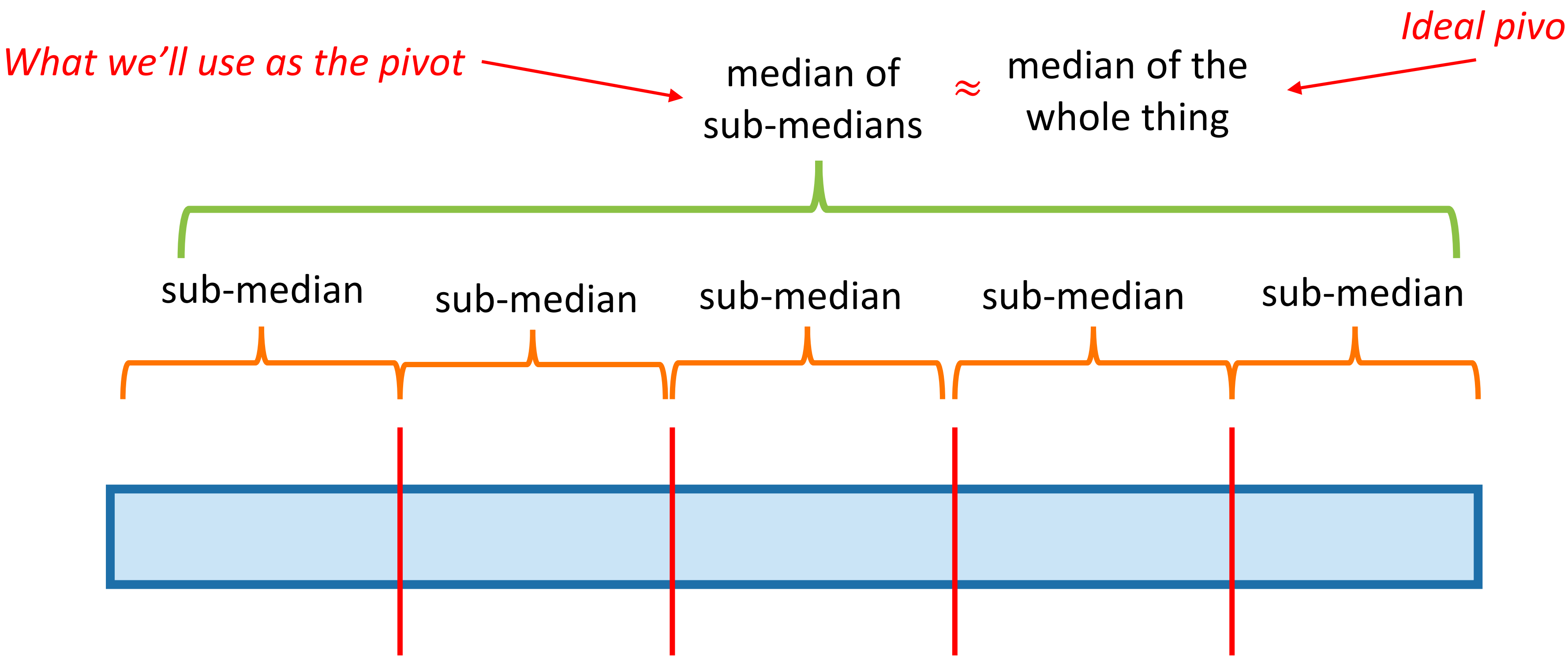
- In time O(n), pick the pivot so that



$$\frac{3n}{10} < \mathrm{len}(L) < \frac{7n}{10} \qquad \frac{3n}{10} < \mathrm{len}(R) < \frac{7n}{10}$$

Another divide-and-conquer alg!

- We can't solve SELECT(A,n/2) (yet)
- But we can divide and conquer and solve SELECT(B,m/2) for smaller values of m (where len(B) = m).
- **Lemma*:** The median of sub-medians is close to the median.



*we will make this a bit more precise.

How to pick the pivot

- **CHOOSEPIVOT(A):**
  - Split A into m = $\left\lceil \frac{n}{5} \right\rceil$ groups, of size <=5 each.
  - **For** i=1, .., m:
    - Find the median within the i'th group, call it p_i
  - p = SELECT($[\, p_1, p_2, p_3, \ldots, p_m \,]$, m/2)
  - **return** p

This takes time O(1), for each group, since each group has size 5. So that's O(m)=O(n) total in the for loop.

8

4

1	8	9	3	15	5	9	1	3	4	12	2	1	5	20	15	13	2	4	6	12	1	15	22	3

6

Pivot is SELECT(8 4 5 6 12 , 3) = 6:

12

6

1	8	9	3	15	5	9	1	3	4	12	2	1	5	20	15	13	2	4	12	1	15	22	3

PARTITION around that 6:

6

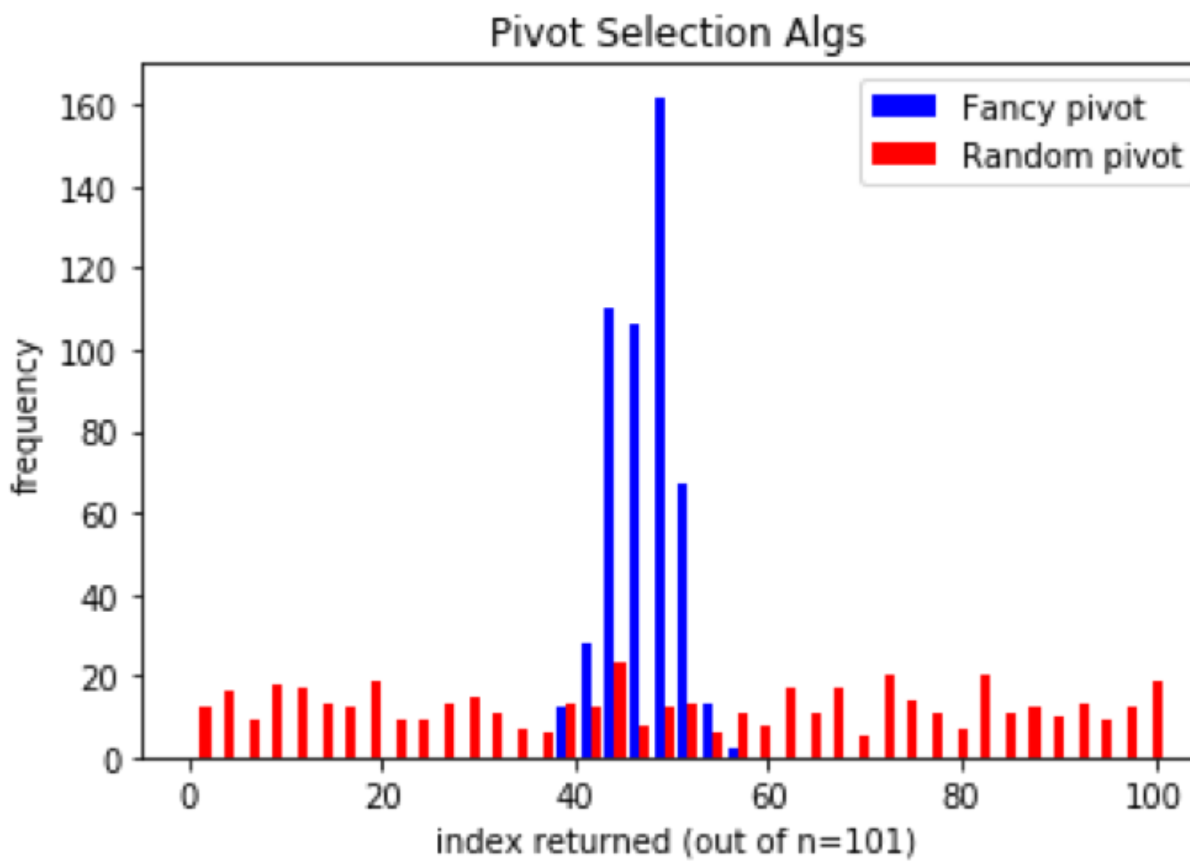
1	3	5	1	3	4	2	1	2	4	1	3	5	8	9	15	9	12	20	15	13	12	15	22

This part is L

This part is R: it's almost the same size as L.

CLAIM: this works
divides the array *approximately* in half

- Empirically (see Lecture 4 IPython Notebook):



CLAIM: this works

divides the array ***approximately*** in half

- Formally, we will prove (later):

Lemma: If we choose the pivots like this, then

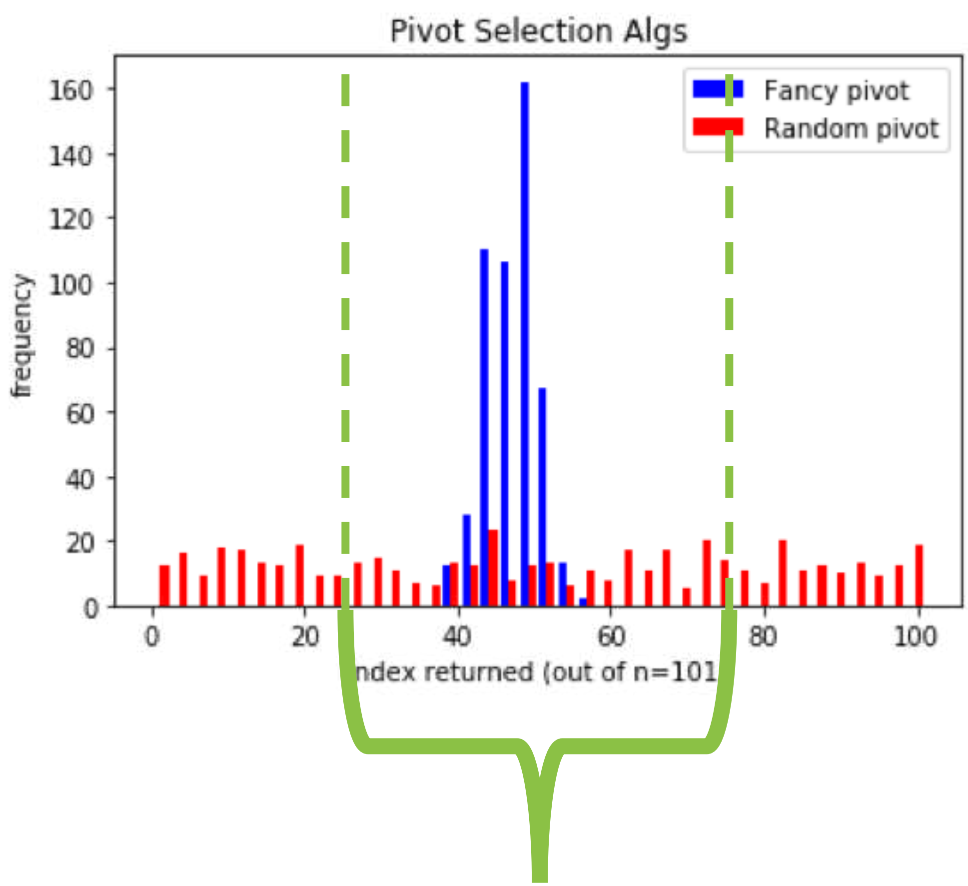
$$|L| \leq \frac{7n}{10} + 5$$

and

$$|R| \leq \frac{7n}{10} + 5$$

Sanity Check

$|L| \leq \frac{7n}{10} + 5$ and $|R| \leq \frac{7n}{10} + 5$



That's this window

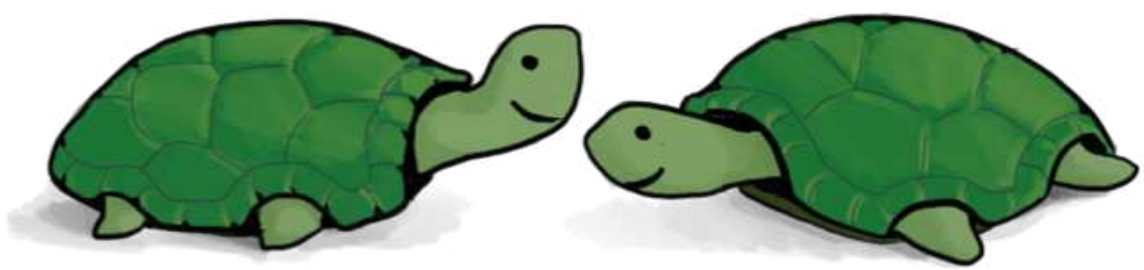
Actually in practice (on randomly chosen arrays) it looks **even better**!

But this is a worst-case bound.

How about the running time?

- Suppose the Lemma is true. (It is).
  - $|L| \leq \frac{7n}{10} + 5$ and $|R| \leq \frac{7n}{10} + 5$
- Recurrence relation:

$$T(n) \leq ?$$



Pseudocode

- **getPivot**(A) returns some pivot for us.
  - How?? We'll see later…
- **Partition**(A,p) splits up A into L, A[p], R.
  - See Lecture 4 notebook for code

- **Select**(A,k):
  - **If** len(A) <= 50:
    - A = **MergeSort**(A)
    - **Return** A[k-1]
  - p = **getPivot**(A)
  - L, pivotVal, R = **Partition**(A,p)
  - **if** len(L) == k-1:
    - return pivotVal
  - **Else if** len(L) > k-1:
    - return **Select**(L, k)
  - **Else if** len(L) < k-1:
    - return **Select**(R, k – len(L) – 1)

Base Case: If the len(A) = O(1), then any sorting algorithm runs in time O(1).

Case 1: We got lucky and found exactly the k'th smallest value!

Case 2: The k'th smallest value is in the first part of the list

Case 3: The k'th smallest value is in the second part of the list

How about the running time?

- Suppose the Lemma is true. (It is).
  - $|L| \leq \frac{7n}{10} + 5$ and $|R| \leq \frac{7n}{10} + 5$
- Recurrence relation:

$$T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n)$$

The call to CHOOSEPIVOT makes one further recursive call to SELECT on an array of size n/5.

Outside of CHOOSEPIVOT, there's at most one recursive call to SELECT on array of size 7n/10 + 5. We're going to drop the "+5" for convenience, but see CLRS for a more careful treatment which includes it.

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.

This sounds like a job for…

The Substitution Method!

Step 1: generate a guess
Step 2: try to prove that your guess is correct
Step 3: profit

$$T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n)$$

That's convenient! We did this at the beginning of lecture!

Conclusion: $T(n) = O(n)$

Technically we only did it for
$T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$,
not when the last term
has a big-Oh…

Plucky the Pedantic Penguin

Recap of approach

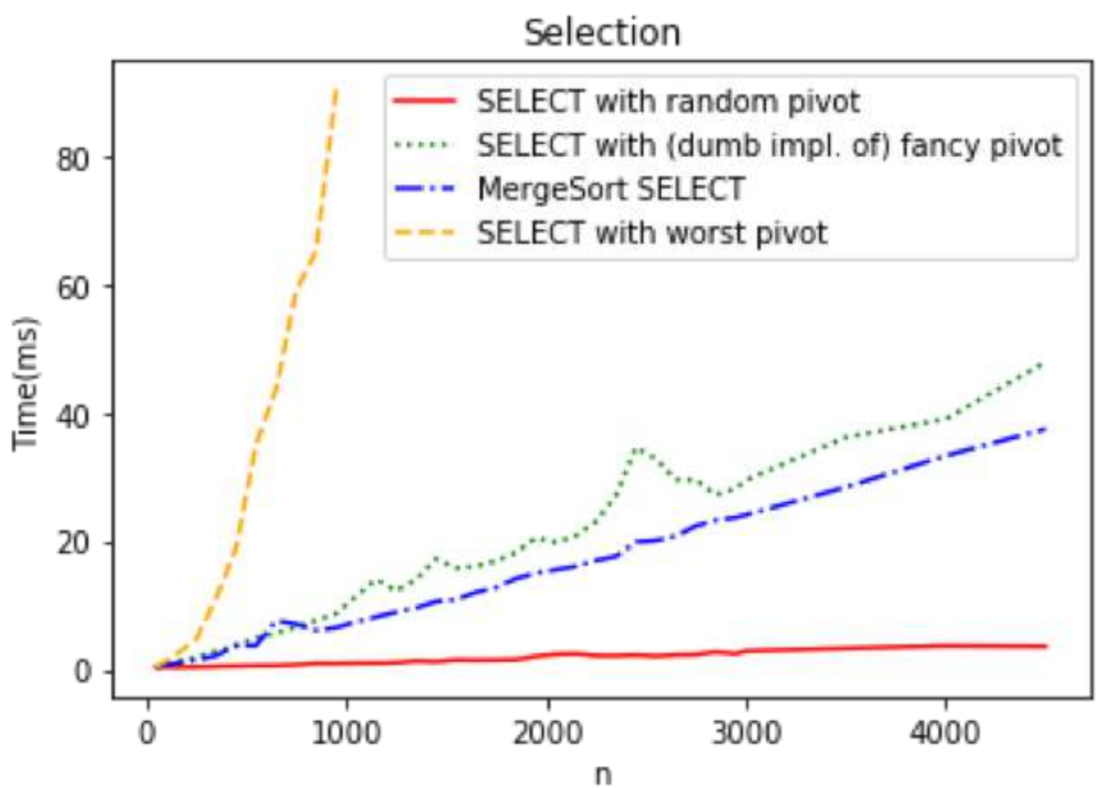
- First, we figured out what the ideal pivot would be.
  - Find the median
- Then, we figured out what a **pretty good** pivot would be.
  - An approximate median
- Finally, we saw how to get our pretty good pivot!
  - Median of medians and divide and conquer!
  - Hooray!

In practice?

- With my dumb implementation, our fancy version of SELECT is worse than the MergeSort-based SELECT ☹
  - But O(n) is better than O(nlog(n))! How can that be?
  - *What's the constant in front of the n in our proof? 20? 30?*

- On **non-adversarial** inputs, random pivot choice is much better.

Moral:
Just pick a random pivot if you don't expect nefarious arrays.



Optimize the implementation of SELECT (with the fancy pivot). Can you beat MergeSort?

Siggi the Studious Stork

What have we learned?

Pending the Lemma

- It is possible to solve SELECT in time O(n).
  - Divide and conquer!
- If you want a deterministic algorithm expect that a bad guy will be picking the list, **choose a pivot cleverly.**
  - More divide and conquer!
- If you don't expect that a bad guy will be picking the list, in practice it's better just to **pick a random pivot.**

The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.
5. (If time) Proof of that Lemma.

If time, back to the Lemma

- **Lemma:** If L and R are as in the algorithm SELECT given above, then

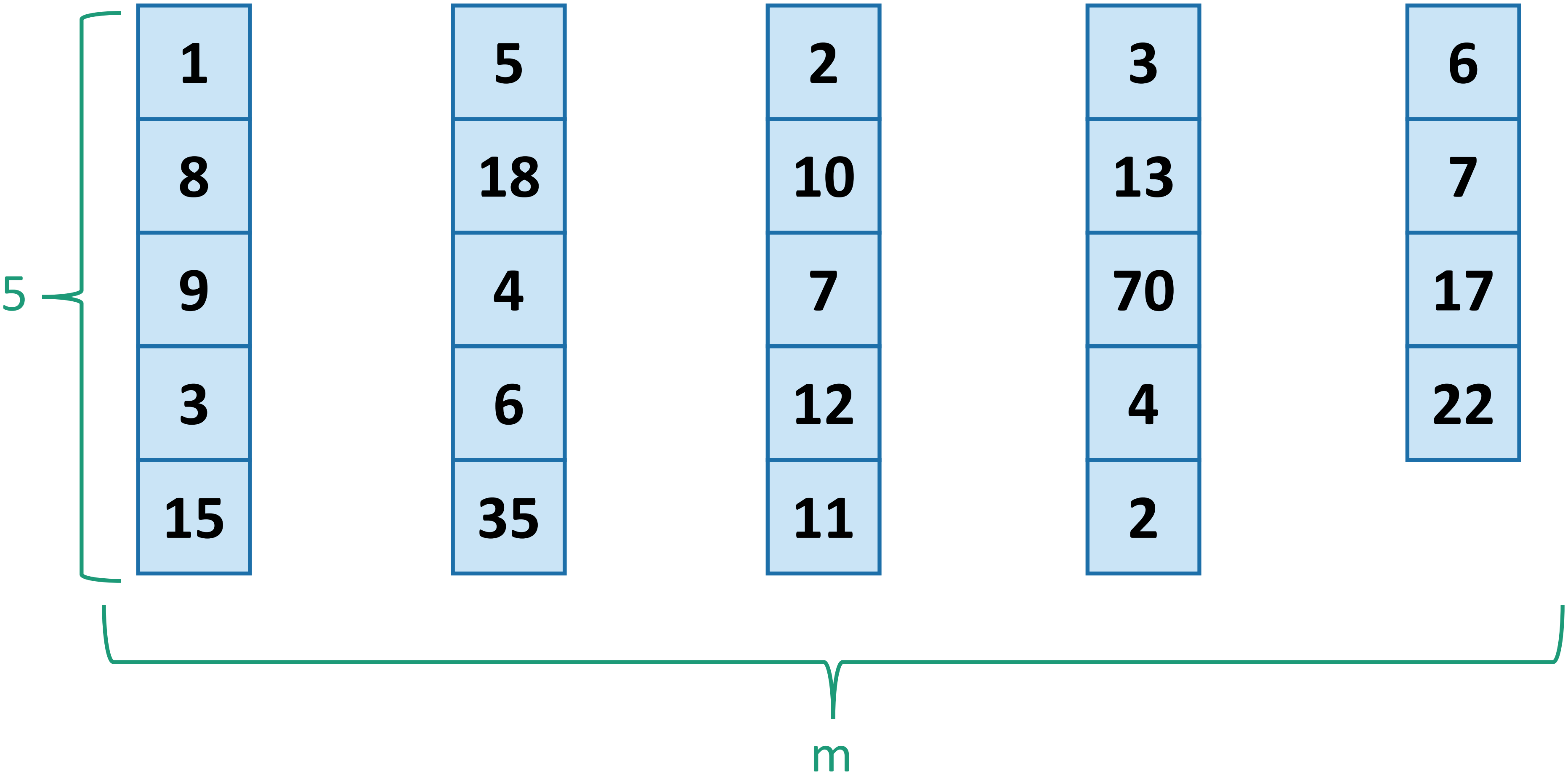
$$|L| \leq \frac{7n}{10} + 5$$

and

$$|R| \leq \frac{7n}{10} + 5$$

- We will see a proof by picture.
- See CLRS for proof by proof.

Proof by picture



Say these are our m = [n/5] sub-arrays of size at most 5.

Proof by picture

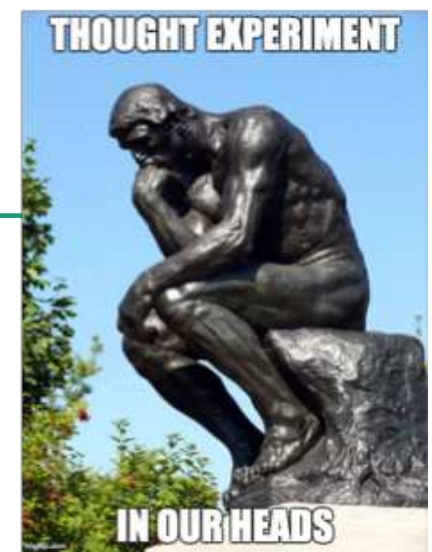
1	4	2	2	6
3	5	7	3	7
8	6	10	4	17
9	18	11	13	22
15	35	12	70	

5

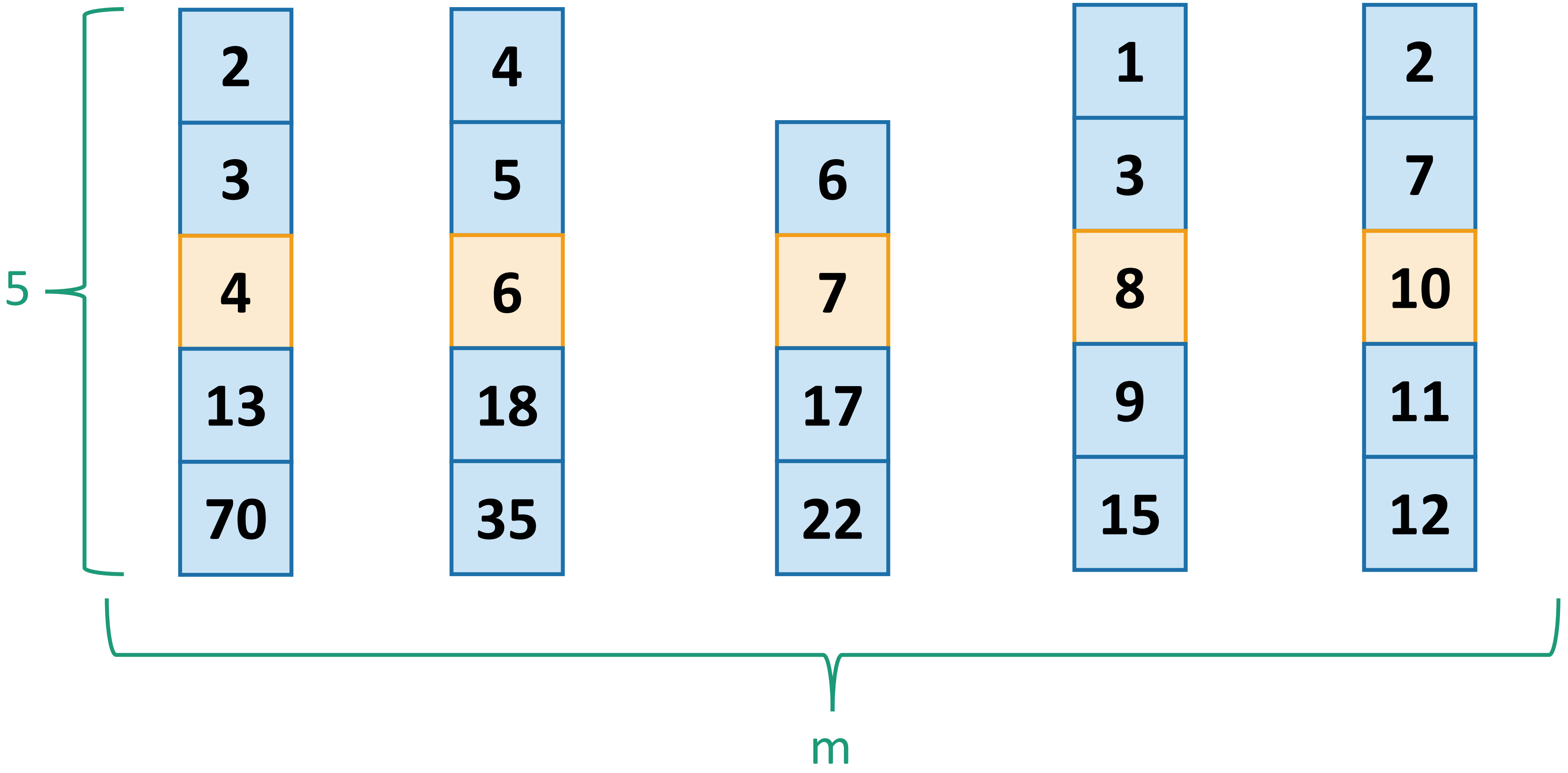
m

In our head, let's sort them.

Then find medians.

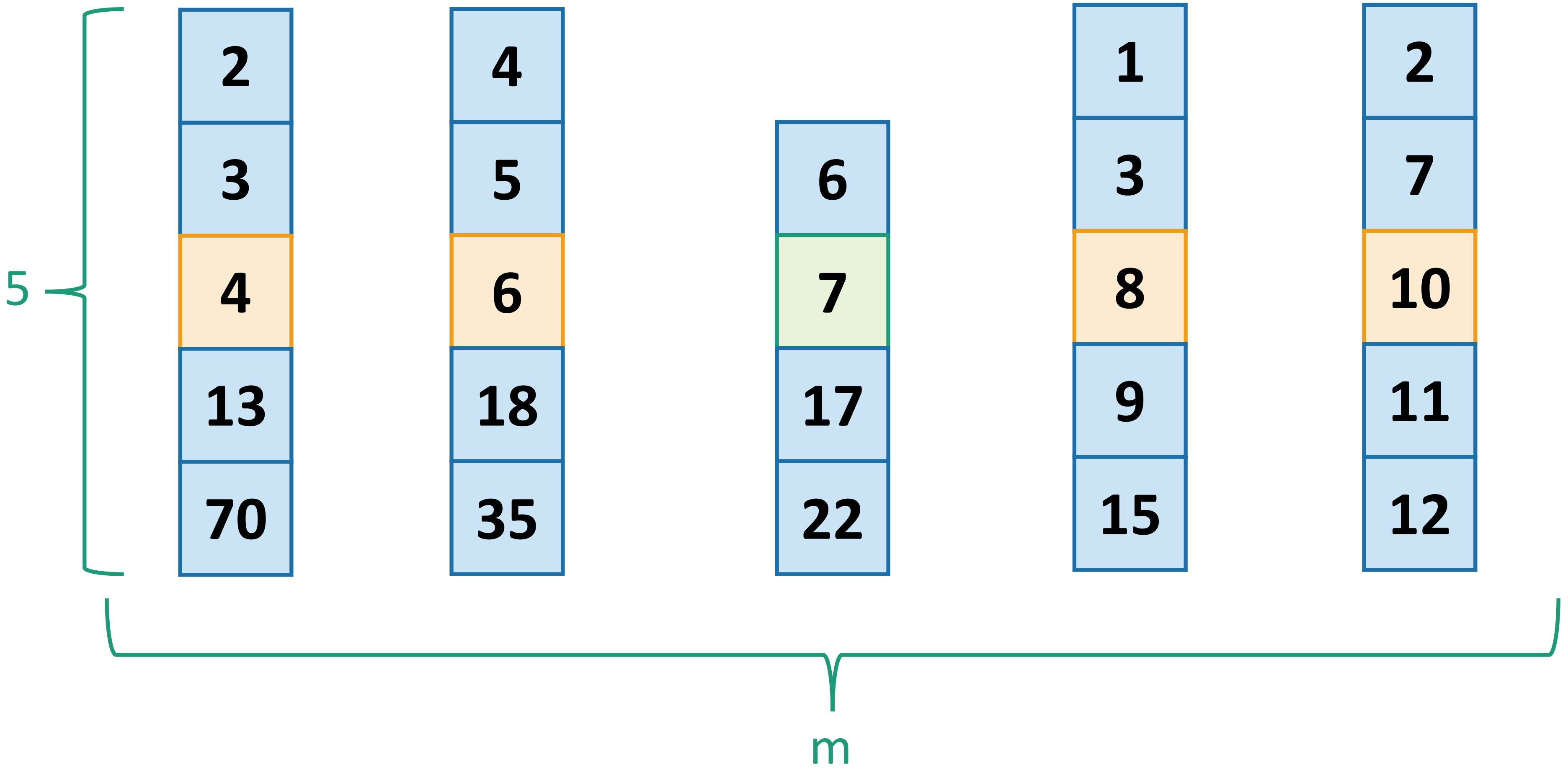


Proof by picture



Then let's sort them by the median

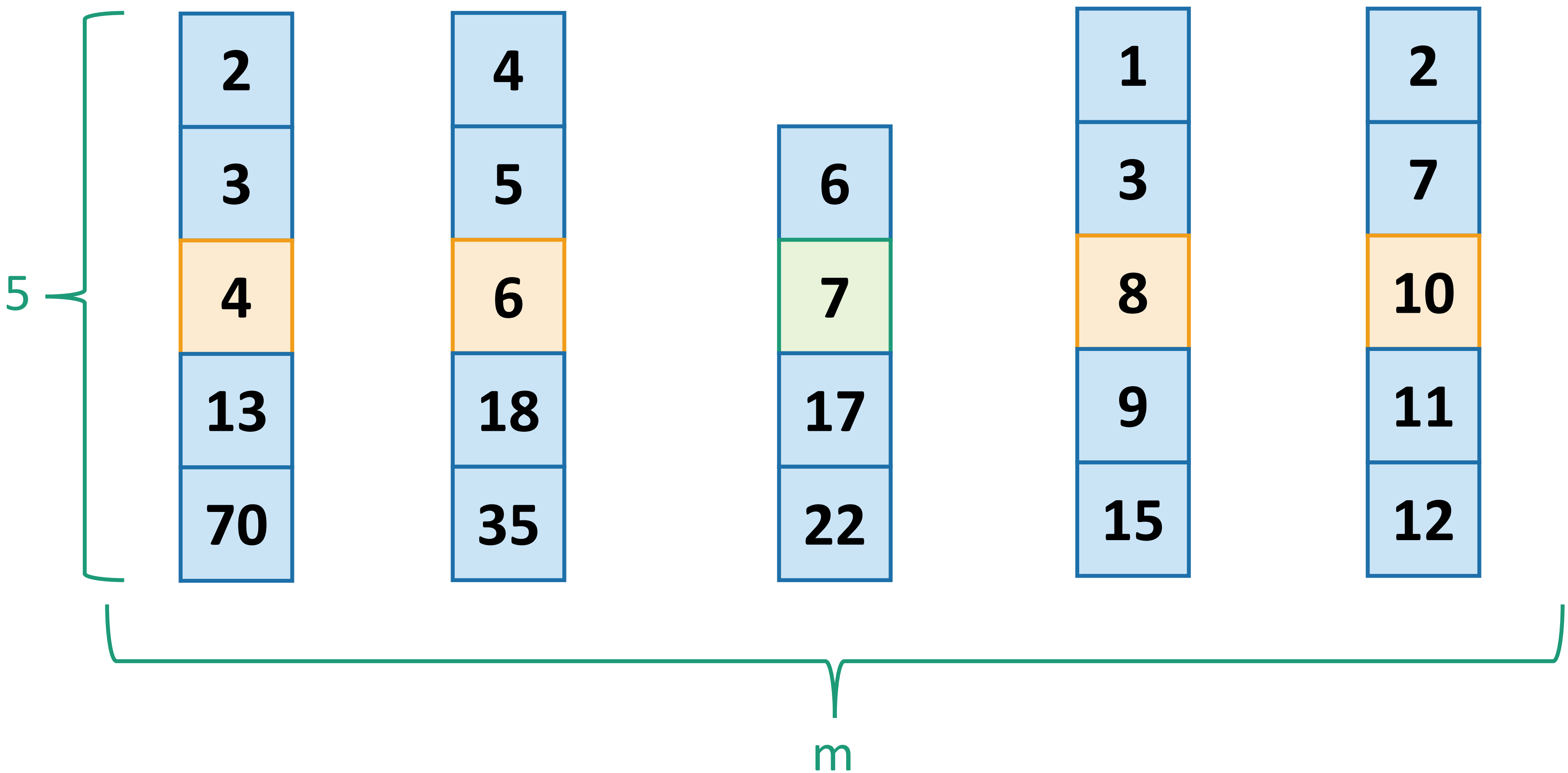
Proof by picture



The median of the medians is 7. That's our pivot!

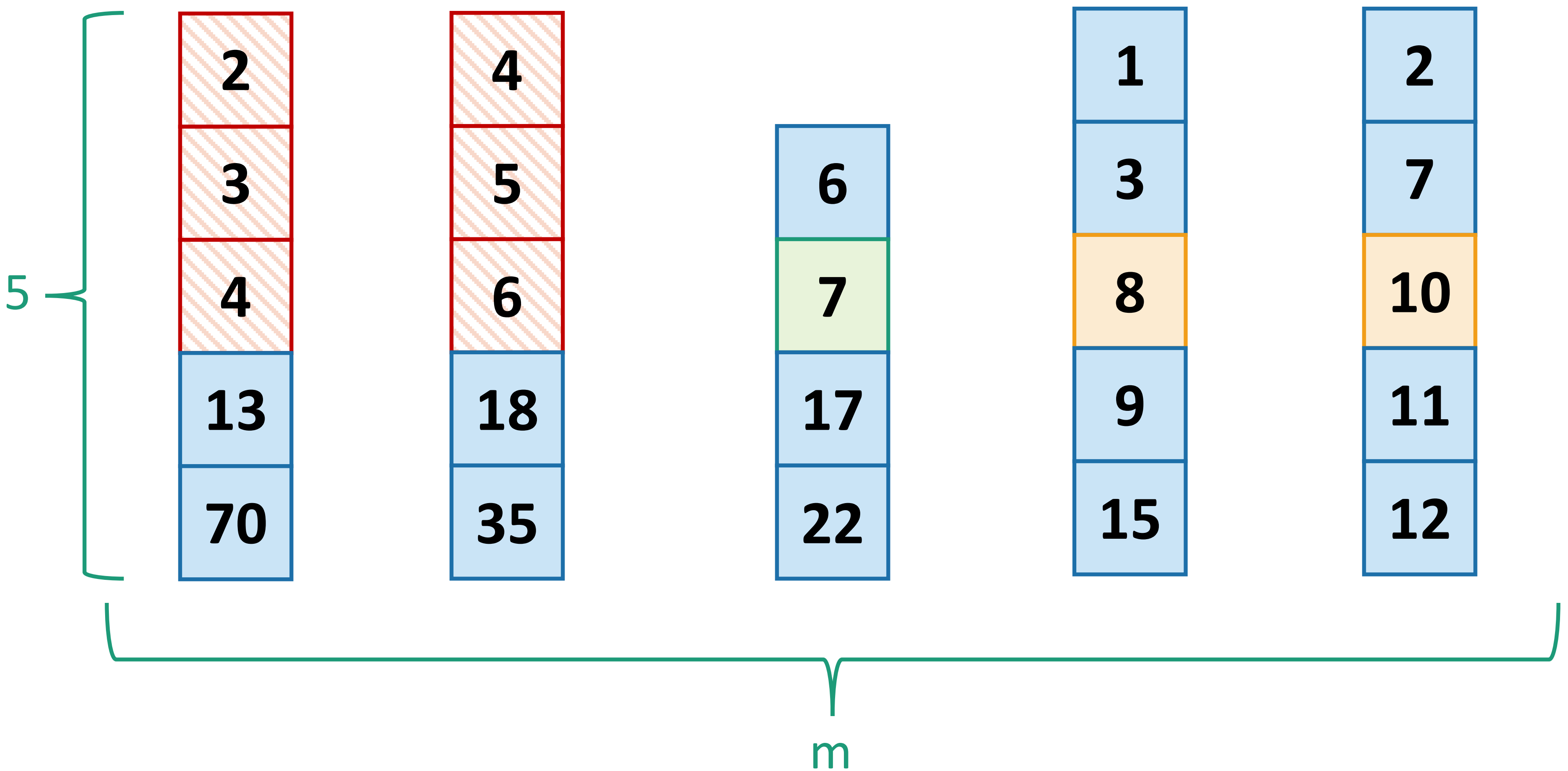
Proof by picture

We will show that lots of elements are smaller than the pivot, hence not too many are larger than the pivot.



How many elements are SMALLER than the pivot?

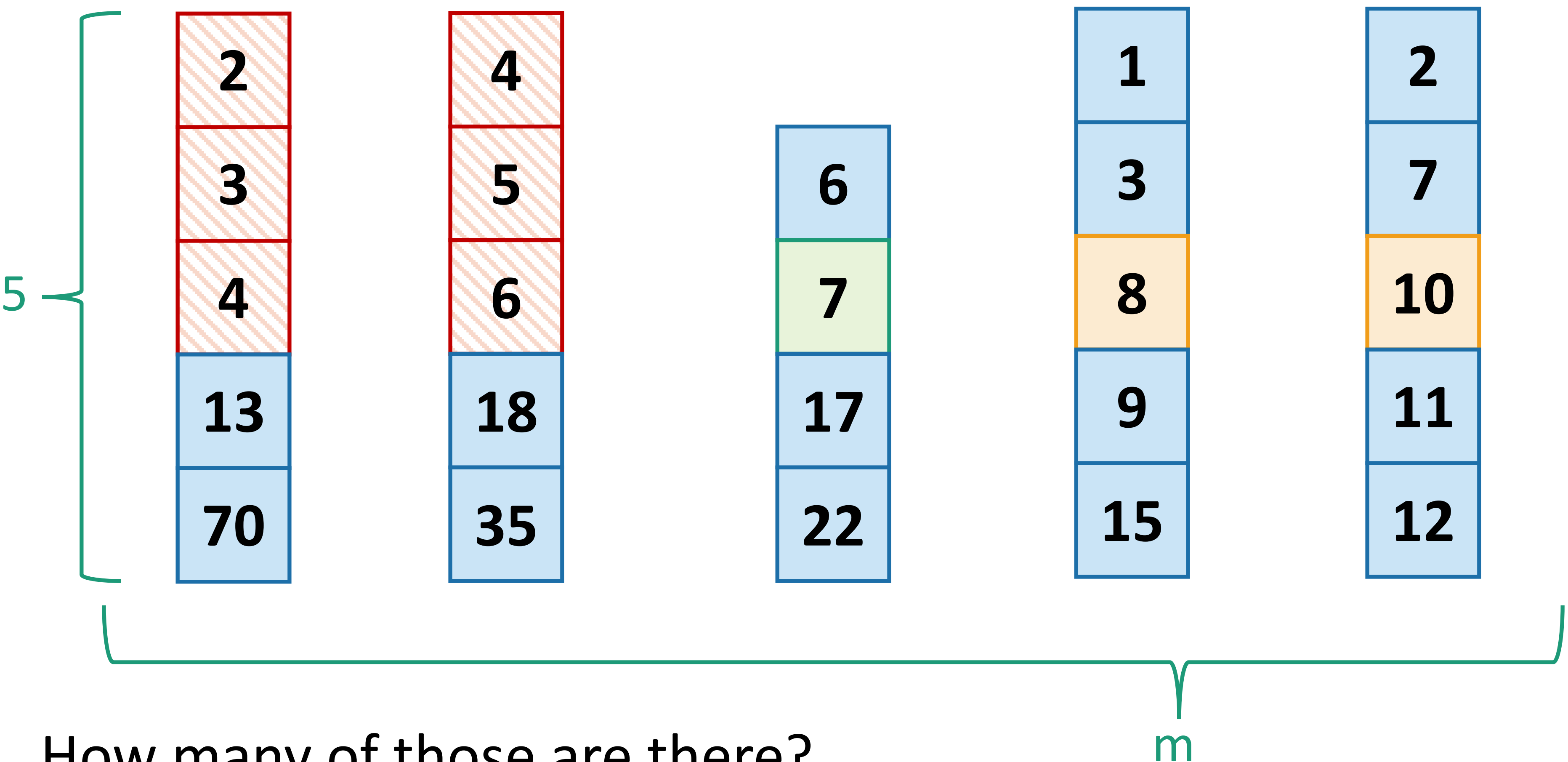
Proof by picture



At least these ones: everything above and to the left.

Proof by picture

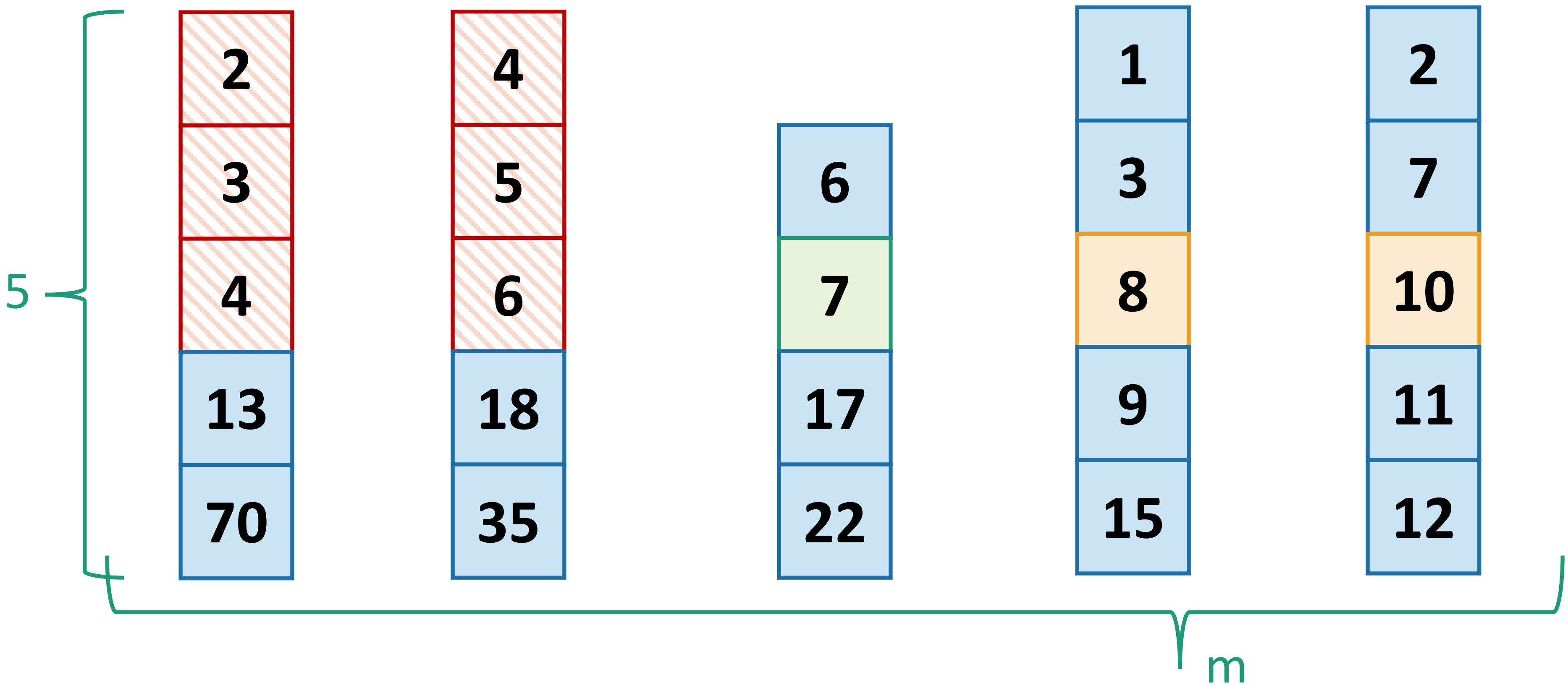
$3 \cdot \left(\left\lceil \frac{m}{2} \right\rceil - 1 \right)$ of these, but then one of them could have been the "leftovers" group.



How many of those are there?

at least $3 \cdot \left(\left\lceil \frac{m}{2} \right\rceil - 2 \right)$

Proof by picture



So how many are LARGER than the pivot? At most…

(derivation on board)

$$n - 1 - 3\left(\left\lceil \frac{m}{2} \right\rceil - 2\right) \leq \frac{7n}{10} + 5$$

Remember $m = \left\lceil \frac{n}{5} \right\rceil$

That was one part of the lemma

- **Lemma:** If L and R are as in the algorithm SELECT given above, then

$$|L| \leq \frac{7n}{10} + 5$$

and

$$|R| \leq \frac{7n}{10} + 5$$

The other part is exactly the same.

The Plan

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Recap

Recap

- Substitution method can work when the master theorem doesn't.

- One place we needed it was for SELECT.
  - Which we can do in time O(n)!

Next time

- Randomized algorithms and QuickSort!

BEFORE next time

- Happy MLK Day!
  - No class Monday!
- Pre-Lecture Exercise 5
  - Remember probability theory?
  - The pre-lecture exercise will jog your memory.