# Lecture 17

What we've done and what's to come

This is super fast!  Do it now!

While we are waiting to start...

***Please*** fill out final Educational Study Survey!

Announcement on Ed for the link, or follow QR code:

# Announcements

- Final exam:
  - Monday 6/12, 3:30pm-6:30pm
  - Your exam location will be emailed to you.

- See website for more details, practice exams, etc!
- See Ed post for resources, study tips, etc!
  - Highlights:
    - **DP Review session**: TODAY 5-7pm on Zoom (link on Ed post)
    - **Past Exam review sessions**! Saturday! 12-2pm AND 4-6PM, Hewlett 201.
    - "HW Party" → **"Exam Party"**! Thursday 6-8pm Gates 403. 🍕🍕🍕

# More announcements

- ***Please*** fill out final Educational Study Survey!
  - Announcement on Ed for link, or
- Course feedback open!
  - A reminder should pop up on Canvas if you sign in!
  - Or fill it out directly here:
    - http://course-evaluations.stanford.edu.
  - Your feedback is super-important!
  - It will help us make the course better!

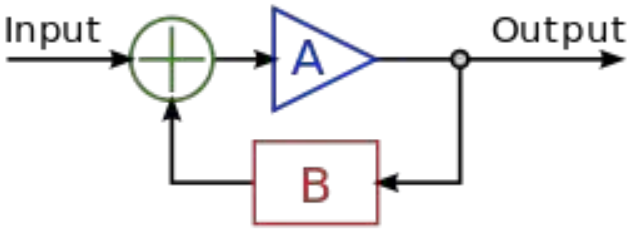



Figure 1: Feedback

# More announcements

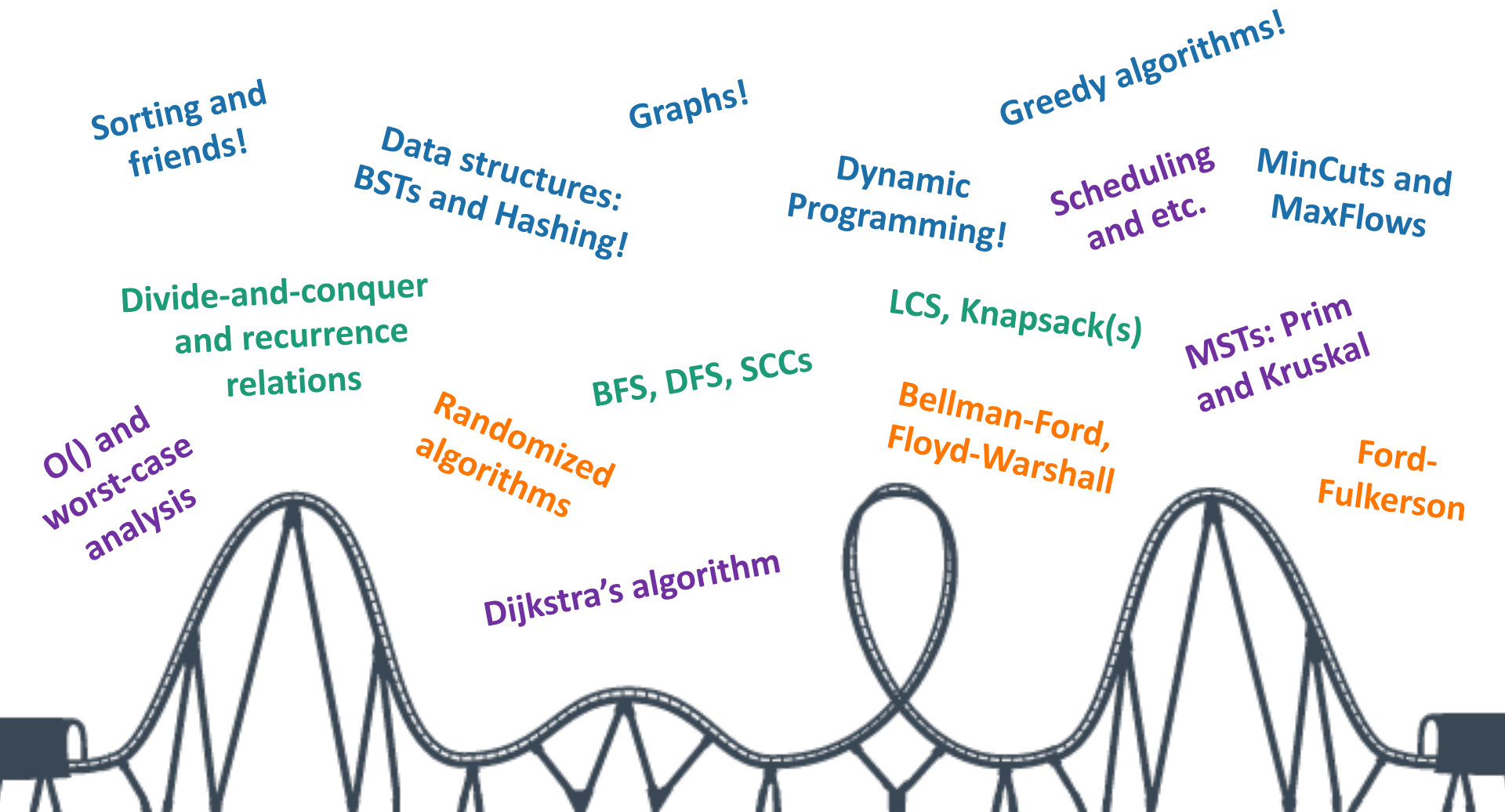
- This is the last class!
  - Time flies when you are having fun...
  - Or doing lots of algorithms problems... ☺
- Please keep in touch!
  - Stop by and say hi!  (I live in Gates 168 most of the time)

# Today

- What just happened?
  - A whirlwind tour of CS161

- What's next?
  - A few gems from future algorithms classes

# It's been a fun ride…

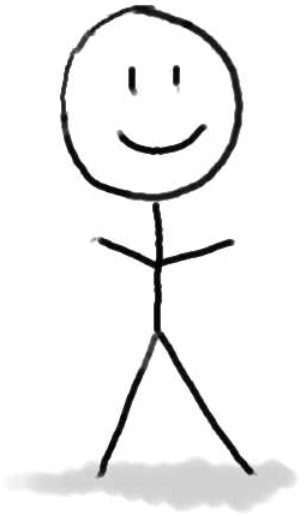


# What have we learned?

17 lectures in 13 slides.

# General approach
## to algorithm design and analysis

**Can I do better?**



Algorithm designer

To answer this question we need both **rigor** and **intuition**:

Plucky the Pedantic Penguin

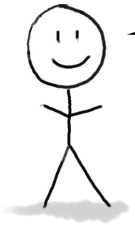
Detail-oriented
Precise
Rigorous

Lucky the Lackadaisical Lemur

Big-picture
Intuitive
Hand-wavey

# We needed more details

python™



Does it work?
Is it fast?

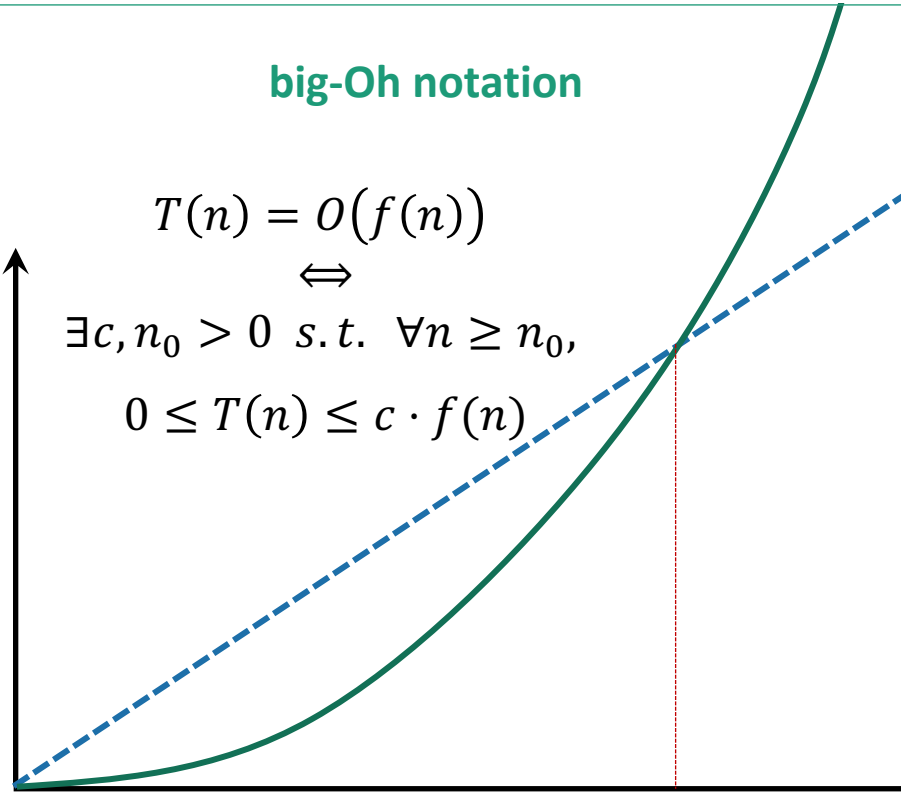
What
does that
mean??

**Worst-case analysis**

**HERE IS AN INPUT!**

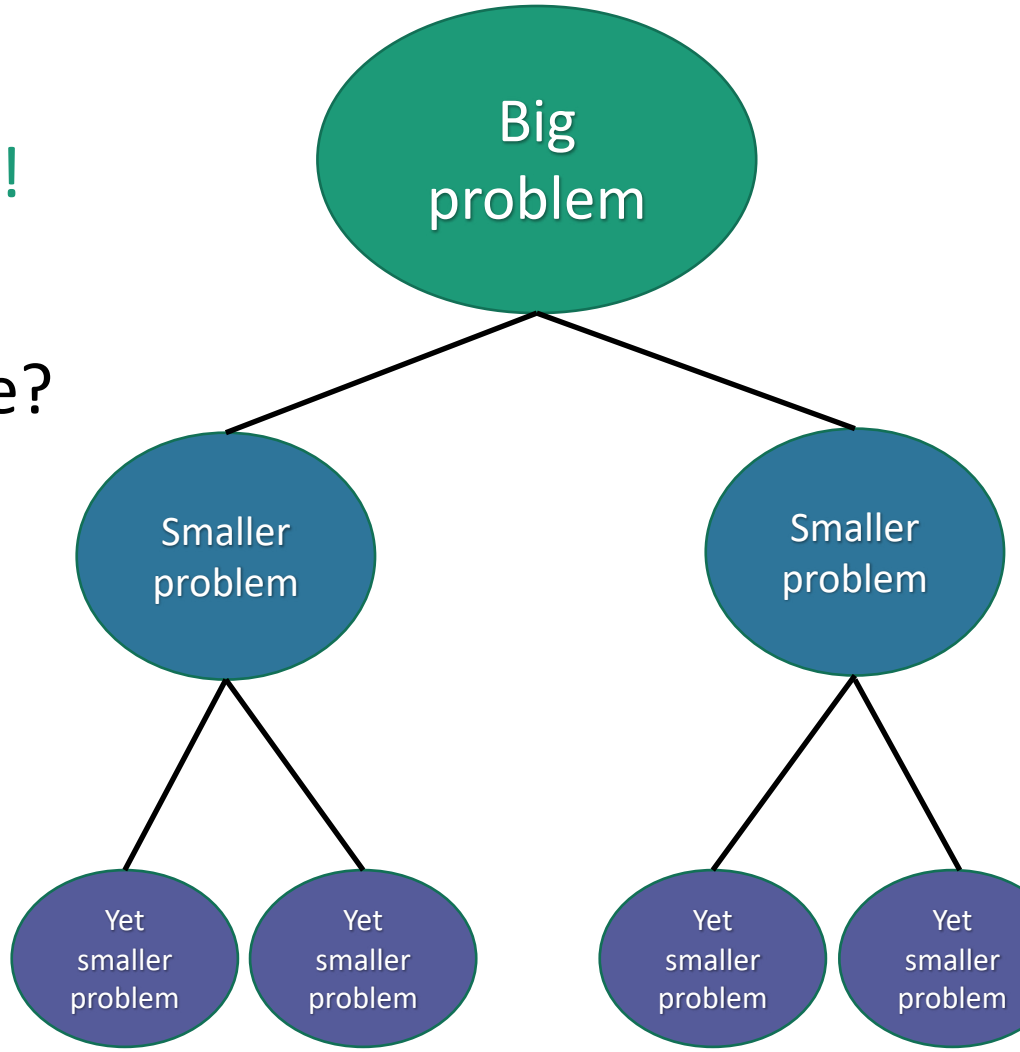
**big-Oh notation**

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$



# Algorithm design paradigm:
# divide and conquer

- Like MergeSort!
- Or Karatsuba's algorithm!
- Or SELECT!
- How do we analyze these?



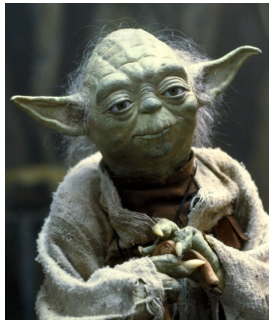

By careful analysis!

Tree method

Substitution Method

Useful shortcut, the **master method** is.

Plucky the Pedantic Penguin



Jedi master Yoda

## While we're on the topic of sorting

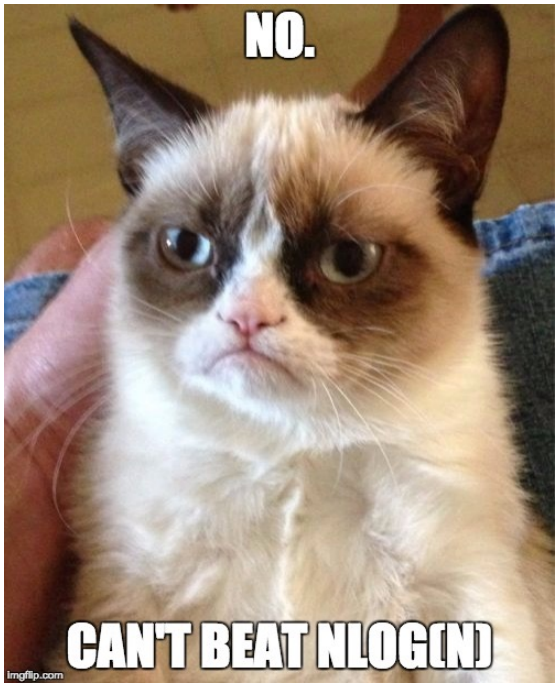
# Why not use randomness?

- We analyzed **QuickSort!**
- Still worst-case input, but we use randomness after the input is chosen.
- Always correct, usually fast.
  - This is a Las Vegas algorithm


# All this sorting is making me wonder…

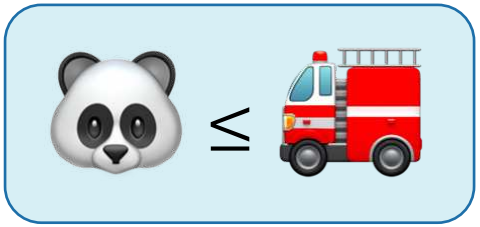
# Can we do better?

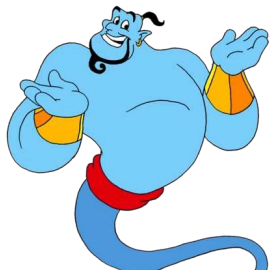
- Depends on who you ask:


- **RadixSort** takes time O(n) if the objects are, for example, small integers!

- Can't do better in a **comparison-based** model.



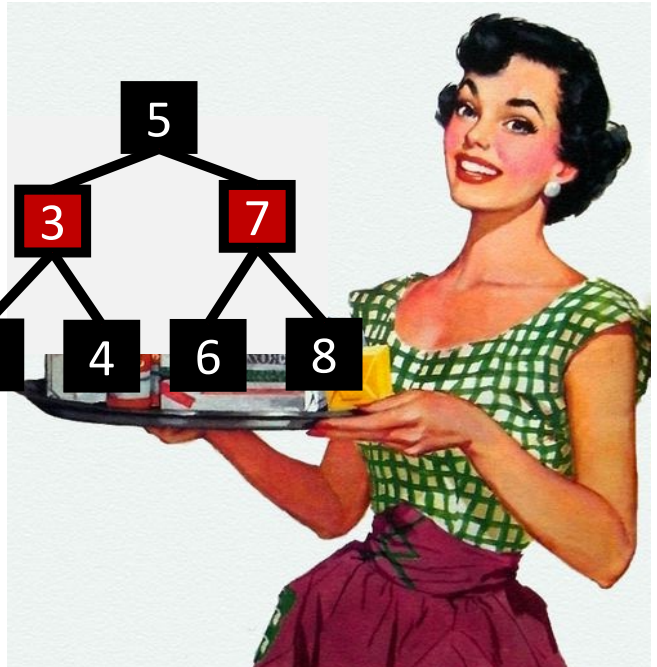
# beyond sorted arrays/linked lists:
# Binary Search Trees!

- Useful data structure!
- Especially the self-balancing ones!

*Red-Black tree!*

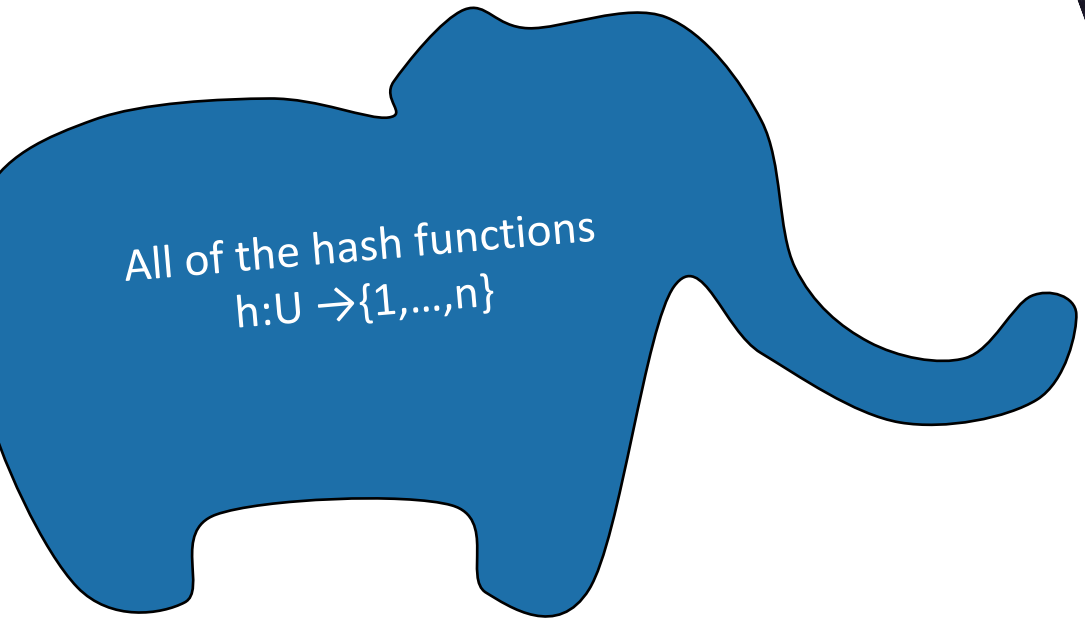
*Maintain balance* by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes.**

*It's just good sense!*



# Another way to store things

# Hash tables!




The universe

hash function h

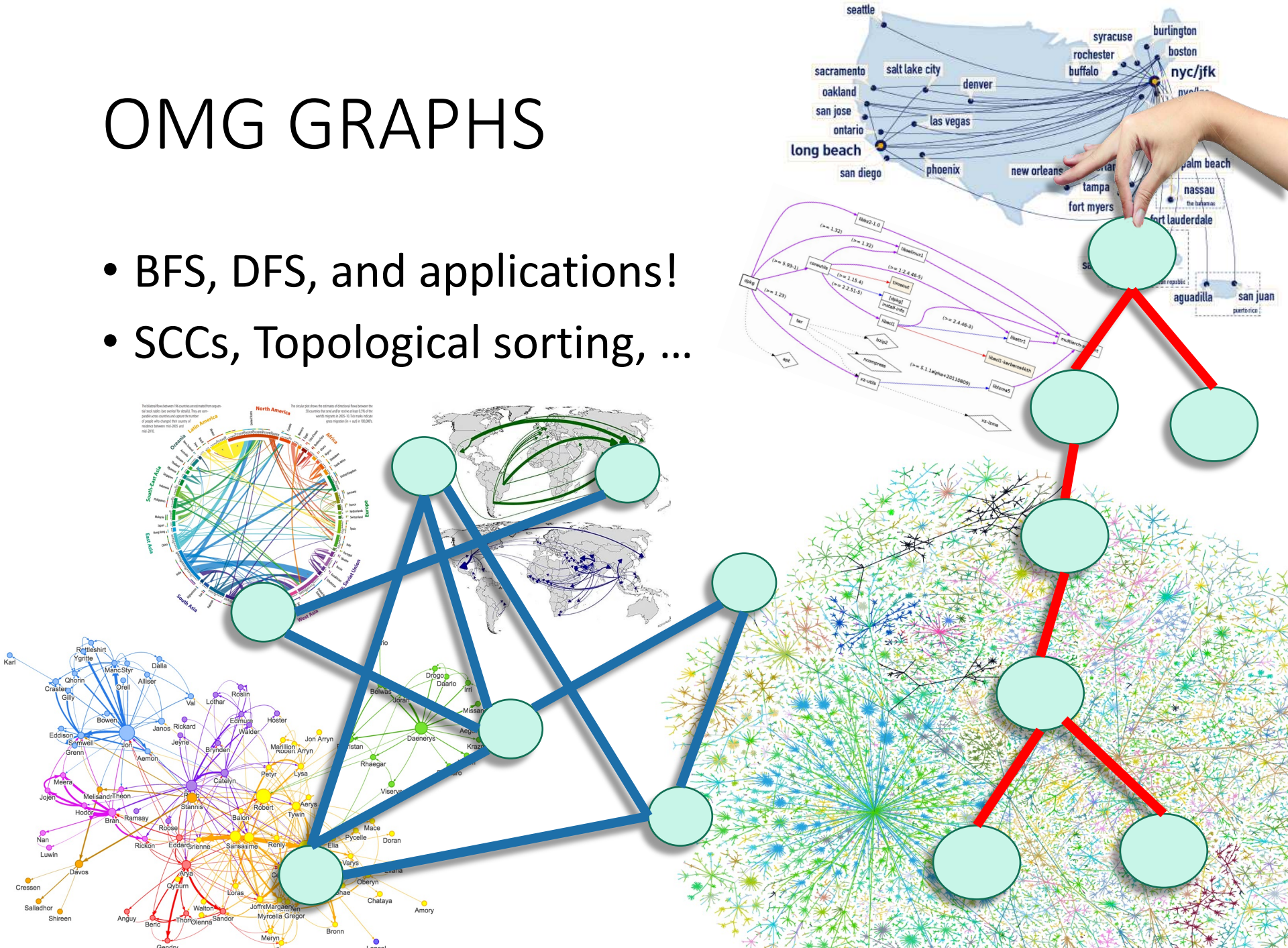
**Choose h randomly from a universal hash family.**

Some buckets

**It's better if the hash family is small! Then it takes less space to store h.**

# OMG GRAPHS

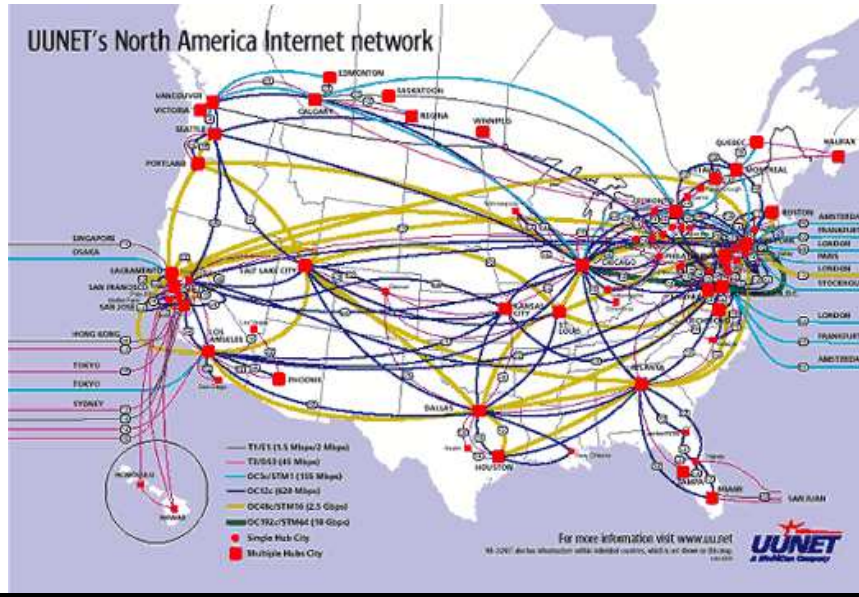
- BFS, DFS, and applications!
- SCCs, Topological sorting, …

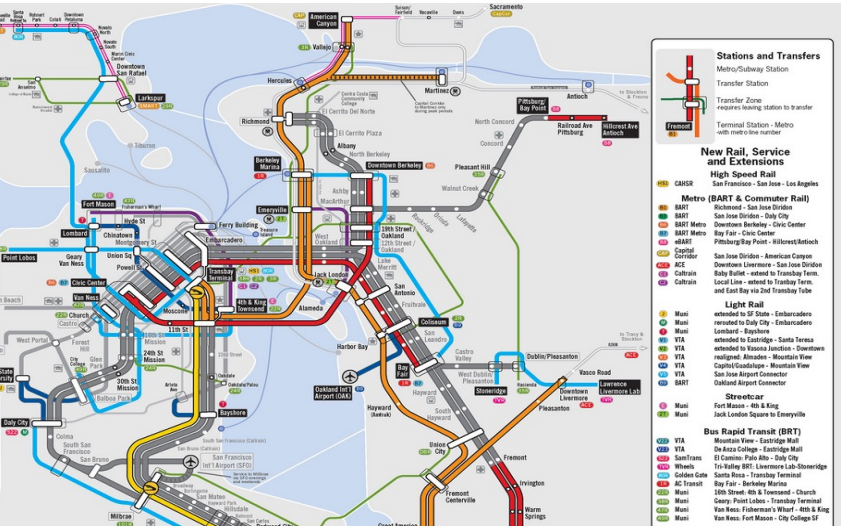


# A fundamental graph problem:

# shortest paths

- Eg, transit planning, packet routing, …
- Dijkstra!
- Bellman-Ford!
- Floyd-Warshall!




```
DN0a22a0e3:~ mary$ traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 10.34.160.2 (10.34.160.2)  38.168 ms  31.272 ms  28.841 ms
 2  [AS0] cwa-vrtr.sunet (10.21.196.28)  33.769 ms  28.245 ms  24.373 ms
 3  [AS32] 171.66.2.229 (171.66.2.229)  24.468 ms  20.115 ms  23.223 ms
 4  [AS32] hpr-svl-rtr-vlan8.sunet (171.64.255.235)  24.644 ms  24.962 ms  17.453 ms
 5  [AS2152] hpr-svl-hpr2--stan-ge.cenic.net (137.164.27.161)  22.129 ms  4.902 ms  3.642 ms
 6  [AS2152] hpr-lax-hpr3--svl-hpr3-100ge.cenic.net (137.164.25.73)  12.125 ms  43.361 ms  32.3
 7  [AS2152] hpr-i2--lax-hpr2-r&e.cenic.net (137.164.26.201)  40.174 ms  38.399 ms  34.499 ms
 8  [AS0] et-4-0-0.4079.sdn-sw.lasv.net.internet2.edu (162.252.70.28)  46.573 ms  23.926 ms  17
 9  [AS0] et-5-1-0.4079.rtsw.salt.net.internet2.edu (162.252.70.31)  30.424 ms  25.770 ms  23.1
10  [AS0] et-4-0-0.4079.sdn-sw.denv.net.internet2.edu (162.252.70.8)  47.454 ms  57.273 ms  73.
11  [AS0] et-4-1-0.4079.rtsw.kans.net.internet2.edu (162.252.70.11)  70.825 ms  67.809 ms  62.1
12  [AS0] et-4-1-0.4070.rtsw.chic.net.internet2.edu (198.71.47.206)  77.937 ms  57.421 ms  63.6
13  [AS0] et-0-1-0.4079.sdn-sw.ashb.net.internet2.edu (162.252.70.60)  77.682 ms  71.993 ms  73
14  [AS0] et-4-1-0.4079.rtsw.wash.net.internet2.edu (162.252.70.65)  71.565 ms  74.988 ms  71.0
15  [AS21320] internet2-gw.mx1.lon.uk.geant.net (62.40.124.44)  154.926 ms  145.606 ms  145.872
16  [AS21320] ae0.mx1.lon2.uk.geant.net (62.40.98.79)  146.565 ms  146.604 ms  146.801 ms
17  [AS21320] ae0.mx1.par.fr.geant.net (62.40.98.77)  153.289 ms  184.995 ms  152.682 ms
18  [AS21320] ae2.mx1.gen.ch.geant.net (62.40.98.153)  160.283 ms  160.104 ms  164.147 ms
19  [AS21320] swice1-100ge-0-3-0-1.switch.ch (62.40.124.22)  162.068 ms  160.595 ms  163.095 ms
20  [AS559] swizh1-100ge-0-1-0-1.switch.ch (130.59.36.94)  165.824 ms  164.216 ms  163.983 ms
21  [AS559] swiez3-100ge-0-1-0-4.switch.ch (130.59.38.109)  164.269 ms  164.370 ms  163.929 ms
22  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  164.082 ms  170.645 ms  165.372
23  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  164.773 ms  165.193 ms  172.158 ms
```

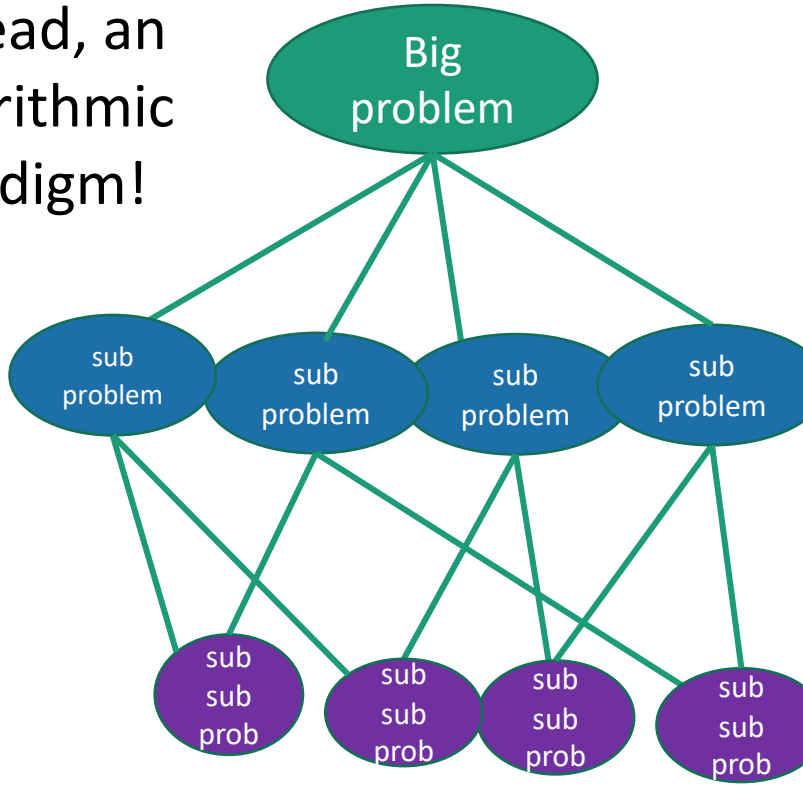
Bellman-Ford and Floyd-Warshall were examples of...

# Dynamic Programming!

We saw many other examples, including Longest Common Subsequence and Knapsack Problems.

- Not programming in an action movie.

Instead, an algorithmic paradigm!




- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Steps 3-5:** Use dynamic programming: fill in a table to find the answer!

Sometimes we can take even better advantage of optimal substructure…with

# Greedy algorithms

- Make a series of choices, and commit!

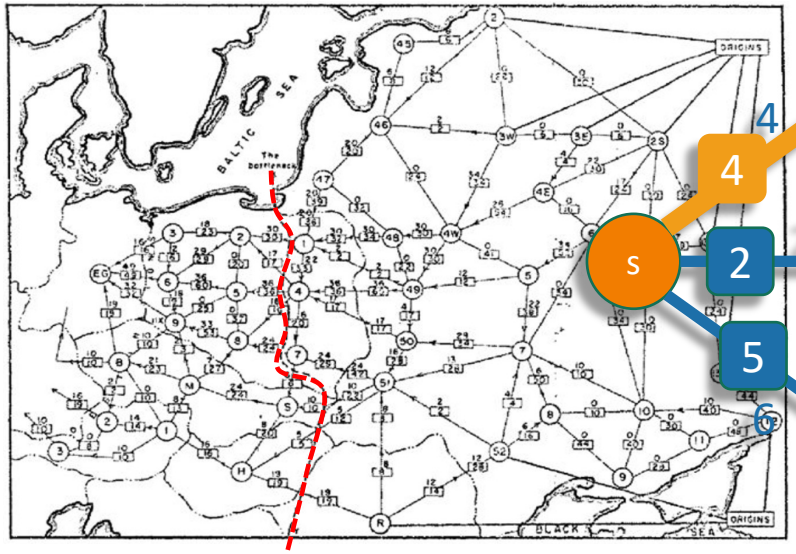
- Intuitively we want to show that our greedy choices never rule out success.
- Rigorously, we usually analyzed these by induction.
- Examples!
  - Activity Selection
  - Job Scheduling
  - Huffman Coding
  - **Minimum Spanning Trees**

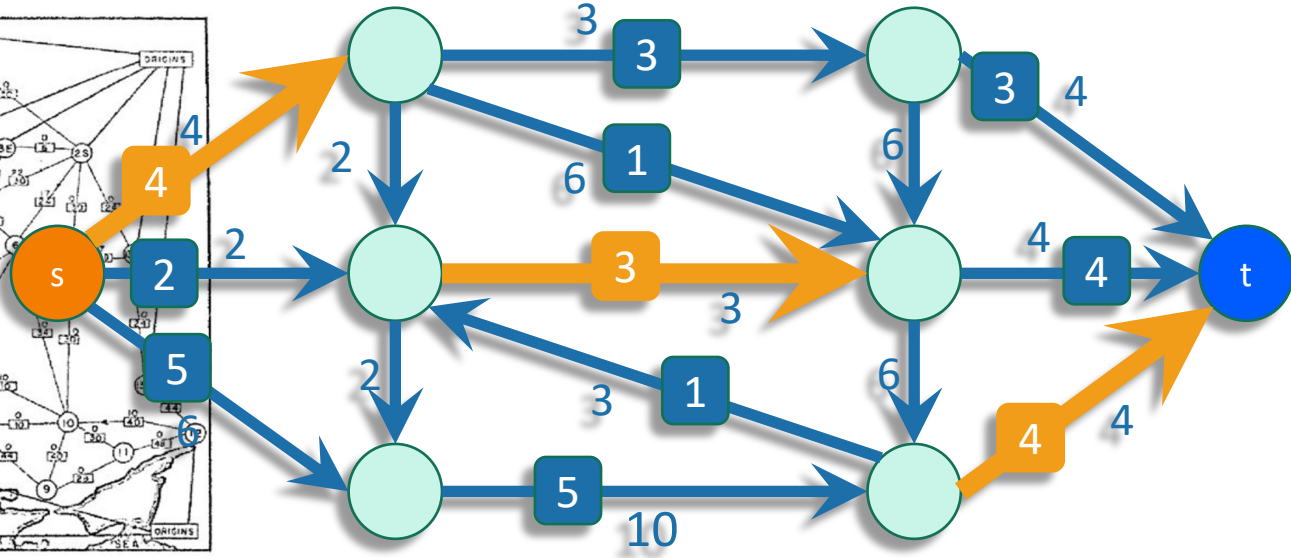
**Prim's algorithm:**
greedily grow a tree

**Kruskal's algorithm:**
greedily grow a forest

# Cuts and flows

- minimum s-t cut is the same as maximum s-t flow!
- Ford-Fulkerson can find them!
  - Increase flow along augmenting paths!
- Applications:
  - routing supplies or interrupting supply chains
  - Giving ice cream to Stanford students

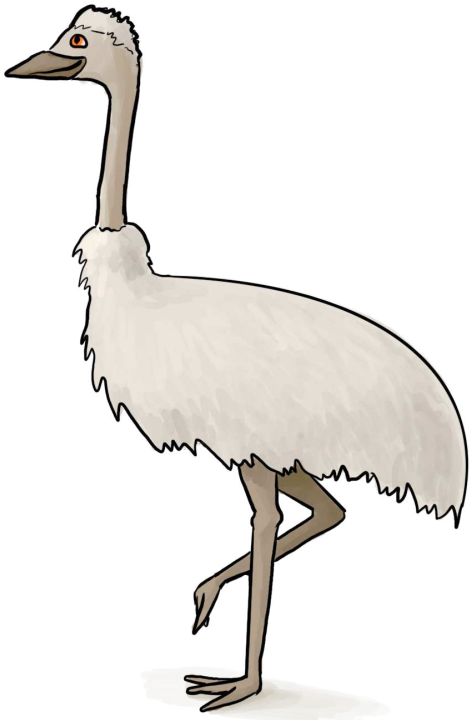



# Along the way

- Embedded EthiCS!
  - Idealization, Abstraction, Measurement
  - Wicked problems and benign problems

- If/when we are developing algorithms in real life, we now have some things to watch out for and think about!

Thanks Diana!



Elan the Ethical Emu

# And now we're here

# What have we learned?

- A few algorithm design paradigms:
  - Divide and conquer, Dynamic Programming, Greedy
- A few analysis tools:
  - Worst-case analysis, asymptotic analysis, recurrence relations, probability tricks, proofs by induction
- A few common objects:
  - Graphs, arrays, trees, hash functions
- A LOT of examples!

# What have we learned?

# We've filled out a toolbox

- Tons of examples give us intuition about what algorithmic techniques might work when.
- The technical skills make sure our intuition works out.

# But there's lots more out there

- What's next???

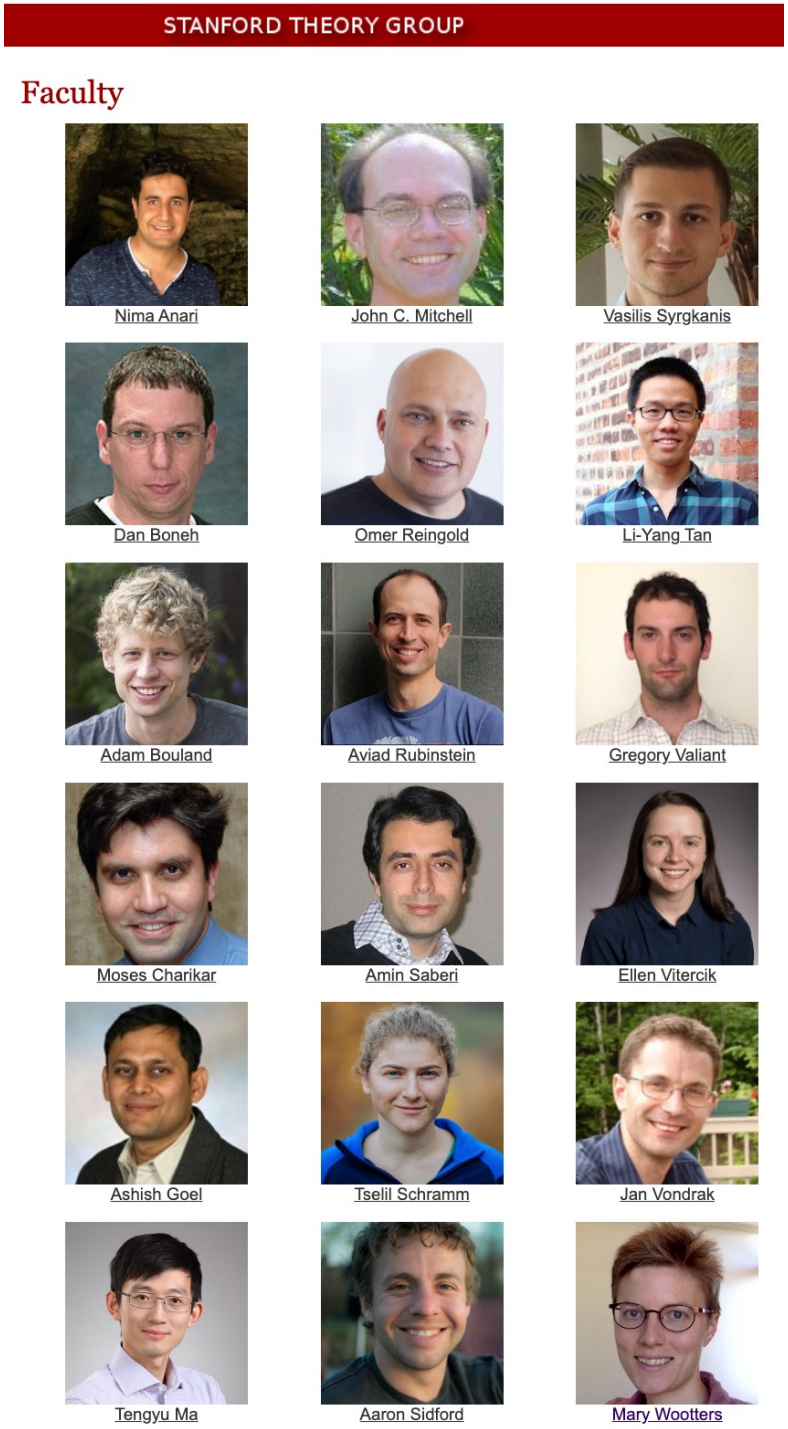
# Want more classes?

**findSomeTheoryCourses**():

- go to theory.stanford.edu
- Click on "People"
- Look at what we're teaching!

- CS154 – Introduction to Complexity
- CS166 – Data Structures
- CS168 – The Modern Algorithmic Toolbox
- MS&E 212 – Combinatorial Optimization
- CS250 – Error Correcting Codes
- CS254 – Computational Complexity
- CS255 – Introduction to Cryptography
- CS261 – Optimization and Algorithmic Paradigms
- CS264 – Beyond Worst-Case Analysis
- CS265 – Randomized Algorithms
- CS269Q – Quantum Computing
- CS269O – Introduction to Optimization Theory
- EE364A/B – Convex Optimization I and II

...and many many more
upper-level topics courses!

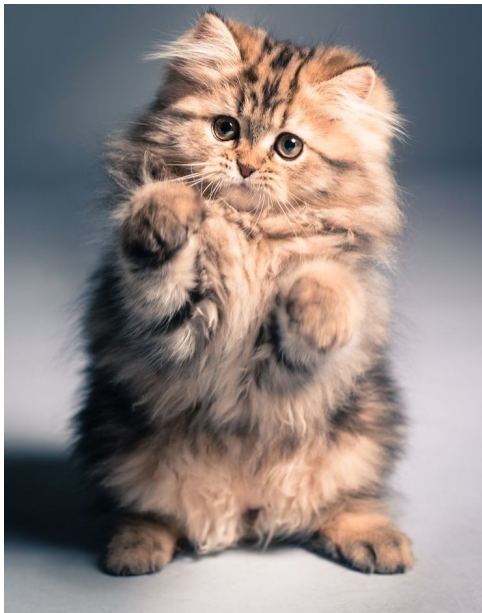


# Today

# A few gems

- Linear programming
- Random projections
- Low-degree polynomials

This will be pretty fluffy, without much detail – take more CS theory classes for more detail!



**NOTHING AFTER THIS POINT WILL BE ON THE FINAL EXAM**

# Linear Programming

- This is a fancy name for optimizing a linear function subject to linear constraints.
- For example:

Maximize

$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

- It turns out the be an extremely general problem.

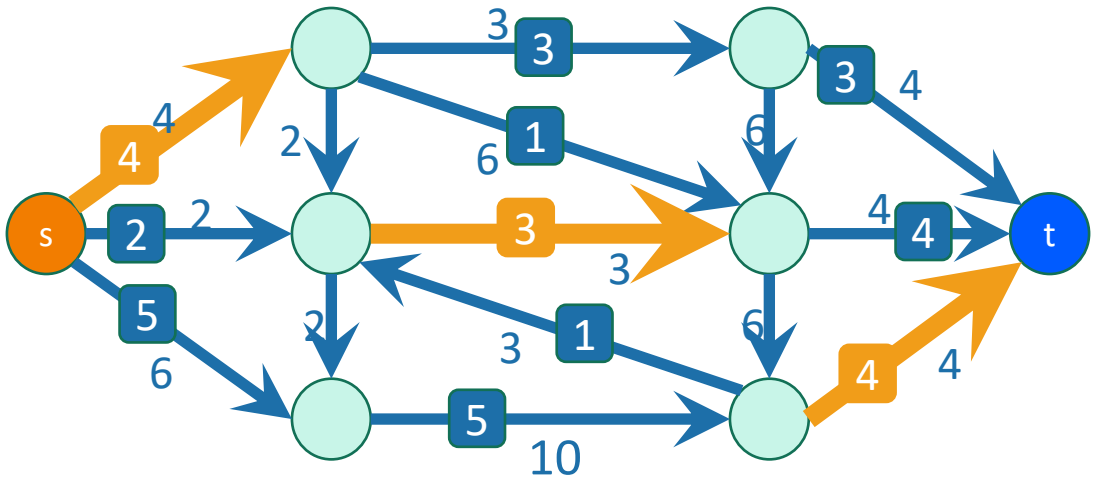
# Actually we just saw it!

Maximize

the sum of the flows leaving s

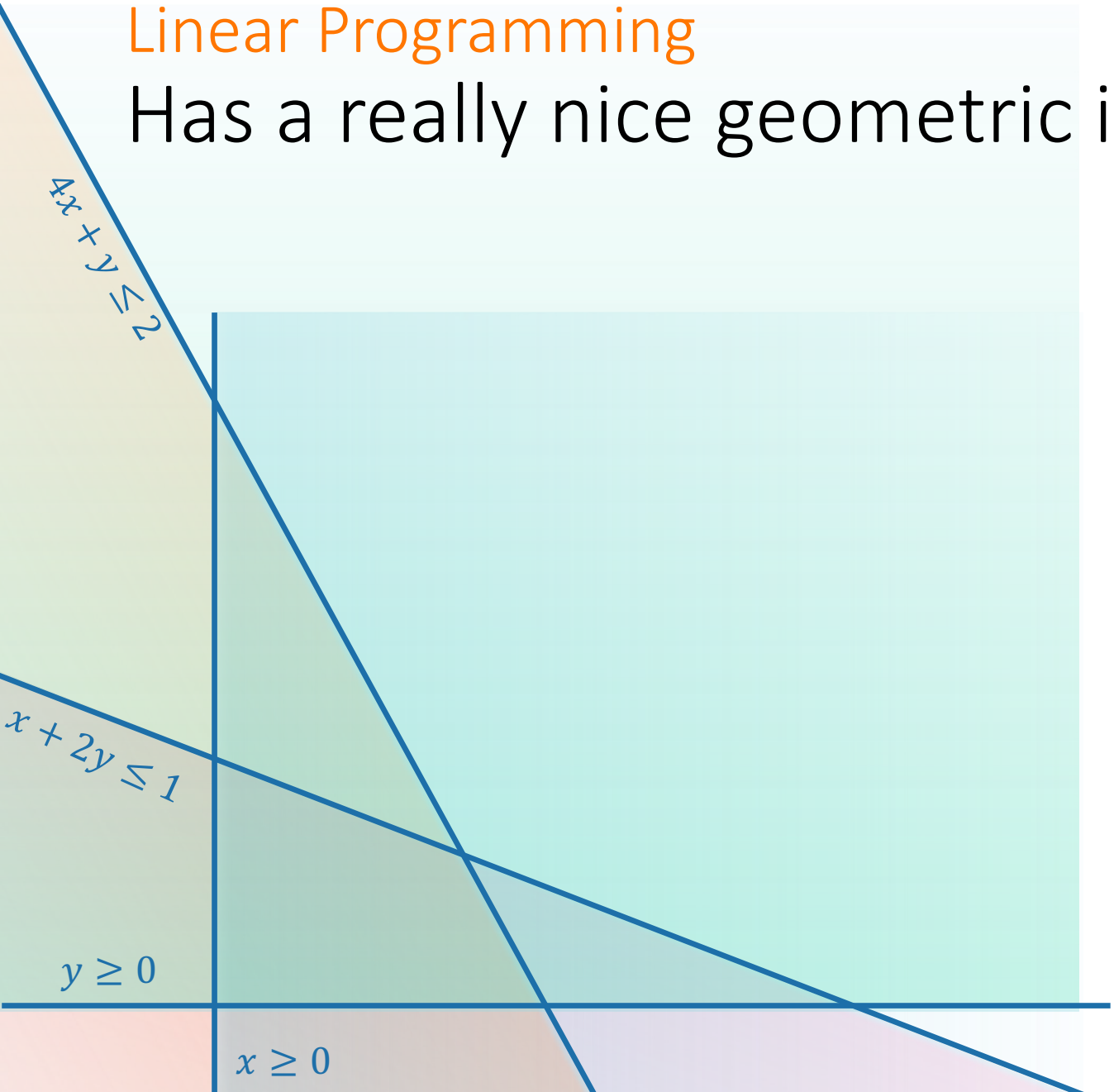
subject to

- None of the flows are bigger than the edge capacities
- At every vertex, stuff going in = stuff going out.



## Linear Programming

# Has a really nice geometric intuition

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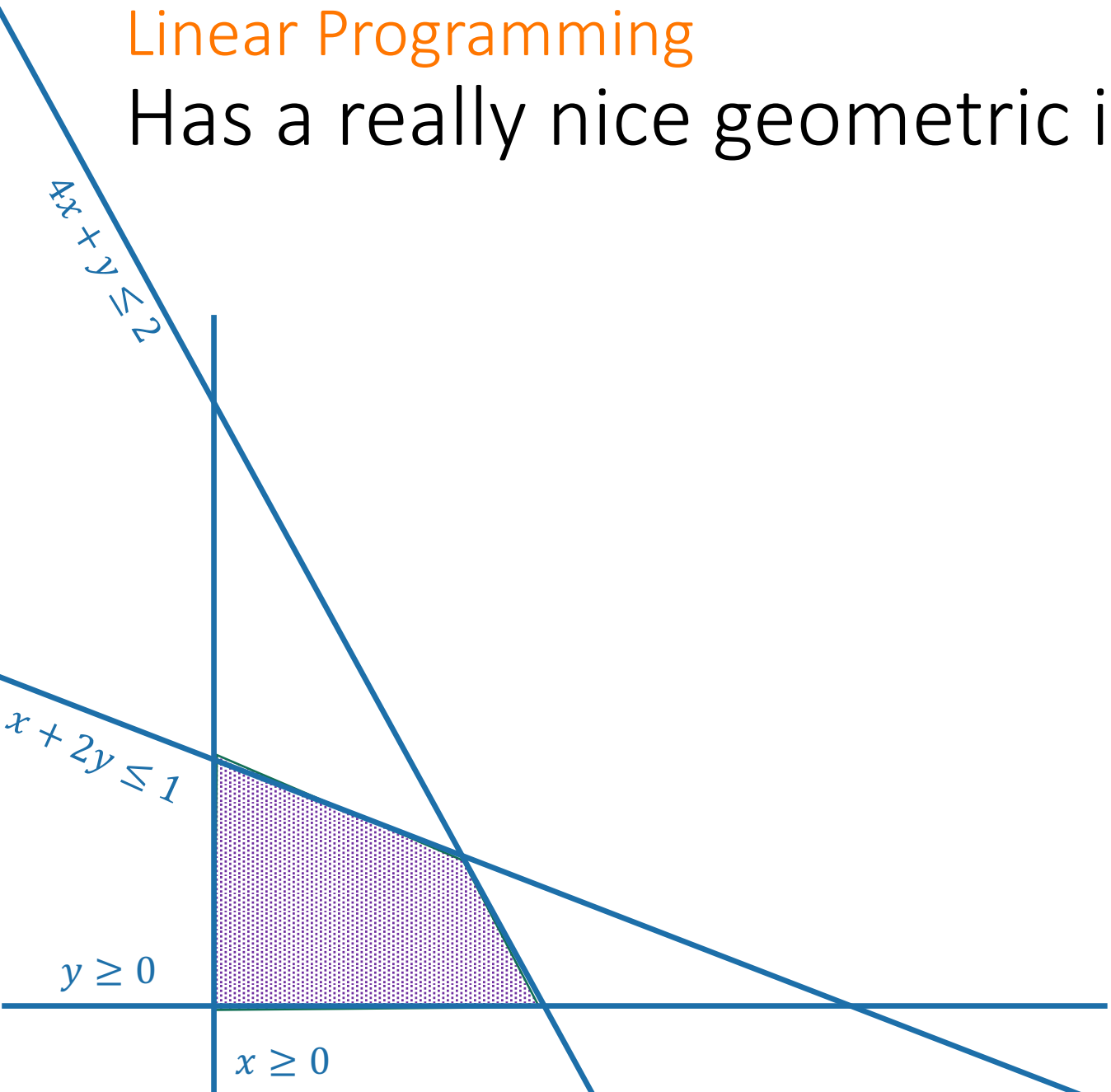


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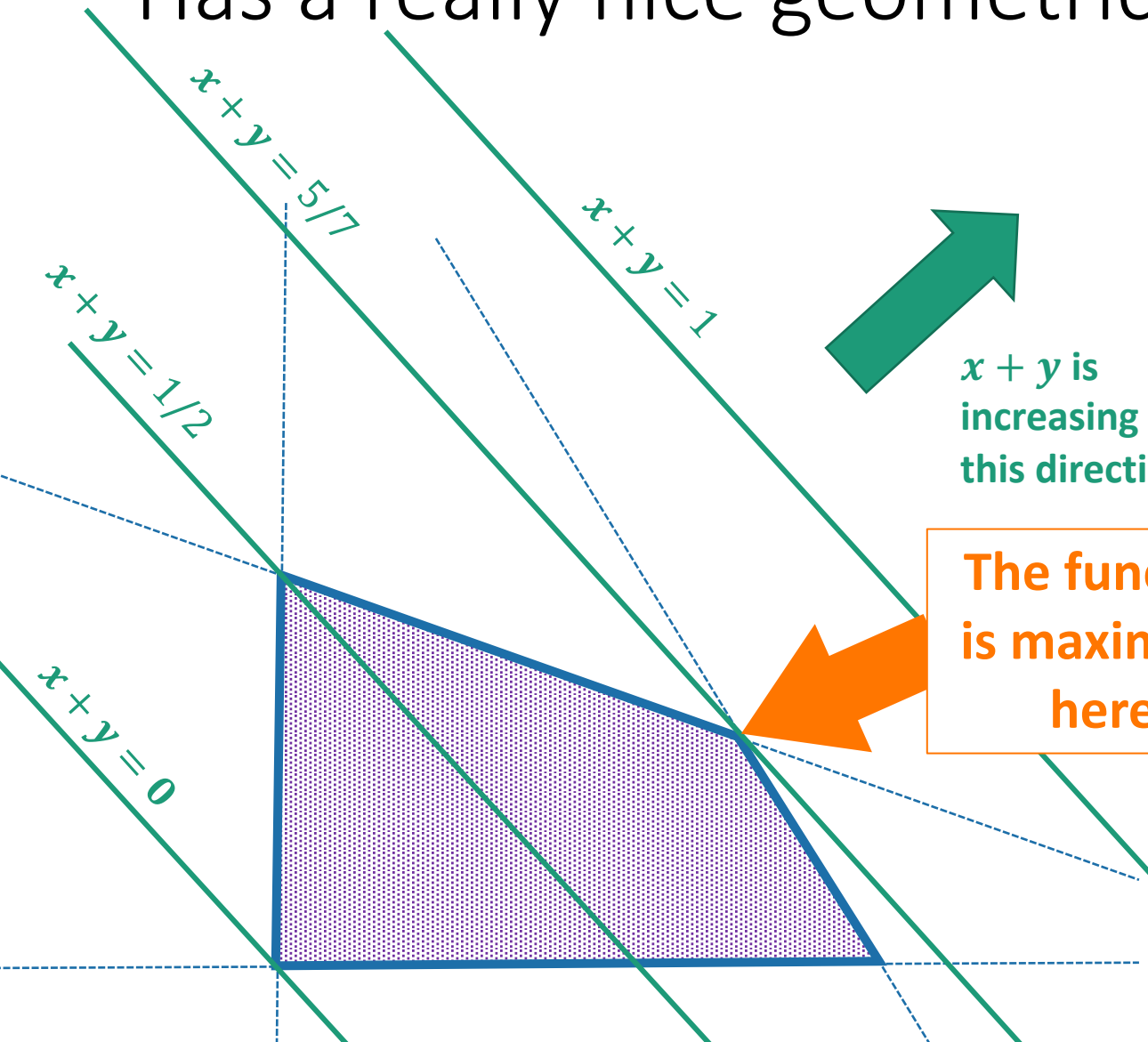
## Linear Programming

# Has a really nice geometric intuition




$x + y$ is increasing in this direction.

**The function is maximized here!**

Maximize

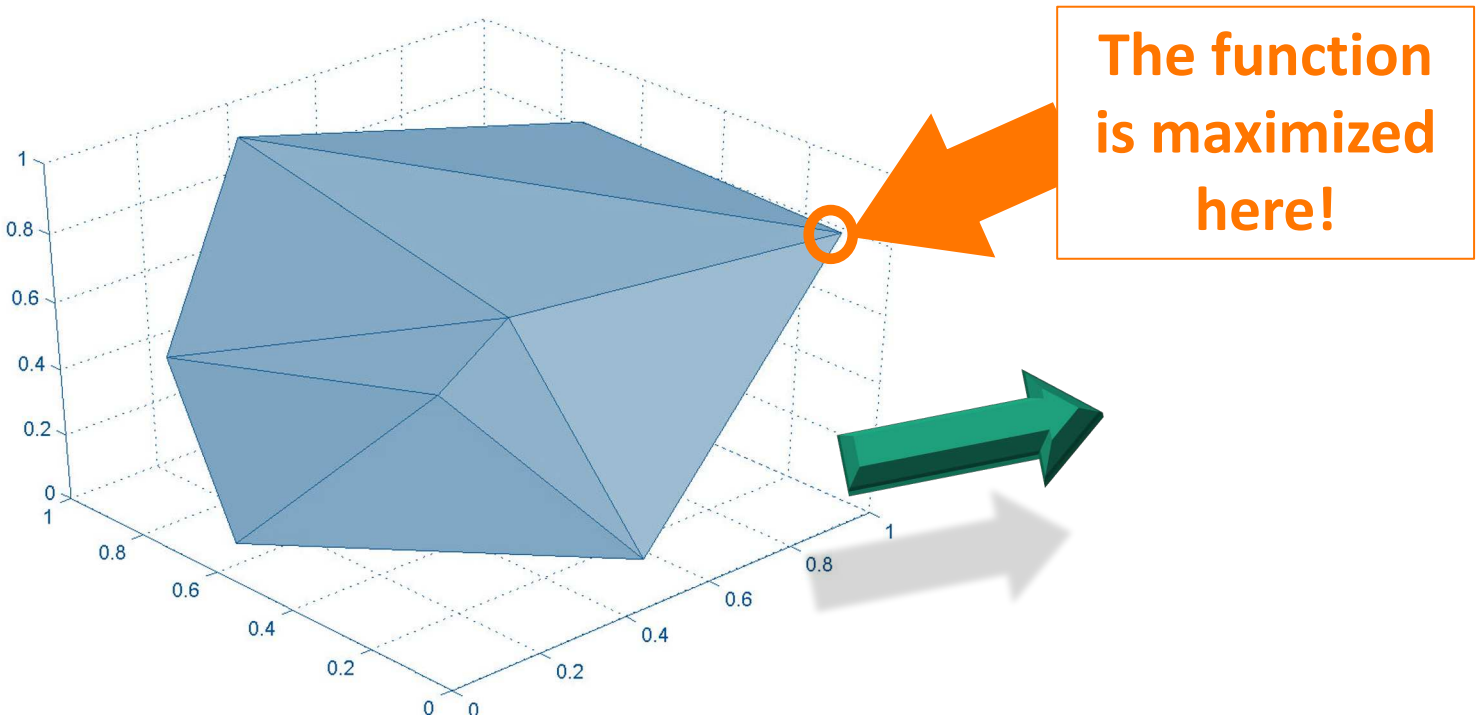
$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

# In general

- The constraints define a **polytope**
- The function defines a **direction**
- We just want to find the vertex that is furthest in that direction.



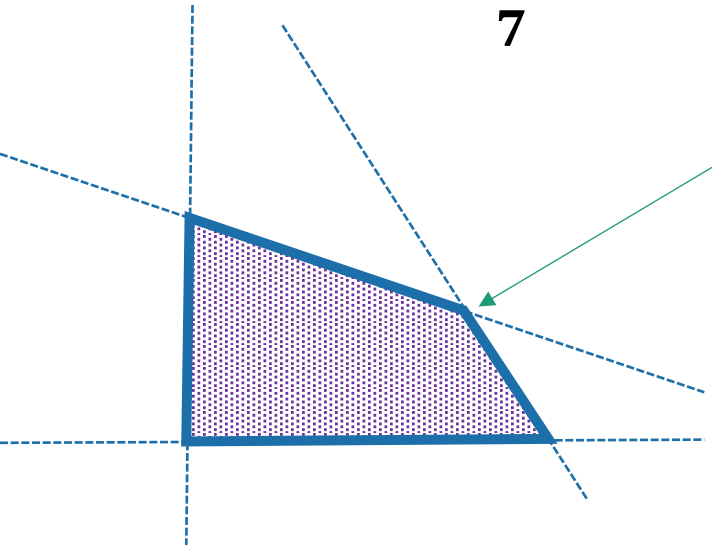
# Duality

## How do we know we have an optimal solution?

**I claim that the optimum is 5/7.**

**Proof:** say x and y satisfy the constraints.

- $\boldsymbol{x + y = \frac{1}{7}(4x + y) + \frac{3}{7}(x + 2y)}$
- $\boldsymbol{\leq \frac{1}{7} \cdot 2 + \frac{3}{7} \cdot 1}$
- $\boldsymbol{= \frac{5}{7}}$



You can check this point has value 5/7…but how would we prove it's optimal other than by eyeballing it?

Maximize

$x + y$

subject to

$x \geq 0$

$y \geq 0$

$4x + y \leq 2$

$x + 2y \leq 1$

cute, but

# How did you come up with 1/7, 3/7?

**I claim that the optimum is 5/7.**

**Proof:** say x and y satisfy the constraints.

- $x + y \leq$ [ ] $(4x + y) +$ [ ] $(x + 2y)$
- $\leq$ [ ] $\cdot\ 2 +$ [ ] $1$
- $=$ [ ]

- I want to choose things to put here
- So that I minimize this
- Subject to these things

Maximize

$x + y$

subject to

$x \geq 0$

$y \geq 0$

$4x + y \leq 2$

$x + 2y \leq 1$

Note: it's not immediately obvious how to turn that into a linear program, this is just meant to convince you that it's plausible.

In this case the dual is:
min $2w + z$ s.t. $w, z \geq 0$,
$4w + z \geq 1$ and $w + 2z \geq 1$

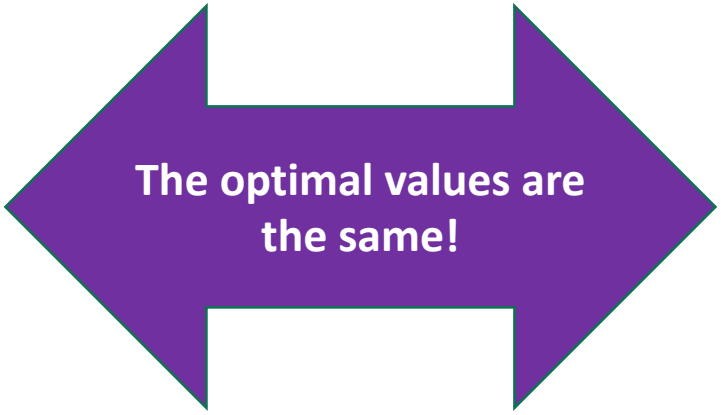
# That's a linear program!

- How did I find those special values 1/7, 3/7?
- I solved some linear program.
- It's called the dual program.

Minimize the upper bound you get, subject to the proof working.

Maximize stuff
subject to stuff

**Primal**




Minimize other stuff
subject to other stuff

**Dual**

We've actually already seen this too

# The Min-Cut Max-Flow Theorem!

**Maximize the sum of the flows leaving s**

s.t

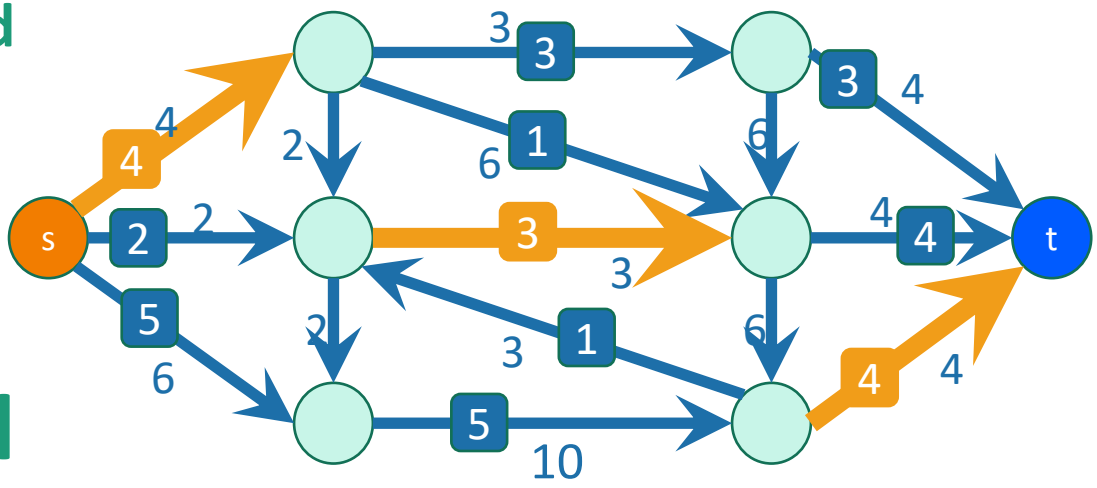
**All the flow constraints are satisfied**

**The optimal values are the same!**

**Minimize the sum of the capacities on a cut**

s.t.

**it's a legit cut**



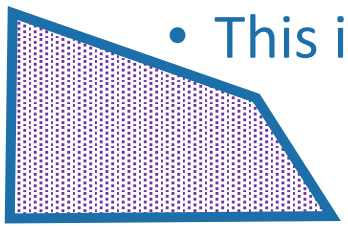

**Primal**

**Dual**

# LPs and Duality are really powerful

- This general phenomenon shows up all over the place
  - Min-Cut Max-Flow is a special case.

- Duality helps us reason about an optimization problem
  - The dual provides a **certificate** that we've solved the primal.
  - eg, if you have a cut and a flow with the same value, you must have found a max flow and a min cut.
- We can solve LPs quickly!
  - For example, by intelligently bouncing around the vertices of the feasible region.
  - This is an **extremely powerful algorithmic primitive**.



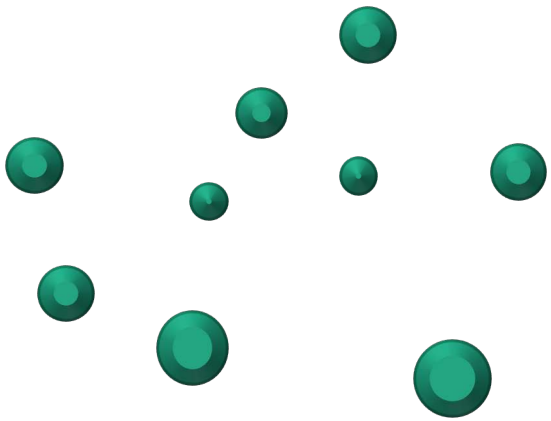
# Today

# A few gems

- Linear programming
- Random projections
- Low-degree polynomials

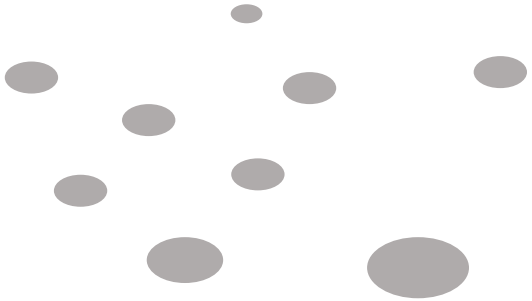
# One of my favorite tricks

## Take a random projection and hope for the best.

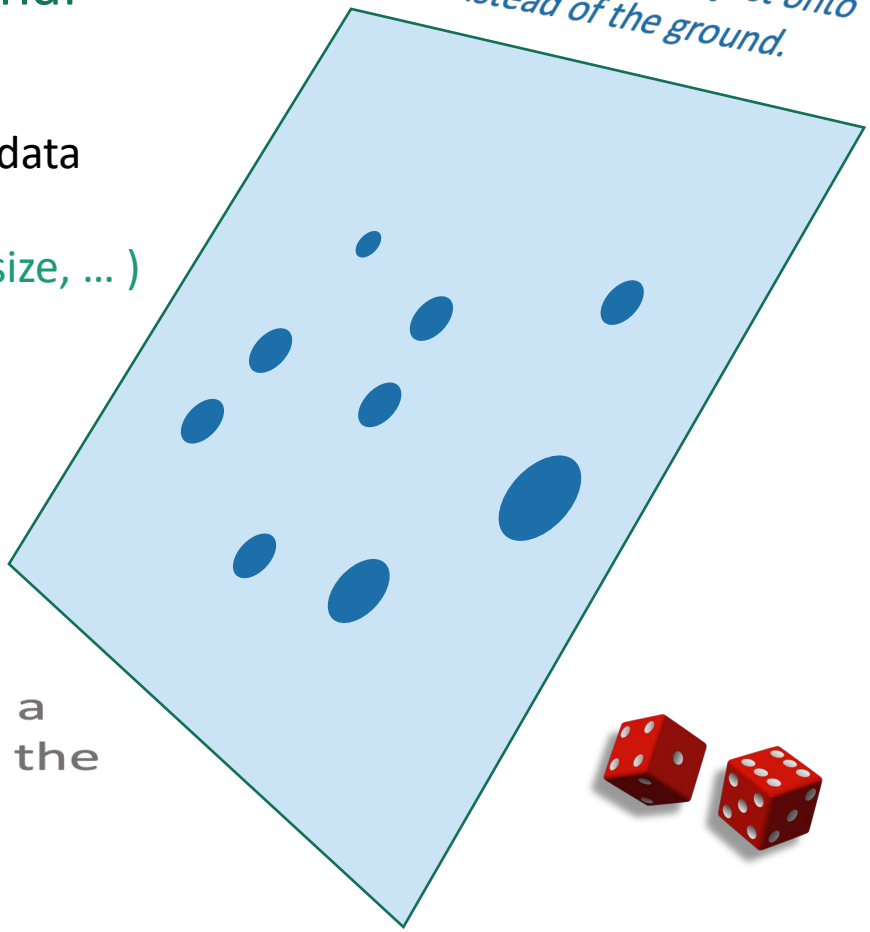


### High-dimensional set of points

For example, each data point is a vector
(age, height, shoe size, … )



Their shadow is a projection onto the ground.

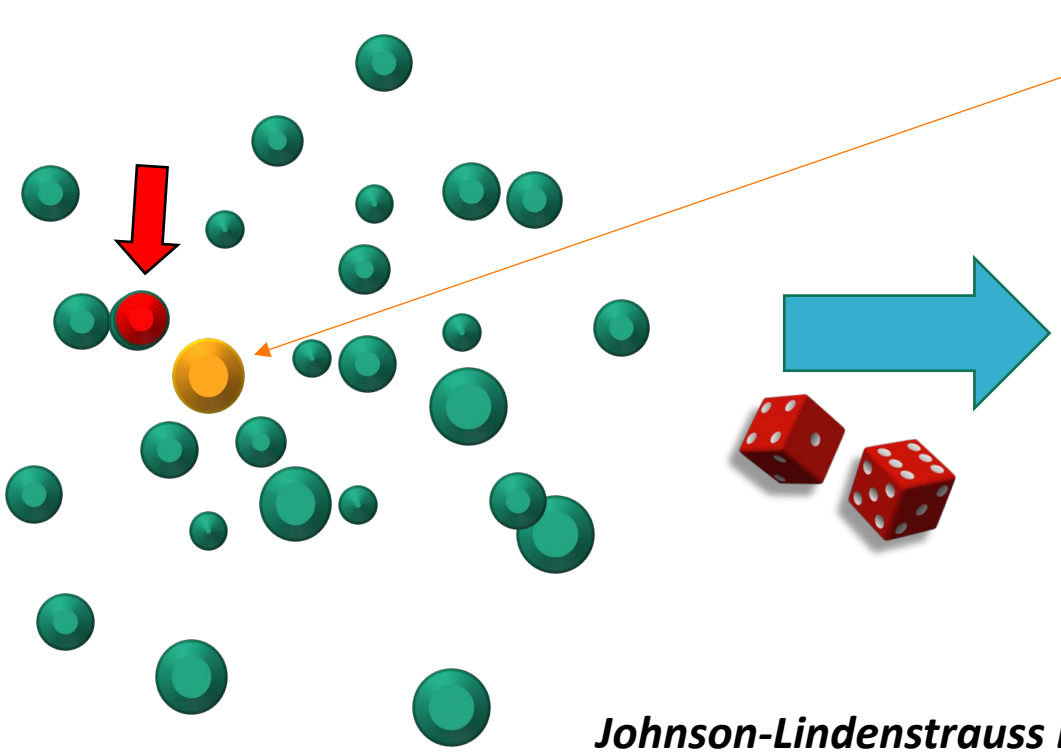


# Why would we do this?

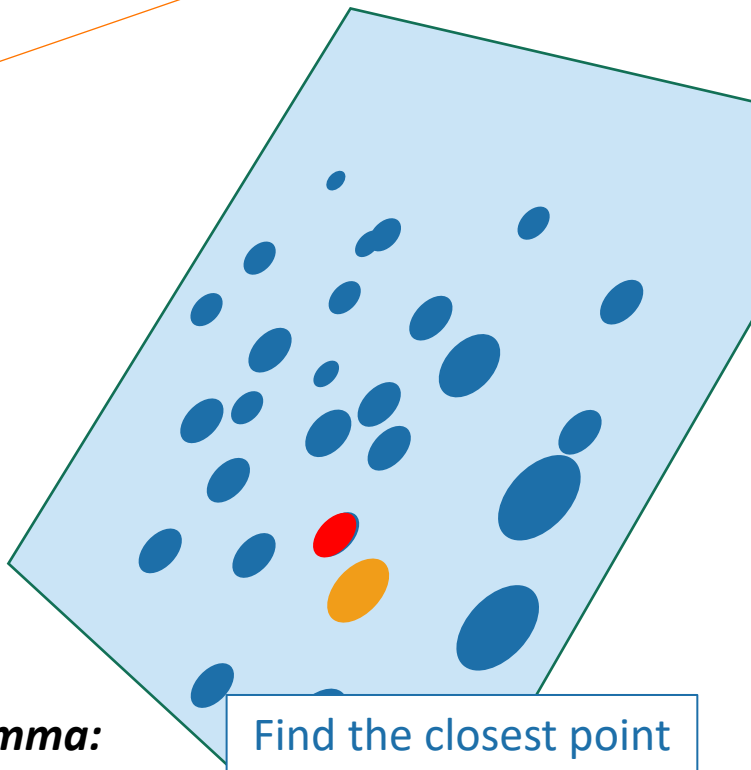
- High dimensional data takes a long time to process.
- Low dimensional data can be processed quickly.
- **“THEOREM”:** Random projections approximately preserve properties of data that you care about.

# Example: nearest neighbors

- I want to find which point is closest to **this one**.







That takes a really long time in high dimensions.

***Johnson-Lindenstrauss Lemma:*** *Euclidean distance is approximately preserved by random projections.*

Find the closest point down here, you're probably pretty correct.

Another example:

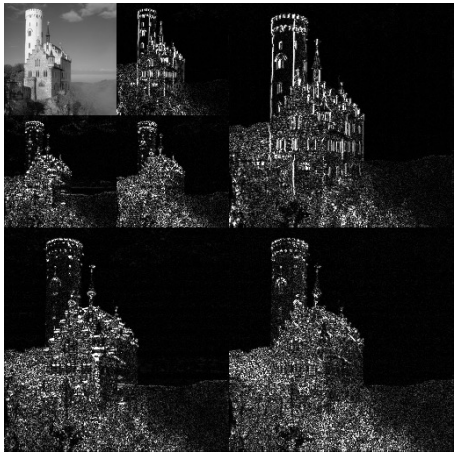
# Compressed Sensing

- Start with a sparse vector
  - Mostly zero or close to zero

(**5** , 0 , 0 , 0 ,0 , 0.01 , 0.01 , **5.8** , **32** , **14** , 0 , 0 , 0 , **12** , 0 , 0 , **5** , 0 , .03)

- For example:

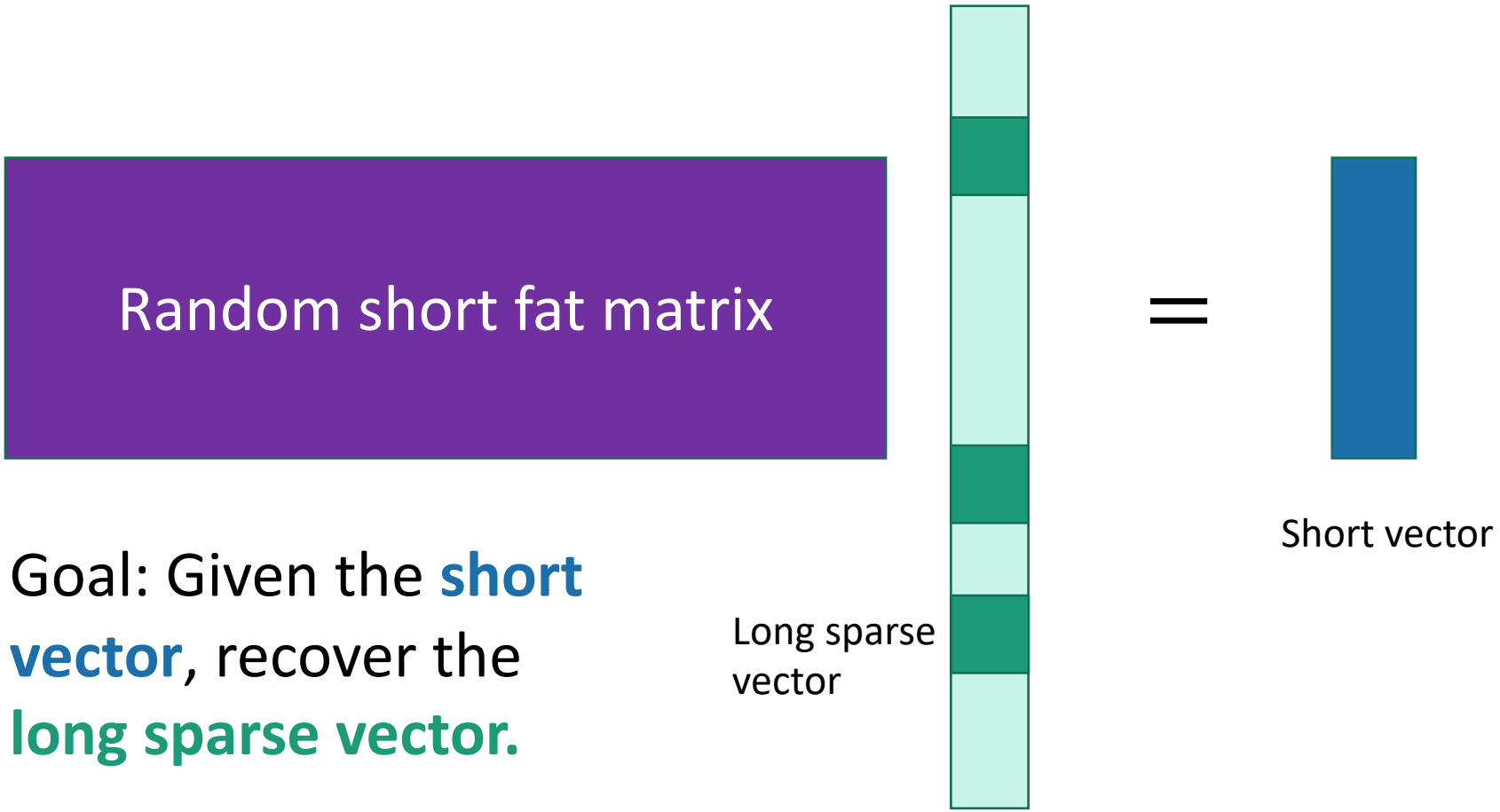
This image is sparse



This image is sparse after I take a wavelet transform.

# Compressed sensing continued

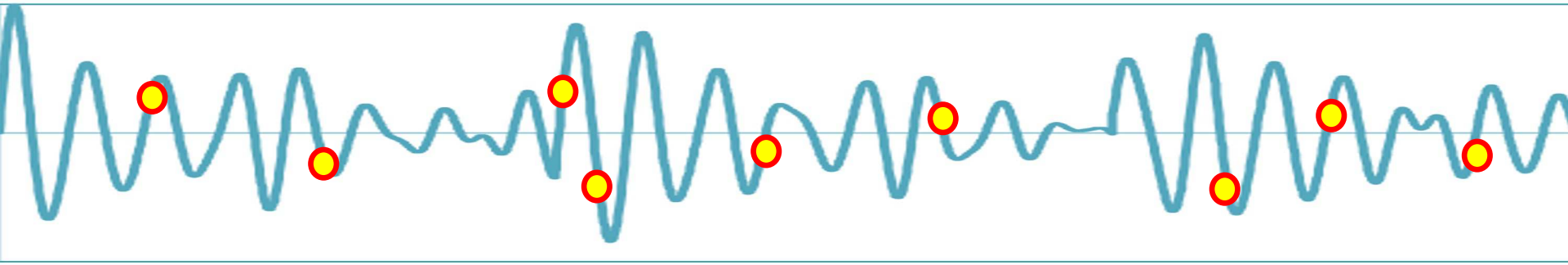
- Take a random projection of that sparse vector:



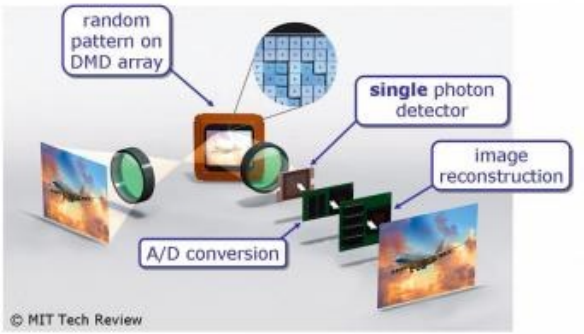

Goal: Given the **short vector**, recover the **long sparse vector.**

# Why would I want to do that?

- Image compression and signal processing
- Especially when you **never have space to store the whole sparse vector to begin with**.

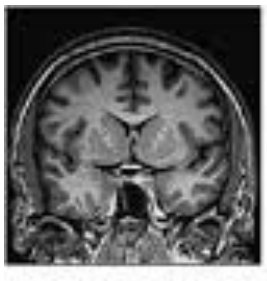


Randomly sampling (in the time domain) a signal that is sparse in the Fourier domain.

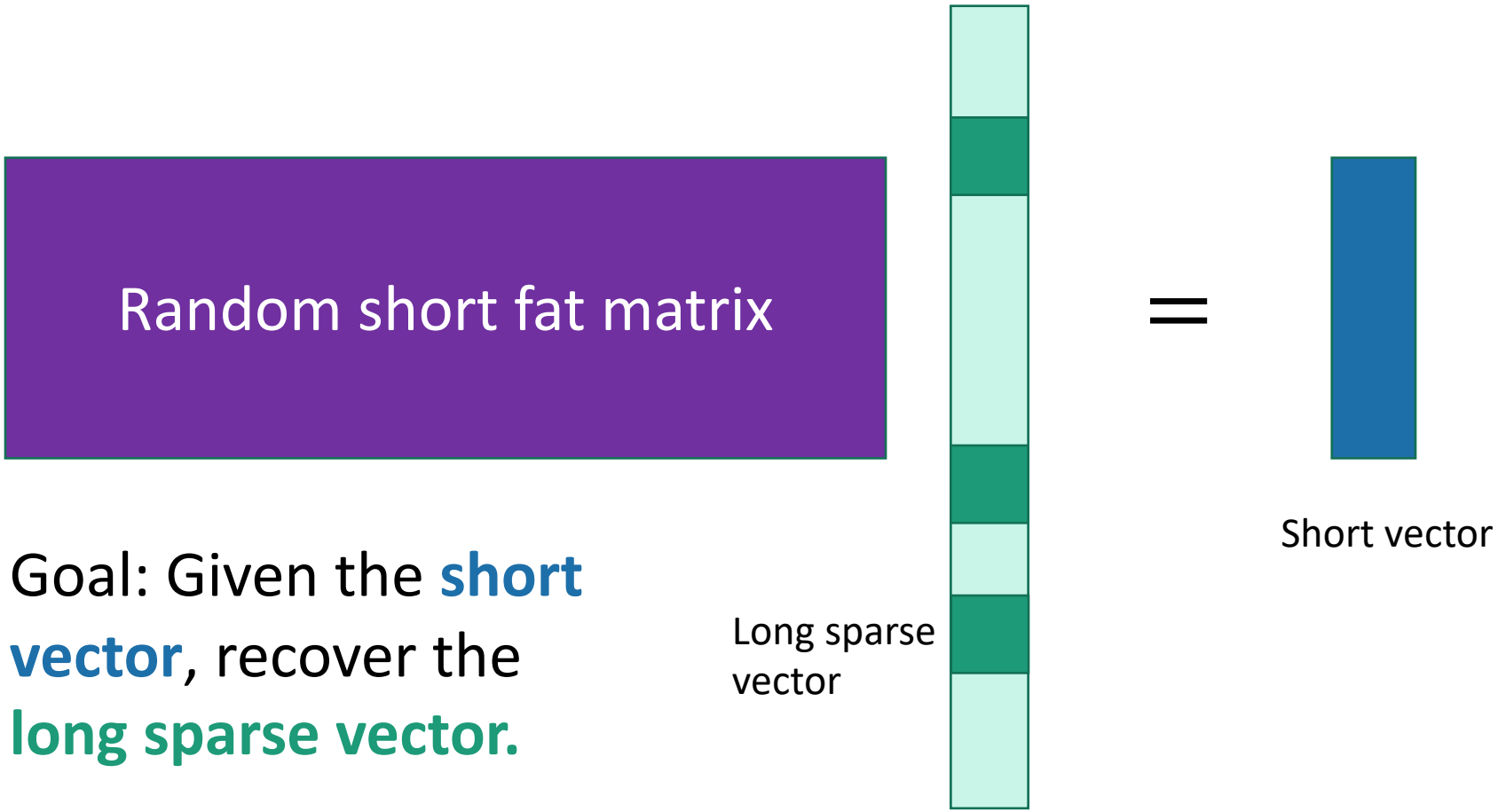



A "single pixel camera" is a thing.

Random measurements in an fMRI means you spend less time inside an fMRI



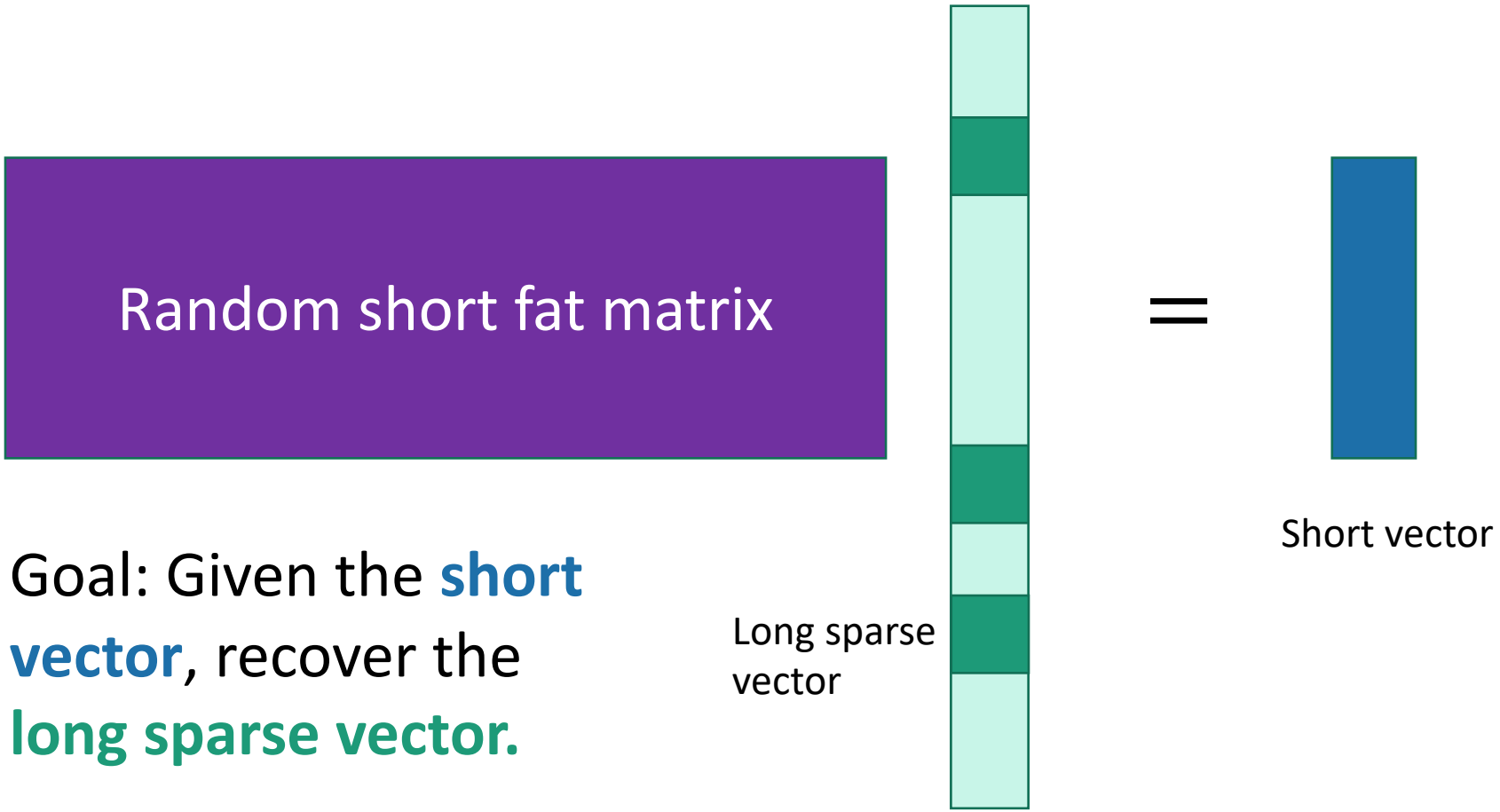
# All examples of this:




Goal: Given the **short vector**, recover the **long sparse vector.**

# But why should this be possible?

- There are tons of long vectors that map to the short vector!

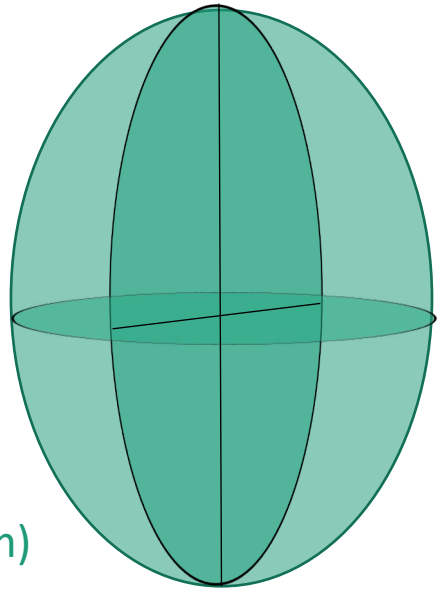


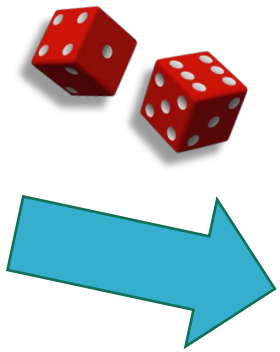

Goal: Given the **short vector**, recover the **long sparse vector.**

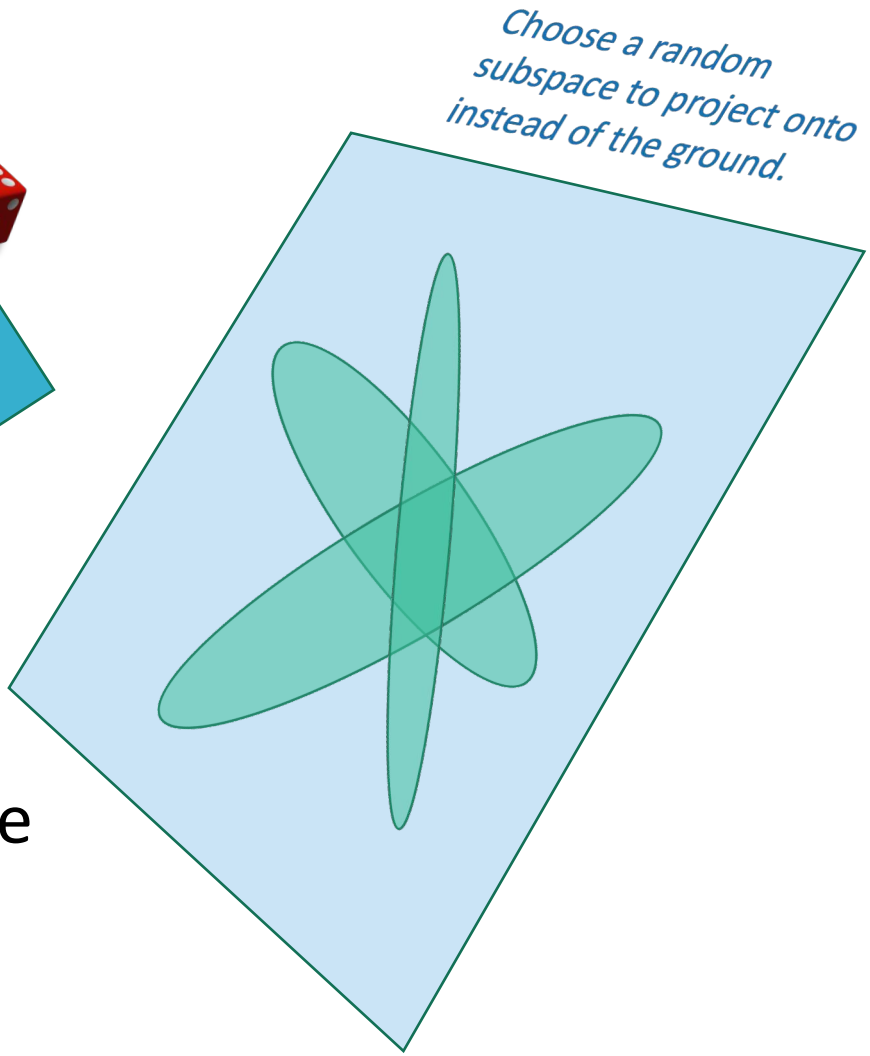
# Back to the geometry

**All of the sparse vectors**

(Infinitely many of them)



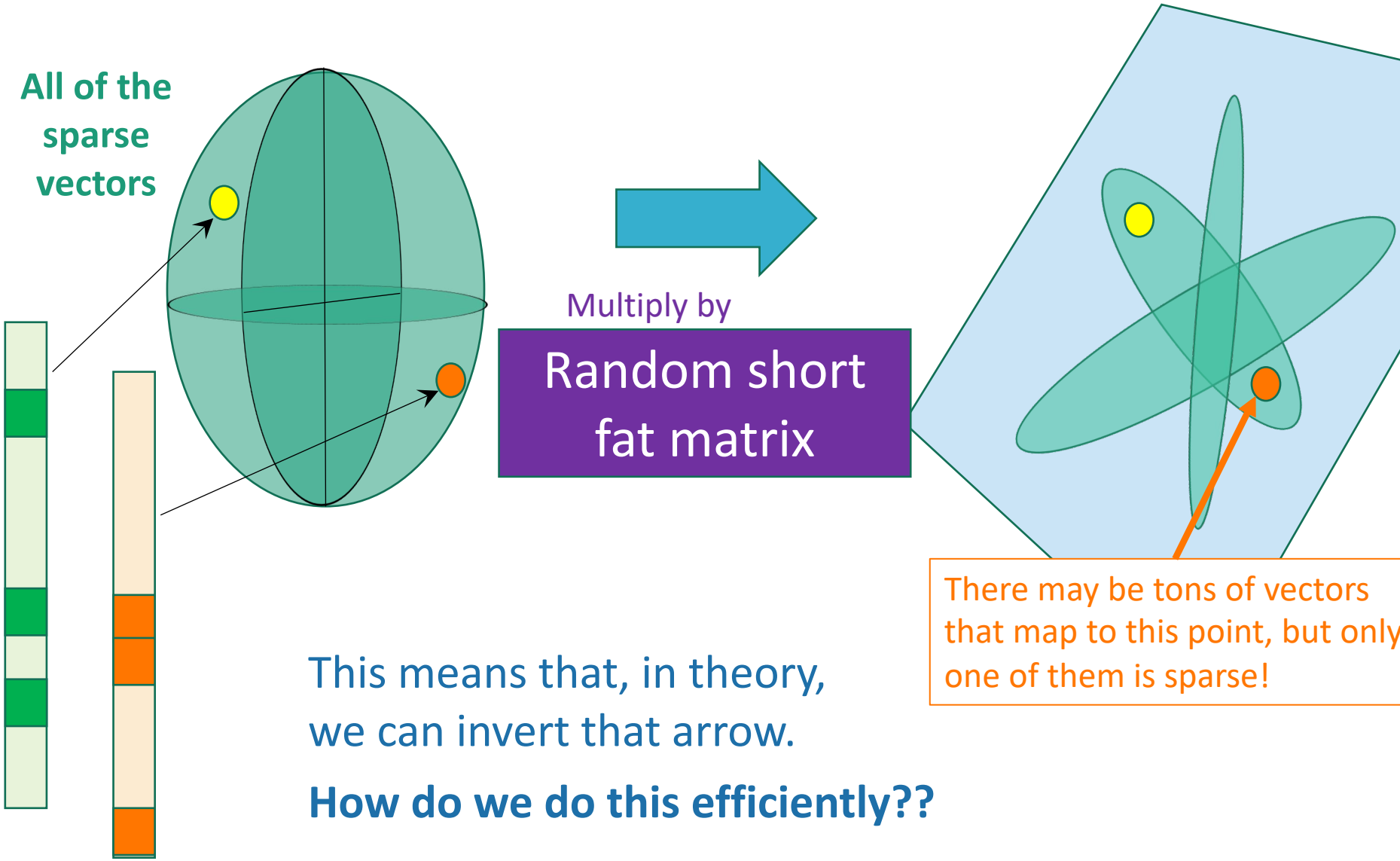





**Theorem:**

random projections preserve the geometry of sparse vectors too.

# We can (in theory) recover the big vector from the small one!

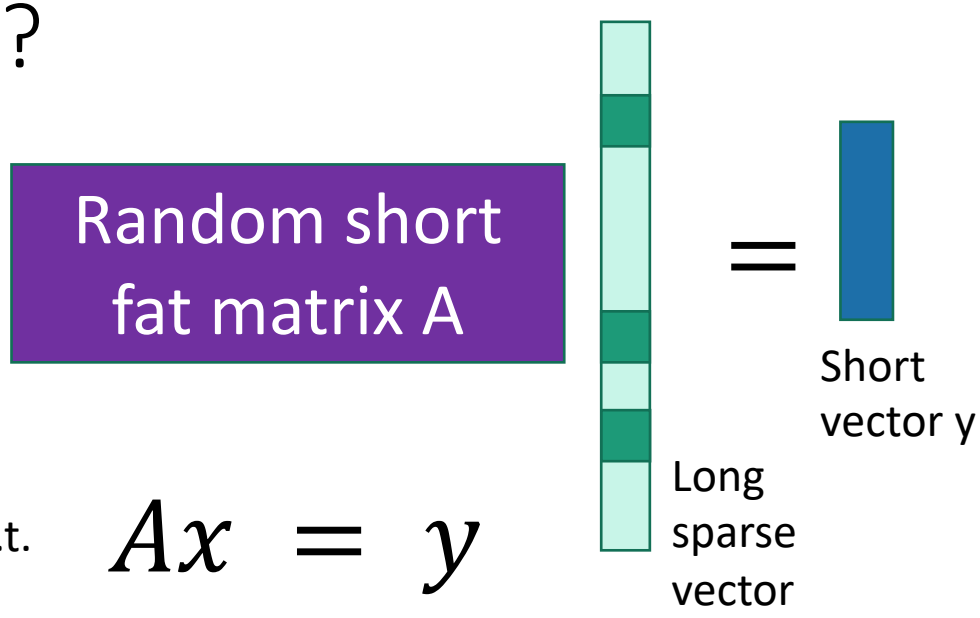



This means that, in theory, we can invert that arrow.

**How do we do this efficiently??**

Goal: Given the **short vector**, recover the **long sparse vector.**

# An efficient algorithm?




What we'd like to do is:

Minimize number of nonzero entries in x

s.t. $Ax = y$

This norm is the sum of the absolute values of the entries of x

This isn't a nice function

**Problem:** I don't know how to do that efficiently!

Instead:

Minimize $\|x\|_1$

s.t. $Ax = y$

- It turns out that because the geometry of sparse vectors is preserved, this optimization problem **gives the same answer**.
- We can use **linear programming** to solve this quickly!

# Today

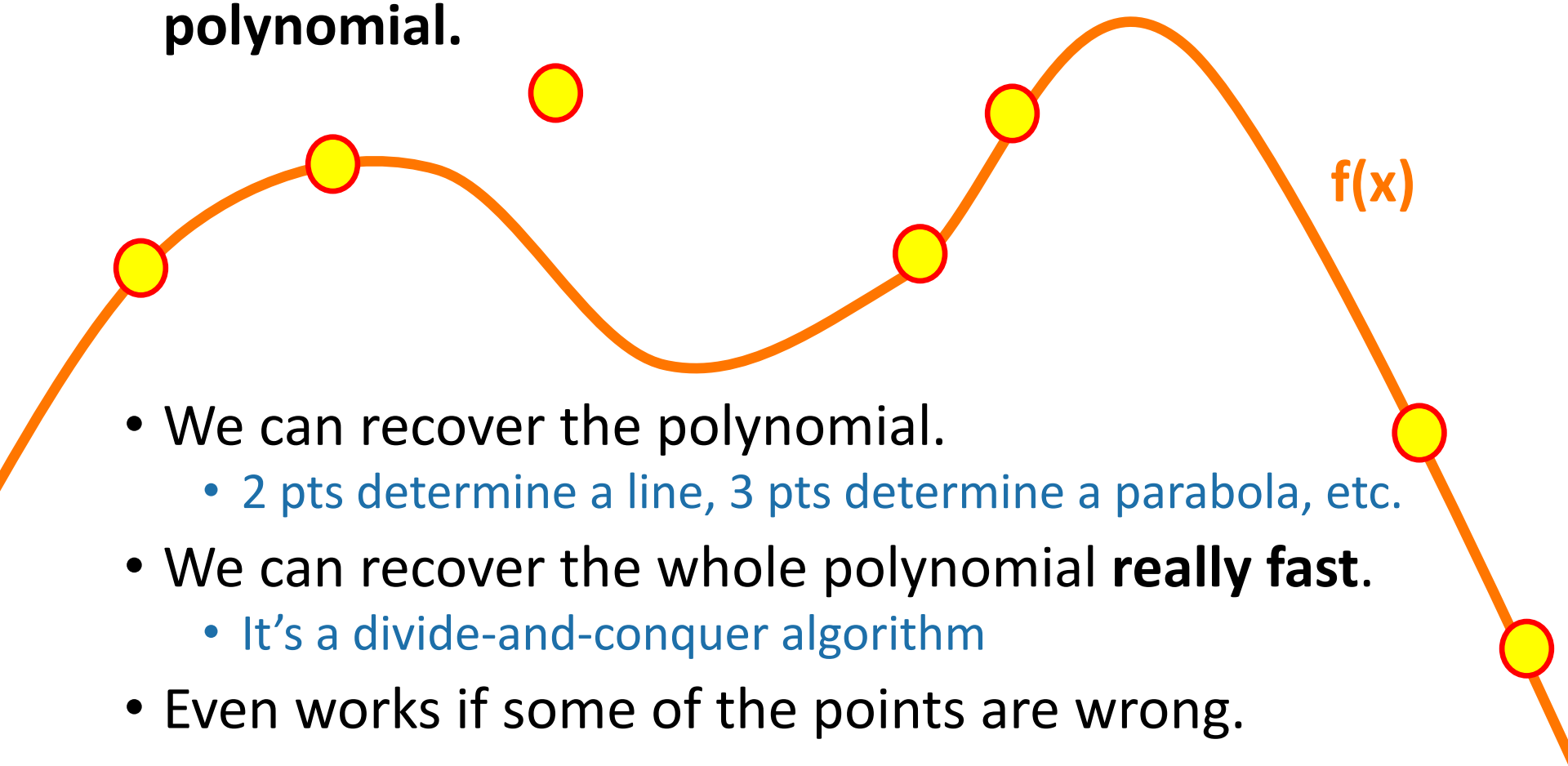
# A few gems

- Linear programming
- Random projections
- Low-degree polynomials

## Another of my favorite tricks

# Polynomial interpolation

- Say we have a few evaluation points of a **low-degree polynomial.**



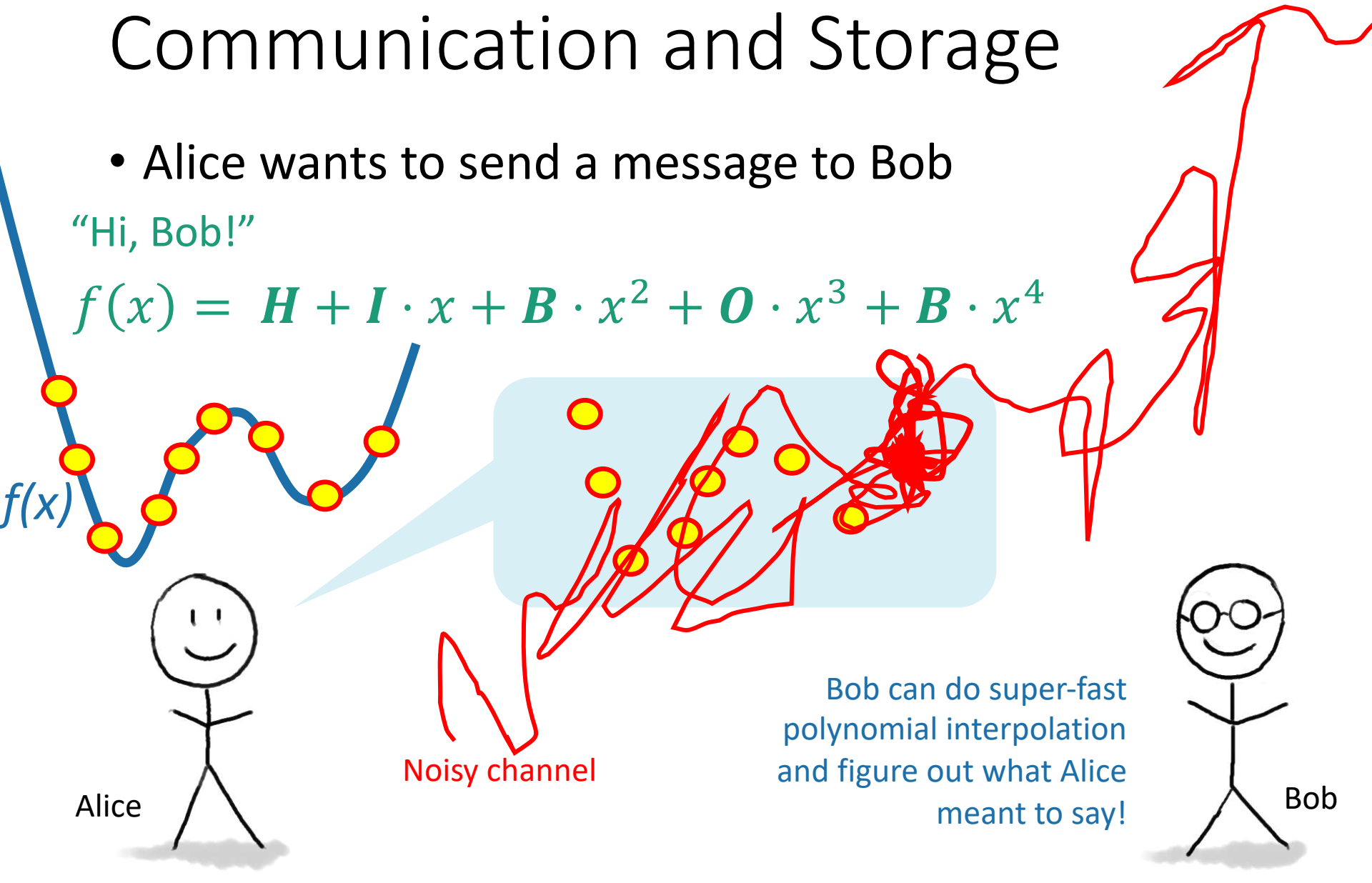

- We can recover the polynomial.
  - 2 pts determine a line, 3 pts determine a parabola, etc.
- We can recover the whole polynomial **really fast**.
  - It's a divide-and-conquer algorithm
- Even works if some of the points are wrong.

# One application:
# Communication and Storage

- Alice wants to send a message to Bob

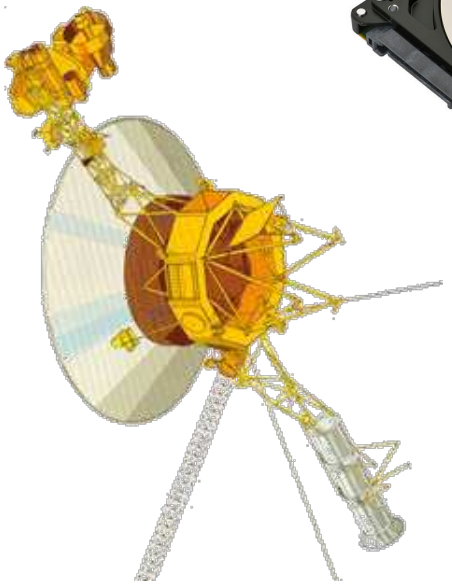
"Hi, Bob!"

$$f(x) = \boldsymbol{H} + \boldsymbol{I} \cdot x + \boldsymbol{B} \cdot x^2 + \boldsymbol{O} \cdot x^3 + \boldsymbol{B} \cdot x^4$$



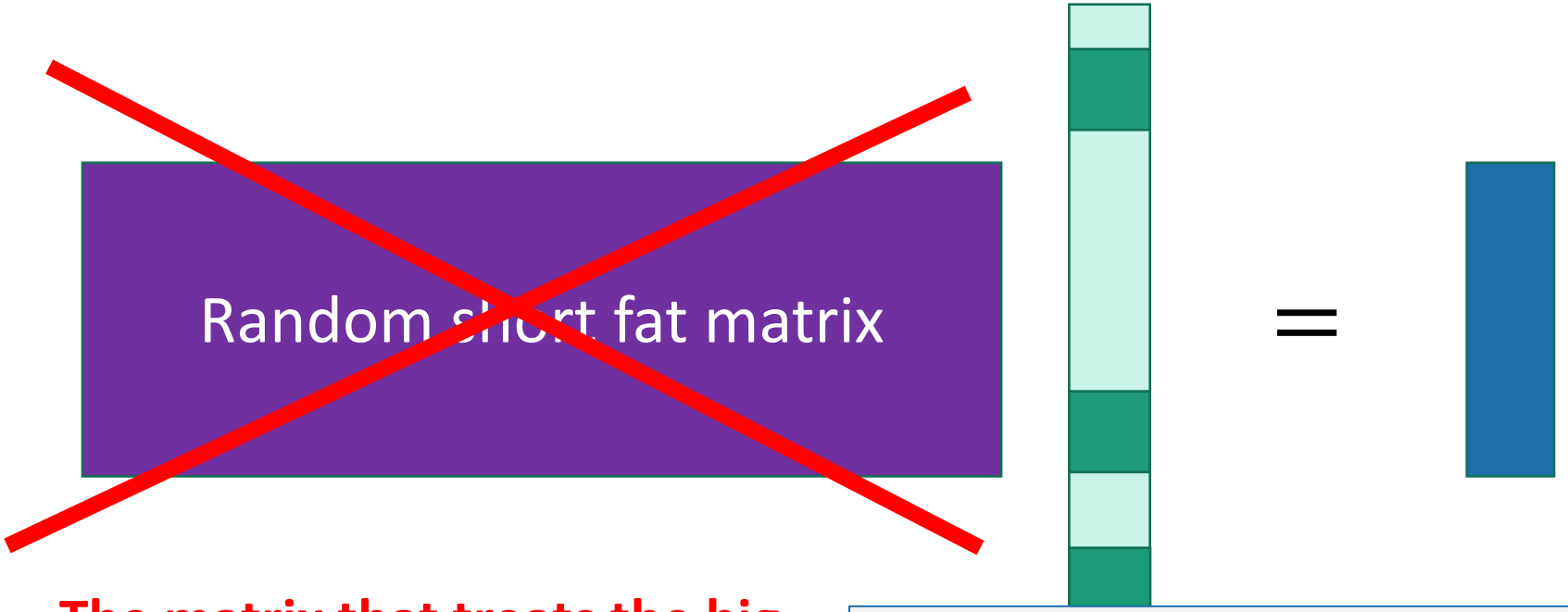
# This is actually used in practice

- It’s called “Reed-Solomon Encoding”



Another application:

# Designing “random” projections that are better than random




**The matrix that treats the big long vector as Alice’s message polynomial and evaluates it REAL FAST at random points.**

- This is still ”random enough” to make the LP solution work.
- It is much more efficient to manipulate and store!

# Today

# A few gems

- Linear programming
- Random projections
- Low-degree polynomials

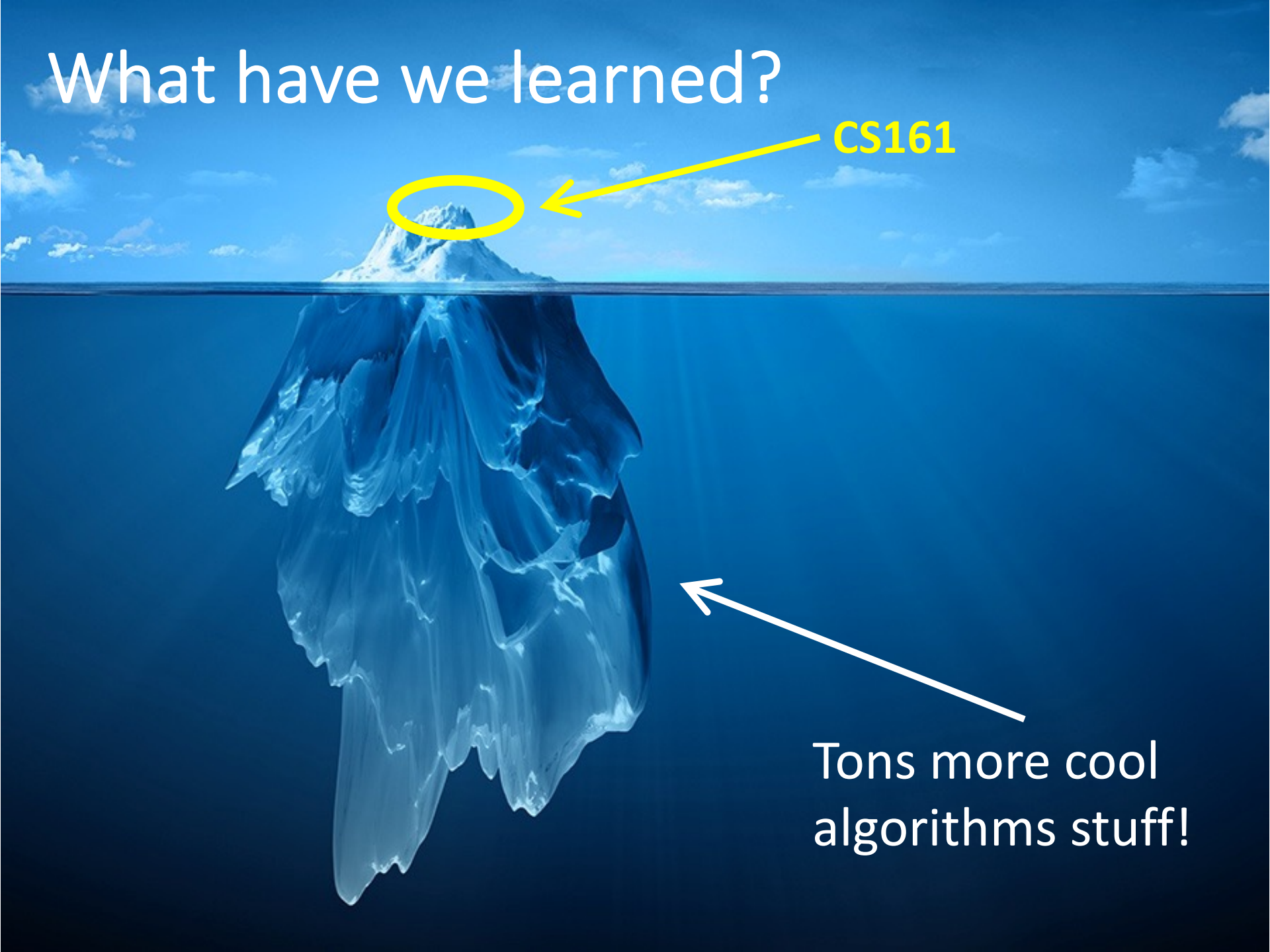
To learn more:

CS168, CS261, …

CS168, CS261, CS265, …

CS168, CS250, …

# What have we learned?



# To see more…

- Take more classes!
- Come hang out with the theory group!
  - Theory lunch, Thursdays at noon
  - Go to theory.stanford.edu and click on “Theory Lunch”

theory.stanford.edu

Stanford theory group:
We are very friendly.

# A few final messages…

# 1. Thanks to the TAs!!!

## tell them you appreciate them!


Amrita


Shubham


Anooshree


Apoorva

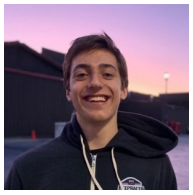

Bharat


Tathagat


Aditya


Jeffrey


Bharath


Callum

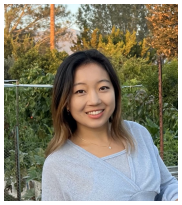

Felipe


Kevin

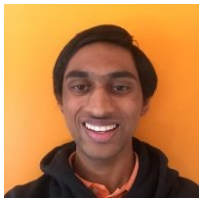

Melinda


WenXin


Sharmila


Michelle Q


Michelle X


Praneeth

Ricky


Rishu


Robert


Wenzheng


Ivan


Jessica


Ruiqi


Saumya


Grant


Sophia

Stephan


Yuzu

2.

# THANKS

## to you!!!!!!